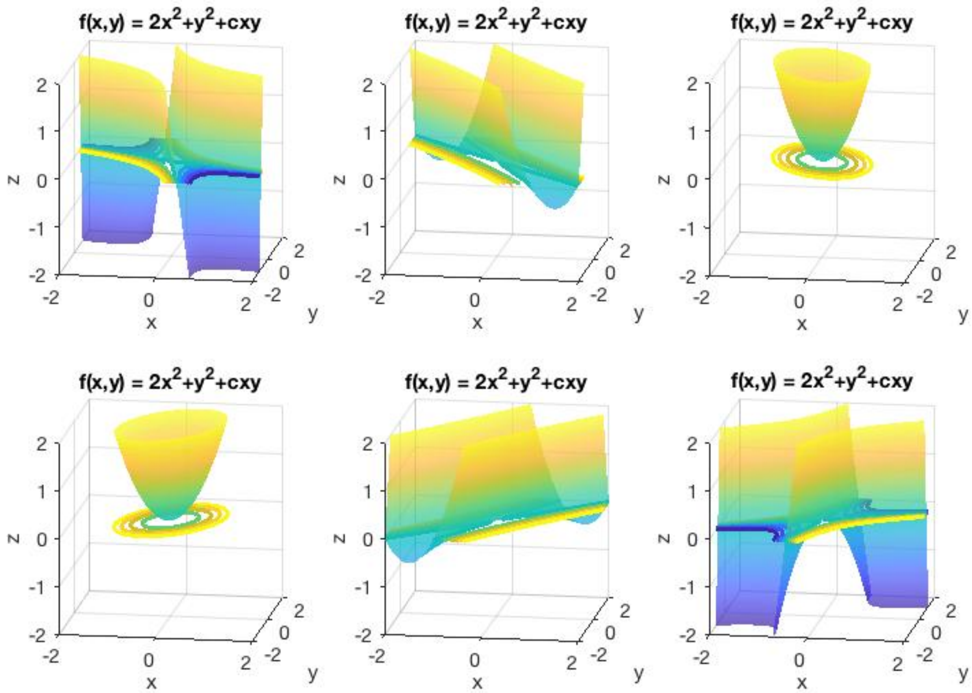
Today:

- ☐ Local extrema: Relate the shape of a quadratic function to the value of the discriminant, $f_{xx}f_{yy} - (f_{xy}^2)$
- ☐ Global extrema: Determine whether global extrema exist by using the extreme value theorem or providing geometric reasoning about the graph of the function.
- ☐ Global extrema: Find global extrema on a domain.

Example (quadratic forms). Below are plots of the graph of the function $f(x, y) = 2x^2 + y^2 + cxy$ for six different values of c. Is c increasing as we move from left to right across the top row and then from left to right across the bottom row? Or is c decreasing? *pollQ*



> **For your reference (section 15.1):** A *quadratic form* is a polynomial where every term is degree 2. (Example: $x^2 + 2xy$ is a quadratic form).
>
> For a function $f(x, y) = ax^2 + by^2 + cxy$ (or for $f(x, y) = f(x_0, y_0) + a(x - x_0)^2 + b(y - y_0)^2 + c(x - x_0)(y - y_0)$),
>
> - If a and b have different signs then the shape will open one way when $x = 0$ and the other way when $y = 0$. The discriminant $4ab - c^2 < 0$ and $(0, 0)$ is a saddle-point.
> - If a and b have the same sign, and ab is large enough relative to c (specifically, the discriminant $4ab - c^2 > 0$), then the shape will be a bowl. It opens upward for $a, b > 0$ and downward for $a, b < 0$. (See two plots above).
> - If a and b have the same sign, but ab is not large enough relative to c (the discriminant $4ab - c^2 < 0$), then the shape will be a saddle. (See four plots above).
>
> For a quadratic approximation at a critical point, where $f(x, y) = f(x_0, y_0) + \frac{f_{xx}}{2}(x - x_0)^2 + \frac{f_{yy}}{2}(y - y_0)^2 + f_{xy}(x - x_0)(y - y_0)$, the discriminant, D, is given by $D = f_{xx}f_{yy} - (f_{xy})^2$.

For your reference (single variable calculus): Suppose P is a point in the domain of f.

- f has a *global minimum* at P if $f(P)$ is less than or equal to all values of f for points in the domain.
- f has a *global maximum* at P if $f(P)$ is greater than or equal to all values of f for points in the domain.

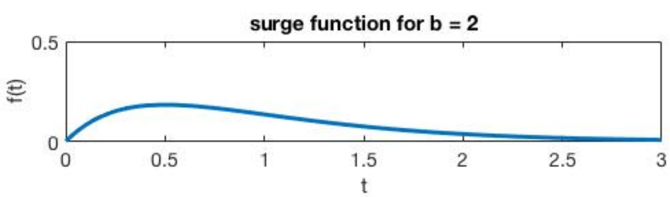
The *extreme value theorem* states that if f is continuous on the closed interval $a \leq x \leq b$ (or on any closed and bounded set), then f has a global maximum and a global minimum on that interval.

An interval is *closed* if it contains its endpoints. $a \leq x \leq b$ is a closed interval. $a < x \leq b$ is not a closed interval: the endpoint of a is not included in the interval.

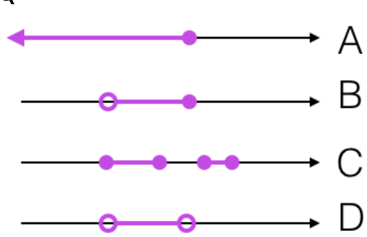
A set is *bounded* if there is some real number less than every value in the set and another real number greater than every value in the set. So there exists N and M such that for x in the set, $N < x < M$.

To find global extrema on a closed interval $a \leq x \leq b$, find the critical points of f in the interval. Evaluate the function at the critical points and at the endpoints. The largest value is the global maximum and the smallest the global minimum.

Example (global extrema). A surge function $f(t) = te^{-bt}$ is one model for the quantity of a drug in the body after time $t \geq 0$. Find the global extrema of $f(t)$ for $t \geq 0$.



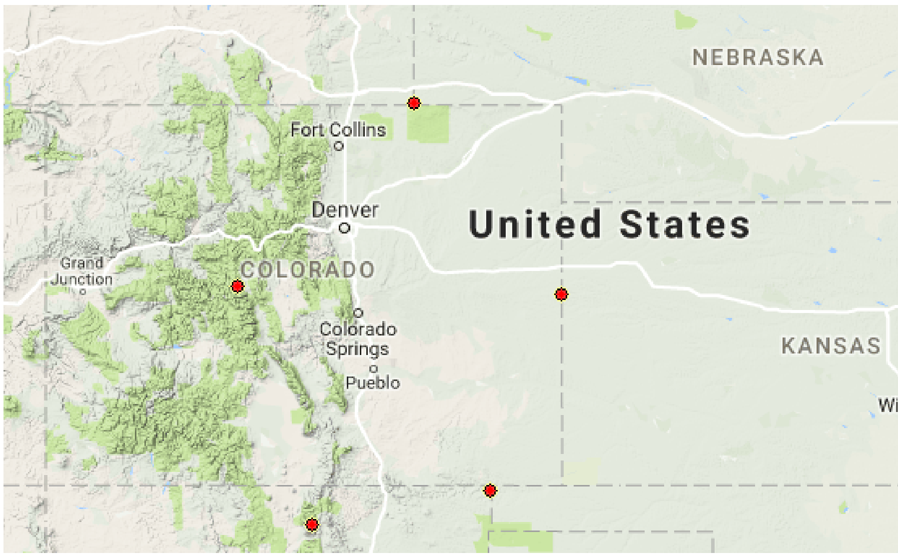
Example (closed and bounded sets). Which of the following sets is both closed and bounded? *pollQ*



Example (extreme value theorem). Does the extreme value theorem apply for the surge-function problem above? *pollQ*

Example (global minimum). Let $f(x) = \frac{2}{x} + x^2$ on the region, C, where $x > 0$. Does $f(x)$ have a global maximum on C? What about a global minimum? *pollQ*

Example (highest points). Notice that the highest point of Colorado is on the peak of a mountain within the state. For Kansas, the highest point is on the border with Colorado (having a high point on the border happens for Iowa, Nebraska, Oklahoma, and a number of other states).



For your reference (section 15.2):

- f has a *global minimum* on R at P_0 if $f(P_0)$ is less than or equal to $f(P)$ for all points P in R.
- f has a *global maximum* on R at P_0 if $f(P_0)$ is greater than or equal to $f(P)$ for all points P in R.

Example (minimum distance). What is the shortest distance from the plane $3x + 2y + z = 1$ to the origin?

To find this distance:

- We will find point(s) on the surface that minimize the square of the distance, $g(x, y, z) = x^2 + y^2 + z^2$.
- Substitute to eliminate one of the input variables to $g(x, y, z)$ so that the square of the distance is a function of two variables, f.
- Find critical points of f.
- Find the points (x, y, z) on the surface that correspond to the critical points of f.
- Determine the distance from those points to the origin.

The *extreme value theorem* states that if f is a continuous function on the closed and bounded region R, then f has a global maximum at some point (x_0, y_0) in R and a global minimum at some point (x_1, y_1) in R.

A *closed* region is one which contains its boundary.

The *boundary* of a region R, denoted ∂R, is the set of boundary points of R.

A point (x_0, y_0) in a region R is a *boundary point* if for every $r > 0$ the disk $(x - x_0)^2 + (y - y_0)^2 < r^2$ contains both points that are in R and points not in R. (No matter how small you make the disk, if it is centered around a boundary point, it will always contain a mix of points inside and points outside the region).

A *bounded* region is one which does not stretch to infinity in any direction.

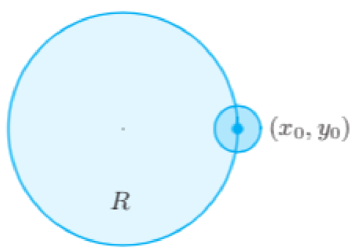


Figure 15.23: Boundary point (x_0, y_0) of R

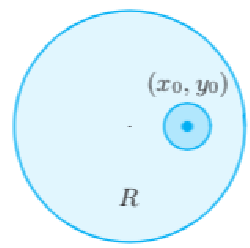
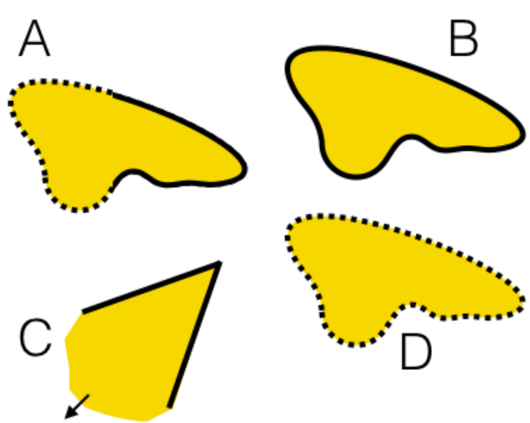


Figure 15.24: Interior point (x_0, y_0) of R

Example (closed and bounded sets). Which of the following sets is both closed and bounded? *pollQ*

Note: later in the semester, identifying the boundary of a region will be an important step in setting up integrals.



Example (global min T/F). If P_0 is a global minimum of f on a closed and bounded region, then P_0 need not be a critical point of f. *pollQ*

Text section	Initial practice (exercises)	PSet practice (problems)
15.2: Optimization: local and global extrema	3, 7, 11	13, 17, 25, 31

Learning Objectives

These objectives are associated with Class 15 + Problem Set 6 questions + Quiz 4 + Office hours. There is no workshop, or section associated with this material because of the exam on October 16th.

Students will be asked to

- Define global minimum and global maximum for a function over a domain. Define the term objective function.
- Estimate or calculate the position and values of global maxima and minima given a contour plot or an equation and a domain. §15.2 1-7, 21
- Determine whether global minima and global maxima exist either via the extreme value theorem or by reasoning about the shape of the functions.
  §15.2 8-12, 29-30
- Show that a local extremum is (or is not) a global extremum.
  §15.2 13, 26
- Translate from a description of an optimization problem to mathematical equations.
  §15.2 14-20, 22-25, 31-32
- Explain the meaning of coefficients of an objective function in the context of an application.
  §15.2 24

Mindsets	
Fixed mindset - *you believe you are born with a certain potential capacity*	Growth mindset - *you believe capacity is acquired through hard work*
Goals	
Performance goal - *works to look good in comparison to others*	Mastery goal - *works to learn / master the material or skill*
Learning Behaviors	
Avoids challenges - *prioritizes working in areas of high competence*	Rises to challenges - *prioritizes areas of new knowledge*
Quits in response to failure - *expends less effort*	Tries harder in response to failure - *puts forth more effort*
Pursues opportunities to bolster self esteem - *seeks affirming social comparisons*	Pursues opportunities to learn more - *seeks more problem-solving strategies*

Modified from Table 6-1 of the National Academies Press 2018: "How People Learn II. Learners, Contexts, and Cultures"