Economics 2450A: Public Economics
Section 1-2: Uncompensated and Compensated Elasticities; Static and Dynamic Labor Supply

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In today's section, we will briefly review the concepts of substitution (compensated) elasticity and uncompensated elasticity. As we will see in the next few weeks compensated and uncompensated labor elasticities play a key role in studies of optimal income taxation. In the second part of the section we will study the context of labor supply choices in a static and dynamic framework.

1 Uncompensated Elasticity and the Utility Maximization Problem

The utility maximization problem: We start by defining the concept of *Walrasian demand* in a standard *utility maximization problem (UMP)*. Suppose the agent chooses a bundle of consumption goods $x_1, \ldots, x_N$ with prices $p_1, \ldots, p_N$ and her endowment is denoted by w. The optimal consumption bundle solves the following:

$$\max_{x_1,\ldots,x_N} u\left(x_1, \ldots, x_N\right)$$

s.t.

$$\sum_{i=1}^{N} p_i x_i \leq w$$

We solve the problem using a Lagrangian approach and we get the following optimality condition (if an interior optimum exists) for every good i:

$$u_i\left(\mathbf{x}^*\right) - \lambda^* p_i = 0$$

Solving this equation for λ^* and doing the same for good j yields:

$$\frac{u_i\left(\mathbf{x}^*\right)}{u_j\left(\mathbf{x}^*\right)} = \frac{p_i}{p_j}$$

This is an important condition in economics and it equates the relative price of two goods to the *marginal rate of substitution (MRS)* between them. The MRS measures the amount of good j that the consumer must be given to compensate the utility loss from a one-unit marginal reduction in her consumption of good i. Graphically, the price ratio is the slope of the budget constraint, while the ratio of marginal utilities represents the slope of the indifference curve.[1]

We call the solution to the utility maximization problem *Walrasian* or *Marshallian demand* and we represent it as a function $\mathbf{x}\left(\mathbf{p}, w\right)$ of the price vector and the endowment. The Walrasian demand has the following two properties:

[1] Notice that in a two goods economy by differentiating the indifference curve $u\left(x_1, x_2\left(x_1\right)\right) = k$ wrt x_1 you get:

- *homogeneity of degree zero*: $x_i(\alpha \mathbf{p}, \alpha w) = x_i(\mathbf{p}, w)$
- *Walras Law*: for every $\mathbf{p} \gg 0$ and $w > 0$ we have $p \cdot \mathbf{x}(\mathbf{p}, w) = w$

We define *uncompensated elasticity* as the percentage change in the consumption of good i when we raise the price p_k. Using the Walrasian demand we can write the uncompensated elasticity as:

$$\varepsilon^u_{i,p_k} = \frac{\partial x_i(\mathbf{p}, w)}{\partial p_k} \frac{p_k}{x_i(\mathbf{p}, w)}$$

Elasticities can also be defined using logarithms such that:

$$\varepsilon^u_{i,p_k} = \frac{\partial \log x_i(\mathbf{p}, w)}{\partial \log p_k}$$

Indirect utility: We introduce the concept of indirect utility that will be useful throughout the class. It also helps interpreting the role of the Lagrange multiplier. The indirect utility is the utility that the agent achieves when consuming the optimal bundle $\mathbf{x}(\mathbf{p}, w)$. It can be obtained by plugging the Walrasian demand into the utility function:

$$v(\mathbf{p}, w) = u(\mathbf{x}(\mathbf{p}, w))$$

The indirect utility has the following properties:

- *homogeneity of degree zero*: since the Walrasian demand is homogeneous of degree zero, it follows that the indirect utility will inherit this property
- $\partial v(\mathbf{p}, w)/\partial w > 0$ and $\partial v(\mathbf{p}, w)/\partial p_k \leq 0$

Roy's Identity and the multiplier interpretation: Using the indirect utility function, the value of the problem can be written as follows at the optimum:

$$v(\mathbf{p}, w) = u(\mathbf{x}^*(\mathbf{p}, w)) + \lambda^*(w - \mathbf{p} \cdot \mathbf{x}^*(\mathbf{p}, w))$$

Applying the Envelope theorem, we can study how the indirect utility responds to changes in the agent's wealth:

$$\frac{\partial v(\mathbf{p}, w)}{\partial w} = \lambda^*$$

The value of the Lagrange multiplier at the optimum is the shadow value of the constraint. Specifically, it is the increase in the value of the objective function resulting from a slight relaxation of the constraint achieved by giving an extra dollar of endowment to the agent. This interpretation of the Lagrangian multiplier is particularly important in the study of optimal Ramsey taxes and transfers. You will see more about it in the second part of the PF sequence.

The Envelope theorem also implies that:

$$u_1 + u_2 \frac{dx_2}{dx_1} = 0$$

which delivers

$$\frac{dx_2}{dx_1} = -\frac{u_1}{u_2}$$

which shows that the ratio of marginal utilities is the slope of the indifference curve at a point (x_1, x_2).

$$\frac{\partial v(\mathbf{p},w)}{\partial p_i} = -\lambda^* x_i(\mathbf{p},w)$$

Using the two conditions together we have:

$$-\frac{\frac{\partial v(\mathbf{p},w)}{\partial p_i}}{\frac{\partial v(\mathbf{p},w)}{\partial w}} = x_i(\mathbf{p},w)$$

This equation is know as the *Roy's Identity* and it derives the Walrasian demand from the indirect utility function.

2 Substitution Elasticity and the Expenditure Minimization Problem

In this section we aim to isolate the substitution effect of a change in price. An increase in the price of good i typically generates two effects:

- *substitution effect*: the relative price of x_i increases, therefore the consumer substitutes away from this good towards other goods,
- *income effect*: the consumer's purchasing power has decreased, therefore she needs to reoptimize her entire bundle. This reduces even more the consumption of good i.

We define *substitution or compensated elasticity* as the percentage change in the demand for a good in response to a change in a price that ignores the income effect. In order to get at this new concept, we focus on a problem that is "dual" to the utility maximization problem: the *expenditure minimization problem (EMP)*. The consumer solves:

$$\min_{x_1,\ldots,x_N} \sum_{i=1}^{N} p_i x_i$$

s.t.

$$u(x_1,\ldots,x_N) \geq \bar{u}$$

The problem asks to solve for the consumption bundle that minimizes the amount spent to achieve utility level $\bar{u}$. The solution delivers two important functions: the *expenditure function* $e(\mathbf{p},\bar{u})$, which measures the total expenditure needed to achieve utility $\bar{u}$ under the price vector $\mathbf{p}$, and the *Hicksian* (or *compensated*) *demand* $\mathbf{h}(\mathbf{p},\bar{u})$, which is the demand vector that solves the minimization problem.

The Walrasian and Hicksian demands answer two different but related problems. The following two statements establish a relationship between the two concepts:

1. If $\mathbf{x}^*$ is optimal in the UMP when wealth is w, then $\mathbf{x}^*$ is optimal in the EMP when $\bar{u} = u(\mathbf{x}^*)$. Moreover, $e(\mathbf{p},\bar{u}) = w$.
2. If $\mathbf{x}^*$ is optimal in the EMP when $\bar{u}$ is the required level of utility, then $\mathbf{x}^*$ is optimal in the UMP when $w = \mathbf{p}\cdot\mathbf{x}^*$. Moreover, $\bar{u} = u(\mathbf{x}^*)$.

The Hicksian demand allows us to isolate the pure substitution effect in response to a price change. We call it compensated since it is derived following the idea that, after a price change, the consumer will be given enough wealth (the "compensation") to maintain the same utility level she experienced before the price change. Suppose that under the price vector $\mathbf{p}$ the consumer demands a bundle $\mathbf{x}$ such that

$\mathbf{p}\cdot\mathbf{x} = w$. When the price vector is $\mathbf{p}'$, the consumer solves the new expenditure minimization problem and switches to $\mathbf{x}'$ such that $u(\mathbf{x}) = u(\mathbf{x}')$ and $\mathbf{p}'\cdot\mathbf{x}' = w'$. The change $\Delta w = w' - w$ is the compensation that the agent receives to be as well off in utility terms after the price change as she was before. Thanks to the compensation there is no income effect coming from the reduction in the agent's purchasing power.

We call the elasticity of the Hicksian demand function *compensated elasticity* and it reads:

$$\varepsilon^c_{i,p_k} = \frac{\partial h_i(\mathbf{p},\bar{u})}{\partial p_k}\frac{p_k}{h_i(\mathbf{p},\bar{u})}$$

3 Relating Walrasian and Hicksian Demand: The Slutsky Equation

We now establish a relationship between the Walrasian and the Hicksian demand elasticities. We know that $u(x_i(\mathbf{p},w)) = \bar{u}$ and $e(\mathbf{p},\bar{u}) = w$. Start from the following identity:

$$x_i(\mathbf{p}, e(\mathbf{p},\bar{u})) = h_i(\mathbf{p},\bar{u})$$

and differentiate both sides wrt p_k to get:

$$\begin{aligned}
\frac{\partial h_i(\mathbf{p},\bar{u})}{\partial p_k} &= \frac{\partial x_i(\mathbf{p}, e(\mathbf{p},\bar{u}))}{\partial p_k} + \frac{\partial x_i(\mathbf{p}, e(\mathbf{p},\bar{u}))}{\partial e(\mathbf{p},\bar{u})}\frac{\partial e(\mathbf{p},\bar{u})}{\partial p_k} \\
&= \frac{\partial x_i(\mathbf{p},w)}{\partial p_k} + \frac{\partial x_i(\mathbf{p},w)}{\partial w} h_k(\mathbf{p},\bar{u}) \\
&= \frac{\partial x_i(\mathbf{p},w)}{\partial p_k} + \frac{\partial x_i(\mathbf{p},w)}{\partial w} x_k(\mathbf{p},, e(\mathbf{p},\bar{u})) \\
&= \frac{\partial x_i(\mathbf{p},w)}{\partial p_k} + \frac{\partial x_i(\mathbf{p},w)}{\partial w} x_k(\mathbf{p},w)
\end{aligned}$$

Rearranging, we derive the following relation:

$$\underbrace{\frac{\partial x_i(\mathbf{p},w)}{\partial p_k}}_{\text{uncompensated change}} = \underbrace{\frac{\partial h_i(\mathbf{p},\bar{u})}{\partial p_k}}_{\text{substitution effect}} - \underbrace{\frac{\partial x_i(\mathbf{p},w)}{\partial w} x_k(\mathbf{p},w)}_{\text{income effect}}$$

we have thus decomposed the uncompensated change into income and substitution effect. Notice also how the income effect is the product of two terms: $\frac{\partial x_i(\mathbf{p},w)}{\partial w}$ is the response of the Walrasian demand for good i to a change in wealth; $x_k(\mathbf{p},w)$ is the *mechanical effect* of an increase in p_k on the agent's purchasing power: an agent whose demand for k was $x_k(\mathbf{p},w)$ experiences a mechanical reduction of her purchasing power amounting to $x_k(\mathbf{p},w)$ when p_k increases by 1. J. R.

4 Static Labor Supply Choice

In this paragraph we study a simple framework of labor supply choice and we derive uncompensated labor elasticities. Assume an agent derives utility from consumption, but disutility from labor. Her preferences are represented by the utility function $u(c,n)$ where $\partial u/\partial c > 0$ and $\partial u/\partial n < 0$. The agent has I amount of wealth and earns salary w. We normalize the price of consumption to 1.[2] The utility maximization problem now is:

[2] Notice that we can normalize the price of consumption in a two goods economy and interpret salary w as the relative price of leisure over consumption.

$$\max_{c,n} u(c,n)$$

s.t.

$$c = wn + I$$

Taking FOCs and rearranging we get the following:

$$-\frac{u_n}{u_c} = w$$

This condition is similar to the one we derived above. It equates the cost of leisure w to the marginal rate of substitution between labor and consumption. Dividing the marginal disutility of labor by the marginal utility of consumption we get the marginal utility cost of labor in consumption units. The condition therefore equates the marginal utility cost of labor to the salary.

We now want to study the labor supply response to a change in salary. Suppose that the wage increases. Since the consumer gets paid more for every hour she works, she will tend to work more (which implies that she will consume less leisure). This is the *substitution effect*. However, since the agent earns more for every hour of work, she gets paid more for the amount of hours she were already working. Since the consumer is wealthier, if leisure is a normal good, she will tend to work less and consume more leisure. This is the *income effect*. Notice that, even if the cost of leisure has increased, the income and substitution effects do not go in the same direction unlike in standard consumer problems where an increase in the price of good i generates a negative income and substitution effect for good i. The reason is that this is an endowment economy where we think about leisure l as the difference between total time endowment T and labor. We have $l = T - n$. In this setup the agent is a net seller of leisure and therefore the income effect is positive for leisure when the salary increases.

Now we get a little more formal and we study analytically the response of labor supply to changes in the wage rate. Totally differentiating the optimality condition wrt w we get:

$$\frac{\partial n}{\partial w} = -\frac{u_c + n(u_{nc} + wu_{cc})}{w^2 u_{cc} + 2wu_{nc} + u_{nn}}$$

Notice that the denominator of the expression is the second order condition of the problem and can therefore be signed. If we assume the problem is concave (in order to get an interior solution), the denominator is negative. This implies that:

$$\frac{\partial n}{\partial w} \propto u_c + n(u_{nc} + wu_{cc})$$

This expression captures the intuition provided above. The first term is the substitution effect, which is always positive and proportional to the marginal utility of consumption: the extent to which the consumer substitutes labor and consumption depends on how attractive consumption is. The second term measures the income effect. It depends on the cross-derivative of consumption and labor and the concavity of the utility function in consumption. The cross-derivative measures how changes in consumption affect the labor disutility. Faster decreasing marginal returns to consumption imply lower incentive to consume when the agent becomes wealthier (remember that $u_{cc} < 0$). The income effect is scaled by n, which is the mechanical effect on endowment of a one unit increase of w.

Example: We now study a functional form for preferences that is particularly convenient for the study of optimal tax problems. Suppose the agent has the following utility:

$$u(c,n) = c - \frac{n^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}}$$

This is a quasi-linear utility function whose property is to rule out income effects. We will come back to this point later.

The optimality condition reads:

$$n^{\frac{1}{\varepsilon}} = w$$

Taking logs we get:

$$\frac{1}{\varepsilon}\log n = \log w$$

Since $\varepsilon^u_{n,w} = \partial \log n / \partial \log w$ we can write:

$$\varepsilon^u_{n,w} = \partial \log n / \partial \log w = \varepsilon$$

Therefore, this utility function has a constant elasticity of labor supply. Also, given the absence of income effects, we know that $\varepsilon^u_{n,w} = \varepsilon^c_{n,w}$.

Compensated Labor Supply Elasticity: We can derive the compensated response of labor supply by using the Slutsky equation. We already know the uncompensated response to wage changes and we therefore need to find $\partial n/\partial I$. Totally differentiating the FOC wrt I we get:

$$\frac{\partial n}{\partial I} = -\frac{u_{nc} + wu_{cc}}{w^2 u_{cc} + 2wu_{nc} + u_{nn}}$$

The Slutsky equation is the following:

$$\frac{\partial n}{\partial w} = \frac{\partial n^c}{\partial w} + \frac{\partial n}{\partial I} n$$

Notice that the sign of the income effect is flipped since w is the price of leisure, while we are studying the response of labor. We therefore conclude:

$$\frac{\partial n^c}{\partial w} = -\frac{u_c}{w^2 u_{cc} + 2wu_{nc} + u_{nn}}$$

By comparing the compensated and uncompensated response we clearly see why quasi-linear preferences imply no income effect: they are separable and linear in consumption. Therefore, $u_{nc} = 0$ and $u_{cc} = 0$.

λ-constant Elasticity: We introduce a concept that will be useful later in the analysis of intertemporal elasticites. The first order conditions for the static labor supply model solved with a Lagrangian approach are:

$$\begin{aligned} u_c &= \lambda \\ u_n &= -\lambda w \end{aligned}$$

Define the λ-constant or Frisch elasticity the elasticity that is computed assuming λ does not change. Totally differentiating we get:

$$\begin{bmatrix} u_{cc} & u_{cn} \\ u_{nc} & u_{nn} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial c^F}{\partial w} \\ \frac{\partial n^F}{\partial w} \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda \end{bmatrix}$$

By inverting the 2×2 matrix we can solve the system. The sultions are:

$$
\begin{aligned}
\begin{bmatrix} \frac{\partial c^F}{\partial w} \\ \frac{\partial n^F}{\partial w} \end{bmatrix} &= \begin{bmatrix} u_{cc} & u_{cn} \\ u_{nc} & u_{nn} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\lambda \end{bmatrix} \\
&= \frac{1}{u_{cc}u_{nn} - u_{cn}^2} \begin{bmatrix} u_{nn} & -u_{cn} \\ -u_{nc} & u_{cc} \end{bmatrix} \begin{bmatrix} 0 \\ -\lambda \end{bmatrix} \\
&= \begin{bmatrix} \frac{\lambda u_{cn}}{u_{cc}u_{nn}-u_{cn}^2} \\ \frac{-\lambda u_{cc}}{u_{cc}u_{nn}-u_{cn}^2} \end{bmatrix}
\end{aligned}
$$

A Comparison Among Elasticities: We now draw a comparison among the three elasticities presented so far. We already know that $\varepsilon^c_{l,w} \geq \varepsilon^u_{l,w}$ since the income effect is negative on labor supply. We therefore need to compare compensated and λ-constant elasticity. We will prove that $\varepsilon^F_{l,w} \geq \varepsilon^c_{l,w}$. Start by writing the following:

$$
\begin{aligned}
\frac{1}{\varepsilon^c_{l,w}} - \frac{1}{\varepsilon^F_{l,w}} &= -\frac{w^2 u_{cc} + 2w u_{nc} + u_{nn}}{u_c} + \frac{u_{cc}u_{nn} - u_{cn}^2}{u_c u_{cc}} \\
&= \frac{1}{u_c}\left(-w^2 u_{cc} - 2w u_{nc} - \frac{u_{cn}^2}{u_{cc}}\right)
\end{aligned}
$$

The definition of λ-constant elasticity implies that $u_{nn} \geq \frac{u_{cn}^2}{u_{cc}}$. It follows that:

$$
\begin{aligned}
-w^2 u_{cc} - 2w u_{nc} - \frac{u_{cn}^2}{u_{cc}} &\geq -w^2 u_{cc} - 2w u_{nc} - u_{nn} \\
&= -SOC \\
&\geq 0
\end{aligned}
$$

Where the last inequality uses the fact that the second order condition must be negative. Hence, we established that $\frac{1}{\varepsilon^c_{l,w}} - \frac{1}{\varepsilon^F_{l,w}} \geq 0$, which implies $\varepsilon^F_{l,w} \geq \varepsilon^c_{l,w}$. Keeping the marginal utility of consumption constant implies that there are no income effects: higher wealth given the same amount of hours of work does not change preferences towards consumption. Thus, the λ-constant elasticity is at least as big as the compensated one.

We therefore conclude that the following is always true:

$$
\varepsilon^F_{l,w} \geq \varepsilon^c_{l,w} \geq \varepsilon^u_{l,w}
$$

5 Dynamic Labor Supply

In the previous paragraph we studied the static labor supply choice. Now we will switch to a dynamic setting that allows us to study labor supply responses to over time changes in salaries. Agents make labor supply decisions in view of their lifetime. Current labor supply depends on current and future wages and income. Compared to static labor supply models, the substitution effect is similar, but the income effect differs since the agent faces a lifetime budget constraint. MaCurdy (1981) provides a useful framework to study labor supply elasticities over the lifecycle. In order to achieve our goal, we need to separate exogenous static changes (such as the ones studied above) from evolutionary changes, due to shifts in the life-cycle wage profile. In this analysis we need to distinguish between expected and unexpected wage changes. While expected changes do not lead to wealth effects, permanent unexpected changes generate strong wealth effects.

We distinguish among three dimensions of labor supply:

1. the pure *lifecycle dimension*. Usually, wages have a hump-shaped pattern over the lifecycle. Agents adjust the hours of work in response to the different salaries they observe along their lifetime.
2. the *macro dimension*. Hours of work vary over the business cycle following unexpected shocks.
3. the *idiosyncratic dimension*. A person may have temporarily higher wages in some period.

In order to isolate the labor supply response to expected changes in wage we need to rule out wealth effects. We will employ the concept of Frisch elasticity, which allows us to keep the marginal utility of consumption constant.

We study intertemporal labor supply in the same framework as before, but we introduce the time dimension. Preferences are now:

$$\sum_{s=t}^{+\infty} \beta^s u(c_s, n_s)$$

The consumer faces the following period-by-period budget constraint:

$$A_{t+1} = (1 + r_t)(A_t + y_t + w_t n_t - c_t)$$

The Bellman equation for the problem is:

$$V(A_t) = \max_{c_t, n_t} u(c_t, n_t) + \beta V((1 + r_t)(A_t + y_t + w_t n_t - c_t))$$

The FOCs for the problem read:

$$\begin{aligned} \lambda_t &= \beta(1 + r_t) V'_{t+1}(A_{t+1}) \\ u_n(c_t, n_t) &= w \lambda_t \end{aligned}$$

By envelope $\lambda_t = V'(A_t)$. Therefore, the conditions become:

$$\begin{aligned} \lambda_t &= \beta(1 + r_t) V'_{t+1}(\lambda_{t+1}) \\ u_n(c_t, n_t) &= \lambda_t w_t \end{aligned}$$

Since $\lambda_t = u_c(c_t, n_t)$ the static labor supply choice is the same as in the previous paragraph:

$$-\frac{u_n(c_t, n_t)}{u_c(c_t, n_t)} = w_t$$

Rearranging the budget constraint we have:

$$c_t = w_t n_t + y_t + \left[\frac{A_{t+1}}{1 + r_t} - A_t\right]$$

Notice that the problem is identical to the previous one where income $I_t = y_t + \left[\frac{A_{t+1}}{1+r_t} - A_t\right]$.

In order to assess the Frisch elasticity, we need to compute the labor responses to changes in w when we keep the λ constant avoiding any wealth effect. The Frisch demands are defined as follows:

$$\begin{aligned} c_t &= c_t^F(w_t, \lambda_t) \\ n_t &= n_t^F(w_t, \lambda_t) \end{aligned}$$

Since the model is identical to the static labor supply choice and we already derived the Frisch elasticity for the latter, we can write:

$$\varepsilon^F_{n_t, w_t} = \frac{\partial n_t^F}{\partial w_t} \frac{w_t}{n_t} = \frac{-\lambda u_{cc}(c_t, n_t)}{u_{cc}(c_t, n_t) u_{nn}(c_t, n_t) - u_{cn}^2(c_t, n_t)} \frac{w_t}{n_t}$$

References

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