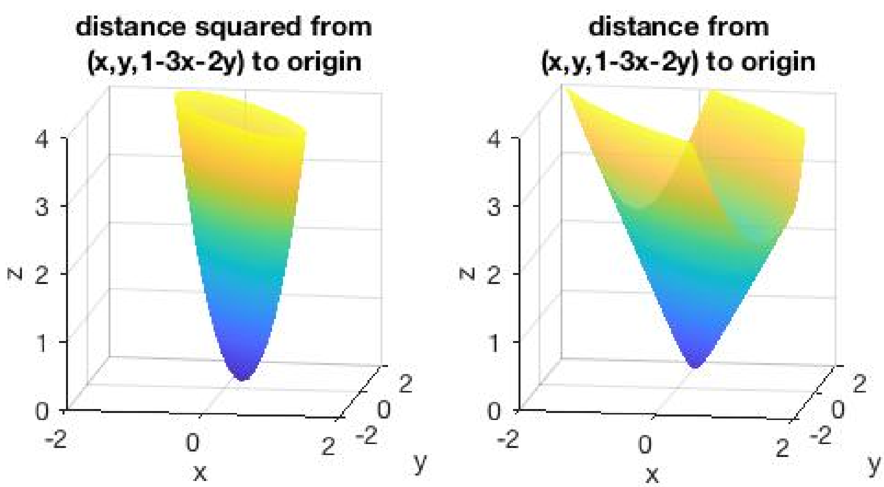
Today:

- ☐ Global extrema: Identify whether a function has a global minimum or maximum on a region.
- ☐ Constrained optimization: Find maximum and minimum values for a constrained problem.
- ☐ Lagrange multipliers: Use the Lagrange multiplier to approximate change in the value of an extremum as a constraint changes.

Example (point to plane distance). This example is from the last class. We want to find the shortest distance from the origin to a point on the plane $3x + 2y + z = 1$.

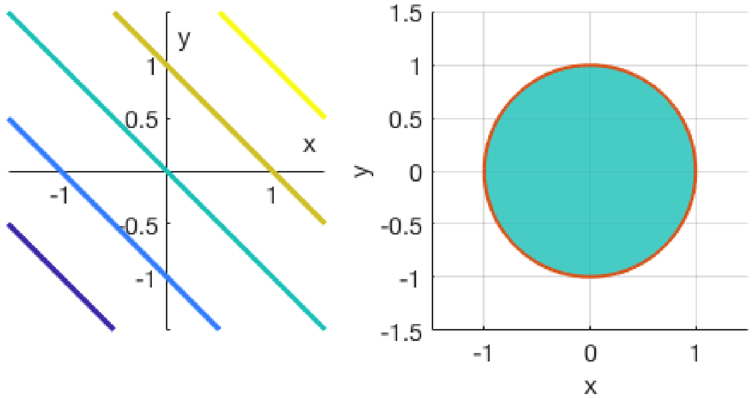
```
syms x y z
g = @(x,y,z) x^2+y^2+z^2;
f = @(x,y) g(x,y,1-3*x-2*y);
fx = diff(f,x);
fy = diff(f,y);
criticalpt = solve(fx==0,fy==0, [x y]);
pt = [criticalpt(1).x, criticalpt(1).y];
distanceval = sqrt(subs(f,[x,y],pt))
```

The distance is $\sqrt{14}/14$. The closest point is $(3/14, 1/7, 1/14)$.



- The extreme value theorem does not apply for this problem. The domain is all of $\mathbb{R}^2$ (the xy-plane) so the domain is *not bounded*. (The domain doesn't have any boundary points. It does include all of the boundary points that it has, so it is considered closed.)
- We can see from the geometry of the square of the distance function (or from the distance function itself) that there is no global maximum.
- The local minimum (identified by finding the critical point of the distance-squared function) will also be the global minimum. We can see this from the geometry of the graph but will not show it algebraically.

Example (global extrema). Find the location of the global minimum of the objective function $f(x,y) = x + y$ on the region $x^2 + y^2 \leq 1$. *pollQ*



Extra. Find the minimum value achieved by the function.

Extreme value theorem notes:

- The region $x^2 + y^2 \leq 1$ is closed (in contrast, the region $x^2 + y^2 < 1$ is not closed).
- The region is bounded (it fits within a rectangle rather than going off to infinity in some direction).
- Since the region is closed and bounded the extreme value theorem applies. There will be a global maximum at some point(s) in the region and a global minimum at some point(s) in the region.

Example (constraint curve). Identify local extrema on the curve below. *pollQ*
Extra. Find the local extremum along the constraint curve that is not labeled. Is it a local minimum or a local maximum?

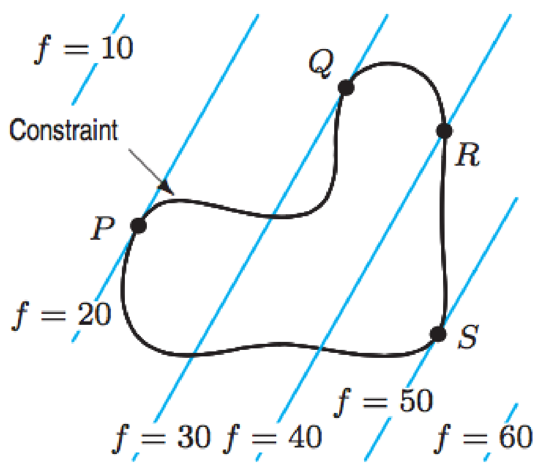


Figure 15.30

Question (extrema). Given an objective function f and a constraint curve $g(x,y) = c$, at a local extremum, what is the value of the directional derivative of f in directions tangent to the constraint curve? *pollQ*

For your reference (section 15.3):

- Local and global extrema along a constraint:
  Suppose P_0 is a point satisfying the *constraint* $g(x, y) = c$ and $f(x, y)$ is an *objective function*.
  - f has a *local minimum* at P_0 subject to the constraint if $f(P_0) \leq f(P)$ for all points P that are near P_0 and satisfy the constraint.
  - f has a *global minimum* at P_0 subject to the constraint if $f(P_0) \leq f(P)$ for all points P that satisfy the constraint.

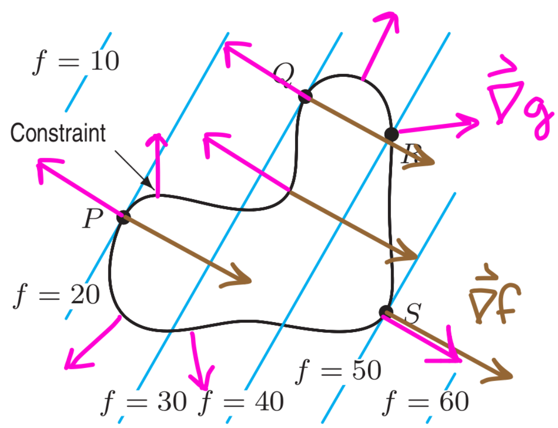
  Local and *global maximum* are defined similarly.

- Rate of change is zero at the extremum (subject to the constraint):
  At a local extremum along the constraint curve, we have $f_{\vec{u}}(P_0) = 0$ for $\vec{u}$ pointing in a direction tangent to the constraint curve.

- Normal vectors are parallel at local extrema:
  If a smooth function $f(x, y)$ has a local maximum or local minimum subject to a smooth constraint $g(x, y) = c$ at a point P_0 then $\vec{\nabla} f(P_0) = \lambda \vec{\nabla} g(P_0)$ and $g = c$. The number λ is called the *Lagrange multiplier*.

- Extreme value theorem subject to a constraint:
  If the set of points satisfying the constraint is closed and bounded, then there must be a global maximum and minimum of f subject to the constraint. The global extremum occurs either at a the location of a local extremum or at an endpoint of the constraint.

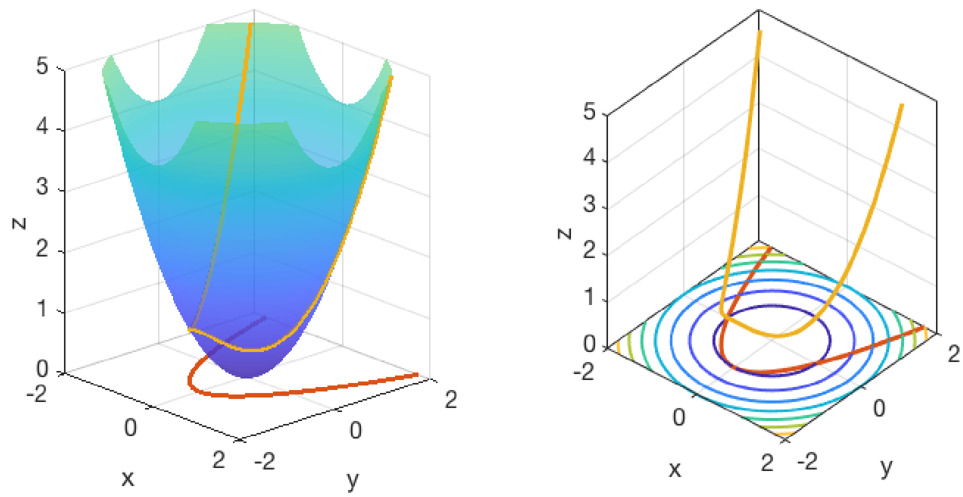
Question (parallel gradients). How can we find points (x, y) where the constraint curve, $g(x, y) = c$, is tangent to a level curve of the objective function, $f(x, y)$?



Example (Lagrange multipliers). Use Lagrange multipliers to find the extrema of $f(x, y) = x + y$ on $x^2 + y^2 = 1$.
To do this in matlab:

```
syms x y L
vals = solve(gradient(x+y)==L*gradient(x^2+y^2),x^2+y^2 == 1,[x y L])
```

Example (unbounded constraint curve). Let $f(x, y) = x^2 + y^2$. If they exist, find the global maximum and global minimum of f subject to the constraint $y - x^2 = -1$. *pollQ*



- The domain of our problem is the curve $y - x^2 = -1$.
- Thinking of this curve as a bent line, it has no boundary points (the line extends forever in each direction rather than being a line segment).
- It is closed (no boundary points so it contains them all) but is not bounded.
- The extreme value theorem does not apply, so we are not guaranteed to have global extrema.

```
syms x y
hold off
fsurf(x^2+y^2,[-2 2 -2 2], 'facealpha', 0.6, 'edgecolor', 'none')
hold on
fplot3(x,x^2-1,sym(0),[-2 2],'linewidth',3)
fplot3(x,x^2-1,x^2+(x^2-1)^2,[-2 2],'linewidth',3)
hold off
xlabel('x'); ylabel('y'); zlabel('z')
set(gca,'fontsize',16)
axis equal
axis([-2 2 -2 2 0 5])
```

Text section	Initial practice (exercises)	PSet practice (problems)
15.3: Constrained optimization: Lagrange multipliers	3, 11, 18	21, 23, 25, 27, 43

Learning Objectives

These objectives are associated with Class 15 + Problem Set 6 questions + Quiz 4 + Office hours. Workshop and section on October 15th and 16th will mix this material with exam review.

Students will be asked to
- Define constraint equation.

- Find maximum and minimum values for a constrained problem using the method of Lagrange multipliers or geometric reasoning.
  §15.3 1-24
- Identify maxima, minima, and saddle points given the gradient vectors of a function.
  §15.3 24
- Determine the sign of a Lagrange multiplier using geometric reasoning given a contour plot with a specified constraint. §15.3 25-26
- Estimate the change in an extreme value with the loosening or tightening of a constraint using the value of the Lagrange multiplier.
  §15.3 27-28, 31
- Interpret the Lagrange multiplier, the objective function, and the constraint equation in the context of an application problem.
  §15.3 29-30
- Translate from a description of an optimization problem to mathematical equations.
  §15.3 32-44, 47

Problem type		
Single variable	Function of two variables	Function of two variables along a constraint curve
Local maximum		
$f(x)$ has a local maximum at P_0 if $f(P_0) \geq f(P)$ for all points P near P_0.	$f(x, y)$ has a local maximum at P_0 if $f(P_0) \geq f(P)$ for all points P near P_0.	f has a local maximum at P_0 subject to the constraint if $f(P_0) \geq f(P)$ for all points P near P_0 satisfying the constraint.
Global maximum		
$f(x)$ has a global maximum at P_0 if $f(P_0) \geq f(P)$ for all points P in the domain.	$f(x, y)$ has a global maximum at P_0 if $f(P_0) \geq f(P)$ for all points P in the domain.	f has a global maximum at P_0 subject to the constraint if $f(P_0) \geq f(P)$ for all points P satisfying the constraint.
Extreme value theorem		
$f(x)$ has a global maximum (and minimum) on a domain if the domain is closed and bounded.	$f(x, y)$ has a global maximum (and minimum) on a domain if the domain is closed and bounded.	$f(x, y)$ has a global maximum (and minimum) on a constraint curve if the set of points satisfying the constraint is closed and bounded.
Closed and bounded domain		
The set of points on the x-axis where $2 \leq x \leq 5$ is an example of a closed and bounded region	The set of points in the xy-plane where $x^2 + y^2 \leq 4$ is an example of a closed and bounded region	The set of points in the xy-plane where $x^2 + y^2 = 4$ is an example of a closed and bounded constraint curve
Closed but unbounded domain		
$2 \leq x$ is an example of a closed but unbounded domain	$y - x \leq 0$ is an example of a closed but unbounded domain	$y = x^2$ is an example of a closed but unbounded constraint curve