Economics 2450A: Public Economics
Technical Note: Optimal Control and Hamiltonians in Tax Problems

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The solution of optimal tax problems in a Mirrleesian framework requires the application of optimal control theory. This short note shows how to setup a Hamiltonian for the tax problem and apply the rules of optimal control to derive equilibrium conditions.

1 Optimal Control

Optimal control theory is typically applied in economics for the solution of dynamic problems in continuous time. One example is the standard Ramsey Growth Model. We will employ the standard notation of growth problems to illustrate the problem setup and the optimality conditions.

Suppose we have the following objective function:

$$\int_0^T v\left[k\left(t\right), c\left(t\right)\right] dt$$

where t is the time dimension and is a continuous variable in $[0, T]$. $v\left(\cdot,\cdot\right)$ represents the instantaneous utility at time t. We call the variable $k\left(t\right)$ state variable, while $c\left(t\right)$ is the control variable. The state variable evolves over time according to an equation of motion:

$$\dot{k} = g\left[k\left(t\right), c\left(t\right), t\right]$$

The change in k over time is dependent on the level of the state variable itself and on the choice of the control at time t. This kind of problems requires a transversality condition that rules out explosive paths and takes the following form:

$$k\left(T\right) e^{-r(T)T} \geq 0$$

$e^{-r(T)T}$ is the discount factor. The condition requires that the net present value of capital in the last time period cannot be negative. Typically, this is equivalent to say that the agent cannot die with liabilities.

The problem is therefore:

$$\max_{c(t)} V\left(0\right) = \int_0^\infty v\left[k\left(t\right), c\left(t\right)\right] dt$$

s.t.

$$\dot{k} = g\left[k\left(t\right), c\left(t\right), t\right]$$

$$k\left(T\right) e^{-r(T)T} \geq 0$$

The optimal control problem is solved using a Hamiltonian that reads:

$$H = v(k, c, t) + \mu(t) g(k, c, t) \tag{1}$$

$\mu(t)$ is the multiplier on the equation of motion. In a classical growth model, it represents the utility value of having one extra unit of capital.

Optimal control theory derives the optimality conditions of the problem. They are:

$$\frac{\partial H}{\partial c(t)} = 0$$

$$\frac{\partial H}{\partial k(t)} = -\mu(t)$$

$$\mu(T) k(T) = 0$$

2 Applying Optimal Control to Tax Problems

The standard Mirrlees problem has objective function:

$$\int_{\underline{n}}^{\bar{n}} G(u_n) f(n)\, dn$$

We treat the utility assigned to individual n in equilibrium as the state variable of the problem and we use the incentive compatibility constraint as an equation of motion for u_n:

$$\dot{u}_n = \frac{z_n v'(z_n/n)}{n^2}$$

The equation of motion describes how utility changes across types in equilibrium. The control variable for the problem is income z_n.

The resource constraint is:

$$\int_{\underline{n}}^{\bar{n}} c_n f(n)\, dn \leq \int_{\underline{n}}^{\bar{n}} n l_n f(n) - E$$

Having chosen z_n as the control variable, we need to rewrite the resource constraint as a function of z_n. Suppose the individual has quasi-linear preferences $u_n = c_n - v(z_n/n)$.[1] We exploit the definition of u_n and note that $c_n = u_n + v(z_n/n)$ to rewrite the resource constraint as follows:

$$\int_{\underline{n}}^{\bar{n}} [u_n + v(z_n/n)] f(n)\, dn \leq \int_{\underline{n}}^{\bar{n}} z_n f(n) - E$$

The Hamiltonian for the tax problem is therefore:

$$H = [G(u_n) + p(n l_n - u_n - v(l_n))] f(n) + \mu(n) \frac{l_n v'(l_n)}{n}$$

where $\mu(n)$ are the multipliers for the incentive constraints representing the shadow value of relaxing the constraint, while p is the multiplier on the resource constraint.

Applying the optimality condition for the optimal control problem, we get:

[1] In Section Notes 3 we will extend the derivation of optimal taxes to a case of separable utility without the quasi-linear form.

$$\frac{\partial H}{\partial l_n} = p\left[n - v'\left(l_n\right)\right] f\left(n\right) + \frac{\mu\left(n\right)}{n}\left[v'\left(l_n\right) + l_n v''\left(l_n\right)\right] = 0$$

$$\frac{\partial H}{\partial u_n} = \left[G'\left(u_n\right) - p\right] f\left(n\right) = -\dot{\mu}\left(n\right)$$

The boundary (transversality) conditions for the problem are:

$$\mu\left(\bar{n}\right) = \mu\left(\underline{n}\right) = 0$$

The Hamiltonian solution requires $\mu\left(\bar{n}\right) u\left(\bar{n}\right) = 0$. However, if we want to provide positive utility to type $\bar{n}$ we must require $\mu\left(\bar{n}\right) = 0$. At the same time, since at the optimum the incentive contraints will be binding downwards, we require $\mu\left(\underline{n}\right) = 0$. As it is standard in this kind of problems, the lowest type does not want to imitate any other agent in equilibrium and her incentive constraint is slack, while everyone else is indifferent between her allocation and the allocation provided to the immediately lower type.

References

[1] *Rober J. Barro and Sala-i-Martin, X. I. (2004), "Economic Growth", MIT Press, 2nd edition*