Today:
□ Cylindrical coordinates: Describe regions of $xyz$-space using the cylindrical coordinate system.
□ Spherical coordinates: Describe regions of $xyz$-space using the spherical coordinate system.
□ Jacobian: Write an integral in cylindrical or spherical coordinates, using the appropriate $dV$ to account for the change of variables.
□ Probability density: Distinguish between probability density and cumulative distribution functions.

**Example (function of integration).** Identify the function of integration for the integral

$$
\int_{0}^{\pi/4} \int_{0}^{1} r^2 \cos \theta \, dr \, d\theta.
$$

pollQ

**Example (cartesian equivalent).** Let $R$ be the region bounded by $x = 1$, $y = 0$, $y = x$. In polar coordinates, is

$$
\int_R x \, dA = \int_{0}^{\pi/4} \int_{0}^{1} r^2 \cos \theta \, dr \, d\theta?
$$

pollQ

**Example (change coordinates).** Rewrite $\int_{0}^{\pi/4} \int_{0}^{1} r^2 \cos \theta \, dr \, d\theta$ in Cartesian coordinates. Use horizontal stripes.

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**For your reference, regions.** A region of integration is called *horizontally simple* the region $R$ is of the form $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, so $\int_R f \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f \, dx \, dy$ (the integral can be written using a single iterated integral using horizontal stripes).

A region of integration is called *vertically simple* the region $R$ is of the form $c \leq x \leq d$, $h_1(x) \leq y \leq h_2(x)$, so $\int_R f \, dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f \, dy \, dx$ (the integral can be written using a single iterated integral using vertical stripes).
Example (cylindrical coordinates).

The region shown in D is the region where $1 \leq r \leq 2$, $\pi/8 \leq \theta \leq \pi/4$, and $1 \leq z \leq 3$.

Match plots A, B, C to the following pairs of surfaces

- $r = 1$, $\pi/8 \leq \theta \leq \pi/4$, $1 \leq z \leq 3$ and $r = 2$, $\pi/8 \leq \theta \leq \pi/4$, $1 \leq z \leq 3$.
- $1 \leq r \leq 2$, $\theta = \pi/8$, $1 \leq z \leq 3$ and $1 \leq r \leq 2$, $\theta = \pi/4$, $1 \leq z \leq 3$.
- $1 \leq r \leq 2$, $\pi/8 \leq \theta \leq \pi/4$, $z = 1$ and $1 \leq r \leq 2$, $\pi/8 \leq \theta \leq \pi/4$, $z = 3$. pollQ

For your reference, section 16.5. Cylindrical coordinates: each point in 3-space is represented using $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$, $-\infty < z < \infty$. $x = r \cos \theta$, $y = r \sin \theta$, $z = z$.

Note that $x^2 + y^2 = r^2$.

A region $a \leq r \leq b$, $c \leq \theta \leq d$, $m \leq z \leq n$, where all bounds are constants, will be a piece of a solid cylinder.

The Jacobian for the change of coordinates from Cartesian to cylindrical is $r$ so the volume element is given by $dV = r dr d\theta dz$.

Example (cylindrical region). Consider a water tank, where the water has depth $h$. Let the volume of water in the tank be given by

$$\int_0^{2\pi} \int_0^h \int_0^{\sqrt{a^2-z^2}} r \, dr \, dz \, d\theta.$$ 

Identify the function of integration and the shape of the tank. pollQ

For cylindrical coordinates, a sketch a cross-section of the region in $rz$-space is usually particularly helpful.
Example (spherical coordinates).

The region shown in D is the region where \(1 \leq \rho \leq 2\), \(\pi/8 \leq \theta \leq 3\pi/8\), and \(\pi/4 \leq \phi \leq 7\pi/6\).

Which of A, B, C is associated with the \(\theta = c\) surfaces (for \(c\) some constant)? pollQ

For your reference, section 16.5.

Spherical coordinates: each point in 3-space is represented using \(0 \leq \rho < \infty\), \(0 \leq \theta \leq 2\pi\), \(0 \leq \phi \leq \pi\).

\[ z = \rho \cos \phi, \quad r = \rho \sin \phi \]

\[ x = r \cos \theta = \rho \sin \phi \cos \theta, \quad y = r \sin \theta = \rho \sin \phi \sin \theta \]

Note that \(x^2 + y^2 + z^2 = \rho^2\).

A region \(a \leq \rho \leq b, c \leq \theta \leq d, m \leq \phi \leq n\), where all bounds are constants, will be a piece of a solid sphere.

The Jacobian for the change of coordinates from Cartesian to cylindrical is \(\rho^2 \sin \phi\) so the volume element is given by \(dV = \rho^2 \sin \phi d\rho d\theta d\phi\).

Example (\(\phi = c\)).

In plot A below are show two surfaces on which \(\phi\) is held constant. On which surface is \(\phi\) greater? pollQ
For your reference, section 8.7. The function $p(x)$ is a probability density function, or pdf, if the fraction of the population for which $a \leq x \leq b = \int_a^b p(x) \, dx$ with $\int_{-\infty}^{\infty} p(x) \, dx = 1$ and $p(x) \geq 0$ for all $x$.

$\int_{-\infty}^{\infty} p(x) \, dx = 1$ means that between $-\infty$ and $\infty$ we’ll find the entire population.

$p(x) \geq 0$ for all $x$ implies there is no interval where there would be a negative fraction of the population.

$P(t) = \int_{-\infty}^{t} p(x) \, dx$ is the fraction of the population having a value of $x$ that is below $t$. $P$ is a nondecreasing function. $\lim_{t \to \infty} P(t) = 1$ and $\lim_{t \to \infty} P(t) = 0$.

The fraction of the population having values of $x$ between $a$ and $b = \int_{a}^{b} p(x) \, dx = P(b) - P(a)$.

Figure 8.90: $P(t)$, the cumulative age distribution function, and its relation to $p(x)$, the age density function

Example (density and distribution functions).
Match the graphs of the density functions, a,b,c, with the graphs of the distribution functions, I, II, III. pollQ
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Learning Objectives

These objectives are associated with Class 20 + Problem Set 8 questions + Quiz 6 + Workshop + Section + Office hours.

Students will be asked to

• spherical
  – define: spherical coordinates, Jacobian for spherical
  – find an equation for a surface in spherical coordinates (§16.5 5-7)
  – identify the sign of a triple integral using information about the function of integration and the region of integration (but without integrating) (§16.5 42-43)
  – evaluate an integral expressed in spherical coordinates (§16.5 10-11)
  – choose appropriate coordinates for a region (§16.5 12-18, 55-56)
  – use triple integrals in Cartesian, cylindrical, or spherical, to compute volume, mass, average value of a function, center of mass, or moment of inertia (§16.5 27-41, 44-72)

• probability
  – define: probability density function (§8.7), cumulative distribution function
  – estimate probability using a probability density function (§8.7 14, 19, 22)
  – construct a probability density function given information about the function (§8.7 12)
  – distinguish between a cumulative distribution function and a probability density function (§8.7 1-10)
  – identify the units of a probability density function (§8.7 11)