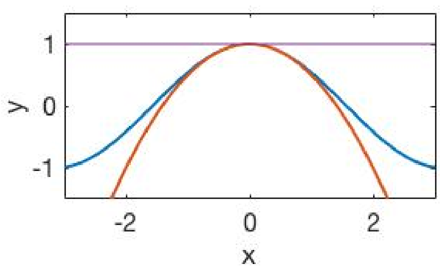
Today:

- ☐ Approximation: Determine the sign of first or second order partials based on a graph, contour plot, or written description/
- ☐ Optimization: Identify and classify critical points of a function of two variables.

Example (soup tastiness). I want to model the tastiness of soup, as I try to get the saltiness right. Assume $T = f(V, s)$ where V is volume and s is grams of salt. At a point P where the soup tastes good, if I add more salt it should taste worse. If I add a little volume (without adding salt) it should also taste worse. If I think add a little salt, that should improve the taste. Given this reasoning, what is the sign of $\dfrac{\partial^2 T}{\partial s \partial V}$?

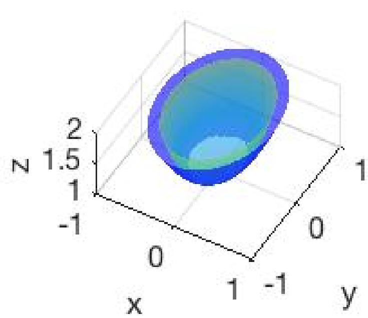
> **For your reference (single variable calculus):** The *second-order Taylor approximation* of $f(x)$ about the point (a) is $f(a) + f_x(a)(x-a) + \frac{1}{2}f_{xx}(a)(x-a)^2$. We can use this Taylor series to construct a quadratic approximation to $f(x, y)$ at (a, b).

Example (approximate $\cos(x)$). Find the first-order and second-order Taylor series of $f(x) = \cos(x)$ about $x = 0$. Use these to write an approximation for the function near $x = 0$.



> **For your reference (section 14.7):** The *second-order Taylor series* of $f(x, y)$ about the point (a, b) is $f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) + \frac{1}{2}f_{xx}(x-a)^2 + f_{xy}(x-a)(y-b) + \frac{1}{2}f_{yy}(y-b)^2$. We can use this Taylor series to construct a quadratic approximation to $f(x, y)$ at (a, b). *pollQ*

Example (approximate $e^{2x^2+y^2}$). Find the first-order and second-order Taylor series of $f(x, y) = e^{2x^2+y^2}$ about $(0, 0)$. Use these to write an approximation for the function near $(0, 0)$.



Example (second-partials). In Figure 14.62, use the graph to identify the signs of $f_{xx}(0,0)$ and of $f_{yy}(0,0)$. *pollQ*

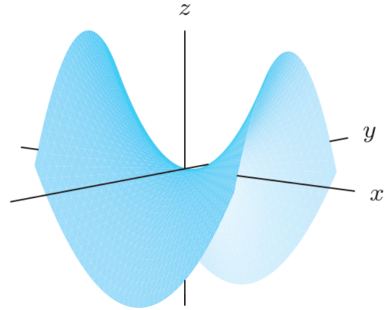


Figure 14.62

For your reference (section 15.1): A *local maximum* is a point P_0 where $f(P_0) \geq f(P)$ for all points P in a neighborhood of P_0. A *local minimum* is a point P_0 where $f(P_0) \leq f(P)$ for all points P in a neighborhood of P_0.

A *neighborhood* is made up of all points in the domain that are within a distance δ of P_0. To be a local extremum, there has to be some distance so that all points within that distance of P_0 satisfy the inequalities above.

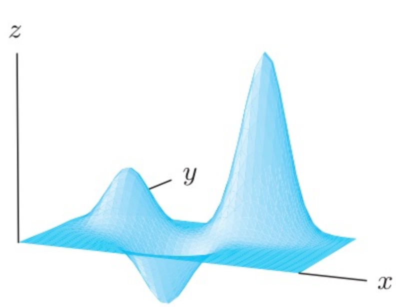
A *global maximum* is a point P_0 where $f(P_0) \geq f(P)$ for all points P in the domain. A *global minimum* is a point P_0 where $f(P_0) \leq f(P)$ for all points P in the domain. We will look more at global extrema next week.

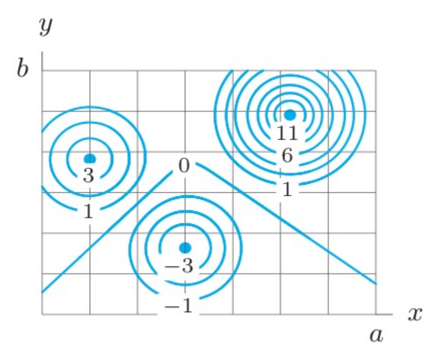
If $f_x(P) = 0$ or is undefined and $f_y(P) = 0$ or is undefined then we call the point P a *critical point* of a function f.

We'll call a critical point where $f(P_0) < f(P)$ at some points and $f(P_0) > f(P)$ at other points in any neighborhood of P_0 a *saddle point*.

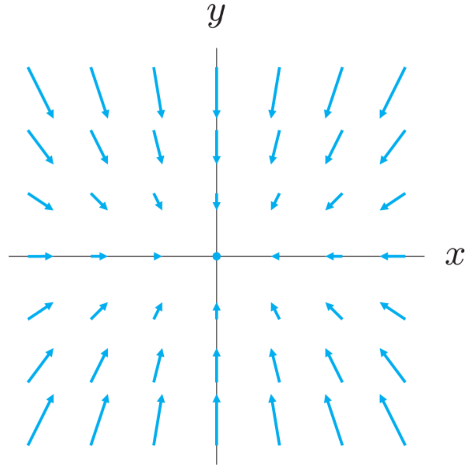
At a point P where $\vec{\nabla} f(P) = \vec{0}$, the *discriminant* $D = f_{xx}(P)f_{yy}(P) - (f_{xy}(P))^2$ tells us whether the quadratic approximation is more like a paraboloid ($D > 0$) or is saddle-shaped ($D < 0$).

Example (local extrema). Identify local minima and local maxima on the contour plot below. What can you say about the first-order Taylor series of the function about these points? What about the second-order Taylor series?





Example (find and classify a critical point). The plot below shows the $\vec{\nabla} f$ for a function $f(x, y)$. Identify the location of the critical point. What kind of critical point is it?



Example (finding critical points). Let $f(x, y) = x^3 + y^3 - 6y^2 - 3x + 9$. Set up a system of equations for finding the critical points of the system.

Example (classifying a critical point 1). Given a function $f(x, y)$ such that $(0, 0)$ is a critical point, could $z = 7 + 2x + x^2 + xy + 2y^2$ be the quadratic approximation to the function about $(0, 0)$? Justify your answer.

Example (classifying a critical point 2). Given a function $f(x, y)$ such that $(1, 5)$ is a critical point and $z = 3 + a(x-1)^2 + b(y-5)^2$ where $a < 0$ and $b > 0$ is the quadratic approximation to $f(x, y)$ at $(1, 5)$, try to classify the critical point as a local minimum, a local maximum, or a saddle point.

Text section	Initial practice (exercises)	PSet practice (problems)
14.7: Second-order partial derivatives	5, 19, 21, 23, 31	33, 39, 41, 51, 55
15.1: Critical points: local extrema and saddle points	1, 5, 15	21, 25, 33, 37, 40

Learning Objectives

These objectives are associated with Class 14 + Problem Set 5 questions + Office hours. There is no workshop or section the week of October 8th because of the Monday holiday. There is no quiz associated with this material because of the exam on October 16th.

Students will be asked to

- second order partials
  - calculate second order partials for a function of multiple variables
  - determine the sign of first or second order partials based on level curves of a function
  - determine the sign of first or second order partials based on a graph of a function
  - use the theorem on the equality of mixed partials to simplify calculations
  - define a smooth function
- Taylor approximation
  - describe the relationship between first- or second-order Taylor approximations of a function at a point and the function itself
  - compute Taylor polynomials of first-order or second-order
  - compare the quality of the approximation between a zeroth-order, first-order, and second-order Taylor approximation
- critical points
  - define: critical point, local maximum, local minimum, saddle point, discriminant
  - identify the critical points of a function of 2-variables from a contour plot, graph, or symbolic formula
  - classify critical points using the discriminant and the second-derivative test as local maxima, local minima, saddle points, or not classified by the test
  - identify when the second-derivative test can be applied and explain why the test works
  - classify a critical point that is not classified by the second-derivative test by reasoning about the graph of the function near the critical point
  - reason about the shape of a graph using partial derivative information at critical points, or using information about the classifications of the critical points