Today:

- ☐ Find (or estimate) the rate of change of a function $f(x, y)$ with respect to time when x, y vary as a function of time t.
- ☐ Find the rate of change of a function $f(x, y)$ with respect to a other variables when the input (x, y) is written as a function of multiple other variables (often (u, v) or (r, θ)).
- ☐ Determine the sign of a second-order partial using a contour plot, graph, table, written description, or other information about the function

For your reference (single variable calculus): Let $z = g(x)$ and $y = f(z)$ with f, g differentiable. According to the *chain rule*,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}.$$

We say y depends *directly* on the independent variable x and the variable z is *intermediate*.

Example (chain rule). The length in micrometers (μm) of steel, L, depends on the air temperature, H °C. The air temperature depends on time, t, measured in hours.

If the length of a steel bridge increases by 0.2 μm for every degree increase in temperature, and the temperature is increasing at 3 °C per hour, how fast is the length of the bridge increasing? What are the units for our answer? *pollQ*

For your reference (section 14.6): Let $z = f(x, y)$, $x = g(t)$, and $y = h(t)$ with f, g, h differentiable. According to the *chain rule*,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

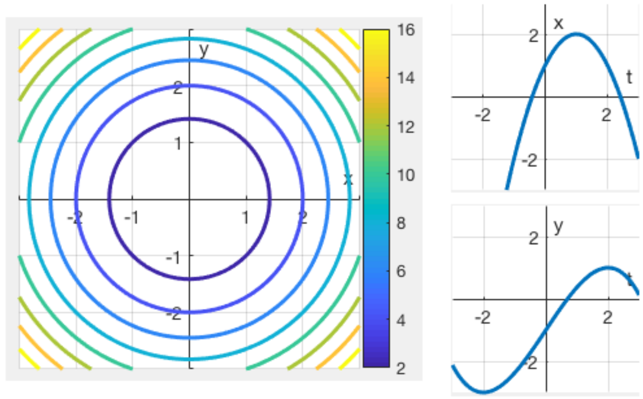
We say z depends *directly* on the independent variable t, while the variables x and y are *intermediate*. Since z is a function of one independent variable we use the notation $\frac{dz}{dt}$.

Let $z = f(x, y)$, $x = g(u, v)$, and $y = h(u, v)$ with f, g, h differentiable. According to the *chain rule*,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

We say z depends *directly* on the independent variables u and v, while the variables x and y are *intermediate*. Since z is a function of two independent variables we use the notation $\frac{\partial z}{\partial u}$.

Example (chain rule). The figure on the left shows contours of $z = f(x, y)$. The figures on the right show x and y as functions of t. Determine the sign of $\left.\frac{dz}{dt}\right|_{t=2}$. *pollQ*



Example (matlab code). In line 14 of the following code, a value is calculated. Is it $\frac{dz}{dt}$ for some $z = f(x, y)$, $x = g(t), y = h(t)$? If it is, what are the functions involved and how do you know? If it isn't, what is being calculated by this code? *pollQ*

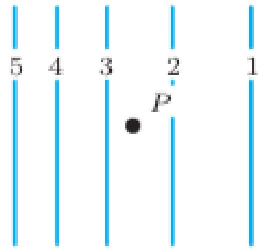
```
1  expr = x*y + 2*sin(y);
2  tval = 3;
3  xoft = t^2;
4  yoft = 2-t;
5  pt = [subs(xoft,t,tval),subs(yoft,t,tval)];
6  fx = diff(expr,x);
7  fxpt = subs(fx,[x,y],pt);
8  fy = diff(expr,y);
9  fypt = subs(fy,[x,y],pt);
10 xtpt = subs(diff(xoft,t),t,tval);
11 ytpt = subs(diff(yoft,t),t,tval);
12 vec = [xtpt,ytpt];
13 uvec = vec/norm(vec);
14 value = fxpt*uvec(1) + fypt*uvec(2)
```

Example (chain rule). Let $z = f(x, y) = \ln(xy)$ with $x = u^2 + v^2$ and $y = u^3 v$. There are two independent variables (u and v) and two intermediate variables (x and y). Find $\frac{\partial z}{\partial u}$. Evaluate it at $u = 2, v = 1$.

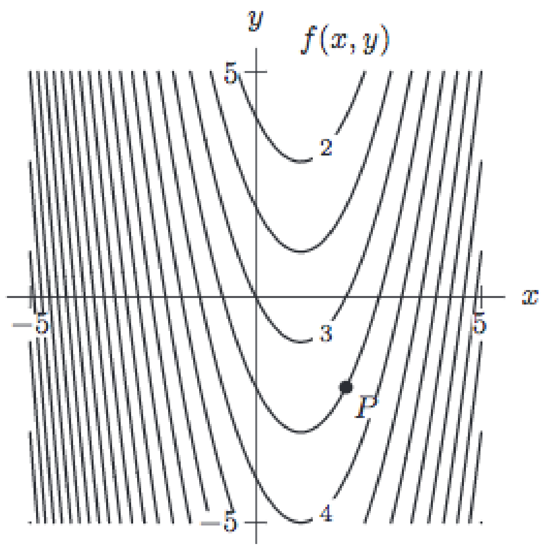
For your reference (section 14.7): The function $z = f(x, y)$ has four *second-order partial derivatives*: $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$. These are also denoted $f_{xx}, f_{xy}, f_{yx}, f_{yy}$. You can think of them as $(f_x)_x, (f_x)_y, (f_y)_x, (f_y)_y$, the first-order partial derivatives of the first-order partial derivatives of $f(x, y)$.

If f_{xy} and f_{yx} are continuous at (a, b), and (a, b) is not on the boundary of the domain, then $f_{xy}(a, b) = f_{yx}(a, b)$.

Example (sign of partials). Identify the signs of $f_x(P)$, $f_y(P)$, $f_{xx}(P)$, $f_{yy}(P)$, $f_{xy}(P)$, $f_{yx}(P)$. *pollQ*



To think more about this, identify the signs of $f_x(P)$, $f_y(P)$, $f_{xx}(P)$, $f_{yy}(P)$, $f_{xy}(P)$, $f_{yx}(P)$ for the diagram below.



Example (second-order partial). The $x = 0$ cross-section of $f(x, y)$ is given by $f(0, y) = y^3$. Is this enough information to determine $f_{yy}(0, 2)$? *pollQ*

After class: Read sections 14.6 and 14.7

Text section	Initial practice (exercises)	PSet practice (problems)
14.6: The chain rule	3, 11, 15	17, 23, 33, 47
14.7: Second-order partial derivatives	5, 19, 21, 23, 31	33, 39, 41, 51, 55

For next time: Read section 15.1

Learning Objectives: Class 13 + pset 5

These objectives are associated with Class 13 + Problem Set 5 questions + Office hours. There is no workshop or section the week of October 8th because of the Monday holiday. There is no quiz associated with this material because of the exam on October 16th.

Students will be asked to

- chain rule

  - state the chain rule and distinguish between the notation $\frac{dz}{dt}$ and $\frac{\partial z}{\partial t}$.
  - describe the relationship of the chain rule to the directional derivative
  - use the chain rule to find (or estimate) the rate of change of a function $f(x, y)$ with respect to time when x, y vary as a function of time t.
    §14.6: 1-5, 16, 23
  - use the chain rule to find the rate of change of a function $f(x, y)$ with respect to a different variable the input (x, y) is a function of multiple other variables (often (u, v)).
    §14.6: 7-15, 17, 20, 25, 34
  - use direct substitution of variables to check a chain rule calculation.
    §14.6: 16
  - identify the meaning of the terms (from the chain rule) that make up the rate of change.
    §14.6: 18
  - translate from a description of a system to a rate of change calculation.
    §14.6: 21-22
  - show a relationship between partial derivatives of a function using the chain rule.
    §14.6: 31-34, 38, 39-44
  - identify when to apply the chain rule in the context of a problem
- second order partials
  - define the second-order partials of $f(x, y)$ and use the notation $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial y^2}$, $f_{xx}, f_{xy}, f_{yx}, f_{yy}$.
  - compute second-order partial derivatives for a function $f(x, y)$
    §14.7: 1-11
  - determine the sign of a second-order partial using a contour plot, graph, table, written description, or other information about the function
    §14.7: 22-31, 41-51, 55-56
  - show a function satisfies a relationship between its partial derivatives
    §14.7: 37-40
- fundamental theorem
  - apply the fundamental theorem of calculus to differentiate a function that involves an integral or to integrate a derivative
    §14.6: 45-47, §14.7: 59-60