

Online Appendix to "Optimal Taxation with Adverse Selection in
the Labor Market"

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I Rothschild and Stiglitz Equilibrium Type

In this Section, I consider the alternative equilibrium definition of Rothschild and Stiglitz (RS), but limit myself to the $N = 2$ model, because of the existence problems with $N > 2$. In RS, firms offer only a single labor contract and have to break even.¹

Definition 1 (*Rothschild-Stiglitz equilibrium*) *A set of contracts offered is a Rothschild-Stiglitz equilibrium if i) firms make zero profit on each contract and ii) there is no other potential contract which would make positive profits, if offered.*

The Rothschild and Stiglitz (1976) equilibrium notion has been used almost exclusively for two type models ($N = 2$). The authors show that no pooling equilibrium can exist, and that, for a sufficiently low fraction of low productivity workers, λ_1 , no equilibrium can exist at all. Whenever a separating equilibrium exists, low types work an efficient number of hours, but high types work excessively, so that firms can separate them from low types. Hence, high ability workers are caught in a “rat race” (Akerlof, 1976).

A Linear taxes in the Rothschild-Stiglitz setting and the possibility of destroying the equilibrium

In a RS-type equilibrium, firms are constrained to break even on each contract offered, so that, at any potential equilibrium allocation, pay is equal to a worker’s total product: $y_i = (1 - t_i)\theta_i h_i$.² Whenever a separating equilibrium exists, the tax formula will be the same as in case AS1 in Section 3 in the main text. However, an equilibrium may not exist, because it might be possible for some firm to offer an alternative pooling contract which would attract all workers. The existence of an equilibrium depends on taxes. Rather than a smooth response of the private market, there may be an abrupt shift, at some tax level, from a separating equilibrium to non-existence, which makes the optimal tax problem more complicated.

One can check that a separating equilibrium (h_2^{RS}, y_2^{RS}) will be stable,³ if the fraction of low types is larger than $\lambda^{RS}(t)$, defined as the threshold value of λ for which the indifference curve of the high type worker going through the equilibrium allocation is just tangent to the pooling line with slope $\theta^{RS}(t) = (\lambda^{RS}(t)\theta_1 + (1 - \lambda^{RS}(t))\theta_2)$. Denote this tangency point by $h(t)$. In this case, there is no possible pooling region, that is, no allocations which are pooling, make non-negative profits, and make both workers better off than the separating equilibrium allocation. If, at a given tax t , λ is already close to the threshold $\lambda^{RS}(t)$ and the government increases the tax, then this *could* open up a possible pooling region and destroy the equilibrium altogether. This phenomenon is illustrated in Figure 1 where, starting from a situation with low taxes (solid indifference curves), taxes are increased (dotted indifference curves) and a pooling region is created. The following proposition describes when this can occur.

¹Since firms need to break even on each contract, they might as well be offering a pair of contracts. But it is without loss of generality to assume that those contracts are offered by two different firms.

²Recall that a pooling equilibrium cannot exist in RS, even in the presence of linear taxes.

³Stability means that the separating equilibrium cannot be broken by a pooling equilibrium, which is the definition used in the original Rothschild and Stiglitz (1976) paper.

Proposition 1 *With a general utility function, if the utility of the high type at the candidate separating equilibrium, denoted u_2^{RS} , is sufficiently strongly decreasing in taxes, that is if*

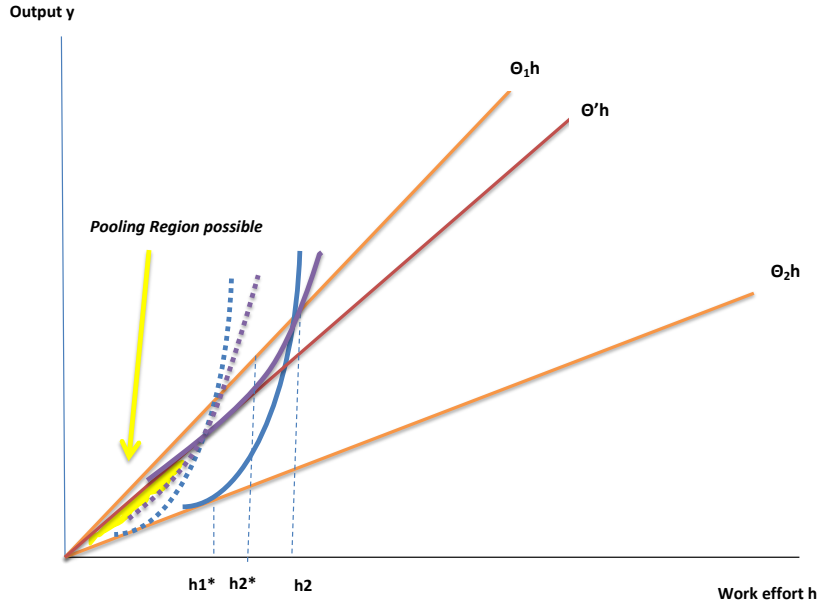
$$du_2^{RS}/dt \leq -\theta^{RS}(t)h(t)$$

then raising taxes could destroy an existing separating equilibrium (in the sense of pushing up the critical threshold $\lambda^{RS}(t)$).

With an isoelastic utility function, the critical threshold $\lambda^{RS}(t)$ is always increasing in t and higher taxes make the existence of a separating equilibrium less likely.

If nonexistence of an equilibrium is an undesirable state, then, the optimal tax rate must be set subject to the additional constraint $\lambda^{RS}(t) \leq \lambda$.⁴

Figure 1: Non existence of the RS equilibrium



B Nonlinear Taxation in the RS Model

Is it still possible to implement all Second Best allocations in an RS setting like it was for the MWS setting with $N = 2$ (i.e, regime 3 in the main text, called “Adverse Selection with unobservable private contracts”), despite the government being unable to see private labor contracts?⁵ We focus on the more interesting case in which the high productivity worker’s incentive constraint would be binding in the Second Best with Adverse Selection and show that this allocation can no longer be an equilibrium with unobservable private contracts.

Suppose again, that the government imposes confiscatory taxes on all income levels other than the recommended ones, y_1 and y_2 . This reduces to only three the potential deviations that firms can make. First, they could try to pool both workers at y_2 . However, no such contract would attract high types, since they would have

⁴The detailed analysis of this problem is not particularly enlightening, given already performed for the MWS case.

⁵Again, any Second Best allocation $\{c_i, h_i\}_{i=1}^2$ at which the incentive compatibility constraint of the low type is binding can still be implemented. It is sufficient to set taxes equal to $T_i = y_i - c_i$ at income levels $y_i = \theta_i h_i$, $i = 1, 2$, and $T = y$ for all other income levels.

to work more for the same pay than under the original contract, to compensate for the low type's poor productivity. Secondly, firm could not possibly "invert" the separating equilibrium by offering contracts $\{y_1, y_1/\theta_2\}$ and $\{y_2, y_2/\theta_1\}$, because that would violate the monotonicity condition on hours.⁶ There is however one other profitable deviation, namely to offer a pooling contract at y_1 . To see this, start from the candidate allocation at which the incentive compatibility constraint of the high type is binding:

$$y_2 - T_2 - \phi_2(h_2) = y_1 - T_1 - \phi_2(h_1) \quad (1)$$

Consider a pooling contract paying y_1 in exchange for h'_1 hours of work, where h'_1 is determined by the zero profit condition:

$$\lambda\theta_1 h'_1 + (1 - \lambda)\theta_2 h'_1 = y_1 \quad (2)$$

There then exists a slightly higher level of hours, $h^* = h'_1 + \varepsilon$ (for some very small $\varepsilon > 0$), such that $\lambda\theta_1 h^* + (1 - \lambda)\theta_2 h^* > y_1$, and a new contract (y_1, h^*) which would yield strictly positive profits if both types accepted it. And indeed, both types will accept it. By (1), the high type was just indifferent between his allocation and the original allocation of the low type, (y_1, h_1) . Furthermore, by (2), it is clear that $h'_1 < h_1$, since $\theta_1 h_1 = y_1$ so that $\theta_2 h_1 > y_1$. If ε is small enough, we also have $h^* < h_1$. Thus, the high type now strictly prefers the allocation (y_1, h^*) to (y_2, h_2) . The low type also prefers this allocation, since he earns the same total pay but works less. The original allocation can thus not have been an equilibrium.⁷ Intuitively, the government is trying to force the private market to do the opposite of what it would normally do, namely to reduce the welfare of the high types for the benefit of the low types. But competition among firms makes them exploit the loophole, created by the inability of the government to see hours worked, to try to make the high type as well off as possible. Therefore, we need to add a new incentive constraint for *firms* to not be able to deviate to that pooling contract, which is more stringent than the standard incentive constraint for the high type:

$$y_2 - T_2 - \phi_2\left(\frac{y_2}{\theta_2}\right) \geq y_1 - T_1 - \phi_2(h'_1) \quad (3)$$

The Pareto frontier for the RS case is thus obtained by solving program $P^{RS}(\mu)$ and is characterized in the next proposition.

$$\begin{aligned} (P^{RS}(\mu)) \quad &: \quad \max_{c_1, c_2, h_1, h_2} \mu(c_1 - \phi_1(h_1)) + (1 - \mu)(c_2 - \phi_2(h_2)) \\ &(IC_{12}) : c_1 - \phi_1(h_1) \geq c_2 - \phi_1(h_2) \\ &(IC_{21}) : c_2 - \phi_2(h_2) \geq c_1 - \phi_2\left(\frac{\theta_1 h_1}{(1 - \lambda)\theta_2 + \lambda\theta_1}\right) \\ &(RC) : \lambda c_1 + (1 - \lambda)c_2 \leq \lambda\theta_1 h_1 + (1 - \lambda)\theta_2 h_2 \end{aligned}$$

Proposition 2 *The Pareto frontier in the RS setting is characterized by:*

Region 1: When $\mu = \lambda$, both (IC_{12}) and (IC_{21}) are slack and both workers work efficient hours. The Pareto frontier is linear.

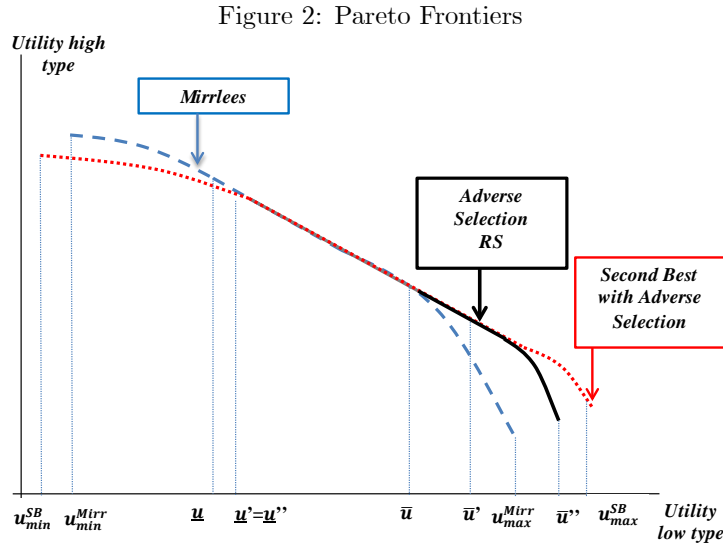
⁶See Appendix 1 for a proof of the necessity of monotonicity at any implementable allocation.

⁷Note that this pooling allocation to which firms are tempted to deviate might not be an equilibrium either, because of the familiar cream-skimming argument which also precludes the existence of a pooling equilibrium in the original Rothschild and Stiglitz paper. But it still is a profitable deviation. Indeed, if it was an equilibrium and it was making both types better off, it would have been on the Pareto frontier.

Region 2: When $\mu > \lambda$, (IC_{21}) is binding, the high type works efficient hours, the low type works too little, and the Pareto frontier is strictly concave. The Rothschild-Stiglitz frontier is above the Mirrlees frontier, but below the Second Best with Adverse Selection frontier.

Region 3: When $\mu < \lambda$, (IC_{12}) is binding, the low type works efficient hours, the high type works too much, and the Pareto frontier is strictly concave. The Rothschild-Stiglitz frontier is below the Mirrlees frontier and coincides with the Second Best with Adverse Selection frontier.

Again, the three Regions can be mapped into three regions for the utility of the low type, u , with the four thresholds defined in the Appendix, $u_{\min}^{RS} < \underline{u}'' < \bar{u}'' < u_{\max}^{RS}$, and whose interpretation is exactly as in the main text. Region 1 corresponds to $\underline{u}'' \leq u \leq \bar{u}''$, Region 2 to $\bar{u}'' \leq u \leq u_{\max}^{RS}$, and Region 3 to $u_{\min}^{RS} \leq u \leq \underline{u}''$. The Section with proofs provides a ranking of all four frontiers with two types (the RS setting, the Mirrlees case, the Second Best with Adverse Selection case, and the Adverse Selection with unobservable private contracts case), while Figure 2 illustrates this relation for when condition **NL1** holds.



It is clear that, while adverse selection still helps the government redistribute, the Rothschild-Stiglitz case lies between the Second Best with Adverse Selection and the Mirrlees case. Firms try to attract the high type worker to a pooling allocation, by offering him an hourly wage equal to the average productivity $\lambda\theta_1 + (1 - \lambda)\theta_2$. This makes a deviation more attractive than in the Second Best, but still less attractive than in the Mirrlees case. Note the difference to regime 3 with $N = 2$ from the main text: it is the fact that firms are constrained to break even on each contract offered which prevents the government from implementing the Second Best allocation.⁸

⁸As a side remark, it is interesting to consider what happens if instead of having the Rothschild-Stiglitz Nash Equilibrium behavior, firms exhibit a behavior characterized by Wilson's (1976) foresight assumption. Under this assumption, firms will consider a deviation to another contract, if and only if that deviation still remains profitable after all contracts which have been rendered unprofitable by it have been dropped. The Wilson notion can be used to justify why a pooling equilibrium may persist in the market. However, it is straightforward to check that it does not alter the Pareto frontier characterization. Indeed, the deviation to pooling at income level y_1 is still profitable, unless the appropriate incentive constraints from the RS setting hold.

II Proofs for Section I

Proof of Proposition (1):

At any intersection of the indifference curve with the pooling line $y = \theta h$ we should have $(u_2^{RS} + \phi_2(h)) / (1 - t) = \theta h$. In addition to obtain an exact tangency, the slopes must be equal so that: $\phi_2'(h_2) / (1 - t) = \theta$.

Special case: power disutility function (under assumption 2) $\phi_i(h_i) = a_i h_i^\eta$.

Then: $u_1^{RS} = [\theta_1(1-t)/a_1\eta]^{\frac{\eta}{\eta-1}} [a_1\eta - 1]$ is the utility of the low type at the candidate RS allocation, (assume that $a_1\eta \geq 1$). Then by the (IC_{12}) being binding:

$$\begin{aligned} u_2^{RS} &= u_1^{RS} + (h_2^{RS})^\eta (a_1 - a_2) \\ u_2^{RS} &= \left[\frac{\theta_1(1-t)}{a_1\eta} \right]^{\frac{\eta}{\eta-1}} [a_1\eta - 1] + (h_2^{RS})^\eta (a_1 - a_2) \end{aligned}$$

The intersection and tangency conditions become: $\eta a_2 h^{\eta-1} = (1-t)\theta$ and $u_2^{RS} + a_2 h^\eta = (1-t)\theta h$. We can rewrite: $h = (u_2^{RS} / (\eta - 1) a_2)^{\frac{1}{\eta}}$ and solve for h :

$$h = \left(\frac{\left[\frac{\theta_1(1-t)}{a_1\eta} \right]^{\frac{\eta}{\eta-1}} [a_1\eta - 1] + (h_2^{RS})^\eta (a_1 - a_2)}{(\eta - 1) a_2} \right)^{\frac{1}{\eta}}$$

To obtain the threshold $\lambda^{RS}(t)$, use that: $\theta = \eta a_2 h^{\eta-1} / (1-t)$, plug in the value for h which leads to, after some algebra:

$$\lambda^{RS}(t) = -\frac{\theta_2}{(\theta_1 - \theta_2)} + \frac{1}{(\theta_1 - \theta_2)} \frac{\eta a_2}{(1-t)} \left(\frac{\left[\frac{\theta_1(1-t)}{a_1\eta} \right]^{\frac{\eta}{\eta-1}} [a_1\eta - 1] + (h_2^{RS})^\eta (a_1 - a_2)}{(\eta - 1) a_2} \right)^{\frac{\eta-1}{\eta}}$$

Taking the derivative of $\lambda^{RS}(t)$ with respect to t :

$$\begin{aligned} \frac{d\lambda}{dt} &= \frac{-1}{(\theta_1 - \theta_2)} \frac{\eta a_2}{(1-t)^2} \left(\frac{\left[\frac{\theta_1(1-t)}{a_1\eta} \right]^{\frac{\eta}{\eta-1}} [a_1\eta - 1] + (h_2^{RS})^\eta (a_1 - a_2)}{(\eta - 1) a_2} \right)^{\frac{\eta-1}{\eta}} \\ &+ \frac{1}{(\theta_1 - \theta_2)} \frac{\eta a_2}{(1-t)} \frac{\eta - 1}{\eta} \\ &\times \left(\frac{1}{(\eta - 1) a_2} \left(-\frac{\eta}{\eta - 1} [a_1\eta - 1] \frac{\theta_1}{a_1\eta} \left[\frac{\theta_1(1-t)}{a_1\eta} \right]^{\frac{1}{\eta-1}} + (a_1 - a_2) \eta \frac{dh_2^{RS}}{dt} (h_2^{RS})^{\eta-1} \right) \right) \\ &\times \left(\frac{\left[\frac{\theta_1(1-t)}{a_1\eta} \right]^{\frac{\eta}{\eta-1}} [a_1\eta - 1] + (h_2^{RS})^\eta (a_1 - a_2)}{(\eta - 1) a_2} \right)^{\frac{-1}{\eta}} \end{aligned}$$

all terms of which are positive. Hence, with isoelastic utility, the critical threshold for existence is increasing in taxes, which means that imposing taxes makes it harder for an equilibrium to exist and the government runs the risk of destroying the equilibrium.

General case (no isoelastic utility):

The hours of type 1 are efficient: $h_1^* = (\phi_1')^{-1}((1-t)\theta_1)$. From (IC_{12}) binding, we get $u_2^{RS} = \theta_2(1-t)h_2 - \phi_2(h_2) = u_1^{RS} + \phi_1(h_2) - \phi_2(h_2)$. The condition for intersection of the indifference curve of the high type at the candidate separating allocation with the pooling line $y = \theta h$ is $u_2^{RS} + \phi_2(h) = (1-t)\theta h$, while the condition for tangency is $\phi_2'(h) = \theta(1-t)$. Hence, combining these two gives a (implicit) solution for h :

$$u_2^{RS} + \phi_2(h) = \phi_2'(h)h \quad (4)$$

$\lambda^{RS}(t)$ is the solution to: $\lambda^{RS} = \phi_2'(h)/(1-t)(\theta_1 - \theta_2) - \theta_2/(\theta_1 - \theta_2)$ and taking the derivative: $d\lambda/dt = -\frac{dh}{dt}\phi_2''(h)(1-t) - \phi_2'(h)/(1-t)^2(\theta_2 - \theta_1)$. Differentiate totally equation (4), to obtain $dh/dt = (du_2^{RS}/dt)(1/(\phi_2''(h)h))$, so that $d\lambda/dt = \left(-\frac{du_2^{RS}}{dt}\frac{1}{h} - \theta\right)/[(1-t)(\theta_2 - \theta_1)]$. Hence, for $d\lambda/dt \geq 0$, we need: $\frac{du_2^{RS}}{dt} \leq -\theta h$, which is the condition in the main text.

Proof of Proposition (2):

The problem, indexed by the utility level of the low type is restated here (multipliers for each constraint are in brackets next to it):

$$\begin{aligned} (P^{RS}(u)) &: \max_{c_1, c_2, h_1, h_2} c_2 - \phi_2(h_2) \\ (IC_{12}) &: c_1 - \phi_1(h_1) \geq c_2 - \phi_1(h_2) \quad [\beta_1] \\ (IC_{21}) &: c_2 - \phi_2(h_2) \geq c_1 - \phi_2\left(\frac{\theta_1 h_1}{(1-\lambda)\theta_2 + \lambda\theta_1}\right) \quad [\beta_2] \\ (RC) &: \lambda c_1 + (1-\lambda)c_2 \leq \lambda\theta_1 h_1 + (1-\lambda)\theta_2 h_2 \quad [\delta] \\ u &\leq c_1 - \phi_1(h_1) \quad [\gamma] \end{aligned}$$

Denote the set of admissible allocations, that is, those allocations which satisfy both incentive compatibility constraints and the resource constraint by $B^{RS}(u)$ and denote by $V_2^{RS}(u) = \max_{c_1, c_2, h_1, h_2 \in B^{RS}(u)} (c_2 - \phi_2(h_2))$. When the value of u varies, the function $V_2^{RS}(u)$ traces out all possible values for the utility of the high type. The Pareto frontier is made of all pairs $(u, V_2^{RS}(u))$ such that $\frac{\partial}{\partial u} V_2^{Mirr}(u) < 0$. The FOCs are:

$$\begin{aligned} [c_1] &: 1 + \beta_2 - \beta_1 - \delta(1-\lambda) = 0 \\ [c_2] &: \beta_1 + \gamma - \beta_2 - \delta\lambda = 0 \\ [h_1] &: -\beta_1\phi_1'(h_1) + \frac{\theta_1}{(1-\lambda)\theta_2 + \lambda\theta_1}\beta_2\phi_2'\left(\frac{\theta_1}{(1-\lambda)\theta_2 + \lambda\theta_1}h_1\right) + \delta\lambda\theta_1 - \gamma\phi_1'(h_1) = 0 \\ [h_2] &: -(1+\beta_2)\phi_2'(h_2) + \beta_1\phi_1'(h_2) + \delta(1-\lambda)\theta_2 = 0 \\ \lambda c_1 + (1-\lambda)c_2 &= \lambda\theta_1 h_1 + (1-\lambda)\theta_2 h_2, \quad c_1 - \phi_1(h_1) = u \end{aligned}$$

Region 1: Suppose that $u \in [\underline{u}'', \bar{u}'']$ where the thresholds are defined by $\underline{u}'' := \underline{c}_2 - \phi_1(h_2^*)$, where $\underline{c}_2 := \lambda\theta_1 h_1^* + (1-\lambda)\theta_2 h_2^* + \lambda(\phi_1(h_2^*) - \phi_1(h_1^*))$ and $\bar{u}'' := \bar{c}_2 - \phi_2(h_2^*) + \phi_2\left(\frac{h_1^*\theta_1}{\lambda\theta_1 + (1-\lambda)\theta_2}\right) - \phi_1(h_1^*)$ where $\bar{c}_2 := \lambda\theta_1 h_1^* + (1-\lambda)\theta_2 h_2^* + \lambda\left(\phi_2(h_2^*) - \phi_2\left(\frac{\theta_1 h_1^*}{(1-\lambda)\theta_2 + \lambda\theta_1}\right)\right)$. Then it is possible to set the hours at their efficient levels, h_1^* and h_2^* and c_1 and c_2 such that:

$$\begin{aligned} \lambda c_1 + (1-\lambda)c_2 &= \lambda\theta_1 h_1^* + (1-\lambda)\theta_2 h_2^* \\ c_1 - \phi_1(h_1^*) &= u \end{aligned}$$

and have both incentive constraints slack. Then, $\gamma = \frac{\lambda}{1-\lambda}$ and $\beta_1 = \beta_2 = 0$. In this region, the Pareto frontier is linear.

Region 2: Suppose $u \geq \bar{u}''$. A parallel reasoning to the one in the proof of Proposition 11 from the main text shows that in this region, (IC_{21}) is binding and (IC_{12}) is slack so that $\beta_1 = 0$. Then, the solution is characterized by:

$$\begin{aligned}\beta_2 &= \gamma - \lambda\gamma - \lambda, \quad \gamma \geq \frac{\lambda}{(1-\lambda)}, \quad \gamma + 1 = \delta \\ \gamma\phi_1'(h_1) &= (\gamma - \lambda\gamma - \lambda) \left(\frac{\theta_1}{(1-\lambda)\theta_2 + \lambda\theta_1} \phi_2' \left(\frac{\theta_1}{(1-\lambda)\theta_2 + \lambda\theta_1} h_1 \right) - \theta_1 \right) + \gamma\theta_1 \\ \theta_2 &= \phi_2'(h_2), \quad u = c_1 - \phi_1(h_1) \\ c_1 &= \lambda\theta_1 h_1 + (1-\lambda)\theta_2 h_2 - (1-\lambda) \left(\phi_2(h_2) - \phi_2 \left(\frac{\theta_1 h_1}{(1-\lambda)\theta_2 + \lambda\theta_1} \right) \right) \\ c_2 &= \lambda\theta_1 h_1 + (1-\lambda)\theta_2 h_2 + \lambda \left(\phi_2(h_2) - \phi_2 \left(\frac{\theta_1 h_1}{(1-\lambda)\theta_2 + \lambda\theta_1} \right) \right)\end{aligned}$$

There is a maximal level of utility for the low type achievable here, denoted u_{\max}^{RS} which is the utility level attained when h_1 is the solution to $\max c_1 - \phi_1(h_1)$ subject to (IC_{21}) , or equivalently, when $\gamma \rightarrow \infty$ in the above FOCs, that is when

$$\phi_1'(h_1) = (1-\lambda) \frac{\theta_1}{(1-\lambda)\theta_2 + \lambda\theta_1} \phi_2' \left(\frac{\theta_1}{(1-\lambda)\theta_2 + \lambda\theta_1} h_1 \right) + \lambda\theta_1$$

A completely symmetric proof to Proposition 11 from the main text shows that the Pareto frontier is concave in this region.

Region 3: Suppose that $u \leq \underline{u}''$. Applying the argument from the proof of Proposition 11 from the main text, we can show that (IC_{12}) is binding and (IC_{21}) is slack, so that $\beta_2 = 0$. Then the solution is characterized by:

$$\begin{aligned}\beta_1 &= (\gamma + 1)\lambda - \gamma, \quad \gamma \leq \frac{\lambda}{(1-\gamma)}, \quad \phi_1'(h_1) = \theta_1, \quad u = c_1 - \phi_1(h_1) \\ \phi_2'(h_2) &= (\gamma\lambda + \lambda - \gamma) (\phi_1'(h_2) - \theta_2) + \theta_2 \\ c_2 &= \lambda\theta_1 h_1 + (1-\lambda)\theta_2 h_2 + \lambda (\phi_1(h_2) - \phi_1(h_1)) \\ c_1 &= \lambda\theta_1 h_1 + (1-\lambda)\theta_2 h_2 - (1-\lambda) (\phi_1(h_2) - \phi_1(h_1))\end{aligned}$$

There is a minimum level of utility for the low type achievable here, denoted u_{\min}^{RS} which is the utility level attained when h_2 is the solution to $\max_h c_2 - \phi_2(h)$ subject to (IC_{12}) , or equivalently, when $\gamma \rightarrow 0$ in the above FOCs, that is when $\phi_2'(h_2) = \lambda\phi_1'(h_2) + (1-\lambda)\theta_2$. A completely symmetric proof to before shows that the Pareto frontier is concave in this region.

The logic for this result is as follows: For u small enough (below \underline{u}'') and hence γ small enough (below $\lambda/(1-\lambda)$), (IC_{12}) is binding and the high type is distorted upwards. As u grows, h_2 is allowed to fall, until the utility u reaches exactly the threshold \underline{u}'' (and correspondingly, γ reaches the threshold $\frac{\lambda}{1-\lambda}$) and h_2 reaches its first best level. After that, as u grows, through the transfer of consumption from agent 2 to agent 1, the (IC_{12}) becomes more and more slack while the (IC_{21}) eventually becomes binding, which occurs exactly when u reaches the upper threshold \bar{u}'' and γ becomes larger than $\lambda/(1-\lambda)$.

Proposition 3 *Let the thresholds $\bar{u}', \bar{u}, \bar{u}'', \underline{u}', \underline{u}''$ and \underline{u} be defined as in the proofs of Propositions (10) and (11) in the main text, and of Proposition (2) in the Online Appendix.*

For $u \leq \bar{u}'$, the Second Best (and hence the Adverse Selection with unobservable private contracts frontier) and Rothschild-Stiglitz frontiers coincide. For $u \geq \bar{u}''$, the Rothschild-Stiglitz frontier is strictly below the Second Best frontier.

Furthermore, there are three regions:

Case 1: *If condition (NL1) holds, the thresholds for utilities are ranked as*

$$\underline{u} \leq \underline{u}' = \underline{u}'' \leq \bar{u} \leq \bar{u}'' \leq \bar{u}'$$

Region 1: For $u \leq \underline{u}'$, the Mirrlees frontier (M) is above the Second Best (SB)⁹ and Rothschild-Stiglitz (AS-RS) frontiers

Region 2: For $\underline{u}' \leq u \leq \bar{u}$, all frontiers coincide and are linear.

Region 3: For $u \geq \bar{u}$, the Second Best and Rothschild Stiglitz frontiers are both above the Mirrlees frontier.

Case 2: *If condition (NL1) does not hold, then:*

$$\underline{u} \leq \bar{u} \leq \underline{u}' = \underline{u}'' \leq \bar{u}'' \leq \bar{u}'$$

Region 1: For $u \leq \bar{u}$, the Mirrlees frontier is above the Second Best and Rothschild-Stiglitz frontiers.

Region 2: For $\bar{u} \leq u \leq \underline{u}'$, either the Second Best and Rothschild-Stiglitz or the Mirrlees frontier could be higher.

Region 3: For $u \geq \underline{u}'$, both the Second Best and the Rothschild-Stiglitz frontiers are above the Mirrlees frontier.

Proof of Proposition (3):

This proof involves only cumbersome algebra on the thresholds for utilities from the various regimes. Then, for a given ranking of those thresholds, the argument for which frontier is above the others is based solely upon considering the binding constraints and under which regime the constrained set is larger. To rank the thresholds, \underline{u}, \bar{u} (for the Mirrlees frontier), \underline{u}', \bar{u}' (for the Second Best), and $\underline{u}'', \bar{u}''$ (for the Adverse Selection case) consider the following calculations: $\underline{u}' - \underline{u} = \lambda(\phi_1(h_2^*) - \phi_1(h_1^*)) - \phi_1(h_2^*) - \lambda\left(\phi_1\left(\frac{\theta_2 h_2^*}{\theta_1}\right) - \phi_1(h_1^*)\right) + \phi_1\left(\frac{h_2^* \theta_2}{\theta_1}\right) = -(1 - \lambda)\left(\phi_1(h_2^*) - \phi_1\left(\frac{\theta_2 h_2^*}{\theta_1}\right)\right) \geq 0$
 $\bar{u} - \bar{u}' = \left(\phi_2(h_1^*) - \phi_2\left(\frac{h_1^* \theta_1}{\theta_2}\right)\right) \lambda + \left(\phi_2\left(\frac{h_1^* \theta_1}{\theta_2}\right) - \phi_2(h_1^*)\right) = -(1 - \lambda)\left(\phi_2(h_1^*) - \phi_2\left(\frac{h_1^* \theta_1}{\theta_2}\right)\right) \leq 0$. Hence, it is always the case that: $\bar{u} \leq \bar{u}'$ and $\underline{u}' \geq \underline{u}$. In addition, we need to compare \underline{u}' to \bar{u} :

$$\begin{aligned} \underline{u}' \leq \bar{u} &\Leftrightarrow \lambda \theta_1 h_1^* + (1 - \lambda) \theta_2 h_2^* + \lambda(\phi_1(h_2^*) - \phi_1(h_1^*)) - \phi_1(h_2^*) \leq \lambda \theta_1 h_1^* + (1 - \lambda) \theta_2 h_2^* \\ &\quad + \lambda \left(\phi_2(h_2^*) - \phi_2\left(\frac{h_1^* \theta_1}{\theta_2}\right) \right) - \phi_2(h_2^*) + \phi_2\left(\frac{h_1^* \theta_1}{\theta_2}\right) - \phi_1(h_1^*) \\ &\Leftrightarrow (1 - \lambda)(\phi_1(h_1^*) - \phi_1(h_2^*)) \leq -(1 - \lambda) \left(\phi_2(h_2^*) - \phi_2\left(\frac{h_1^* \theta_1}{\theta_2}\right) \right) \\ &\Leftrightarrow (\phi_1(h_1^*) - \phi_1(h_2^*)) \leq \phi_2\left(\frac{h_1^* \theta_1}{\theta_2}\right) - \phi_2(h_2^*) \end{aligned}$$

⁹Recall that the Second Best Frontier coincides with the Adverse Selection with unobservable private contracts frontier.

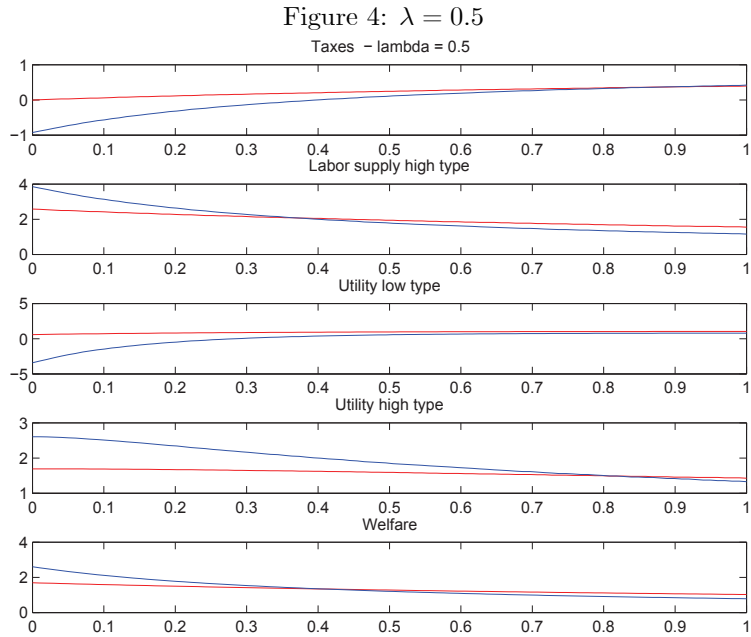
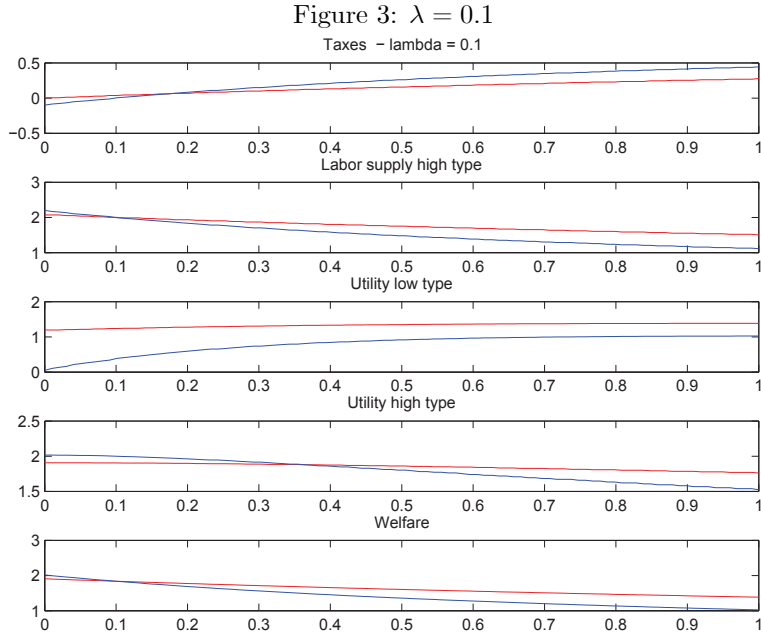
If this condition holds, (condition **NL1** in the main text) then $\underline{u}' \leq \bar{u}$. Comparing \bar{u}'' and \bar{u}' , as well as \bar{u}'' and \bar{u} yields:

$$\begin{aligned}\bar{u}'' \leq \bar{u}' &\Leftrightarrow \phi_2 \left(\frac{h_1^* \theta_1}{\lambda \theta_1 + (1 - \lambda) \theta_2} \right) \leq \phi_2 (h_1^*) \\ \bar{u}'' \geq \bar{u} &\Leftrightarrow \phi_2 \left(\frac{\theta_1 h_1^*}{(1 - \lambda) \theta_2 + \lambda \theta_1} \right) \geq \phi_2 \left(\frac{h_1^* \theta_1}{\theta_2} \right)\end{aligned}$$

which are both always true. In addition, note that $\underline{u}' = \underline{u}''$. Hence, there are two possible cases. If *(NL1)* holds, then $\underline{u} \leq \underline{u}' = \underline{u}'' < \bar{u} \leq \bar{u}'' \leq \bar{u}'$. Else, $\underline{u} \leq \bar{u} \leq \underline{u}' = \underline{u}'' \leq \bar{u}'' \leq \bar{u}'$.

III Numerical Analysis of the Linear Tax in Section 3

In this Section, I provide some numerical illustrations of the taxes and allocations for the $N = 2$ types analysis in Section 3 of the main text. The simulations assume the following parameter values: $\theta_1 = 1$, $\theta_2 = 2$, $\phi_1(h) = \frac{2}{3}h^2$, $\phi_2(h) = \frac{1}{2}h^2$, $\lambda = 0.1$ and $\lambda = 0.5$, respectively.



IV Proofs from the Main text

Lemma 1 Monotonicity: *At an implementable solution, we have $h_2(t) \geq h_1(t)$:*

Proof:

Combining the two (IC) constraints yields:

$$\phi_2(h_1) - \phi_2(h_2) \geq (1-t)(y_1 - y_2) \geq \phi_1(h_1) - \phi_1(h_2)$$

which requires: $\phi_2(h_2) - \phi_2(h_1) \leq \phi_1(h_2) - \phi_1(h_1)$, and hence: $\int_{h_1}^{h_2} \phi_2'(h) dh \leq \int_{h_1}^{h_2} \phi_1'(h) dh$. But since by the Spence-Mirrlees single-crossing condition, $\phi_1'(h) > \phi_2'(h)$ at every h , implementability requires that $h_2(t) \geq h_1(t)$.

Proof of Proposition 2:

i) Let us show that (IC₁₂) is binding, if condition (4) and assumption (2) hold. Suppose by contradiction that at some level of taxes t , (IC₁₂) was slack. In that case, necessarily $\varphi > 0$, as explained above. With both incentive constraints slack, we know that both h_1 and h_2 would be at their efficient levels so that: $h_i^*(t) = (\theta_i(1-t)/(a_i\eta))^{\frac{1}{\eta-1}}$ and a slack (IC₁₂) would imply, whenever $t < 1$:

$$\begin{aligned} & (1-t)\theta_1 \left(\frac{\theta_1(1-t)}{a_1\eta} \right)^{\frac{1}{\eta-1}} - a_1 \left(\frac{\theta_1(1-t)}{a_1\eta} \right)^{\frac{\eta}{\eta-1}} > (1-t)\theta_2 \left(\frac{\theta_2(1-t)}{a_2\eta} \right)^{\frac{1}{\eta-1}} - a_1 \left(\frac{\theta_2(1-t)}{a_2\eta} \right)^{\frac{\eta}{\eta-1}} \\ \Leftrightarrow & \theta_1 \left(\frac{\theta_1}{a_1\eta} \right)^{\frac{1}{\eta-1}} - a_1 \left(\frac{\theta_1}{a_1\eta} \right)^{\frac{\eta}{\eta-1}} > \theta_2 \left(\frac{\theta_2}{a_2\eta} \right)^{\frac{1}{\eta-1}} - a_1 \left(\frac{\theta_2}{a_2\eta} \right)^{\frac{\eta}{\eta-1}} \end{aligned}$$

which exactly violates condition (4).

ii) With isoelastic utility of the form $\phi_i(h) = a_i h^\eta$, the (IC₁₂) constraint implies that: $\theta_2(1-t)h_2 - a_1 h_2^\eta = K(t)$ where $K(t) = (1-t)\theta_1 h_1^* - a_1 h_1^{\eta}$. But from the FOC of the low type, $\eta a_1 h_1^*(t)^{\eta-1} = (1-t)\theta_1$ hence $h_1^*(t) = [(1-t)\theta_1/a_1\eta]^{\frac{1}{\eta-1}}$ so that $K(t) = (\eta-1)a_1 \left(\frac{\theta_1}{a_1\eta} \right)^{\frac{\eta}{\eta-1}} (1-t)^{\frac{\eta}{\eta-1}}$. Hence the (IC₁₂) constraint implies that: $\theta_2(1-t)h_2 - a_1 h_2^\eta = (\eta-1)a_1 \left(\frac{\theta_1}{a_1\eta} \right)^{\frac{\eta}{\eta-1}} (1-t)^{\frac{\eta}{\eta-1}}$. For this to hold at any value of t , we require that: $h_2 = M(1-t)^{\frac{1}{\eta-1}}$, for a constant M , independent of t . In that case, $\tilde{\lambda}$ becomes independent of t :

$$\tilde{\lambda}(t) = \frac{(1-t)\theta_2 - a_2 M^{\eta-1}(1-t)}{(1-t)\theta_2 - a_1 M^{\eta-1}(1-t)} = \frac{\theta_2 - a_2 M^{\eta-1}}{\theta_2 - a_1 M^{\eta-1}}$$

Proof of Proposition 10:

This proof is similar to Bierbrauer and Boyer (2010), adapted to the case at hand. I first reformulate the problem as maximizing type 2's utility subject to the low type's utility constraint. Multipliers are in brackets on the line of the constraint they apply to:

$$\begin{aligned} (P^{Mirr}(u)) : & \max_{\{c_1, c_2, h_1, h_2\}} c_2 - \phi_2(h_2) \\ (IC_{12}) : & c_1 - \phi_1(h_1) \geq c_2 - \phi_1\left(\frac{h_2\theta_2}{\theta_1}\right) \quad [\beta_1] \\ (IC_{21}) : & c_2 - \phi_2(h_2) \geq c_1 - \phi_2\left(\frac{h_1\theta_1}{\theta_2}\right) \quad [\beta_2] \\ (RC) : & \lambda c_1 + (1-\lambda)c_2 \leq \lambda\theta_1 h_1 + (1-\lambda)\theta_2 h_2 \quad [\delta] \\ & c_1 - \phi_1(h_1) \geq u \quad [\gamma] \end{aligned}$$

The Pareto weights from the main text can be mapped into the multipliers of the utility constraints using $\gamma = \mu_1/\mu_2 = \mu/(1-\mu)$. Note that if the Pareto frontier is linear in some regions, then the same set of Pareto weights could correspond to several different levels of u . I simultaneously solve for the thresholds for μ in the Proposition and the thresholds for u from the text.

Denote the set of admissible allocations, i.e., the allocations $\{c_1, c_2, h_1, h_2\}$ which satisfy both incentive compatibility and the resource constraint, by $B^{Mirr}(u)$ and let $V_2^{Mirr}(u) = \max_{c_1, c_2, h_1, h_2 \in B^{Mirr}(u)} (c_2 - \phi_2(h_2))$. When the value of u varies, the function $V_2^{Mirr}(u)$ traces out all possible values for the utility of the high type. The Pareto frontier consists of all pairs $(u, V_2^{Mirr}(u))$ such that $\frac{\partial}{\partial u} V_2^{Mirr}(u) < 0$.

The general solution to this problem is then characterized by the following necessary conditions:

$$\begin{aligned}
[c_2] : 1 + \beta_2 - \beta_1 - (1 - \lambda) \delta &= 0 \\
[c_1] : \beta_1 - \beta_2 + \delta(1 - \lambda) - 1 &= 0 \\
\gamma &= \delta - 1 \\
[h_1] : -(\beta_1 + (\delta - 1)) \phi_1'(h_1) + \beta_2 \frac{\theta_1}{\theta_2} \phi_2' \left(\frac{h_1 \theta_1}{\theta_2} \right) + \delta \theta_1 \lambda &= 0 \\
[h_2] : -\phi_2'(h_2) + \beta_1 \frac{\theta_2}{\theta_1} \phi_1' \left(\frac{h_2 \theta_2}{\theta_1} \right) + \delta \theta_2 (1 - \lambda) &= 0 \\
\lambda c_1 + (1 - \lambda) c_2 = \lambda \theta_1 h_1 + (1 - \lambda) \theta_2 h_2, c_1 - \phi_1(h_1) &= u
\end{aligned}$$

There are three possible cases to consider: either one of (IC_{12}) or (IC_{21}) is binding, or none is. It is never optimal to have both binding, as has been shown several times in the literature (e.g., Bierbrauer and Boyer, 2010).

Region 1. Suppose that both constraints are slack. This case occurs when $\beta_1 = 0, \beta_2 = 0$, and hence $\gamma = \bar{\gamma} \equiv \frac{\lambda}{(1-\lambda)}$. In this region, hours of work are at their efficient levels, defined by $\phi_i'(h_i) = \theta_i$. The interval of levels of utility u for which this can occur, denoted by $[\underline{u}, \bar{u}]$, is derived as follows. Suppose that u increases. At some level, it will become attractive for the high type to pretend to be a low type, and the (IC_{21}) will just start binding. One can easily check that this will occur exactly at level \bar{u} , defined by: $\bar{u} \equiv \bar{c}_2 - \phi_2(h_2^*) + \phi_2 \left(\frac{h_1^* \theta_1}{\theta_2} \right) - \phi_1(h_1^*)$, where $\bar{c}_2 \equiv \lambda \theta_1 h_1^* + (1 - \lambda) \theta_2 h_2^* + \lambda \left(\phi_2(h_2^*) - \phi_2 \left(\frac{h_1^* \theta_1}{\theta_2} \right) \right)$. Similarly, if u decreases too much, the low type will start wanting to pretend to be a high type and (IC_{12}) will just become binding. This occurs exactly at utility level \underline{u} defined by $\underline{u} \equiv \underline{c}_2 - \phi_1 \left(\frac{h_2^* \theta_2}{\theta_1} \right)$, where $\underline{c}_2 \equiv \lambda \theta_1 h_1^* + (1 - \lambda) \theta_2 h_2^* + \lambda \left(\phi_1 \left(\frac{\theta_2 h_2^*}{\theta_1} \right) - \phi_1(h_1^*) \right)$. Note that in this region, the Pareto frontier is linear, since $V_2^{Mirr}(u) = c_2 - \phi_2(h_2^*) = \frac{\lambda}{(1-\lambda)} \theta_1 h_1 + \theta_2 h_2 - \frac{\lambda}{(1-\lambda)} u - \frac{\lambda}{(1-\lambda)} \phi_1(h_1) - \phi_2(h_2^*)$, so that $\partial V_2^{Mirr}(u) / \partial u = -\lambda / (1 - \lambda)$.

Region 2. Suppose that (IC_{21}) is binding, then (IC_{12}) must be slack, so that $\beta_1 = 0$. The solution is then characterized by:

$$\gamma\phi'_1(h_1) = \gamma\theta_1 + [\gamma - \gamma\lambda - \lambda] \left[\frac{\theta_1}{\theta_2}\phi'_2\left(\frac{h_1\theta_1}{\theta_2}\right) - \theta_1 \right] \quad (5)$$

$$c_1 = \lambda\theta_1 h_1 + (1 - \lambda)\theta_2 h_2 - (1 - \lambda) \left(\phi_2(h_2) - \phi_2\left(\frac{h_1\theta_1}{\theta_2}\right) \right)$$

$$c_2 = \lambda\theta_1 h_1 + (1 - \lambda)\theta_2 h_2 + \lambda \left(\phi_2(h_2) - \phi_2\left(\frac{h_1\theta_1}{\theta_2}\right) \right)$$

$$h_2 = h_2^* \equiv (\phi'_2)^{-1}(\theta_2)$$

$$u + \phi_1(h_1) = \lambda\theta_1 h_1 + (1 - \lambda)\theta_2 h_2^* - (1 - \lambda) \left(\phi_2(h_2^*) - \phi_2\left(\frac{h_1\theta_1}{\theta_2}\right) \right) \quad (6)$$

Note that this case is consistent (i.e., $\beta_2 \geq 0$) if and only if $\gamma \geq \bar{\gamma}$, or equivalently, if and only if $u \geq \bar{u}$. There is a downward distortion in the labor supply of the low type, as can be seen from the FOCs for h_1 .

The maximal value for u in this region, denoted by u_{\max}^{Mirr} , is achieved when h_1 is at the level which maximizes $c_1 - \phi_1(h_1)$ subject to (IC_{21}) , or equivalently, at the level which obtains when $\gamma \rightarrow \infty$ and $\phi'_1(h_1) = \lambda\theta_1 + (1 - \lambda)\frac{\theta_1}{\theta_2}\phi'_2\left(\frac{h_1\theta_1}{\theta_2}\right)$.

In this region, $\frac{\partial}{\partial u} V_2^{Mirr}(u) = -\gamma < 0$. The Lagrange multiplier γ is given by:

$$\gamma = \lambda \frac{\left[\theta_1 - \frac{\theta_1}{\theta_2}\phi'_2\left(\frac{h_1\theta_1}{\theta_2}\right) \right]}{\left(\phi'_1(h_1) - \lambda\theta_1 - [1 - \lambda]\frac{\theta_1}{\theta_2}\phi'_2\left(\frac{h_1\theta_1}{\theta_2}\right) \right)}$$

Note that since $\theta_2 = \phi'_2(h_2)$, we have: $\phi'_2\left(\frac{h_1\theta_1}{\theta_2}\right) \leq \phi'_2(h_1) \leq \phi'_2(h_2) = \theta_2$ so that the numerator of γ is positive and hence the denominator must also be positive.

To show that the Pareto frontier is concave in this region, note that:

$$\frac{\partial^2}{\partial u^2} V_2^{Mirr}(u) = \lambda \frac{\frac{\partial h_1}{\partial u} \left(\frac{\theta_1}{\theta_2} \right) \left(\frac{\theta_1}{\theta_2} \right) \phi''_2\left(\frac{h_1\theta_1}{\theta_2}\right) \left(\phi'_1(h_1) - \theta_1 \right) + \left[\theta_2 - \phi'_2\left(\frac{h_1\theta_1}{\theta_2}\right) \right] \phi''_1(h_1)}{\left(\phi'_1(h_1) + \lambda\theta_1 - [1 - \lambda]\frac{\theta_1}{\theta_2}\phi'_2\left(\frac{h_1\theta_1}{\theta_2}\right) \right)^2} \quad (7)$$

Using expression in (6), the fact that $dh_2/du = 0$ in this region, and that the denominator of γ is positive, we can see that $dh_1/du < 0$. In addition, the numerator in (7) is positive from the SOC with respect to h_1 . Hence, $\partial^2 V_2^{Mirr}(u)/\partial u^2 < 0$.

Region 3. Suppose that (IC_{12}) is binding and (IC_{21}) is slack, with $\beta_2 = 0$. The solution is characterized by:

$$h_1 = h_1^* \equiv (\phi'_1)^{-1}(\theta_1), \quad \phi'_2(h_2) = \theta_2 + (-\gamma + \lambda\gamma + \lambda) \left[\frac{\theta_2}{\theta_1}\phi'_1\left(\frac{\theta_2 h_2}{\theta_1}\right) - \theta_2 \right]$$

$$c_1 = \lambda\theta_1 h_1 + (1 - \lambda)\theta_2 h_2 - (1 - \lambda) \left(\phi_1\left(\frac{\theta_2 h_2}{\theta_1}\right) - \phi_1(h_1) \right)$$

$$c_2 = \lambda\theta_1 h_1 + (1 - \lambda)\theta_2 h_2 + \lambda \left(\phi_1\left(\frac{\theta_2 h_2}{\theta_1}\right) - \phi_1(h_1) \right)$$

$$c_1 - \phi_1(h_1) = u$$

This case is consistent (i.e., $\beta_1 \geq 0$) if and only if $\gamma \leq \bar{\gamma}$ which is equivalent to $u \leq \underline{u}$. The minimal level of utility u in this regime, denoted by u_{\min}^{Mirr} , is achieved when h_2 is solution to $\max(c_2 - \phi_2(h_2))$ subject to (IC_{12}) , or

equivalently, when $\gamma \rightarrow 0$ and $\phi'_2(h_2) = (1 - \lambda)\theta_2 + \lambda \left[\frac{\theta_2}{\theta_1} \phi'_1 \left(\frac{\theta_2 h_2}{\theta_1} \right) \right]$. The argument to show concavity is as for Region 2, except that the SOC for h_2 is used.

The proof is complete by recalling that $\gamma = \mu/(1 - \mu)$ so that $\gamma > \bar{\gamma} = \lambda/(1 - \lambda) \Leftrightarrow \mu > \lambda$.

Proof of Proposition 11 :

The problem, indexed by the utility level of the low type and specialized to $N = 2$ is reformulated here, with multipliers in brackets after each constraint:

$$\begin{aligned} (P^{SB}(u)) &: \max c_2 - \phi_2(h_2) \\ (IC_{12}) &: c_1 - \phi_1(h_1) \geq c_2 - \phi_1(h_2) \quad [\beta_1] \\ (IC_{21}) &: c_2 - \phi_2(h_2) \geq c_1 - \phi_2(h_1) \quad [\beta_2] \\ (RC) &: \lambda c_1 + (1 - \lambda) c_2 \leq \lambda \theta_1 h_1 + (1 - \lambda) \theta_2 h_2 \quad [\delta] \\ c_1 - \phi_1(h_1) &\geq u \quad (\gamma) \quad [\gamma] \end{aligned}$$

Again, we can map problem $P^{SB}(\mu)$ to $P^{SB}(u)$ using $\gamma = \mu/(1 - \mu)$, taking into account that in linear regions of the Pareto frontier, several values of u correspond to the same value of μ . Denote the set of allocations satisfying both incentive compatibility and the resource constraint by $B^{SB}(u)$. Let $V_2^{SB}(u) = \max_{c_1, c_2, h_1, h_2 \in B^{SB}(u)} (c_2 - \phi_2(h_2))$. When the value of u varies, the function $V_2^{SB}(u)$ traces out all possible values for the utility of the high type. The Pareto frontier consists of all pairs $(u, V_2^{SB}(u))$ such that $\frac{\partial}{\partial u} V_2^{SB}(u) < 0$. In the general case, the FOCs are:

$$\begin{aligned} [c_2] &: 1 + \beta_2 - \beta_1 - \delta(1 - \lambda) = 0 \\ [c_1] &: \beta_1 + \gamma - \beta_2 - \delta\lambda = 0 \\ [h_1] &: -\beta_1 \phi'_1(h_1) + \beta_2 \phi'_2(h_1) + \delta\lambda\theta_1 - \gamma\phi'_1(h_1) = 0 \\ [h_2] &: -(1 + \beta_2) \phi'_2(h_2) + \beta_1 \phi'_1(h_2) + \delta(1 - \lambda)\theta_2 = 0 \\ &\lambda c_1 + (1 - \lambda) c_2 = \lambda \theta_1 h_1 + (1 - \lambda) \theta_2 h_2, \quad c_1 - \phi_1(h_1) = u \end{aligned}$$

Region 1: Suppose that the utility level u is in $[\underline{u}', \bar{u}']$ where $\underline{u}' \equiv \underline{c}_2 - \phi_1(h_2^*)$, $\underline{c}_2 \equiv \lambda\theta_1 h_1^* + (1 - \lambda)\theta_2 h_2^* + \lambda(\phi_1(h_2^*) - \phi_1(h_1^*))$, $\bar{u}' \equiv \bar{c}_2 - \phi_2(h_2^*) + \phi_2(h_1^*) - \phi_1(h_1^*)$, and $\bar{c}_2 \equiv c_2 = \lambda\theta_1 h_1^* + (1 - \lambda)\theta_2 h_2^* + \lambda(\phi_2(h_2^*) - \phi_2(h_1^*))$. One can check that it is possible to set hours at their efficient levels, and c_1 and c_2 such that:

$$\lambda c_1 + (1 - \lambda) c_2 = \lambda \theta_1 h_1^* + (1 - \lambda) \theta_2 h_2^*, \quad c_1 = u + \phi_1(h_1)$$

and to have both incentive constraints slack. Any distortion in the hours of work would imply a reduced welfare. Hence, both $\beta_1 = \beta_2 = 0$, and $\gamma = \bar{\gamma} = \lambda/(1 - \lambda)$. The Pareto frontier is linear and decreasing in u , since $\frac{\partial}{\partial u} V_2^{SB}(u) = \frac{-\lambda}{1 - \lambda}$.

Region 2: Suppose that $u \geq \bar{u}'$. Suppose by contradiction that (IC_{21}) is slack, so that $h_1 = h_1^*$. Since $u_1 = c_1 - \phi_1(h_1^*)$, the inequality on u implies that: $c_1 \geq \lambda\theta_1 h_1^* + (1 - \lambda)\theta_2 h_2^* + \lambda(\phi_2(h_2^*) - \phi_2(h_1^*)) - \phi_2(h_2^*) + \phi_2(h_1^*)$.

The utility of the high type from pretending to be low type would be:

$$c_1 - \phi_2(h_1^*) \geq \lambda \theta_1 h_1^* + (1 - \lambda) \theta_2 h_2^* + \lambda (\phi_2(h_2^*) - \phi_2(h_1^*)) - \phi_2(h_2^*) = \bar{c}_2 - \phi_2(h_1^*) \geq c_2 - \phi_2(h_2^*)$$

where the last inequality follows from the inequality on c_1 , which implies that $c_2 \leq \bar{c}_2$ by the resource constraint. Hence, the high type agent has an incentive to deviate if $h_1 = h_1^*$, so that in fact, we need to have $h_1 < h_1^*$. But then, (IC_{21}) needs to be binding, or we could increase h_1 by some small dh_1 without violating (IC_{21}) and generate more output than would be necessary to compensate the low type for the increased effort (that is, $\theta_1 dh_1 > \phi_1'(h_1) dh_1$). In addition, the constraint of the low type, (IC_{12}) , is slack since:

$$c_2 - \phi_1(h_2^*) \leq \lambda \theta_1 h_1^* + (1 - \lambda) \theta_2 h_2^* + \lambda (\phi_2(h_2^*) - \phi_2(h_1^*)) - \phi_1(h_2^*) \leq c_1 + \phi_2(h_2^*) - \phi_2(h_1^*) - \phi_1(h_2^*) < c_1 - \phi_1(h_1^*)$$

where we used that $\phi_2(h_2^*) - \phi_2(h_1^*) < \phi_1(h_2^*) - \phi_1(h_1^*)$, implied by the Spence-Mirrlees condition. Hence $\beta_1 = 0$ and the solution in Region 2 is characterized by:

$$\begin{aligned} \beta_2 &= \gamma - \lambda\gamma - \lambda, \quad \gamma \geq \frac{\lambda}{(1 - \lambda)}, \quad \delta = \gamma + 1, \quad u = c_1 - \phi_1(h_1) \\ \gamma \phi_1'(h_1) &= (\gamma - \lambda\gamma - \lambda) (\phi_2'(h_1) - \theta_1) + \gamma \theta_1 \\ \theta_2 &= \phi_2'(h_2), \\ c_2 &= \lambda \theta_1 h_1 + (1 - \lambda) \theta_2 h_2 + \lambda (\phi_2(h_2) - \phi_2(h_1)) \\ c_1 &= \lambda \theta_1 h_1 + (1 - \lambda) \theta_2 h_2 - (1 - \lambda) (\phi_2(h_2) - \phi_2(h_1)) \end{aligned}$$

The maximal level of utility u in this region, denoted by u_{\max}^{SB} , is the level achieved when h_1 is set to maximize $c_1 - \phi_1(h_1)$ subject to (IC_{21}) , or equivalently, the level of u when $\gamma \rightarrow \infty$ and $\phi_1'(h_1) = (1 - \lambda) \phi_2'(h_1) + \lambda \theta_1$. The Pareto frontier is decreasing since $\partial V_2^{SB}(u) / \partial u = -\gamma < 0$. In addition, it is concave. Indeed,

$$\frac{\partial^2 V_2^{SB}(u)}{\partial u^2} = -\frac{\partial \gamma}{\partial u} = -\lambda \frac{\partial h_1}{\partial u} \frac{\phi_2''(h_1) (\theta_1 - \phi_1'(h_1)) - (\theta_1 - \phi_2'(h_1)) \phi_1''(h_1)}{[\phi_1'(h_1) - (1 - \lambda) \phi_2'(h_1) - \lambda \theta_1]^2}$$

First note that $\gamma = \lambda \frac{\theta_1 - \phi_2'(h_1)}{[\phi_1'(h_1) - (1 - \lambda) \phi_2'(h_1) - \lambda \theta_1]}$. Since the numerator is positive, the denominator must be positive too. From the constraints: $u = c_1 - \phi_1(h_1) = \lambda \theta_1 h_1 + (1 - \lambda) \theta_2 h_2^* - (1 - \lambda) (\phi_2(h_2^*) - \phi_2(h_1)) - \phi_1(h_1)$. Given that $\frac{\partial h_2}{\partial u} = 0$ in this region and that the denominator of γ is positive, the previous expression shows that $\frac{\partial h_1}{\partial u} < 0$. From the SOC in h_1 and the positive denominator of γ , we have $\partial^2 V_2^{SB}(u) / \partial u^2 < 0$.

Region 3: Suppose that u is such that $u \leq \underline{u}'$, where \underline{u}' is defined as $\underline{u}' \equiv \underline{c}_2 - \phi_1(h_2^*)$ and $\underline{c}_2 \equiv \lambda \theta_1 h_1^* + (1 - \lambda) \theta_2 h_2^* + \lambda (\phi_1(h_2^*) - \phi_1(h_1^*))$. Define $\underline{c}_1 \equiv \lambda \theta_1 h_1^* + (1 - \lambda) \theta_2 h_2^* - (1 - \lambda) (\phi_1(h_2^*) - \phi_1(h_1^*))$ from the budget constraint. In this case, it must be that (IC_{12}) is violated at the first best hour levels, since:

$$\begin{aligned} c_1 - \phi_1(h_1^*) &\leq \underline{c}_1 - \phi_1(h_1^*) = \lambda \theta_1 h_1^* + (1 - \lambda) \theta_2 h_2^* - (1 - \lambda) (\phi_1(h_2^*) - \phi_1(h_1^*)) - \phi_1(h_1^*) \\ &= \underline{c}_2 - \lambda (\phi_1(h_2^*) - \phi_1(h_1^*)) - (1 - \lambda) (\phi_1(h_2^*) - \phi_1(h_1^*)) - \phi_1(h_1^*) = \underline{c}_2 - \phi_1(h_2^*) \end{aligned}$$

Hence, h_2^* is distorted upwards, and (IC_{12}) must be binding. If it were not, we could decrease h_2^* and c_2 simultaneously so as not to change the utility of the high type and still create a surplus to be given to the low type. Thus, $\beta_1 \geq 0$. In addition, (IC_{21}) is slack. From the (IC_{12}) , replace c_2 as a function of c_1 and note

that $c_2 - \phi_2(h_2) = c_1 + \phi_1(h_2) - \phi_1(h_1^*) - \phi_2(h_2) \geq c_1 - \phi_2(h_1^*)$, where the last inequality follows from the Spence-Mirrlees single crossing condition. Hence $\beta_2 = 0$. The solution in this case is fully characterized by:

$$\begin{aligned}\beta_1 &= (\gamma + 1)\lambda - \gamma, \quad \gamma < \frac{\lambda}{(1 - \lambda)}, \quad \phi_1'(h_1) = \theta_1, \quad u = c_1 - \phi_1(h_1) \\ \phi_2'(h_2) &= (\gamma\lambda + \lambda - \gamma)(\phi_1'(h_2) - \theta_2) + \theta_2 \\ c_2 &= \lambda\theta_1 h_1 + (1 - \lambda)\theta_2 h_2 + \lambda(\phi_1(h_2) - \phi_1(h_1)) \\ c_1 &= \lambda\theta_1 h_1 + (1 - \lambda)\theta_2 h_2 - (1 - \lambda)(\phi_1(h_2) - \phi_1(h_1))\end{aligned}$$

The minimal level of utility u in this region, denoted by u_{\min}^{SB} , is the utility level achieved when h_2 is set to maximize $c_2 - \phi_2(h_2)$ subject to (IC_{12}) , or equivalently, the level of u when $\gamma \rightarrow 0$ and $\phi_2'(h_2) = \lambda\phi_1'(h_2) + (1 - \lambda)\theta_2$. The proof of concavity of the frontier is exactly as for Region 2.

Finally, the proof is complete by recalling that $\gamma = \mu / (1 - \mu)$ so that $\gamma > \bar{\gamma} = \lambda / (1 - \lambda) \Leftrightarrow \mu > \lambda$.