Today:

- Lagrange multipliers: Find the location of extrema along a constraint curve.
- Integration: Estimate an integral by thinking of it as a weighted sum.
- Iterated integrals: Set up iterated integrals to find $\int_{R} f \ dA$ for a function of two variables over a region $R$.

**Example (unbounded constraint curve).** Let $f(x, y) = x^2 + y^2$. If they exist, find the global maximum and global minimum of $f$ subject to the constraint $y - x^2 = -1$. poll/Q

- The domain of our problem is the curve $y - x^2 = -1$.
- Thinking of this curve as a bent line, it has no boundary points (the line extends forever in each direction rather than being a line segment).
- It is closed (no boundary points so it contains them all) but is not bounded.
- The extreme value theorem does not apply, so we are not guaranteed to have global extrema.

**Example (integral).** Imagine the figure below shows the density of an object in kilograms per meter squared. The $x$- and $y$-axes are measured in meters.

Find an estimate for the mass of the object. poll/Q
Example (mean value). Assume the mass of the object is 400 kilograms. If it were, find the mean density of the object in kilograms per meter squared.

Example (sign of an integral - single variable). Let $C$ be the region of the $x$-axis such that $-1 \leq x \leq 1$. Without calculating the integral, find the signs of

$$\int_C dx, \int_C x \, dx, \int_C (x - 1) \, dx. \text{ pollQ}$$

Example (sign of an integral - two variables). Let $R$ be the unit disk in $\mathbb{R}^2$ (the $xy$-plane). Without calculating the integral, find the signs of

$$\int_R dA, \int_R x \, dA, \int_R e^{xy} \, dA. \text{ pollQ}$$

For your reference (section 16.1): Suppose the function $f$ is continuous on $R$, the rectangle $a \leq x \leq b, c \leq y \leq d$. Cut $R$ into subrectangles of size $\Delta A$. If $(u_{ij}, v_{ij})$ is a point in the $ij$-th subrectangle, we define the definite integral of $f$ over $R$

$$\int_R f \, dA = \lim_{\Delta A \to 0} \sum_{i,j} f(u_{ij}, v_{ij}) \Delta A.$$

The sum $\sum_{i,j} f(u_{ij}, v_{ij}) \Delta A$ is called a Riemann sum. (Riemann did this work around 1854. Published in 1867, after his death. ([Weblink to “Elements of the History of Mathematics”]))

Let $x, y, f(x, y)$ all represent lengths, with $f$ positive. $\int_R f \, dA$ is the volume under the graph of $f$ above the region $R$.

Let $x, y$ represent lengths. $\int_R 1 \, dA = \int_R dA$ is the area of the region $R$. (The 1 is unitless).

The mean value of $f$ on the region $R$ is $\frac{1}{\text{Area}(R)} \int_R f \, dA$. 

Example (volume). The solid below $f(x, y) = 3 + x - \frac{y}{2}$ and above the region $R$ where $x^2 + y^2 \leq 1$ in the $xy$-plane has volume $\int_R (3 + x - \frac{y}{2}) \, dA$.

For your reference (section 16.2): For the density example, total mass $\approx \sum_{i,j} f(u_{ij}, v_{ij}) \Delta x \Delta y$. Assume $0 \leq x \leq 20$ is split into $n$ intervals, $1 \leq i \leq n$ and $0 \leq y \leq 10$ is split into $m$, $1 \leq j \leq m$, with $(u_{ij}, v_{ij})$ a location in the $ij$th box. We can write $\sum_{i,j} f(u_{ij}, v_{ij}) \Delta x \Delta y = \sum_{j=1}^{m} \left( \sum_{i=1}^{n} f(u_{ij}, v_{ij}) \Delta x \right) \Delta y$.

If $R$ is the rectangle $a \leq x \leq b, c \leq y \leq d$ and $f$ is a continuous function on $R$ then the integral of $f$ over $R$ exists and is equal to the iterated integral

$$\int_{R} f \, dA = \int_{y=c}^{y=d} \left( \int_{x=a}^{x=b} f(x, y) \, dx \right) \, dy.$$

We usually write $\int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy$ for this.

The order of integration (for the integrals we’ll do in this class) can be exchanged.

$$\int_{y=c}^{y=d} \left( \int_{x=a}^{x=b} f(x, y) \, dx \right) \, dy = \int_{x=a}^{x=b} \left( \int_{y=c}^{y=d} f(x, y) \, dy \right) \, dx$$ (see Fubini’s theorem for conditions, 1907).

On non-rectangular regions, the inner limits capture the shape of the region while the outer limits are the outer bounds of the shadow of the object on the axis. The limits of the inner integral may depend on the variable in the outer integral. The limits of the outer integral do not depend on the variables in integration.
Example (iterated integrals). For a function of one variable, $\int_C f \, dx$ over a region $C$ where $a \leq x \leq b$ can be computed via the definite integral $\int_a^b f \, dx$. By convention, $\int_C f \, dx$ always denotes that the integral is taken from the left to the right.

For a function of two variables, $\int_R f \, dA$ is usually computed one of two ways:

- (horizontal strips) integrating first in $x$ and then integration the result in $y$
- integration first in $y$ and then integration the result in $x$

By convention, $\int_R f \, dA$ always denotes that the integral is taking from left to right in $x$ and from bottom to top in $y$.

Example (triangular region). Set up an iterated integral for $\int_R f \, dA$ where $f$ is an unknown function of two variables and $R$ is the triangular region shown below.
Learning Objectives

These objectives are associated with Class 17 + Class 18 + Problem Set 7 questions + Quiz 5 + Workshop + Section + Office hours.

Students will be asked to
• integration
  – define: Riemann sum, double integral
  – approximate an integral of a function of multiple variables over a region using a contour plot of the function or a table of values
  – determine the sign of an integral (without integrating) given information about the function of integration and the region of integration
• double integrals
  – define: iterated integral,
  – explain the relationship between iterated integrals and \( \int_R f \, dA \)
  – sketch the region of integration corresponding to the bounds of an iterated integral
  – evaluate an iterated integral
  – find bounds of integration given a description of a region of integration in 2-space
  – set up and evaluate appropriate iterated integrals given a function of integration and a description of a region
  – reverse the order of integration for an iterated integral
  – use double integrals to find the average value of a function, or to find area, volume, mass, population, or force
Matlab code for first figure

```matlab
syms x y
hold off
fsurf(x^2+y^2,[-2 2 -2 2], 'facealpha', 0.6, 'edgecolor', 'none')
hold on
fplot3(x,x^2-1,sym(0),[-2 2], 'linewidth', 3)
fplot3(x,x^2-1,x^2+(x^2-1)^2,[-2 2], 'linewidth', 3)
hold off
xlabel('x'); ylabel('y'); zlabel('z')
set(gca,'fontsize',16)
axis equal
axis([-2 2 -2 2 0 5])
```

<table>
<thead>
<tr>
<th>Problem type</th>
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<tbody>
<tr>
<td>Function of one variable ((x)) with (x) indicating a distance.</td>
<td>Function of two variables ((x, y)) with (x) and (y) indicating distances.</td>
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</tbody>
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<table>
<thead>
<tr>
<th>Size of region</th>
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<tbody>
<tr>
<td>(\int_C dx) gives the length of the line segment (C).</td>
<td>(\int_R dA) give the area of the region (R).</td>
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<tr>
<th>Integral of a height function</th>
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<td>For (f \geq 0) indicating a height, (\int_C f , dx) is the area of the region above (C) with height given by (f(x)).</td>
<td>For (f \geq 0) indicating a height, (\int_C f , dA) is the volume of the region above (C) with height given by (f(x, y)).</td>
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<th>Integral of a density function</th>
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<td>For (f \geq 0) indicating a density in mass per unit of distance, population per unit of distance, or probability per unit of distance, (\int_C f , dx) is the total mass, total population, or total probability (respectively).</td>
<td>For (f \geq 0) indicating a density in mass per distance squared, population per distance squared, or probability per distance squared, (\int_C f , dA) is the total mass, total population, or total probability (respectively).</td>
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<th>Mean value of a function</th>
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<tr>
<td>For (f) any function, (\frac{1}{\text{Length}(C)} \int_C f , dx) where (\langle f \rangle) is referred to as the mean value of (f) on the region (C). Rearranging, the mean value of (f) on the region (C) is given by (\frac{1}{\text{Area}(R)} \int_R f , dA)</td>
<td>For (f) any function, (\frac{1}{\text{Area}(R)} \int_R f , dA) where (\langle f \rangle) is referred to as the mean value of (f) on the region (R). Rearranging, the mean value of (f) on the region (R) is given by (\frac{1}{\text{Area}(R)} \int_R f , dA).</td>
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