Appendices to

Empirical Bayes Forecasts of One Time Series Using Many Predictors

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Appendix A: Proofs of Theorems

Appendix B: Berry-Esseen Theorems for Densities and their Derivatives

Appendix C: Description of Time Series Data Used in the Empirical Analysis

1 Appendix A: Proofs of Theorems

The first part of this appendix contains proofs of Theorems 1 and 2. The second part contains proofs of Theorems 3 and 4. Please note that when an equation number is referred to in this appendix, the reference is to the equation in this appendix (rather than in the body of the paper) which has that number, unless otherwise stated.

1.1 A.1 Proofs of Theorems 1 and 2

Before we begin, note that Assumption 2 clearly implies the following summability inequalities (due to the exponentially decaying upper bound on the ν_n)

$$\begin{aligned} \sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\nu_n &\leq D < \infty \\ \sum_{n=1}^{\infty} n\nu_n &\leq D < \infty \end{aligned} \tag{1}$$

which evidently yield (since ν_n is nonnegative by definition)

$$\sum_{n=1}^{\infty}\nu_n \leq D < \infty. \tag{2}$$

Proof of Theorem 1: First, we show unbiasedness:

$$\begin{aligned} E\left[\hat{\sigma}_{\varepsilon}^2\right] &= \frac{1}{T-K}E\left[\varepsilon'\varepsilon\right] - \frac{1}{T\left(T-K\right)}E\left[\varepsilon' X X' \varepsilon\right] \\ &= \frac{T}{T-K}\sigma_{\varepsilon}^2 - \frac{1}{T\left(T-K\right)}\sum_{i=1}^{K}\sum_{t=1}^{T}\sum_{s=1}^{T}E\left[X_{is}X_{it}\varepsilon_s\varepsilon_t\right] \\ &= \frac{T}{T-K}\sigma_{\varepsilon}^2 - \frac{\sigma_{\varepsilon}^2}{T\left(T-K\right)}\sum_{i=1}^{K}E\left[\sum_{t=1}^{T}X_{it}^2\right] \end{aligned} \tag{3}$$

$$= \frac{T}{T-K}\sigma_\varepsilon^2 - \frac{KT\sigma_\varepsilon^2}{T(T-K)}$$
$$= \sigma_\varepsilon^2$$

where the first equality follows easily from the definition of $\hat{\sigma}_\varepsilon^2$, the second is by Assumption 1 and calculation, for the first and second terms respectively, the third equality is due to the fact that $E[X_{is}X_{it}\varepsilon_s\varepsilon_t] = 0$ if $s \neq t$, and $= \sigma_\varepsilon^2 E[X_{it}^2]$ if $s = t$. The fourth equality follows from the fact that $\sum_{t=1}^{T} X_{it}^2 = T$ and from trivial summation. The final equality follows by simple cancellation. Note that the unbiasedness does *not* depend on the mixing assumption, although it is crucial later in the proof.

Before entering into the proof of the squared-error bound, it is worth noting that, for any random variables $Z_1, \ldots, Z_m$ each of which has an n^{th} absolute moment, and if $p_i \geq 0$, $\sum_{i=1}^{m} p_i = n$, then, by iterating Hölder's inequality,

$$|E[\Pi_{i=1}^m Z_i^{p_i}]| \leq \Pi_{i=1}^m (E[|Z_i|^n])^{p_i / n}. \tag{4}$$

This fact will prove useful when we apply Assumption 1 to various cross-moments below, and we will not explicitly cite it when it is applied, in the interest of brevity.

Now, using unbiasedness,

$$E\left[\left(\hat{\sigma}_\varepsilon^2 - \sigma_\varepsilon^2\right)^2\right] \tag{5}$$
$$= E\left[\left(\hat{\sigma}_\varepsilon^2\right)^2\right] - \sigma_\varepsilon^4$$

$$= E\left[\frac{1}{(T-K)^2}\left(\varepsilon'\left(I-\frac{XX'}{T}\right)\varepsilon\right)\left(\varepsilon'\left(I-\frac{XX'}{T}\right)\varepsilon\right)\right]-\sigma_\varepsilon^4$$

$$= E\left[\frac{1}{(T-K)^2}\left[\begin{array}{c}(\varepsilon'\varepsilon)^2-2(\varepsilon'\varepsilon)\left(\varepsilon'\frac{XX'}{T}\varepsilon\right)+\\ \left(\varepsilon'\frac{XX'}{T}\varepsilon\right)\left(\varepsilon'\frac{XX'}{T}\varepsilon\right)\end{array}\right]\right]-\sigma_\varepsilon^4$$

$$= \left\{\begin{array}{c} E\left[\frac{1}{(T-K)^2}\sum_{s=1}^T\sum_{t=1}^T\varepsilon_s^2\varepsilon_t^2\right] \\ +E\left[\frac{-2}{T(T-K)^2}\sum_{t=1}^T\sum_{i=1}^K\sum_{s=1}^T\sum_{u=1}^T\varepsilon_t^2X_{is}X_{iu}\varepsilon_s\varepsilon_u\right] \\ +E\left[\begin{array}{c}\frac{1}{T^2(T-K)^2}\sum_{i=1}^K\sum_{j=1}^K\sum_{s=1}^T\sum_{t=1}^T\sum_{u=1}^T\sum_{w=1}^T \\ X_{is}X_{it}X_{ju}X_{jw}\varepsilon_s\varepsilon_t\varepsilon_u\varepsilon_w\end{array}\right]\end{array}\right\}$$
$$-\sigma_\varepsilon^4$$

and we shall address each of the three expectations in the final expression in turn. The first expectation can be written as

$$\begin{aligned} E\left[\frac{1}{(T-K)^2}\sum_{s=1}^T\sum_{t=1}^T\varepsilon_s^2\varepsilon_t^2\right] &= \frac{1}{(T-K)^2}\sum_{s=1}^T\sum_{t=1}^T E\left[\varepsilon_s^2\varepsilon_t^2\right] \\ &= \frac{1}{(T-K)^2}\sum_{s=1}^T\left[(T-1)\sigma_\varepsilon^4+E\left[\varepsilon_s^4\right]\right] \\ &= \frac{T(T-1)}{(T-K)^2}\sigma_\varepsilon^4+\frac{\sum_{s=1}^T E\left[\varepsilon_s^4\right]}{(T-K)^2}. \end{aligned} \quad (6)$$

so that

$$\begin{aligned} &\left|E\left[\frac{1}{(T-K)^2}\sum_{s=1}^T\sum_{t=1}^T\varepsilon_s^2\varepsilon_t^2\right]-\frac{T(T-1)}{(T-K)^2}\sigma_\varepsilon^4\right| \\ \leq\ & \frac{\sum_{s=1}^T E\left[\varepsilon_s^4\right]}{(T-K)^2} \\ \leq\ & \frac{DT}{(T-K)^2} \\ \leq\ & \frac{C_1}{K} \end{aligned} \quad (7)$$

where the first inequality is clear, the second follows from Assumption 1, and the third is by the asymptotic nesting which has been assumed, in which K is an asymptotically constant fraction of T.

The second expectation may be evaluated as follows:

$$\begin{aligned} & E\left[\frac{-2}{T(T-K)^2}\sum_{t=1}^{T}\sum_{i=1}^{K}\sum_{s=1}^{T}\sum_{u=1}^{T}\varepsilon_t^2 X_{is}X_{iu}\varepsilon_s\varepsilon_u\right] \\ = & \frac{-2}{T(T-K)^2}\sum_{t=1}^{T}\sum_{i=1}^{K}\sum_{s=1}^{T}\sum_{u=1}^{T}E\left[\varepsilon_t^2 X_{is}X_{iu}\varepsilon_s\varepsilon_u\right] \end{aligned} \tag{8}$$

Now we must break out 6 cases:

Case 1: $s = t = u$.

This contributes, in *absolute value*,

$$\begin{aligned} & \frac{2}{T(T-K)^2}\sum_{i=1}^{K}\sum_{t=1}^{T}E\left[X_{it}^2\varepsilon_t^4\right] \\ \leq & \frac{2D}{T(T-K)^2}\sum_{i=1}^{K}\sum_{t=1}^{T}E\left[X_{it}^2\right] \\ = & \frac{2D}{T(T-K)^2}\sum_{i=1}^{K}E\left[\sum_{t=1}^{T}X_{it}^2\right] \\ = & \frac{2KTD}{T(T-K)^2} \\ \leq & \frac{C_2}{K} \end{aligned} \tag{9}$$

where the first inequality is by Assumption 1, the first equality is by the linearity of expectations, the second equality is by $\sum_{t=1}^{T} X_{it}^2 = T$, and the second inequality is by the asymptotic nesting, in which K is asymptotically a constant fraction of T.

Case 2: All time subscripts are distinct.

Then $E\left[X_{is}X_{iu}\varepsilon_s\varepsilon_u\varepsilon_t^2\right] = 0$ by using the m. d. s. property of ε, or, if t is the greatest subscript, by the homoskedastic m. d. s. property of ε followed by the m. d. s. property.

Case 3: $s = t > u$ or $u = t > s$.

Suppose w. l. o. g. that $s = t > u$; then

$$\begin{aligned} & \left|E\left[X_{it}\varepsilon_t^3 X_{iu}\varepsilon_u\right]\right| && (10) \\ = \; & \left|E\left[X_{it}\varepsilon_t^3 X_{iu}\varepsilon_u\right] - E\left[X_{it}\varepsilon_t^3\right]E\left[X_{iu}\varepsilon_u\right]\right| \\ \leq \; & \nu_{t-u}\left(E\left[X_{it}^2\varepsilon_t^6\right]\right)^{1/2}\left(E\left[X_{iu}^2\varepsilon_u^2\right]\right)^{1/2} \\ \leq \; & M_1\nu_{t-u} && (11) \end{aligned}$$

where the equality follows from the fact that $E\left[X_{iu}\varepsilon_u\right] = 0$, and the first inequality follows from Doukhan (1994, Theorem 3 (5) on page 9), which states that, if r and z are $\mathcal{H}_1^m$-measurable and $\mathcal{H}_{m+n}^\infty$-measurable random variables, respectively, and if $E\left[r^2\right], \; E\left[z^2\right] < \infty$, then

$$\left|Cov\left(r, z\right)\right| \leq \nu_n\sqrt{E\left[r^2\right]}\sqrt{E\left[z^2\right]}.$$

Note in particular that the moment bounds hold by Assumption 1, and that the σ-field $\mathcal{H}_a^b$ is generated by the random variables $\{X_a, \varepsilon_a, \ldots, X_b, \varepsilon_b\}$. The second inequality follows from the uniform (over i, u, and t) bounds on the moments guaranteed by Assumption 1. Thus, the absolute value of the contribution of these terms to the expectation is

$$\begin{aligned}
& \frac{2}{T\left(T-K\right)^{2}}\left|\sum_{i=1}^{K}\sum_{u=1}^{T}\sum_{t=u+1}^{T}E\left[X_{it}\varepsilon_{t}^{3}X_{iu}\varepsilon_{u}\right]\right| \qquad (12)\\
\leq\ & \frac{2}{T\left(T-K\right)^{2}}\sum_{i=1}^{K}\sum_{u=1}^{T}\sum_{t=u+1}^{T}\left|E\left[X_{it}\varepsilon_{t}^{3}X_{iu}\varepsilon_{u}\right]\right|\\
\leq\ & \frac{2M_1}{T\left(T-K\right)^{2}}\sum_{i=1}^{K}\sum_{u=1}^{T}\sum_{t=u+1}^{T}\nu_{t-u}\\
=\ & \frac{2M_1}{T\left(T-K\right)^{2}}\sum_{i=1}^{K}\sum_{u=1}^{T}\sum_{n=1}^{T-u}\nu_{n}\\
\leq\ & \frac{2M_1}{T\left(T-K\right)^{2}}\sum_{i=1}^{K}\sum_{u=1}^{T}\sum_{n=1}^{\infty}\nu_{n}\\
\leq\ & \frac{2M_1C^*}{T\left(T-K\right)^{2}}\sum_{i=1}^{K}\sum_{u=1}^{T}1\\
=\ & \frac{2M_1C^*KT}{T\left(T-K\right)^{2}}\\
\leq\ & \frac{C_3}{K}
\end{aligned}$$

where the first inequality is by the triangle inequality, the second inequality is by the preceding display, the first equality is by setting $n = t - u$, the third inequality is due to the fact that $\nu_n \geq 0$ by definition, the fourth inequality is by expression (2), and second equality is by trivial summation, and the final inequality is due to the asymptotic nesting, in which K is asymptotically a constant fraction of T.

Case 4: $s = t < u$.

Here $E\left[X_{it}\varepsilon_t^3 X_{iu}\varepsilon_u\right] = 0$ by the m. d. s. property of ε.

Case 5: $s = u > t$.

Here $E\left[X_{is}^2\varepsilon_s^2\varepsilon_t^2\right] = \sigma_\varepsilon^2 E\left[X_{is}^2\varepsilon_t^2\right]$ by the homoskedastic m. d. s. property of ε, and

$$\begin{aligned} & \left|E\left[X_{is}^2\varepsilon_t^2\right] - E\left[X_{is}^2\right]E\left[\varepsilon_t^2\right]\right| \\ \leq\ & \nu_{s-t}\left(E\left[X_{is}^4\right]\right)^{1/2}\left(E\left[\varepsilon_t^4\right]\right)^{1/2} \\ \leq\ & M_2\nu_{s-t} \end{aligned} \tag{13}$$

where the first inequality is that of Doukhan (1994, Theorem 3 (5) on page 9) used above (noting that Assumption 1 guarantees moment existence), while the second follows from Assumption 1's uniform (over i, s, and t) bound on the moments. But $E\left[X_{is}^2\right]E\left[\varepsilon_t^2\right] = \sigma_\varepsilon^2 E\left[X_{is}^2\right]$. Thus

$$\begin{aligned} & \left|\begin{array}{l} \frac{-2}{T(T-K)^2}\sum_{i=1}^K\sum_{t=1}^T\sum_{s=t+1}^T E\left[X_{is}^2\varepsilon_s^2\varepsilon_t^2\right] \\ -\left(\frac{-2\sigma_\varepsilon^4}{T(T-K)^2}\sum_{i=1}^K\sum_{t=1}^T\sum_{s=t+1}^T E\left[X_{is}^2\right]\right) \end{array}\right| \\ \leq\ & \frac{2}{T\left(T-K\right)^2}\sum_{i=1}^K\sum_{t=1}^T\sum_{s=t+1}^T\left|E\left[X_{is}^2\varepsilon_s^2\varepsilon_t^2\right] - \sigma_\varepsilon^4 E\left[X_{is}^2\right]\right| \\ \leq\ & \frac{2M_2}{T\left(T-K\right)^2}\sum_{i=1}^K\sum_{t=1}^T\sum_{s=t+1}^T\nu_{s-t} \\ \leq\ & \frac{2M_2}{T\left(T-K\right)^2}\sum_{i=1}^K\sum_{t=1}^T\sum_{n=1}^\infty\nu_n \\ \leq\ & \frac{2M_2C^*}{T\left(T-K\right)^2}\sum_{i=1}^K\sum_{t=1}^T 1 \end{aligned} \tag{14}$$

$$
\begin{aligned}
&= \frac{2M_2C^*KT}{T\left(T-K\right)^2} \\
&\leq \frac{C_4}{K}.
\end{aligned}
$$

Case 6: $t > s = u$.

Here $E\left[X_{is}^2\varepsilon_s^2\varepsilon_t^2\right] = \sigma_\varepsilon^4 E\left[X_{is}^2\right]$ by the homoskedastic m. d. s. property of ε as given in Assumption 1. Thus the overall contribution of these terms is:

$$
\frac{-2\sigma_\varepsilon^4}{T\left(T-K\right)}\sum_{i=1}^{K}\sum_{t=1}^{T}\sum_{s=1}^{t-1} E\left[X_{is}^2\right]. \tag{15}
$$

Now that we have broken out the cases, we may pull them back together again:

$$
\left| E\left[\frac{-2}{T\left(T-K\right)^2}\sum_{t=1}^{T}\sum_{i=1}^{K}\sum_{s=1}^{T}\sum_{u=1}^{T}\varepsilon_t^2 X_{is}X_{iu}\varepsilon_s\varepsilon_u\right] - \left(\frac{-2KT}{\left(T-K\right)^2}\sigma_\varepsilon^2\right)\right| \tag{16}
$$

$$
\begin{aligned}
&\leq \frac{C_1+C_2+C_3+C_4}{K} + \left|\begin{array}{c} \frac{-2\sigma_\varepsilon^4}{T(T-K)^2}\sum_{i=1}^{K}\sum_{t=1}^{T}\sum_{s=t+1}^{T} E\left[X_{is}^2\right] \\ +\frac{-2\sigma_\varepsilon^4}{T(T-K)}\sum_{i=1}^{K}\sum_{t=1}^{T}\sum_{s=1}^{t-1} E\left[X_{is}^2\right] \\ -\left(\frac{-2KT}{(T-K)^2}\sigma_\varepsilon^2\right)\end{array}\right| \\
&\leq \frac{C_1+C_2+C_3+C_4}{K} + \frac{2\sigma_\varepsilon^4}{T\left(T-K\right)^2}\sum_{i=1}^{K}\sum_{t=1}^{T} E\left[X_{it}^2\right] + \\
&\quad \left|\frac{-2\sigma_\varepsilon^4}{T\left(T-K\right)^2}\sum_{i=1}^{K}\sum_{t=1}^{T}\sum_{s=1}^{T} E\left[X_{is}^2\right] - \left(\frac{-2KT}{\left(T-K\right)^2}\sigma_\varepsilon^2\right)\right| \\
&= \frac{C_1+C_2+C_3+C_4}{K} + \frac{2\sigma_\varepsilon^4}{T\left(T-K\right)^2}\sum_{i=1}^{K} E\left[\sum_{t=1}^{T} X_{it}^2\right] + \\
&\quad \left|\frac{-2\sigma_\varepsilon^4}{T\left(T-K\right)^2}\sum_{i=1}^{K}\sum_{t=1}^{T} E\left[\sum_{s=1}^{T} X_{is}^2\right] - \left(\frac{-2KT}{\left(T-K\right)^2}\sigma_\varepsilon^2\right)\right| \\
&= \frac{C_1+C_2+C_3+C_4}{K} + \frac{2\sigma_\varepsilon^4 K}{\left(T-K\right)^2} +
\end{aligned}
$$

$$
\begin{aligned}
& \left|\frac{-2\sigma_\varepsilon^4 K T^2}{T\left(T-K\right)^2}-\left(\frac{-2KT}{\left(T-K\right)^2}\sigma_\varepsilon^2\right)\right| \\
= \quad & \frac{C_1+C_2+C_3+C_4}{K}+\frac{2\sigma_\varepsilon^4 K}{\left(T-K\right)^2} \\
\leq \quad & \frac{C_5}{K}
\end{aligned}
$$

Finally, the third expectation is

$$
\begin{aligned}
& E\left[\begin{array}{c} \frac{1}{T^2(T-K)^2}\sum_{i=1}^K\sum_{j=1}^K\sum_{s=1}^T\sum_{t=1}^T\sum_{u=1}^T\sum_{w=1}^T \\ X_{is}X_{it}X_{ju}X_{jw}\varepsilon_s\varepsilon_t\varepsilon_u\varepsilon_w \end{array}\right] \qquad (17) \\
= \quad & \frac{1}{T^2\left(T-K\right)^2}\sum_{i=1}^K\sum_{j=1}^K\sum_{s=1}^T\sum_{t=1}^T\sum_{u=1}^T\sum_{w=1}^T \\
& E\left[X_{is}X_{it}X_{ju}X_{jw}\varepsilon_s\varepsilon_t\varepsilon_u\varepsilon_w\right]
\end{aligned}
$$

for which we will need to consider 11 different cases.

Case 1: $s=t=u=w$.

$E\left[X_{is}^2X_{js}^2\varepsilon_s^4\right] \leq DE\left[X_{is}^2X_{js}^2\right] \leq D^2$ by Assumption 1 (and nonnegativity, for the first inequality), so the absolute value of the contribution of these terms is

$$
\begin{aligned}
& \frac{1}{T^2\left(T-K\right)^2}\sum_{i=1}^K\sum_{j=1}^K\sum_{s=1}^T E\left[X_{is}^2X_{js}^2\varepsilon_s^4\right] \qquad (18) \\
\leq \quad & \frac{D^2}{T^2\left(T-K\right)^2}\sum_{i=1}^K\sum_{j=1}^K\sum_{s=1}^T 1 \\
= \quad & \frac{K^2D^2}{T\left(T-K\right)^2} \\
\leq \quad & \frac{C_6}{K}.
\end{aligned}
$$

Case 2: All time subscripts are distinct.

Here $E\left[X_{is}X_{it}X_{ju}X_{jw}\varepsilon_s\varepsilon_t\varepsilon_u\varepsilon_w\right] = 0$ by the m. d. s. property of ε (Assumption 1).

Case 3: $s = t > u > w$ (w. l. o. g.; also, $t = u > w > s$, etc.).

The terms here are of the form $E\left[X_{it}^2\varepsilon_t^2X_{ju}\varepsilon_uX_{jw}\varepsilon_w\right]$. Noting that, by the m. d. s. property of ε (Assumption 1), $E\left[X_{ju}\varepsilon_uX_{jw}\varepsilon_w\right] = 0$, we see that

$$\begin{aligned} & \left|E\left[X_{it}^2\varepsilon_t^2X_{ju}\varepsilon_uX_{jw}\varepsilon_w\right]\right| \\ \leq\; & \left|E\left[X_{it}^2\varepsilon_t^2X_{ju}\varepsilon_uX_{jw}\varepsilon_w\right] - E\left[X_{it}^2\varepsilon_t^2\right]E\left[X_{ju}\varepsilon_uX_{jw}\varepsilon_w\right]\right| \\ \leq\; & \nu_{t-u}\left(E\left[X_{it}^4\varepsilon_t^4\right]\right)^{1/2}\left(E\left[X_{ju}^2\varepsilon_u^2X_{jw}^2\varepsilon_w^2\right]\right)^{1/2} \\ \leq\; & M_3\nu_{t-u} \end{aligned} \tag{19}$$

where the first inequality follows by the observation preceding the display, the second inequality is by Doukhan (1994, Theorem 3 (5) on page 9) and Assumption 2, and the third inequality follows from Assumption 1.

Observing that $E\left[X_{jw}\varepsilon_w\right] = 0$ by the m. d. s. property of ε (Assumption 1), we also obtain

$$\begin{aligned} & \left|E\left[X_{it}^2\varepsilon_t^2X_{ju}\varepsilon_uX_{jw}\varepsilon_w\right]\right| \\ \leq\; & \left|E\left[X_{it}^2\varepsilon_t^2X_{ju}\varepsilon_uX_{jw}\varepsilon_w\right] - E\left[X_{it}^2\varepsilon_t^2X_{ju}\varepsilon_u\right]E\left[X_{jw}\varepsilon_w\right]\right| \end{aligned} \tag{20}$$

$$
\begin{aligned}
&\leq \quad \nu_{u-w}\left(E\left[X_{it}^{4}\varepsilon_{t}^{4}X_{ju}^{2}\varepsilon_{u}^{2}\right]\right)^{1/2}\left(E\left[X_{jw}^{2}\varepsilon_{w}^{2}\right]\right)^{1/2} \\
&\leq \quad M_{4}\nu_{u-w}
\end{aligned}
$$

in an entirely similar fashion.

Thus, the total absolute-value of the contribution of the terms covered by this case satisfies

$$
\begin{aligned}
&\left|\frac{1}{T^{2}\left(T-K\right)^{2}}\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{w=1}^{T}\sum_{u=w+1}^{T}\sum_{t=u+1}^{T}E\left[X_{it}^{2}\varepsilon_{t}^{2}X_{ju}\varepsilon_{u}X_{jw}\varepsilon_{w}\right]\right| \qquad (21)\\
\leq\ & \frac{1}{T^{2}\left(T-K\right)^{2}}\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{w=1}^{T}\sum_{u=w+1}^{T}\sum_{t=u+1}^{T}\left|E\left[X_{it}^{2}\varepsilon_{t}^{2}X_{ju}\varepsilon_{u}X_{jw}\varepsilon_{w}\right]\right| \\
\leq\ & \frac{1}{T^{2}\left(T-K\right)^{2}}\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{w=1}^{T}\sum_{u=w+1}^{T}\sum_{t=u+1}^{T}\min\left\{M_{3}\nu_{t-u},M_{4}\nu_{u-w}\right\} \\
\leq\ & \frac{\max\left\{M_{3},M_{4}\right\}}{T^{2}\left(T-K\right)^{2}}\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{w=1}^{T}\sum_{u=w+1}^{T}\sum_{t=u+1}^{T}\min\left\{\nu_{t-u},\nu_{u-w}\right\} \\
\leq\ & \frac{D\max\left\{M_{3},M_{4}\right\}}{T^{2}\left(T-K\right)^{2}}\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{w=1}^{T}\sum_{u=w+1}^{T}\sum_{t=u+1}^{T}\min\left\{e^{-\frac{\lambda}{2}(t-u)},e^{-\frac{\lambda}{2}(u-w)}\right\} \\
=\ & \frac{D\max\left\{M_{3},M_{4}\right\}}{T^{2}\left(T-K\right)^{2}}\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{w=1}^{T}\sum_{u=w+1}^{T}\left\{\sum_{t=u+1}^{2u-w}e^{-\frac{\lambda}{2}(u-w)}+\sum_{t=2u-w+1}^{T}e^{-\frac{\lambda}{2}(t-u)}\right\} \\
=\ & \frac{D\max\left\{M_{3},M_{4}\right\}}{T^{2}\left(T-K\right)^{2}}\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{w=1}^{T}\left\{\begin{array}{c}\sum_{u=w+1}^{T}\left(u-w\right)e^{-\frac{\lambda}{2}(u-w)}\\ +\sum_{u=w+1}^{T}\sum_{t=2u-w+1}^{T}e^{-\frac{\lambda}{2}(t-u)}\end{array}\right\} \\
=\ & \frac{D\max\left\{M_{3},M_{4}\right\}}{T^{2}\left(T-K\right)^{2}}\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{w=1}^{T}\left\{\begin{array}{c}\sum_{n=1}^{T-w}ne^{-\frac{\lambda}{2}n}\\ +\sum_{n=1}^{T-w}\sum_{m=n+1}^{T-(n+w)}e^{-\frac{\lambda}{2}m}\end{array}\right\} \\
\leq\ & \frac{D\max\left\{M_{3},M_{4}\right\}}{T^{2}\left(T-K\right)^{2}}\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{w=1}^{T}\left\{\begin{array}{c}\sum_{n=1}^{\infty}ne^{-\frac{\lambda}{2}n}\\ +\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}e^{-\frac{\lambda}{2}m}\end{array}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{2D^2 \max\{M_3, M_4\}}{T^2 (T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{w=1}^{T} 1 \\
&= \frac{2D^2 \max\{M_3, M_4\} K^2}{T (T-K)^2} \\
&\leq \frac{C_7}{K}
\end{aligned}
$$

where the first inequality is clear, the second inequality follows from the two bounds given above (since each of the previous two bounds holds for each term, the smaller of them must hold), the third inequality is evident, the fourth inequality follows from Assumption 2, the first equality follows from the fact that $t-u > u-w$ if and only if $t > 2u-w$ (and the monotonicity of e^{-n}), the second equality follows from trivial summation and rearrangement, the third equality follows upon setting $m = t-u$ and $n = u-w$, the fifth inequality follows from the nonnegativity of ν_n and n, the sixth inequality follows directly from the bounds of expression (1), the third equality is due to simple summation, and the seventh and final inequality follows from the asymptotic nesting, in which K is asymptotically a constant fraction of T.

Case 4: $s > t = u > w$ (w. l. o. g.; this case includes all terms whose time subscripts take this form, with one greatest, two equal, and one least).

Here we obtain, by the m. d. s. property of ε (Assumption 1), $E\left[X_{is}\varepsilon_s X_{it} X_{jt} \varepsilon_t^2 X_{jw} \varepsilon_w\right] = 0$.

Case 5: $s > t > u = w$ (w. l. o. g.; this case includes all terms whose time subscripts take this form, with one greatest, one intermediate, and two equal and minimal).

Just as in Case 4, $E\left[X_{is}\varepsilon_s X_{it}\varepsilon_t X_{ju}^2\varepsilon_u^2\right] = 0$ by the m. d. s. property of ε.

Case 6: $s = t = u > w$ (w. l. o. g.; this case includes all terms which have three equal time subscripts and one lesser time subscript).

Noting that $E\left[X_{jw}\varepsilon_w\right] = 0$ by the m. d. s. property of ε (Assumption 1), we have that

$$\begin{aligned} & \left|E\left[X_{it}^2 X_{jt}\varepsilon_t^3 X_{jw}\varepsilon_w\right]\right| && (22) \\ \leq\ & \left|E\left[X_{it}^2 X_{jt}\varepsilon_t^3 X_{jw}\varepsilon_w\right] - E\left[X_{it}^2 X_{jt}\varepsilon_t^3\right] E\left[X_{jw}\varepsilon_w\right]\right| \\ \leq\ & \nu_{t-w}\left(E\left[X_{it}^4 X_{jt}^2\varepsilon_t^6\right]\right)^{1/2}\left(E\left[X_{jw}^2\varepsilon_w^2\right]\right)^{1/2} \\ \leq\ & M_5\nu_{t-w} \end{aligned}$$

where the first inequality follows from the preceding observation, the second is due to Doukhan (1994, Theorem 3 (5) on page 9) and Assumption 2, and the final inequality is due to the uniform moment bounds of Assumption 1.

The total absolute-value contribution of the terms handled in this case is thus

$$\left|\frac{1}{T^2\left(T-K\right)^2}\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{t=1}^{T}\sum_{w=t+1}^{T} E\left[X_{it}^2 X_{jt}\varepsilon_t^3 X_{jw}\varepsilon_w\right]\right| \tag{23}$$

$$
\begin{aligned}
&\leq \frac{1}{T^2 (T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} \sum_{w=t+1}^{T} \left| E\left[X_{it}^2 X_{jt} \varepsilon_t^3 X_{jw} \varepsilon_w \right] \right| \\
&\leq \frac{M_5}{T^2 (T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} \sum_{w=t+1}^{T} \nu_{t-w} \\
&\leq \frac{DM_5}{T^2 (T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} 1 \\
&= \frac{DM_5 K^2}{T (T-K)^2} \\
&\leq \frac{C_8}{K}
\end{aligned}
$$

where the first inequality is standard, the second follows from the above bound on each summand, the third inequality follows from the summability condition of expression (2), the equality follows by simple summation, and the final inequality is due to the asymptotic nesting, in which K is a constant fraction of T asymptotically.

Case 7: $s > t = u = w$ (w. l. o. g.; this case handles all terms which have three equal time subscripts and one greater time subscript).

By the m. d. s. property of the ε, we have that $E\left[X_{is}\varepsilon_s X_{it} X_{jt}^2 \varepsilon_t^3\right] = 0$.

Case 8: $s = t > u = w$ (note that here, we deal only with the specific subscript ordering given).

We have that $E\left[X_{it}^2 \varepsilon_t^2 X_{ju}^2 \varepsilon_u^2\right] = \sigma_\varepsilon^2 E\left[X_{it}^2 X_{ju}^2 \varepsilon_u^2\right]$, $E\left[X_{it}^2 \varepsilon_t^2\right] = \sigma_\varepsilon^2 E\left[X_{it}^2\right]$, and $E\left[X_{ju}^2 \varepsilon_u^2\right] = \sigma_\varepsilon^2 E\left[X_{ju}^2\right]$, so

$$
\left| E\left[X_{it}^2 \varepsilon_t^2 X_{ju}^2 \varepsilon_u^2\right] - \sigma_\varepsilon^4 E\left[X_{it}^2\right] E\left[X_{ju}^2\right] \right| \tag{24}
$$

$$
\begin{aligned}
&= \sigma_\varepsilon^2 \left| E\left[X_{it}^2 X_{ju}^2 \varepsilon_u^2\right] - \sigma_\varepsilon^2 E\left[X_{it}^2\right] E\left[X_{ju}^2\right] \right| \\
&= \sigma_\varepsilon^2 \left| E\left[X_{it}^2 X_{ju}^2 \varepsilon_u^2\right] - E\left[X_{it}^2\right] E\left[X_{ju}^2 \varepsilon_u^2\right] \right| \\
&\leq \sigma_\varepsilon^2 \nu_{t-u} \left(E\left[X_{it}^4\right]\right)^{1/2} \left(E\left[X_{ju}^4 \varepsilon_u^4\right]\right)^{1/2} \\
&\leq M_6 \nu_{t-u}
\end{aligned}
$$

where the first two equalities follow from the observations made immediately above the display, the first inequality comes from Hall and Heyde (1980, Theorem A.6 on page 278) and Assumption 2, and the final inequality comes from the uniform moment bounds given by Assumption 1.

We can now bound the absolute value of the difference between the total contribution of the terms handled by this case and

$$
\frac{\sigma_\varepsilon^4}{T^2 (T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{u=1}^{T} \sum_{t=u+1}^{T} E\left[X_{it}^2\right] E\left[X_{ju}^2\right]. \tag{25}
$$

This is done as follows:

$$
\begin{aligned}
&\left| \begin{array}{l} \frac{1}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{u=1}^{T} \sum_{t=u+1}^{T} E\left[X_{it}^2 \varepsilon_t^2 X_{ju}^2 \varepsilon_u^2\right] \\ -\frac{\sigma_\varepsilon^4}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{u=1}^{T} \sum_{t=u+1}^{T} E\left[X_{it}^2\right] E\left[X_{ju}^2\right] \end{array} \right| \qquad (26) \\
\leq\ & \frac{1}{T^2 (T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{u=1}^{T} \sum_{t=u+1}^{T} \left| E\left[X_{it}^2 \varepsilon_t^2 X_{ju}^2 \varepsilon_u^2\right] - \sigma_\varepsilon^4 E\left[X_{it}^2\right] E\left[X_{ju}^2\right] \right| \\
\leq\ & \frac{M_6}{T^2 (T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{u=1}^{T} \sum_{t=u+1}^{T} \nu_{t-u} \\
\leq\ & \frac{M_6}{T^2 (T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{u=1}^{T} \sum_{n=1}^{\infty} \nu_n \\
\leq\ & \frac{M_6 D}{T^2 (T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{u=1}^{T} 1
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{M_6 D K^2}{T\left(T-K\right)^2} \\
&\leq \frac{C_9}{K}
\end{aligned}
$$

where the first inequality is standard, the second is by the above bound on each summand, the third follows from nonnegativity of the ν_n and setting $n = t - u$, the fourth follows from expression (2), the equality is by simple summation, and the final inequality is due to the asymptotic nesting, in which K is an asymptotically constant fraction of T.

Case 9: $u = w > s = t$ (note that here, we deal only with the specific subscript ordering given).

This is identical to Case 8 above, except that we get the time terms $t < u$ rather than the terms $t > u$. Thus, the result is that the total contribution of the terms to which this case applies is close to

$$
\frac{\sigma_\varepsilon^4}{T^2\left(T-K\right)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{u=1}^{T} \sum_{t=1}^{u-1} E\left[X_{it}^2\right] E\left[X_{ju}^2\right] \tag{27}
$$

in the sense of the following bound:

$$
\begin{aligned}
&\left| \begin{array}{l} \frac{1}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{u=1}^{T} \sum_{t=1}^{u-1} E\left[X_{it}^2 \varepsilon_t^2 X_{ju}^2 \varepsilon_u^2\right] \\ -\frac{\sigma_\varepsilon^4}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{u=1}^{T} \sum_{t=1}^{u-1} E\left[X_{it}^2\right] E\left[X_{ju}^2\right] \end{array} \right| \\
&\leq \frac{C_{10}}{K}.
\end{aligned} \tag{28}
$$

Case 10: $s = u > t = w$ (note that here, we deal only with the specific ordering of the subscripts given).

$E\left[X_{is}X_{js}\varepsilon_s^2 X_{it}X_{jt}\varepsilon_t^2\right] = \sigma_\varepsilon^2 E\left[X_{is}X_{js}X_{it}X_{jt}\varepsilon_t^2\right]$ by the homoskedastic m. d. s. property of ε (from Assumption 1). Now, $E\left[X_{it}X_{jt}\varepsilon_t^2\right] = \sigma_\varepsilon^2 E\left[X_{it}X_{jt}\right]$ for the same reason, so

$$
\begin{aligned}
& \left|E\left[X_{is}X_{js}\varepsilon_s^2 X_{it}X_{jt}\varepsilon_t^2\right] - \sigma_\varepsilon^4 E\left[X_{it}X_{jt}\right]E\left[X_{is}X_{js}\right]\right| \qquad (29) \\
= \;& \left|\sigma_\varepsilon^2 E\left[X_{is}X_{js}X_{it}X_{jt}\varepsilon_t^2\right] - \sigma_\varepsilon^2 E\left[X_{it}X_{jt}\varepsilon_t^2\right]E\left[X_{is}X_{js}\right]\right| \\
= \;& \sigma_\varepsilon^2 \left|E\left[X_{is}X_{js}X_{it}X_{jt}\varepsilon_t^2\right] - E\left[X_{it}X_{jt}\varepsilon_t^2\right]E\left[X_{is}X_{js}\right]\right| \\
\leq \;& \sigma_\varepsilon^2 \nu_{s-t}\left(E\left[X_{it}^2 X_{jt}^2 \varepsilon_t^4\right]\right)^{1/2}\left(E\left[X_{is}^2 X_{js}^2\right]\right)^{1/2} \\
\leq \;& M_7 \nu_{s-t}
\end{aligned}
$$

where the first two equalities are by the identities noted immediately above the display, the first inequality is by Doukhan (1994, Theorem 3 (5) on page 9) and Assumption 2, and the final inequality is by the uniform moment bounds given in Assumption 1. Thus, we can obtain, exactly as in Cases 8 and 9 above, a bound on the absolute value of the difference between the total contribution of the terms handled here and the object

$$
\frac{\sigma_\varepsilon^4}{T^2\left(T-K\right)^2}\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{t=1}^{T}\sum_{s=t+1}^{T} E\left[X_{it}X_{jt}\right]E\left[X_{is}X_{js}\right] \qquad (30)
$$

where the bound is:

$$\left| \begin{array}{c} \frac{1}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} \sum_{s=t+1}^{T} E\left[X_{is}X_{js}\varepsilon_s^2 X_{it}X_{jt}\varepsilon_t^2\right] \\ -\frac{\sigma_\varepsilon^4}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} \sum_{s=t+1}^{T} E\left[X_{it}X_{jt}\right] E\left[X_{is}X_{js}\right] \end{array} \right| \quad (31)$$

$$\leq \frac{C_{11}}{K}.$$

Case 11: $t = w > s = u$ (note that here, we deal only with the specific ordering of the subscripts given).

This is entirely similar to Case 10, except that the bound is derived on the distance between the total contribution of the terms handled here and

$$\frac{\sigma_\varepsilon^4}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} \sum_{s=1}^{t-1} E\left[X_{it}X_{jt}\right] E\left[X_{is}X_{js}\right]. \quad (32)$$

The bound is:

$$\left| \begin{array}{c} \frac{1}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} \sum_{s=1}^{t-1} E\left[X_{is}X_{js}\varepsilon_s^2 X_{it}X_{jt}\varepsilon_t^2\right] \\ -\frac{\sigma_\varepsilon^4}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} \sum_{s=1}^{t-1} E\left[X_{it}X_{jt}\right] E\left[X_{is}X_{js}\right] \end{array} \right| \quad (33)$$

$$\leq \frac{C_{12}}{K}.$$

Now we may pull all the cases back together again to get:

$$\left| \begin{array}{c} \frac{1}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{s=1}^{T} \sum_{t=1}^{T} \sum_{u=1}^{T} \sum_{w=1}^{T} \\ E\left[X_{is}X_{it}X_{ju}X_{jw}\varepsilon_s\varepsilon_t\varepsilon_u\varepsilon_w\right] \\ -\frac{\sigma_\varepsilon^4 K^2}{(T-K)^2} \end{array} \right| \quad (34)$$

$$
\begin{aligned}
\leq\quad & \frac{C_7+C_8+C_9}{K} \\
& + \left| \begin{array}{c} \frac{1}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{u=1}^{T} \sum_{t=u+1}^{T} E\left[X_{it}^2 \varepsilon_t^2 X_{ju}^2 \varepsilon_u^2\right] \\ -\frac{\sigma_\varepsilon^4}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{u=1}^{T} \sum_{t=u+1}^{T} E\left[X_{it}^2\right] E\left[X_{ju}^2\right] \end{array} \right| \\
& + \left| \begin{array}{c} \frac{1}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{u=1}^{T} \sum_{t=1}^{u-1} E\left[X_{it}^2 \varepsilon_t^2 X_{ju}^2 \varepsilon_u^2\right] \\ -\frac{\sigma_\varepsilon^4}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{u=1}^{T} \sum_{t=1}^{u-1} E\left[X_{it}^2\right] E\left[X_{ju}^2\right] \end{array} \right| \\
& + \left| \begin{array}{c} \frac{1}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} \sum_{s=t+1}^{T} E\left[X_{is} X_{js} \varepsilon_s^2 X_{it} X_{jt} \varepsilon_t^2\right] \\ -\frac{\sigma_\varepsilon^4}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} \sum_{s=t+1}^{T} E\left[X_{it} X_{jt}\right] E\left[X_{is} X_{js}\right] \end{array} \right| \\
& + \left| \begin{array}{c} \frac{1}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} \sum_{s=1}^{t-1} E\left[X_{is} X_{js} \varepsilon_s^2 X_{it} X_{jt} \varepsilon_t^2\right] \\ -\frac{\sigma_\varepsilon^4}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} \sum_{s=1}^{t-1} E\left[X_{it} X_{jt}\right] E\left[X_{is} X_{js}\right] \end{array} \right| \\
& + \left| \begin{array}{c} \frac{\sigma_\varepsilon^4}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} \sum_{s=1}^{T} E\left[X_{it} X_{jt}\right] E\left[X_{is} X_{js}\right] \\ -\frac{\sigma_\varepsilon^4}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} E\left[X_{it} X_{jt}\right] E\left[X_{it} X_{jt}\right] \end{array} \right| \\
& + \left| \begin{array}{c} \frac{\sigma_\varepsilon^4}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{u=1}^{T} \sum_{t=1}^{T} E\left[X_{it}^2\right] E\left[X_{ju}^2\right] \\ -\frac{\sigma_\varepsilon^4}{T^2(T-K)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} E\left[X_{it}^2\right] E\left[X_{jt}^2\right] \\ -\frac{\sigma_\varepsilon^4 K^2}{(T-K)^2} \end{array} \right| \\
\leq\quad & \frac{C_7+C_8+C_9+C_{10}+C_{11}+C_{12}}{K} \\
& + \left| \frac{\sigma_\varepsilon^4}{T^2\left(T-K\right)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} \sum_{s=1}^{T} E\left[X_{it} X_{jt}\right] E\left[X_{is} X_{js}\right] \right| \\
& + \left| \frac{\sigma_\varepsilon^4}{T^2\left(T-K\right)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} E\left[X_{it} X_{jt}\right] E\left[X_{it} X_{jt}\right] \right| \\
& + \left| \frac{\sigma_\varepsilon^4}{T^2\left(T-K\right)^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{t=1}^{T} E\left[X_{it}^2\right] E\left[X_{jt}^2\right] \right|
\end{aligned}
$$

$$
\begin{aligned}
& + \left| \begin{array}{c} \frac{\sigma_\varepsilon^4}{T^2(T-K)^2} \sum_{i=1}^K \sum_{j=1}^K \sum_{u=1}^T \sum_{t=1}^T E\left[X_{it}^2\right] E\left[X_{ju}^2\right] \\ -\frac{\sigma_\varepsilon^4 K^2}{(T-K)^2} \end{array} \right| \\
\leq \quad & \frac{C_7 + C_8 + C_9 + C_{10} + C_{11} + C_{12} + C_{13} + C_{14}}{K} \\
& + \left| \frac{\sigma_\varepsilon^4}{T^2 (T-K)^2} \sum_{i=1}^K \sum_{j=1}^K \sum_{t=1}^T \sum_{s=1}^T E\left[X_{it} X_{jt}\right] E\left[X_{is} X_{js}\right] \right| \\
\leq \quad & \frac{C_7 + C_8 + C_9 + C_{10} + C_{11} + C_{12} + C_{13} + C_{14}}{K} \\
& + \left| \frac{\sigma_\varepsilon^4}{T^2 (T-K)^2} \sum_{i=1}^K \sum_{\substack{j=1 \\ j \neq i}}^K \sum_{t=1}^T \sum_{s=1}^T E\left[X_{it} X_{jt}\right] E\left[X_{is} X_{js}\right] \right| \\
& + \left| \frac{\sigma_\varepsilon^4}{T^2 (T-K)^2} \sum_{i=1}^K \sum_{t=1}^T \sum_{s=1}^T E\left[X_{it}^2\right] E\left[X_{is}^2\right] \right| \\
\leq \quad & \frac{C_7 + C_8 + C_9 + C_{10} + C_{11} + C_{12} + C_{13} + C_{14} + C_{15}}{K} \\
= \quad & \frac{C_{16}}{K}.
\end{aligned}
$$

We may finally combine the bounds we have obtained for each of the three expectations involved, and notice that $\sigma_\varepsilon^4 = \frac{T^2 - 2KT + K^2}{(T-K)^2} \sigma_\varepsilon^4$, to yield:

$$
\begin{aligned}
& \left| E\left[\left(\hat{\sigma}_\varepsilon^2\right)^2\right] - \sigma_\varepsilon^4 \right| \qquad (35) \\
\leq \quad & \left| E\left[\frac{1}{(T-K)^2} \sum_{s=1}^T \sum_{t=1}^T \varepsilon_s^2 \varepsilon_t^2\right] - \frac{T(T-1)}{(T-K)^2} \sigma_\varepsilon^4 \right| \\
& + \left| \frac{T}{(T-K)^2} \sigma_\varepsilon^4 \right| \\
& + \left| \begin{array}{c} E\left[\frac{-2}{T(T-K)^2} \sum_{t=1}^T \sum_{i=1}^K \sum_{s=1}^T \sum_{u=1}^T \varepsilon_t^2 X_{is} X_{iu} \varepsilon_s \varepsilon_u\right] \\ -\left(\frac{-2KT}{(T-K)^2} \sigma_\varepsilon^2\right) \end{array} \right|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\begin{array}{c}\frac{1}{T^2(T-K)^2}\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{s=1}^{T}\sum_{t=1}^{T}\sum_{u=1}^{T}\sum_{w=1}^{T}\\ E\left[X_{is}X_{it}X_{ju}X_{jw}\varepsilon_s\varepsilon_t\varepsilon_u\varepsilon_w\right]\\ -\frac{\sigma_\varepsilon^4K^2}{(T-K)^2}\end{array}\right| \\
\leq\; & \frac{C_1}{K} + \left|\frac{T}{(T-K)^2}\sigma_\varepsilon^4\right| + \frac{C_5}{K} + \frac{C_{16}}{K} \\
\leq\; & \frac{C_1}{K} + \frac{C_{17}}{K} + \frac{C_5}{K} + \frac{C_{16}}{K} \\
\leq\; & \frac{C}{K}
\end{aligned}
$$

which completes the proof of Theorem 1. Q.E.D.

Proof of Theorem 2: Because we have w. l. o. g. set $H = I_K$,

$$
\begin{aligned}
R\left(\hat{b},b\right) &= \frac{1}{T}E\left[\left(\hat{b}-b\right)'\left(\hat{b}-b\right)\right] \\
&= \frac{1}{T}E\left[\left(\frac{1}{\sqrt{T}}X'\varepsilon\right)'\left(\frac{1}{\sqrt{T}}X'\varepsilon\right)\right] \\
&= \frac{1}{T^2}E\left[\sum_{i=1}^{K}\sum_{t=1}^{T}\sum_{s=1}^{T}X_{it}X_{is}\varepsilon_t\varepsilon_s\right] \\
&= \frac{1}{T^2}\sum_{i=1}^{K}\sum_{t=1}^{T}\sum_{s=1}^{T}E\left[X_{it}X_{is}\varepsilon_t\varepsilon_s\right] \\
&= \frac{1}{T^2}\sum_{i=1}^{K}\sum_{t=1}^{T}E\left[X_{it}^2\varepsilon_t^2\right] \\
&= \frac{\sigma_\varepsilon^2}{T^2}\sum_{i=1}^{K}E\left[\sum_{t=1}^{T}X_{it}^2\right] \\
&= \frac{\sigma_\varepsilon^2}{T^2}\sum_{i=1}^{K}T \\
&= \frac{K}{T}\sigma_\varepsilon^2
\end{aligned} \tag{36}
$$

$$\rightarrow \quad \rho\sigma_\varepsilon^2$$

where the first four equalities follow from simple calculation, the fifth equality holds because if $s \neq t$, then $E[X_{is}X_{it}\varepsilon_s\varepsilon_t] = 0$ by the m. d. s. property of ε (from Assumption 1), the sixth equality is due to the fact that $E[X_{it}^2\varepsilon_t^2] = \sigma_\varepsilon^2 E[X_{it}^2]$, the seventh equality follows from $\sum_{t=1}^T X_{it}^2 = T$ (since $X'X = TI_K$), the eighth equality follows from simple summation, and the limit is due to the asymptotic nesting we have specified, in which $\frac{K}{T} \rightarrow \rho$ as $T \rightarrow \infty$. All of the other results are immediate consequences of the above calculation. Q. E. D.

1.2 A.2 Proofs of Theorems 3, 4, and 5

We start by collecting some additional definitions.

Definition 1

$$\begin{aligned}
d_K &\equiv \sqrt{\frac{\sigma_\varepsilon^2}{64}\log K} \\
\hat{b}_{-i} &\equiv \left(\hat{b}_1, \ldots, \hat{b}_{i-1}, \hat{b}_{i+1}, \ldots, \hat{b}_K\right) \\
\phi(u) &\equiv \textit{the univariate normal density with} \\
&\quad \textit{mean } 0 \textit{ and variance } \sigma_\varepsilon^2 \\
f_{iK}(u_i) &\equiv \textit{the marginal likelihood of } \hat{b}_i \textit{ given } b_i \\
&= \int_{u_{-i}} f_K(u)\, du_{-i}
\end{aligned}$$

$$\bar{f}_K(x) \equiv \frac{1}{K}\sum_{i=1}^{K} f_{iK}(x)$$

$$f_{ijK}\left(\hat{b}_i - b_i, \hat{b}_j - b_j\right) \equiv \textit{the likelihood of } \hat{b}_i \textit{ and } \hat{b}_j \textit{ given } b_i \textit{ and } b_j$$

$$= \int_{u_{-(i,j)}} f_K(u)\, du_{-(i.j)}$$

$$m_{iK}\left(\hat{b}_i\right) \equiv \int_{-\infty}^{\infty} f_{iK}\left(\hat{b}_i - b_i\right) dG(b_i)$$

$$m'_{iK}\left(\hat{b}_i\right) \equiv \frac{d}{d\hat{b}_i}\int_{-\infty}^{\infty} f_{iK}\left(\hat{b}_i - b_i\right) dG(b_i)$$

$$\bar{m}_K(x) \equiv \frac{1}{K}\sum_{i=1}^{K} m_{iK}(x)$$

$$m_\phi\left(\hat{b}_1\right) \equiv \int_{-\infty}^{\infty} \phi\left(\hat{b}_1 - b_1\right) dG(b_1)$$

$$m_{ijK}^{C}\left(\hat{b}_i \mid \hat{b}_j\right) \equiv \frac{m_{ijK}\left(\hat{b}_i, \hat{b}_j\right)}{m_{iK}\left(\hat{b}_j\right)}$$

$$\equiv \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{ijK}\left(\hat{b}_i - b_i, \hat{b}_j - b_j\right) dG(b_i)\, dG(b_j)}{m_{iK}\left(\hat{b}_j\right)}$$

$$\bar{m}_{iK}^{C}(x \mid y) \equiv \frac{1}{K-1}\sum_{\substack{j=1,\\ j\neq i}}^{K} m_{ijK}^{C}(x \mid y)$$

Note that Assumption 7 implies the following limits:

$$\begin{aligned} s_K q_K &\to \infty \\ K s_K^8 &\to \infty \\ s_K^2 \log K &\to \infty \\ \frac{K^{-5/48}\log^2 K}{s_K^2} &\to 0 \\ \frac{K^{-5/96} h_K \log K}{s_K^2} &\to 0 \\ \frac{h_K}{s_K} &\to 0 \end{aligned} \tag{37}$$

$$q_K^2 \;/\; K \quad \to \quad 0$$

The following lemmas are used in proving the main results. The first three lemmas collect Berry-Esseen-type results about convergence of certain densities and their derivatives to local limits.

Lemma 1 (Berry-Esseen Results for Densities) *Under Assumptions 1, 2, and 3, we have the following local limit rate results:* $\exists$ *finite* C, K_0 *s.t.* $\forall\ K \geq K_0$,

$$\sup_{i,s} |f_{iK}(s) - \phi(s)| \quad \leq \quad CK^{-\frac{1}{4}} \log K \tag{38}$$

$$\sup_{i,s} |f'_{iK}(s) - \phi'(s)| \quad \leq \quad CK^{-\frac{1}{8}} \log K \tag{39}$$

Proof of Lemma 1: Let $\eta_{it} = \frac{X_{it}\varepsilon_t}{\sigma_\varepsilon}$. Suppose that the sequence $\eta_{i1}, \eta_{i2}, \ldots$ satisfies Conditions A and C of Appendix B, where the constants in those conditions do not depend on i. Then, by Theorem 2 of Appendix B and a simple change of scale (recalling that σ_ε^2 is bounded away from both zero and infinity),

$$\sup_{s} |f_{iK}(s) - \phi(s)| \quad \leq \quad CK^{-\frac{1}{4}} \log K \tag{40}$$

$$\sup_{s} |f'_{iK}(s) - \phi'(s)| \quad \leq \quad CK^{-\frac{1}{8}} \log K$$

for each i, where the constant C does not depend on i. It follows that these inequalities hold uniformly in $i = 1, \ldots, K$, i. e., that (38) and (39) hold. To prove the theorem, it therefore suffices to prove that $\eta_{i1}, \eta_{i2}, \ldots$ satisfy Conditions A and C with constants that do not depend on i.

We first verify that Condition C is satisfied. By Assumption 1, $E[\eta_{it}] = 0$ and

$$E\left[\left(\sqrt{\frac{1}{T}}\sum_{t=1}^{T}\eta_{it}\right)^2\right] = \frac{1}{T\sigma_\varepsilon^2}\sum_{s=1}^{T}\sum_{t=1}^{T}E\left[X_{it}X_{is}\varepsilon_t\varepsilon_s\right] \tag{41}$$

$$= \frac{1}{T\sigma_\varepsilon^2}\sum_{t=1}^{T}E\left[X_{it}^2\varepsilon_t^2\right]$$

$$= \frac{1}{T}E\left[\sum_{t=1}^{T}X_{it}^2\right]$$

$$= 1$$

where the first equality is by definition, the second is by the m. d. s. property of the ε's (Assumption 1), the third is by the homoskedasticity of the ε's (Assumption 1) and the fourth equality is by the orthonormality of the X's.

Also, by Assumption 1,

$$\sup_i E\left[\eta_{it}^6\right] = \sup_i E\left[X_{it}^6\varepsilon_t^6\right] \tag{42}$$

$$\leq \sup_i \left(E\left[X_{it}^{12}\right]E\left[\varepsilon_t^{12}\right]\right)^{1/2}$$

$$= \left(\sup_i E\left[X_{it}^{12}\right]E\left[\varepsilon_t^{12}\right]\right)^{1/2}$$

$$\leq C$$

so the moment conditions in Condition C hold uniformly in i. The condition on the (time series) mixing coefficients in Condition C and the fact that it hold uniformly in i follow directly from Assumption 2. This verifies that Assumptions 1 and 2 imply Condition C.

It remains to show that Condition A of Appendix B is implied by Assumption 3 uniformly in i. Let

$$\psi_{it}^{X\varepsilon}(s) \equiv \int_{-\infty}^{\infty} e^{is\eta_{it}} p_{it}^{\eta}\left(\eta_{it} \mid \underline{X}^{(t-1)}, \varepsilon^{(t-1)}\right) d\eta_{it} \tag{43}$$

be the characteristic function of η_{it} (conditional on all past X and ε), and define

$$\psi_{t}^{\varepsilon}(s) \equiv \int_{-\infty}^{\infty} e^{is\varepsilon_{t}} p_{t}^{\varepsilon}\left(\varepsilon_{t} \mid \underline{X}^{(t-1)}, \varepsilon^{(t-1)}\right) d\varepsilon_{t} \tag{44}$$

to be the characteristic function of ε_t (conditional on all past X and ε).

We need to show that $\exists \alpha > 0, C_0 < \infty$, and $M_0 < \infty$ such that

$$\forall |s| \geq C_0,\ \sup_{it} \left|\psi_{it}^{X\varepsilon}(s)\right| \leq M_0 |s|^{-\alpha}. \tag{45}$$

Set $C_0 = 1$. Note that Assumption 3 implies that

$$\begin{aligned}
\sup_t \left|\psi_t^{\varepsilon}(s)\right| &= \sup_t \left|\int_{-\infty}^{\infty} e^{is\varepsilon_t} p_t^{\varepsilon}\left(\varepsilon_t \mid \underline{X}^{(t-1)}, \varepsilon^{(t-1)}\right) d\varepsilon_t\right| \\
&= \sup_t \left|-\frac{1}{is}\int_{-\infty}^{\infty} e^{is\varepsilon_t}\left[\frac{d}{d\varepsilon_t} p_t^{\varepsilon}\left(\varepsilon_t \mid \underline{X}^{(t-1)}, \varepsilon^{(t-1)}\right)\right] d\varepsilon_t\right| \\
&= \sup_t \left|-\frac{1}{s^2}\int_{-\infty}^{\infty} e^{is\varepsilon_t}\left[\frac{d^2}{d\varepsilon_t^2} p_t^{\varepsilon}\left(\varepsilon_t \mid \underline{X}^{(t-1)}, \varepsilon^{(t-1)}\right)\right] d\varepsilon_t\right| \\
&\leq |s|^{-2} \sup_t \int_{-\infty}^{\infty} \left|e^{is\varepsilon_t}\right| \left|\frac{d^2}{d\varepsilon_t^2} p_t^{\varepsilon}\left(\varepsilon_t \mid \underline{X}^{(t-1)}, \varepsilon^{(t-1)}\right)\right| d\varepsilon_t \\
&= |s|^{-2} \sup_t \int_{-\infty}^{\infty} \left|\frac{d^2}{d\varepsilon_t^2} p_t^{\varepsilon}\left(\varepsilon_t \mid \underline{X}^{(t-1)}, \varepsilon^{(t-1)}\right)\right| d\varepsilon_t \\
&\leq M_7 |s|^{-2}
\end{aligned} \tag{46}$$

where the first equality is by definition, the second is by integration by parts, the third is by another integration by parts, the first inequality is obvious, the fourth equality is by $|e^{-is\varepsilon_t}| = 1$, and the final inequality is by $\sup_t \int_{-\infty}^{\infty} \left| \frac{d^2}{d\varepsilon_t^2} p_t^{\varepsilon} \left(\varepsilon_t \mid \underline{X}^{(t-1)}, \varepsilon^{(t-1)} \right) \right| d\varepsilon_t < \infty$ according to Assumption 3.

By Feller (1971, page 527) (or simply a short calculation), we have the first inequality in the following display, and the rest follow from (46) and our assumptions: for $|s| \geq 1$,

$$
\begin{aligned}
\sup_{it} \left| \psi_{it}^{X\varepsilon}(s) \right| &\leq \sup_{it} \left| \int_{-\infty}^{\infty} \psi_t^{\varepsilon}(sx) \, p_{it}^X(x) \, dx \right| \qquad (47) \\
&\leq \sup_{it} \left| \int_{1}^{\infty} \psi_t^{\varepsilon}(sx) \, p_{it}^X(x) \, dx \right| + \sup_{it} \left| \int_{1/|s|}^{1} \psi_t^{\varepsilon}(sx) \, p_{it}^X(x) \, dx \right| \\
&\quad + \sup_{it} \left| \int_{-1/|s|}^{1/|s|} \psi_t^{\varepsilon}(sx) \, p_{it}^X(x) \, dx \right| + \sup_{it} \left| \int_{-1}^{-1/|s|} \psi_t^{\varepsilon}(sx) \, p_{it}^X(x) \, dx \right| \\
&\quad + \sup_{it} \left| \int_{-\infty}^{-1} \psi_t^{\varepsilon}(sx) \, p_{it}^X(x) \, dx \right| \\
&\leq \int_{1}^{\infty} \frac{M_7}{|sx|^2} p_{it}^X(x) \, dx + \int_{1/|s|}^{1} \frac{M_6 M_7}{|sx|^2} dx + \int_{-1/|s|}^{1/|s|} M_6 dx \\
&\quad + \int_{-1}^{-1/|s|} \frac{M_6 M_7}{|sx|^2} dx + \int_{-\infty}^{-1} \frac{M_7}{|sx|^2} p_{it}^X(x) \, dx \\
&\leq \frac{2M_7}{|s|^2} + \frac{2M_6}{|s|} + \frac{2M_7 M_6}{|s|^2} + \frac{2M_7 M_6}{|s|}
\end{aligned}
$$

so $\alpha = 1$ and we are finished.

Lemma 2 (Berry-Esseen Results for Joint Densities) *Under Assumptions 1, 2, and 3, we have that* $\exists$ *finite* C, K_0 *s.t.* $\forall$ $K \geq K_0$,

$$
\begin{aligned}
\sup_{i,j,s,u} \left| f_{ijK}(s,u) - \phi(s) \, \phi(u) \right| &\leq CK^{-\frac{1}{6}} \log K \qquad (48) \\
\sup_{i,j,s,u} \left| \frac{\partial}{\partial s} f_{ijK}(s,u) - \phi'(s) \, \phi(u) \right| &\leq CK^{-\frac{1}{12}} \log K \qquad (49)
\end{aligned}
$$

Proof of Lemma 2: Let $\eta_{ijt} = \frac{1}{\sigma_\varepsilon} \begin{pmatrix} X_{it}\varepsilon_t \\ X_{jt}\varepsilon_t \end{pmatrix}$. If the sequence of bivariate random variables $\eta_{ij1}, \eta_{ij2}, \ldots$ satisfies Conditions A and C in Appendix B uniformly in (i,j), then Theorem 3 of Appendix B holds uniformly in (i,j) and Lemma 2 follows. The argument that Assumptions 1 and 2 imply Condition C for $\eta_{ij1}, \eta_{ij2}, \ldots$ uniformly in (i,j) parallels the corresponding argument in the proof of Lemma 1 and is omitted.

It remains only to show that Assumption 3 implies Condition A of Appendix B uniformly in (i,j). Let

$$\psi_{it}^{X\varepsilon}(s_1,s_2) \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\left(s_1\eta_{ijt}^{(1)}+s_2\eta_{ijt}^{(2)}\right)} p_{ijt}^{\eta}\left(\eta_{ijt} \mid \underline{X}^{(t-1)}, \varepsilon^{(t-1)}\right) d\eta_{ijt}^{(1)} d\eta_{ijt}^{(2)} \qquad (50)$$

be the characteristic function of η_{ijt}. We need to show that $\exists \alpha > 0, C_0 < \infty$, and $M_0 < \infty$ such that

$$\forall\, |s| \geq C_0,\ \sup_{it} \left|\psi_{it}^{X\varepsilon}(s)\right| \leq M_0 \,|s|^{-\alpha}\,. \qquad (51)$$

As in the proof of the corresponding part of Lemma 1, choose $C_0 = 1$.

We cannot use the rather simple method of Lemma 1 above to prove (51), because of the possibility that the vectors (s_1,s_2) and (x_i,x_j) might be orthogonal, causing $\frac{1}{(s_1x_i+s_2x_j)^2}$ to be undefined (infinite) even when both $|s|$ and $|x|$ are large.

To avoid this problem, define by A the set of points in the plane such that the angle between the vectors s and x is within γ of either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ (that is, the vectors are "γ-close" to being orthogonal), and the magnitude of the x vector is between $\frac{1}{|s|}$ and $|s|^{1/4}$. Recall that $(s_1x_i+s_2x_j)^2 = |s|^2\,|x|^2 \cos^2\theta_{s,x}$, where $\theta_{s,x}$ is the angle between s

and x. Also, note that the density $p^X_{ijt}(x_i, x_j)$ is bounded by Assumption 3. Finally, recall that $|\cos\gamma|$ dominates a sawtoothed function of γ (draw a line from each zero of $|\cos\gamma|$ to the nearest maximum of $|\cos\gamma|$ to the left of the zero, and another line to the nearest maximum to the right of the zero, and you will have drawn the sawtoothed function). Thus, $\cos^{-2}\left(\frac{\pi}{2} \pm \gamma\right) \leq \frac{\pi^2}{4}\gamma^{-2}$ for $0 < \gamma \leq \frac{\pi}{2}$, and the same expression holds for $\cos^{-2}\left(\frac{3\pi}{2} \pm \gamma\right)$.

Set $C_0 = \left(\frac{\pi}{2}\right)^{-4/3}$, so we will demonstrate that the inequality in display (51) holds for s such that $|s| \geq \left(\frac{\pi}{2}\right)^{-4/3}$, which implies that $|s|^{-3/4} \leq \frac{\pi}{2}$. Set $\gamma = |s|^{-3/4}$, so that $\gamma \leq \frac{\pi}{2}$ for $|s| \geq C_0$ and we may apply the cosine inequality developed in the preceding paragraph. Let B_r denote a ball of radius r centered at 0 in $\Re^2$, and let $\Re^2 - B_r$ denote $\Re^2$ excluding this ball. If $r_1 > r_2$, let $B_{r_1} - B_{r_2}$ denote B_{r_1} excluding B_{r_2}. By Feller (1971, page 527) (or simply a short calculation), we have the first inequality in the following display, where the rest follow from (46) and our assumptions.

$$
\begin{aligned}
\sup_{ijt}\left|\psi^{X\varepsilon}_{ijt}(s_1, s_2)\right| &\leq \sup_{ijt}\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi^{\varepsilon}_t(s_1x_i + s_2x_j)\, p^X_{ijt}(x_i, x_j)\, dx_i dx_j\right| \qquad (52) \\
&\leq \sup_{ijt}\left|\int_{\Re^2 - B_{|s|^{1/4}}}\psi^{\varepsilon}_t(s_1x_i + s_2x_j)\, p^X_{ijt}(x_i, x_j)\, dx\right| + \\
&\quad \sup_{ijt}\left|\int_{B_{1/|s|}}\psi^{\varepsilon}_t(s_1x_i + s_2x_j)\, p^X_{ijt}(x_i, x_j)\, dx\right| + \\
&\quad \sup_{ijt}\left|\int_{B_{|s|^{1/4}} - B_{1/|s|}}\psi^{\varepsilon}_t(s_1x_i + s_2x_j)\, p^X_{ijt}(x_i, x_j)\, dx\right| \\
&\leq \sup_{ijt}\int_{\Re^2 - B_{|s|^{1/4}}}\left|\psi^{\varepsilon}_t(s_1x_i + s_2x_j)\right| p^X_{ijt}(x_i, x_j)\, dx +
\end{aligned}
$$

$$
\begin{aligned}
& \sup_{ijt} \int_{B_{1/|s|}} \left|\psi_t^{\varepsilon}\left(s_1 x_i + s_2 x_j\right)\right| p_{ijt}^X\left(x_i, x_j\right) dx + \\
& \sup_{ijt} \int_{B_{|s|^{1/4}} - B_{1/|s|}} \left|\psi_t^{\varepsilon}\left(s_1 x_i + s_2 x_j\right)\right| p_{ijt}^X\left(x_i, x_j\right) dx \\
\leq\ & \sup_{ijt} \int_{\Re^2 - B_{|s|^{1/4}}} p_{ijt}^X\left(x_i, x_j\right) dx + \\
& \sup_{ijt} \int_{B_{1/|s|}} p_{ijt}^X\left(x_i, x_j\right) dx + \\
& \sup_{ijt} \int_{A} p_{ijt}^X\left(x_i, x_j\right) dx + \\
& \sup_{ijt} \int_{B_{|s|^{1/4}} - B_{1/|s|} - A} \left|\psi_t^{\varepsilon}\left(s_1 x_i + s_2 x_j\right)\right| p_{ijt}^X\left(x_i, x_j\right) dx \\
\leq\ & \frac{C}{|s|^{1/2}} + \frac{C_2}{|s|} + 2\,|s|^{1/2}\,\gamma + \\
& \sup_{ijt} \int_{B_{|s|^{1/4}} - B_{1/|s|} - A} \left(s_1 x_i + s_2 x_j\right)^{-2} p_{ijt}^X\left(x_i, x_j\right) dx \\
=\ & \frac{C}{|s|^{1/2}} + \frac{C_2}{|s|} + 2\,|s|^{1/2}\,\gamma + \\
& \sup_{ijt} \int_{B_{|s|^{1/4}} - B_{1/|s|} - A} |s|^{-2}\,|x|^{-2} \cos^{-2}\theta_{s,x} p_{ijt}^X\left(x_i, x_j\right) dx \\
\leq\ & \frac{C}{|s|^{1/2}} + \frac{C_2}{|s|} + C_3\,|s|^{1/2}\,\gamma + \\
& \frac{\pi^2}{4}\,|s|^{-2}\,\gamma^{-2} \sup_{ijt} \int_{B_{|s|^{1/4}} - B_{1/|s|} - A} |x|^{-2}\, p_{ijt}^X\left(x_i, x_j\right) dx \\
\leq\ & \frac{C}{|s|^{1/2}} + \frac{C_2}{|s|} + C_3\,|s|^{1/2}\,\gamma + \\
& \frac{\pi^2}{4}\,|s|^{-2}\,\gamma^{-2} C_4 \int_{B_{|s|^{1/4}}} \frac{1}{r} dr d\theta \\
\leq\ & \frac{C}{|s|^{1/2}} + \frac{C_2}{|s|} + C_3\,|s|^{1/2}\,\gamma + \\
& \frac{\pi^2}{4}\,|s|^{-2}\,\gamma^{-2} C_4 2\pi \frac{1}{4} \ln\left(|s|\right) \\
\leq\ & \frac{C}{|s|^{1/2}} + \frac{C_2}{|s|} + \frac{C_3}{|s|^{1/4}} + \frac{C_5}{|s|^{1/2}} \ln\left(|s|\right)
\end{aligned}
$$

$$\leq \quad \frac{C}{|s|^{1/2}} + \frac{C_2}{|s|} + \frac{C_3}{|s|^{1/4}} + \frac{C_5}{|s|^{1/3}}$$

where the second-to-last inequality follows from our choice of $\gamma = |s|^{-3/4}$. Thus, Condition A is satisfied with $\alpha = \frac{1}{4}$, and we are finished.

Lemma 3 (Rates for Additional Densities) *Under Assumptions 1, 2, and 3,* $\exists\, C, K_0 < \infty$ *s.t.* $\forall\, K \geq K_0$,

$$\sup_{s} \left| \bar{f}_K(s) - \phi(s) \right| \leq CK^{-\frac{1}{4}} \log K \qquad (a)$$

$$\sup_{i,\hat{b}_i} \left| m_{iK}\left(\hat{b}_i\right) - m_{\phi}\left(\hat{b}_i\right) \right| \leq CK^{-\frac{1}{4}} \log K \qquad (b)$$

$$\sup_{i,\hat{b}_i} \left| \bar{m}_K\left(\hat{b}_i\right) - m_{\phi}\left(\hat{b}_i\right) \right| \leq CK^{-\frac{1}{4}} \log K \qquad (c)$$

$$\sup_{i,\hat{b}_i} \left| m'_{iK}\left(\hat{b}_i\right) - m'_{\phi}\left(\hat{b}_i\right) \right| \leq CK^{-\frac{1}{8}} \log K \qquad (d)$$

$$\sup_{i,\hat{b}_i} \left| \bar{m}'_K\left(\hat{b}_i\right) - m'_{\phi}\left(\hat{b}_i\right) \right| \leq CK^{-\frac{1}{8}} \log K \qquad (e)$$

$$\sup_{i,j,\hat{b}_i,\hat{b}_j} \left| m_{ijK}\left(\hat{b}_i, \hat{b}_j\right) - m_{\phi}\left(\hat{b}_i\right) m_{\phi}\left(\hat{b}_j\right) \right| \leq CK^{-\frac{1}{6}} \log K \qquad (f)$$

$$\sup_{i,j,\hat{b}_i,\hat{b}_j} \left| \frac{\partial}{\partial \hat{b}_i} m_{ijK}\left(\hat{b}_i, \hat{b}_j\right) - m'_{\phi}\left(\hat{b}_i\right) m_{\phi}\left(\hat{b}_j\right) \right| \leq CK^{-\frac{1}{12}} \log K \qquad (g)$$

Proof: Part (a) follows from Lemma 1 and

$$\begin{aligned}
& \sup_{s} \left| \bar{f}_K(s) - \phi(s) \right| \qquad (53) \\
= \; & \sup_{s} \left| \frac{1}{K} \sum_{i=1}^{K} \left(f_{iK}(s) - \phi(s) \right) \right| \\
\leq \; & \frac{1}{K} \sum_{i=1}^{K} \sup_{s} \left| f_{iK}(s) - \phi(s) \right| \\
\leq \; & CK^{-1/4} \log K.
\end{aligned}$$

Part (b) follows from Lemma 1 and

$$\begin{aligned}
& \sup_{i,\hat{b}_i} \left| m_{iK}\left(\hat{b}_i\right) - m_\phi\left(\hat{b}_i\right) \right| && (54) \\
= & \sup_{i,\hat{b}_i} \left| \int_{-\infty}^{\infty} \left[f_{iK}\left(\hat{b}_i - b_i\right) - \phi\left(\hat{b}_i - b_i\right) \right] dG\left(b_i\right) \right| \\
\leq & \int_{-\infty}^{\infty} \sup_{i,\hat{b}_i} \left| f_{iK}\left(\hat{b}_i - b_i\right) - \phi\left(\hat{b}_i - b_i\right) \right| dG\left(b_i\right) \\
\leq & \; CK^{-1/4} \log K
\end{aligned}$$

Part (c) follows from part (b) in exactly the same way that part (a) follows from Lemma 1. Part (d) follows from Lemma 1 and

$$\begin{aligned}
& \sup_{i,\hat{b}_i} \left| m'_{iK}\left(\hat{b}_i\right) - m'_\phi\left(\hat{b}_i\right) \right| && (55) \\
= & \sup_{i,\hat{b}_i} \left| \frac{d}{d\hat{b}_i} \int_{-\infty}^{\infty} \left[f_{iK}\left(\hat{b}_i - b_i\right) - \phi\left(\hat{b}_i - b_i\right) \right] dG\left(b_i\right) \right| \\
= & \sup_{i,\hat{b}_i} \left| \int_{-\infty}^{\infty} \left[f'_{iK}\left(\hat{b}_i - b_i\right) - \phi'\left(\hat{b}_i - b_i\right) \right] dG\left(b_i\right) \right| \\
\leq & \int_{-\infty}^{\infty} \sup_{i,\hat{b}_i} \left| f'_{iK}\left(\hat{b}_i - b_i\right) - \phi'\left(\hat{b}_i - b_i\right) \right| dG\left(b_i\right) \\
\leq & \; CK^{-1/8} \log K
\end{aligned}$$

where we can interchange differentiation and integration because of the uniformly bounded derivatives of both of the likelihood functions (see Appendix B for the proof that $\exists K_0 < \infty$ such that f'_{iK} is uniformly bounded for all $K \geq K_0$, under our assumptions). Part (e) follows from Part (d) just as Part (a) follows from Lemma 1.

Part (f) follows from Lemma 2 and

$$
\begin{aligned}
& \sup_{i,\hat{b}_i,\hat{b}_j} \left| m_{ijK}\left(\hat{b}_i,\hat{b}_j\right) - m_\phi\left(\hat{b}_i,\hat{b}_j\right) \right| && (56) \\
= & \sup_{i,\hat{b}_i,\hat{b}_j} \left| \int_{-\infty}^{\infty} \left[f_{ijK}\left(\hat{b}_i - b_i, \hat{b}_j - b_j\right) - \phi\left(\hat{b}_i - b_i\right)\phi\left(\hat{b}_j - b_j\right)\right] dG\left(b_i\right) dG\left(b_j\right) \right| \\
\leq & \int_{-\infty}^{\infty} \sup_{i,\hat{b}_i,\hat{b}_j} \left| f_{ijK}\left(\hat{b}_i - b_i, \hat{b}_j - b_j\right) - \phi\left(\hat{b}_i - b_i\right)\phi\left(\hat{b}_j - b_j\right)\right| dG\left(b_i\right) dG\left(b_j\right) \\
\leq & CK^{-1/6}\log K.
\end{aligned}
$$

Part (g) follows from Lemma 2 and

$$
\begin{aligned}
& \sup_{i,\hat{b}_i,\hat{b}_j} \left| \frac{\partial}{\partial \hat{b}_j} m_{ijK}\left(\hat{b}_i,\hat{b}_j\right) - \frac{\partial}{\partial \hat{b}_j} m_\phi\left(\hat{b}_i,\hat{b}_j\right) \right| && (57) \\
= & \sup_{i,\hat{b}_i,\hat{b}_j} \left| \frac{\partial}{\partial \hat{b}_i} \int_{-\infty}^{\infty} \left[f_{iK}\left(\hat{b}_i - b_i, \hat{b}_j - b_j\right) - \phi\left(\hat{b}_i - b_i\right)\phi\left(\hat{b}_j - b_j\right)\right] dG\left(b_i\right) \right| \\
= & \sup_{i,\hat{b}_i,\hat{b}_j} \left| \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial \hat{b}_i} f_{iK}\left(\hat{b}_i - b_i\right) - \phi'\left(\hat{b}_i - b_i\right)\phi\left(\hat{b}_j - b_j\right)\right] dG\left(b_i\right) \right| \\
\leq & \int_{-\infty}^{\infty} \sup_{i,\hat{b}_i,\hat{b}_j} \left| \frac{\partial}{\partial \hat{b}_i} f_{iK}\left(\hat{b}_i - b_i\right) - \phi'\left(\hat{b}_i - b_i\right)\phi\left(\hat{b}_j - b_j\right)\right| dG\left(b_i\right) \\
\leq & CK^{-1/12}\log K
\end{aligned}
$$

where we can interchange differentiation and integration because of the uniformly bounded derivatives of both of the likelihood functions (see Appendix B for the proof that $\exists K_0 < \infty$ such that $\frac{\partial}{\partial \hat{b}_i} f_{ijK}$ is uniformly bounded for all $K \geq K_0$, under our assumptions).

Lemma 4 (Information and Related Bounds) *Assumptions 1, 2, 3, 6, and 7* *imply that*

$$\int_{-\infty}^{\infty} \frac{\left(m'_\phi(x)\right)^2}{m_\phi(x)} dx \quad < \quad \infty \tag{58}$$

$$\int_{-\infty}^{\infty} \left(\frac{\bar{m}'_K(x)}{\bar{m}_K(x) + s_K}\right)^2 \bar{m}_K(x)\, dx \quad \to \quad \int_{-\infty}^{\infty} \frac{\left(m'_\phi(x)\right)^2}{m_\phi(x)} dx \tag{59}$$

$$\int_{-\infty}^{\infty} \left(\frac{m'_\phi(x)}{m_\phi(x) + s_K}\right)^2 m_\phi(x)\, dx \quad \to \quad \int_{-\infty}^{\infty} \frac{\left(m'_\phi(x)\right)^2}{m_\phi(x)} dx \tag{60}$$

$$\int_{-\infty}^{\infty} \left(\frac{m'_\phi(x)}{m_\phi(x) + s_K}\right)^2 \bar{m}_K(x)\, dx \quad \to \quad \int_{-\infty}^{\infty} \frac{\left(m'_\phi(x)\right)^2}{m_\phi(x)} dx \tag{61}$$

$$r_G\left(\hat{b}^{NB}, \phi_K\right) \quad < \quad \infty \tag{62}$$

Further, if in addition Assumption 5 holds,

$$\sup_{x,\,\theta\in\Theta} \left|m'_\phi(x;\theta)\right| \;<\; \infty \tag{63}$$

Proof: We shall prove the above lines in order, beginning with the inequality of expression (58).

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\left(m'_\phi(x)\right)^2}{m_\phi(x)} dx \quad &= \quad \int_{-\infty}^{\infty} \left(\frac{m'_\phi(x)}{m_\phi(x)}\right)^2 m_\phi(x)\, dx \\
&= \quad \int_{-\infty}^{\infty} \left(\frac{x - \hat{b}^{NB}(x)}{\sigma_\varepsilon^2}\right)^2 m_\phi(x)\, dx \\
&\leq \quad \frac{2}{\sigma_\varepsilon^4} \int_{-\infty}^{\infty} x^2 m_\phi(x)\, dx \\
&\quad\; + \frac{2}{\sigma_\varepsilon^4} \int_{-\infty}^{\infty} \left(\hat{b}^{NB}(x)\right)^2 m_\phi(x)\, dx \\
&= \quad \frac{2}{\sigma_\varepsilon^4} \int_{-\infty}^{\infty} x^2 m_\phi(x)\, dx \\
&\quad\; + \frac{2}{\sigma_\varepsilon^4} \int_{-\infty}^{\infty} \left(\frac{\int_{-\infty}^{\infty} b\phi(x-b)\, dG(b)}{\int_{-\infty}^{\infty} \phi(x-b)\, dG(b)}\right)^2 m_\phi(x)\, dx
\end{aligned} \tag{64}$$

$$
\begin{aligned}
&\leq \frac{2}{\sigma_\varepsilon^4}\int_{-\infty}^{\infty} x^2 m_\phi(x)\,dx \\
&\quad + \frac{2}{\sigma_\varepsilon^4}\int_{-\infty}^{\infty} \frac{\int_{-\infty}^{\infty} b^2\phi(x-b)\,dG(b)}{\int_{-\infty}^{\infty}\phi(x-b)\,dG(b)} m_\phi(x)\,dx \\
&= \frac{2}{\sigma_\varepsilon^4}\int_{-\infty}^{\infty} x^2 m_\phi(x)\,dx \\
&\quad + \frac{2}{\sigma_\varepsilon^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b^2\phi(x-b)\,dG(b)\,dx \\
&= \frac{2}{\sigma_\varepsilon^4}\int_{-\infty}^{\infty} x^2 m_\phi(x)\,dx \\
&\quad + \frac{2}{\sigma_\varepsilon^4}\int_{-\infty}^{\infty} b^2\left\{\int_{-\infty}^{\infty}\phi(x-b)\,dx\right\} dG(b) \\
&= \frac{2}{\sigma_\varepsilon^4}\int_{-\infty}^{\infty} x^2 m_\phi(x)\,dx \\
&\quad + \frac{2}{\sigma_\varepsilon^4}\int_{-\infty}^{\infty} b^2 dG(b) \\
&< \infty
\end{aligned}
$$

where the first equality follows from trivial manipulation, the second equality is due to the fact that $\hat{b}^{NB}(x) = x - \sigma_\varepsilon^2 \frac{m'_\phi(x)}{m_\phi(x)}$ due to the normal likelihood, the first inequality comes from the fact that $(a-b)^2 \leq 2a^2 + 2b^2$, the third equality is by definition, the second inequality is due to the convexity of the squaring function and an almost-sure-in-x Jensen's inequality result for conditional expectations (since $\hat{b}^{NB}(x)$ is the conditional expectation of b given x) (see Billingsley [1995, page 449, equation (34.7)]; note that the distribution for x for which the result holds almost surely is precisely the distribution with density m_ϕ, so the stated inequality holds), the fourth equality holds since $m_\phi(x) = \int_{-\infty}^{\infty}\phi(x-b)\,dG(b)$, the fifth equality follows from the Tonelli-Fubini theorem, the sixth equality holds since $\int_{-\infty}^{\infty}\phi(x-b)\,dx = 1\ \forall b \in \mathcal{R}$, and the final inequality is a direct result of the moment bounds given in Assumptions 1 and 6.

Now we shall prove the convergence in (59).

$$
\begin{aligned}
& \int_{-\infty}^{\infty} \left(\frac{\bar{m}_K'(x)}{\bar{m}_K(x) + s_K} \right)^2 \bar{m}_K(x)\, dx \; - \; \int_{-\infty}^{\infty} \frac{\left(m_\phi'(x)\right)^2}{m_\phi(x)} dx && (65) \\
= \; & \int_{-\infty}^{\infty} \left(\frac{\bar{m}_K'(x)}{\bar{m}_K(x) + s_K} \right)^2 \left(\bar{m}_K(x) - m_\phi(x)\right) dx && \text{(Term I)} \\
& + \int_{-\infty}^{\infty} \left[\left(\frac{\bar{m}_K'(x)}{\bar{m}_K(x) + s_K} \right)^2 - \left(\frac{m_\phi'(x)}{m_\phi(x)} \right)^2 \right] m_\phi(x)\, dx && \text{(Term II)}
\end{aligned}
$$

follows by adding and subtracting $\int_{-\infty}^{\infty} \left(\frac{\bar{m}_K'(x)}{\bar{m}_K(x)+s_K} \right)^2 m_\phi(x)\, dx$. First we show that Term I converges to zero. Let $z_K \to \infty$ such that $s_K^{-2} z_K K^{-1/4} \log K \to 0$ and $s_K^{-2} z_K^{-2} \to 0$, which is certainly possible in light of Assumption 7. Then

$$
\begin{aligned}
& \int_{-\infty}^{\infty} \left(\frac{\bar{m}_K'(x)}{\bar{m}_K(x) + s_K} \right)^2 \left(\bar{m}_K(x) - m_\phi(x)\right) dx && (66) \\
\leq \; & C_0 s_K^{-2} \int_{-\infty}^{\infty} \left(\bar{m}_K(x) - m_\phi(x)\right) dx \\
= \; & C_0 s_K^{-2} \left\{ \begin{array}{c} \int_{-z_K}^{z_K} \left(\bar{m}_K(x) - m_\phi(x)\right) dx \\ + \int_{-\infty}^{-z_K} \left(\bar{m}_K(x) - m_\phi(x)\right) dx \\ + \int_{z_K}^{\infty} \left(\bar{m}_K(x) - m_\phi(x)\right) dx \end{array} \right\} \\
\leq \; & C_0 s_K^{-2} \left\{ \begin{array}{c} 2 z_K C K^{-1/4} \log K \\ + \int_{-\infty}^{-z_K} \left(\bar{m}_K(x) - m_\phi(x)\right) dx \\ + \int_{z_K}^{\infty} \left(\bar{m}_K(x) - m_\phi(x)\right) dx \end{array} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad C_0 s_K^{-2} \left\{ \begin{array}{c} 2z_K C K^{-1/4} \log K \\ + C_2 z_K^{-2} \end{array} \right\} \\
&\rightarrow \quad 0
\end{aligned}
$$

where the first inequality follows from the positivity of $\bar{m}_K(x)$ and the boundedness of $\bar{m}'_K(x)$ (from Lemma 3(c)), the first equality is trivial, the second inequality follows by using Lemma 3(c) to get $\int_{-z_K}^{z_K} (\bar{m}_K(x) - m_\phi(x))\, dx \leq \int_{-z_K}^{z_K} CK^{-1/4} \log K dx = 2z_K CK^{-1/4} \log K$, the third inequality follows from Markov's inequality and the (uniformly in K) finite second moments (due to Assumptions 1 and 6) of the distributions with densities $\bar{m}_K$ and m_ϕ. Finally, the convergence to zero is by the construction of z_K.

Second we demonstrate that Term II converges to zero. Use the fact that $(a^2 - b^2) = (a-b)^2 + 2b(a-b)$ to obtain

$$
\int_{-\infty}^{\infty} \left[\left(\frac{\bar{m}'_K(x)}{\bar{m}_K(x) + s_K} \right)^2 - \left(\frac{m'_\phi(x)}{m_\phi(x)} \right)^2 \right] m_\phi(x)\, dx \tag{67}
$$

$$
\begin{aligned}
= \quad & \int_{-\infty}^{\infty} \left(\frac{\bar{m}'_K(x)}{\bar{m}_K(x) + s_K} - \frac{m'_\phi(x)}{m_\phi(x)} \right)^2 m_\phi(x)\, dx && \text{(Term IIA)} \\
& +2 \int_{-\infty}^{\infty} \left(\frac{m'_\phi(x)}{m_\phi(x)} \right) \left(\frac{\bar{m}'_K(x)}{\bar{m}_K(x) + s_K} - \frac{m'_\phi(x)}{m_\phi(x)} \right) m_\phi(x)\, dx. && \text{(Term IIB)}
\end{aligned}
$$

We deal with Term IIA, then Term IIB. Add and subtract $\frac{m'_\phi(x)}{m_\phi(x) + s_K}$ inside the square, then use the fact that $(a+b) \leq 2a^2 + 2b^2$, to obtain

$$
\int_{-\infty}^{\infty} \left(\frac{\bar{m}'_K(x)}{\bar{m}_K(x) + s_K} - \frac{m'_\phi(x)}{m_\phi(x)} \right)^2 m_\phi(x)\, dx \tag{68}
$$

$$\leq \quad 2\int_{-\infty}^{\infty}\left(\frac{\bar{m}'_K(x)}{\bar{m}_K(x)+s_K}-\frac{m'_\phi(x)}{m_\phi(x)+s_K}\right)^2 m_\phi(x)\,dx \qquad \text{(Term IIAi)}$$

$$+\,2\int_{-\infty}^{\infty}\left(\frac{m'_\phi(x)}{m_\phi(x)+s_K}-\frac{m'_\phi(x)}{m_\phi(x)}\right)^2 m_\phi(x)\,dx. \qquad \text{(Term IIAii)}$$

Now Term IIAi can be shown to converge to zero by the following argument, which uses the fact that $\left(\frac{a}{b}-\frac{c}{d}\right)^2=\left(\frac{a}{b}\left(\frac{d-b}{d}\right)+\frac{a-c}{d}\right)^2\leq 2\left(\frac{a}{b}\right)^2\left(\frac{d-b}{d}\right)^2+2\left(\frac{a-c}{d}\right)^2$ to obtain the first inequality below:

$$
\begin{aligned}
& 2\int_{-\infty}^{\infty}\left(\frac{\bar{m}'_K(x)}{\bar{m}_K(x)+s_K}-\frac{m'_\phi(x)}{m_\phi(x)+s_K}\right)^2 m_\phi(x)\,dx \qquad (69)\\
\leq\; & 4\int_{-\infty}^{\infty}\left(\frac{\bar{m}'_K(x)}{\bar{m}_K(x)+s_K}\right)^2\left(\frac{m_\phi(x)-\bar{m}_K(x)}{m_\phi(x)+s_K}\right)^2 m_\phi(x)\,dx\\
& +4\int_{-\infty}^{\infty}\left(\frac{\bar{m}'_K(x)-m'_\phi(x)}{m_\phi(x)+s_K}\right)^2 m_\phi(x)\,dx\\
\leq\; & 4\int_{-\infty}^{\infty}\left(\frac{\bar{m}'_K(x)}{\bar{m}_K(x)+s_K}\right)^2\left(\frac{CK^{-1/4}\log K}{m_\phi(x)+s_K}\right)^2 m_\phi(x)\,dx\\
& +4\int_{-\infty}^{\infty}\left(\frac{CK^{-1/8}\log K}{m_\phi(x)+s_K}\right)^2 m_\phi(x)\,dx\\
\leq\; & 4s_K^{-4}C_2K^{-1/2}\log K \;+\; 4s_K^{-2}C^2K^{-1/4}\log K\\
\to\; & 0
\end{aligned}
$$

where the second inequality follows from Lemma 3(c,e) and the third inequality follows from the nonnegativity of $\bar{m}_K(x)$ and the boundedness (by Lemma 3(c)) of $\bar{m}'_K(x)$. Finally, the convergence to zero is by rate bounds given in equation (37).

To see that Term IIAii converges to zero, note that $\left(\frac{m'_\phi(x)}{m_\phi(x)+s_K}-\frac{m'_\phi(x)}{m_\phi(x)}\right)^2$ clearly converges to zero pointwise (since $s_K\to 0$). Thus, if there is an integrable (with respect to the measure having density m_ϕ) dominating function for $\left(\frac{m'_\phi(x)}{m_\phi(x)+s_K}-\frac{m'_\phi(x)}{m_\phi(x)}\right)^2$, then we may apply the Dominated Convergence Theorem and be finished. But since $(a-b)^2 \leq 2a^2 + 2b^2$, we have that

$$\begin{aligned} & \left(\frac{m_\phi'(x)}{m_\phi(x) + s_K} - \frac{m_\phi'(x)}{m_\phi(x)} \right)^2 \\ \leq \quad & 2 \left(\frac{m_\phi'(x)}{m_\phi(x) + s_K} \right)^2 + 2 \left(\frac{m_\phi'(x)}{m_\phi(x)} \right)^2 \\ \leq \quad & 2 \left(\frac{m_\phi'(x)}{m_\phi(x)} \right)^2 + 2 \left(\frac{m_\phi'(x)}{m_\phi(x)} \right)^2 \\ = \quad & 4 \left(\frac{m_\phi'(x)}{m_\phi(x)} \right)^2 \end{aligned} \tag{70}$$

and we know from our above proof of the inequality (58) that $\int_{-\infty}^{\infty} \left(\frac{m_\phi'(x)}{m_\phi(x)} \right)^2 m_\phi(x)\, dx = \int_{-\infty}^{\infty} \frac{(m_\phi'(x))^2}{m_\phi(x)} dx < \infty$. Thus, Term IIAii converges to zero.

We must still deal with Term IIB:

$$\begin{aligned} & 2 \int_{-\infty}^{\infty} \left(\frac{m_\phi'(x)}{m_\phi(x)} \right) \left(\frac{\bar{m}_K'(x)}{\bar{m}_K(x) + s_K} - \frac{m_\phi'(x)}{m_\phi(x)} \right) m_\phi(x)\, dx \\ \leq \quad & 2 \left(\int_{-\infty}^{\infty} \left(\frac{m_\phi'(x)}{m_\phi(x)} \right)^2 m_\phi(x)\, dx \right)^{1/2} \\ & \times \left(\int_{-\infty}^{\infty} \left(\frac{\bar{m}_K'(x)}{\bar{m}_K(x) + s_K} - \frac{m_\phi'(x)}{m_\phi(x)} \right)^2 m_\phi(x)\, dx \right)^{1/2} \end{aligned} \tag{71}$$

and we see that the first factor is, by our above proof of (58), simply a finite constant, while the second factor is just the square root of Term IIA. But this means that Term IIB converges to zero, so we have demonstrate the convergence displayed in (59).

To show that the convergence (60) holds, simply note that $\left(\frac{m_\phi'(x)}{m_\phi(x)+s_K} \right)^2 \to \left(\frac{m_\phi'(x)}{m_\phi(x)} \right)^2$ pointwise certainly holds, and further that $\left(\frac{m_\phi'(x)}{m_\phi(x)+s_K} \right)^2 \leq \left(\frac{m_\phi'(x)}{m_\phi(x)} \right)^2$ and

$\int_{-\infty}^{\infty}\left(\frac{m'_\phi(x)}{m_\phi(x)}\right)^2 m_\phi(x)\,dx = \int_{-\infty}^{\infty}\frac{\left(m'_\phi(x)\right)^2}{m_\phi(x)}dx < \infty$, so we may apply the Dominated Convergence Theorem to obtain the desired conclusion.

To demonstrate the convergence (61) simply add and subtract $\int_{-\infty}^{\infty}\left(\frac{m'_\phi(x)}{m_\phi(x)+s_K}\right)^2 m_\phi(x)\,dx$ to obtain

$$\begin{aligned} & \int_{-\infty}^{\infty}\left(\frac{m'_\phi(x)}{m_\phi(x)+s_K}\right)^2 \bar{m}_K(x)\,dx - \int_{-\infty}^{\infty}\frac{\left(m'_\phi(x)\right)^2}{m_\phi(x)}dx \\ = & \int_{-\infty}^{\infty}\left(\frac{m'_\phi(x)}{m_\phi(x)+s_K}\right)^2 \left(\bar{m}_K(x) - m_\phi(x)\right)dx \\ & + \int_{-\infty}^{\infty}\left(\frac{m'_\phi(x)}{m_\phi(x)+s_K}\right)^2 m_\phi(x)\,dx - \int_{-\infty}^{\infty}\frac{\left(m'_\phi(x)\right)^2}{m_\phi(x)}dx \\ \to & \ 0 \end{aligned} \tag{72}$$

where the convergence to zero follows because the first integral can be treated in exactly the same way that Term I above was, while the difference of the second and third integrals goes to zero by the convergence (60) shown immediately above.

The finiteness of the Bayes risk in the normal problem, as claimed in (62), can be shown as follows:

$$\begin{aligned} r_G\left(\hat{b}^{NB},\phi_K\right) &= \rho\int\int\frac{1}{K}\sum_{i=1}^{K}\left(\hat{b}_i^{NB}\left(\hat{b}_i\right)-b_i\right)^2\phi_K\left(\hat{b}-b\right)d\hat{b}dG(b) \\ &= \rho\frac{1}{K}\sum_{i=1}^{K}\int\int\left(\hat{b}_i^{NB}\left(\hat{b}_i\right)-b_i\right)^2\phi\left(\hat{b}_i-b_i\right)d\hat{b}_i dG(b_i) \\ &= \rho\int\int\left(\hat{b}_1^{NB}\left(\hat{b}_1\right)-b_1\right)^2\phi\left(\hat{b}_1-b_1\right)d\hat{b}_1 dG(b_1) \\ &\leq \rho\int\int b_1^2\phi\left(\hat{b}_1-b_1\right)d\hat{b}_1 dG(b_1) \\ &= \rho\int b_1^2\left\{\int\phi\left(\hat{b}_1-b_1\right)d\hat{b}_1\right\}dG(b_1) \end{aligned} \tag{73}$$

$$
\begin{aligned}
&= \rho \int b_1^2 dG(b_1) \\
&< \infty
\end{aligned}
$$

where the first equality is by definition, the second equality is trivial, the third equality follows because each term in the sum of the expression on the second line is identical, and we may rename all of the variables we are integrating over b_1 and $\hat{b}_1$, then take the average over K such identical terms, and the first inequality holds because $\hat{b}^{NB}$ is a Bayes decision rule, so it produces a Bayes risk no higher than that of any other decision rule; in particular, it performs no worse, in Bayes-risk terms, than the constant estimator 0. The fourth equality follows from the Tonelli-Fubini theorem. Finally, the second inequality holds by Assumption 6.

To see that the finite-derivative claim (63) holds in the parametric case, use the following argument:

$$
\begin{aligned}
\sup_{x,\,\theta\in\Theta} \left| m'_\phi(x;\theta) \right| &= \sup_{x,\,\theta\in\Theta} \left| \frac{d}{dx} \int \phi(x-b)\, dG(b;\theta) \right| \\
&= \sup_{x,\,\theta\in\Theta} \left| \int \phi'(x-b)\, dG(b;\theta) \right| \\
&\leq \sup_{\theta\in\Theta} \int \sup_x \left| \phi'(x-b) \right| dG(b;\theta) \\
&= \sup_{\theta\in\Theta} \int C_0 dG(b;\theta) \\
&= C_0
\end{aligned} \tag{74}
$$

where the first equality is by definition, the second equality is by the form of the normal likelihood, which satisfies the conditions of Dudley (1999, Corollary A.12 on page 394), the inequality is due to the convexity of the sup function, and the third equality follows by the direct calculation that $\sup_x |\phi'(x-b)| = \frac{1}{\sigma_\varepsilon^2\sqrt{2\pi}} e^{-1/2}\ \forall b$, and by defining $C_0 = \frac{1}{\sigma_\varepsilon^2\sqrt{2\pi}} e^{-1/2}$. The fourth equality is trivial, since C_0 is a constant with respect to both b and θ.

Lemma 5 (Lower Bounds for Densities) *Under Assumptions 1, 2, 3, 6, and 7,* $\exists C_1, C_2, K_0 < \infty$ *s.t.* $\forall\, K \geq K_0$

$$\inf_{x\,\in\,[-d_K,d_K]} m_\phi(x) \geq C_1 K^{-1/32} \qquad (a)$$

$$\inf_{i,\,x\,\in\,[-d_K,d_K]} m_{iK}(x) \geq C_2 K^{-1/32} \qquad (b)$$

$$\sup_{x\,\in\,[-d_K,d_K]} \left\{\frac{1}{m_\phi(x)}\right\} \leq \frac{1}{C_1} K^{1/32} \qquad (c)$$

$$\sup_{i,\,x\,\in\,[-d_K,d_K]} \left\{\frac{1}{m_{iK}(x)}\right\} \leq \frac{1}{C_2} K^{1/32} \qquad (d)$$

where d_K is defined in Definition 1 in this appendix.

Proof: We prove the parts of the lemma in order, from (a) to (d).

$$\begin{aligned}
\inf_{x\,\in\,[-d_K,d_K]} m_\phi(x) &= \inf_{x\,\in\,[-d_K,d_K]} \int_{-\infty}^{\infty} \phi(x-u)\, dG(u) \\
&\geq \int_{-\infty}^{\infty} \left\{ \inf_{x\,\in\,[-d_K,d_K]} \phi(x-u) \right\} dG(u) \\
&= \int_{-\infty}^{\infty} \left\{ \inf_{x\,\in\,[-d_K,d_K]} \frac{1}{\sigma_\varepsilon\sqrt{2\pi}} e^{-\frac{1}{2\sigma_\varepsilon^2}(x-u)^2} \right\} dG(u) \\
&\geq \int_{-d_K}^{d_K} \left\{ \inf_{x\,\in\,[-d_K,d_K]} \frac{1}{\sigma_\varepsilon\sqrt{2\pi}} e^{-\frac{1}{2\sigma_\varepsilon^2}(x-u)^2} \right\} dG(u) \\
&\geq \int_{-d_K}^{d_K} \frac{1}{\sigma_\varepsilon\sqrt{2\pi}} e^{-\frac{1}{2\sigma_\varepsilon^2}4d_K^2}\, dG(u)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sigma_\varepsilon\sqrt{2\pi}} e^{-\frac{1}{2\sigma_\varepsilon^2}4d_K^2} \int_{-d_K}^{d_K} dG(u) \\
&\geq \frac{1}{D\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma_\varepsilon^2}4d_K^2\right) \int_{-d_K}^{d_K} dG(u) \\
&\geq \frac{1}{D\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma_\varepsilon^2}4d_K^2\right) \left[1 - \frac{C_3}{d_K^2}\right] \\
&\geq C_1 \exp\left(-\frac{1}{2\sigma_\varepsilon^2}4d_K^2\right) \\
&= C_1 \exp\left(-\frac{1}{2\sigma_\varepsilon^2}4\sigma_\varepsilon^2\frac{\log K}{64}\right) \\
&= C_1 K^{-1/32}
\end{aligned}
$$

where the first equality is by definition, the first inequality holds because the infimum function is concave, the second equality is again by definition, the second inequality is due to the fact that the integrand is nonnegative, the third inequality follows because the infimum of the normal density over x and u both in $[-d_K, d_K]$ can be no smaller than if x and u were $2d_K$ apart, the third equality follows because the integrand in the previous line is not a function of u, the fourth inequality follows from the fact that $\sigma_\varepsilon^2 \leq D < \infty$ by Assumption 1, the fifth inequality is by the existence of a second moment of G (Assumption 6) and Markov's inequality, the sixth equality holds because, for K beyond some K_0 large enough, we can simply note that $1 - \frac{C_3}{d_K^2}$ is greater than some positive constant, the fourth equality is by the definition of d_K in Assumption 7, and the last equality follows by simple calculation.

To see that the inequality of part (b) holds, note that $\inf_{i,\, x \in [-d_K, d_K]} m_{iK}(x) \geq \inf_{x \in [-d_K, d_K]} m_\phi(x) - CK^{-1/4}\log K \geq C_1 K^{-1/32} - CK^{-1/4}\log K$, where the first inequality comes from Lemma 3(b) and the second comes from part (a) of this lemma, which we just proved. Now, for K beyond some K_0 which is sufficiently large, we can simply absorb the $K^{-1/4} \log K$ term into the constant on the $K^{-1/32}$ term, because the latter goes to zero more slowly. Doing so, we obtain the inequality of part (b). To prove (c) and (d), simply invert the relations in (a) and (b).

Lemma 6 (Kernel MSE Rates) *Under Assumptions 1, 2, 3, 4, 6, and 7,* $\exists\, C, K_0 < \infty$ *s.t.* $\forall\, K \geq K_0$,

$$\begin{aligned}
&\sup_{i,\left|\hat{b}_i\right| \leq d_K} E\left\{\left[\hat{m}_{iK}\left(\hat{b}_i\right) - \bar{m}_K\left(\hat{b}_i\right)\right]^2 \mid \hat{b}_i\right\} && (a)\\
\leq\ & C\left(\begin{array}{c} \frac{1}{h_K(K-1)} + h_K^2 + K^{-13/48}\log^2 K \\ + 2K^{-13/96} h_K \log K \ + \ K^{-1/2}\log^2 K \end{array}\right) \\
&\sup_{i,\left|\hat{b}_i\right| \leq d_K} E\left\{\left[\hat{m}'_{iK}\left(\hat{b}_i\right) - \bar{m}'_K\left(\hat{b}_i\right)\right]^2 \mid \hat{b}_i\right\} && (b)\\
\leq\ & C\left(\begin{array}{c} \frac{1}{h_K^3(K-1)} + h_K^2 + K^{-5/48}\log^2 K \\ + 2K^{-5/96} h_K \log K \ + \ K^{-1/4}\log^2 K \end{array}\right) \\
&\sup_{i,\left|\hat{b}_i\right| \leq d_K} \Pr\left[\left|\breve{l}_i\left(\hat{b}_i\right)\right| > q_K \mid \hat{b}_i\right] && (c)\\
\to\ & 0
\end{aligned}$$

Proof: If we have $\hat{b}_i \in [-d_K, d_K]$ then

$$\begin{aligned}
& E\left\{\left[\hat{m}_{iK}\left(\hat{b}_i\right) - \bar{m}_K\left(\hat{b}_i\right)\right]^2 \mid \hat{b}_i\right\} \qquad (75)\\
=\ & Var\left(\hat{m}_{iK}\left(\hat{b}_i\right) \mid \hat{b}_i\right)
\end{aligned}$$

$$+\left(E\left[\hat{m}_{iK}\left(\hat{b}_i\right) \mid \hat{b}_i\right] - \bar{m}_K\left(\hat{b}_i\right)\right)^2$$

$$\leq \quad Var\left(\hat{m}_{iK}\left(\hat{b}_i\right) \mid \hat{b}_i\right) \qquad \text{(Term I)}$$

$$+ 2\left(E\left[\hat{m}_{iK}\left(\hat{b}_i\right) \mid \hat{b}_i\right] - m_\phi\left(\hat{b}_i\right)\right)^2 \qquad \text{(Term II)}$$

$$+ 2\left(m_\phi\left(\hat{b}_i\right) - \bar{m}_K\left(\hat{b}_i\right)\right)^2. \qquad \text{(Term III)}$$

First consider Term I. Now,

$$Var\left(\hat{m}_{iK}\left(\hat{b}_i\right) \mid \hat{b}_i\right) \tag{76}$$

$$= \quad Var\left(\frac{1}{h_K(K-1)}\sum_{j\neq i} w\left(\frac{\hat{b}_i - \hat{b}_j}{h_K}\right) \mid \hat{b}_i\right)$$

$$\leq \quad \frac{1}{h_K^2(K-1)^2}\sum_{j\neq i}\sum_{n\neq i}\left|Cov\left(w\left(\frac{\hat{b}_i - \hat{b}_j}{h_K}\right), w\left(\frac{\hat{b}_i - \hat{b}_n}{h_K}\right) \mid \hat{b}_i\right)\right|$$

$$\leq \quad \frac{1}{h_K^2(K-1)^2}\sum_{j\neq i}\sum_{n\neq i}\tau\left(|j-n|\right)\sqrt{E\left(w\left(\frac{\hat{b}_i - \hat{b}_j}{h_K}\right)^2 \mid \hat{b}_i\right)} \times \sqrt{E\left(w\left(\frac{\hat{b}_i - \hat{b}_j}{h_K}\right)^2 \mid \hat{b}_i\right)}$$

$$\leq \quad \frac{C}{h_K(K-1)}$$

where the first equality is by definition, the first inequality is familiar, and the second inequality holds by the following argument: first, note that the σ-fields generated by random variables which are linear combinations of the elements of the $T \times 1$ vectors $\underline{X}_i$ (where the weights in the linear combinations are fixed) are certainly sub-σ-fields of the σ-fields generated by the $\underline{X}_i$ vectors themselves (intuitively, we may lose information by taking linear combinations, but we will certainly never gain information by doing so). Thus, if we recall the notation of Assumption 4, and in addition define $\mathcal{G}_a^c$ as the σ-field generated by the random variables $\left\{\hat{b}_i = \frac{1}{\sqrt{T}}\underline{X}_i \cdot \varepsilon + b_i : a \leq i \leq c\right\}$, we see that for any b and for any ε, $\mathcal{G}_a^c$ is a sub-σ-field of $\mathcal{F}_a^c$, so that for any ε and for any b,

$$\begin{aligned}
\tau(n) &\geq \sup_m \sup_{x \in \mathcal{F}_1^m, y \in \mathcal{F}_{m+n}^\infty} \left| Corr\left(x,\ y \mid \underline{X}_j, \varepsilon\right)\right| \\
&\geq \sup_m \sup_{x \in \mathcal{G}_1^m(\varepsilon), y \in \mathcal{G}_{m+n}^\infty(\varepsilon)} \left| Corr\left(x,\ y \mid \frac{1}{\sqrt{T}}\underline{X}_j \cdot \varepsilon + b_j\right)\right|
\end{aligned}$$

but this means that the mixing coefficients $\tau(n)$ apply to the $\hat{b}_i$ as well as the $\underline{X}_i$, since we can certainly use the following bound (letting the σ-fields generated by the $\hat{b}_i$ be denoted $\mathcal{G}_a^c$ as before):

$$\begin{aligned}
&\sup_m \sup_{x \in \mathcal{G}_1^m, y \in \mathcal{G}_{m+n}^\infty} \left| Corr\left(x,\ y \mid \hat{b}_j, \varepsilon\right)\right| \\
&= \sup_m \sup_{x \in \mathcal{G}_1^m, y \in \mathcal{G}_{m+n}^\infty} E\left[\left| Corr\left(x,\ y \mid \hat{b}_j, \varepsilon, b\right)\right|\right] \\
&\leq \sup_{\varepsilon, b} \sup_m \sup_{x \in \mathcal{G}_1^m, y \in \mathcal{G}_{m+n}^\infty} \left| Corr\left(x,\ y \mid \hat{b}_j, \varepsilon, b\right)\right| \\
&\leq \tau(n)
\end{aligned}$$

which allows us to make use of Doukhan (1994, Theorem 3 (5) on page 9) to conclude that, if u and v are $\mathcal{G}_1^m$-measurable and $\mathcal{G}_{m+n}^\infty$-measurable random variables, respectively, and if $E[u^2],\ E[v^2] < \infty$, then

$$|Cov(u, v)| \leq \tau(n) \sqrt{E[u^2]}\sqrt{E[v^2]}.$$

The final inequality in display (76) follows from a change of variables (so that the argument of the kernel is no longer scaled by h_K), the boundedness of the kernel, and the summability of the $\tau(n)$ (from Assumption 4).

Next, turn to Term II in display (75). Because $|a|\,|b-c| + |ab - c^2| \geq |c|\,|a-c|$,

$$
\begin{aligned}
& \sup_{i,x\in[-d_K,d_K],y\in\Re} m_\phi(x)\left|\bar{m}_{iK}^C(y|x) - m_\phi(y)\right| \\
= \quad & \sup_{i,x\in[-d_K,d_K],y\in\Re} m_\phi(x)\left|\frac{1}{(K-1)}\sum_{j\neq i}\left(m_{ijK}^C(y|x) - m_\phi(y)\right)\right| \\
\leq \quad & \sup_{i,x\in[-d_K,d_K],y\in\Re} \frac{1}{(K-1)}\sum_{j\neq i} m_\phi(x)\left|m_{ijK}^C(y|x) - m_\phi(y)\right| \\
\leq \quad & \frac{1}{(K-1)}\sum_{j\neq i}\sup_{i,x\in[-d_K,d_K],y\in\Re} m_\phi(x)\left|m_{ijK}^C(y|x) - m_\phi(y)\right| \\
\leq \quad & \frac{1}{(K-1)}\sum_{j\neq i}\sup_{i,x\in[-d_K,d_K],y\in\Re}\left\{\begin{array}{c} m_{ijK}^C(y|x)\,|m_{iK}(x) - m_\phi(x)| \\ + |m_{ijK}(y,x) - m_\phi(y)\,m_\phi(x)| \end{array}\right\} \\
\leq \quad & \frac{1}{(K-1)}\sum_{j\neq i}\left\{\begin{array}{c} \sup_{i,x\in[-d_K,d_K],y\in\Re} m_{ijK}^C(y|x)\,|m_{iK}(x) - m_\phi(x)| \\ + \ \sup_{i,x\in[-d_K,d_K],y\in\Re} |m_{ijK}(y,x) - m_\phi(y)\,m_\phi(x)| \end{array}\right\} \\
\leq \quad & \left\{\frac{1}{(K-1)}\sum_{j\neq i}\sup_{i,x\in[-d_K,d_K],y\in\Re} m_{ijK}^C(y|x)\,|m_{iK}(x) - m_\phi(x)|\right\} \\
& + CK^{-1/6}\log K \\
\leq \quad & \left\{\frac{1}{(K-1)}\sum_{j\neq i}\left[\sup_{i,x\in[-d_K,d_K],y\in\Re} m_{ijK}^C(y|x)\right]\left[\sup_{i,x\in[-d_K,d_K],y\in\Re} |m_{iK}(x) - m_\phi(x)|\right]\right\} \\
& + CK^{-1/6}\log K \\
\leq \quad & \left\{\frac{1}{(K-1)}\sum_{j\neq i}\sup_{i,x\in[-d_K,d_K],y\in\Re} m_{ijK}^C(y|x)\,CK^{-1/4}\log K\right\} \\
& + CK^{-1/6}\log K
\end{aligned}
$$

$$\leq \left\{ \frac{1}{(K-1)} \sum_{j \neq i} C^* K^{1/32} C K^{-1/4} \log K \right\}$$
$$+ C K^{-1/6} \log K$$
$$= C^* K^{1/32} C K^{-1/4} \log K + C K^{-1/6} \log K$$
$$\leq C_2^* K^{-1/6} \log K$$

where the first equality is by definition, the first inequality is by the convexity of the absolute value function, the second inequality is by the convexity of the sup function, the third inequality is by the fact stated immediately above the display, the fourth inequality is again by the convexity of the sup function, the fifth inequality is by Lemma 3(f), the sixth inequality is again by the properties of the sup function (the sup of the product may never be greater than the product of the sups when the arguments are nonnegative), the seventh inequality follows from Lemma 3(b), and the eighth inequality follows because $m_{ijK}^{C}(y|x) = \frac{m_{ijK}(y,x)}{m_{iK}(x)}$ by definition, and the numerator of this fraction is bounded, since the variance of ϕ is bounded below, so the density ϕ is bounded, so the density m_ϕ is bounded, and Lemma 3(f) implies that then the numerator of the fraction is bounded. But the denominator is bounded by $\frac{1}{C_2} K^{1/32}$, as we know from Lemma 5(d). The second equality follows since the summand does not depend on j, and the final inequality is due to the fact that $K^{(1/32)-(1/4)} \log K = K^{-7/32} \log K$ converges to zero more quickly than $K^{-1/6} \log K$, so the $K^{-7/32} \log K$ term may be absorbed into the constant on the

$K^{-1/6} \log K$ term for sufficiently large K.

But from the above calculation, it rapidly follows that

$$
\begin{aligned}
& \sup_{i, x \in [-d_K, d_K], y \in \Re} \left| \bar{m}_{iK}^{C}(y|x) - m_\phi(y) \right| \qquad (77) \\
= & \sup_{i, x \in [-d_K, d_K], y \in \Re} \frac{m_\phi(x)}{m_\phi(x)} \left| \bar{m}_{iK}^{C}(y|x) - m_\phi(y) \right| \\
\leq & \left\{ \sup_{i, x \in [-d_K, d_K], y \in \Re} \frac{1}{m_\phi(x)} \right\} \\
& \times \left\{ \sup_{i, x \in [-d_K, d_K], y \in \Re} m_\phi(x) \left| \bar{m}_{iK}^{C}(y|x) - m_\phi(y) \right| \right\} \\
\leq & \frac{1}{C_1} K^{1/32} C_2^* K^{-1/6} \log K \\
= & C_3^* K^{-13/96} \log K
\end{aligned}
$$

where the first equality follows by multiplication by one, the first inequality is due to the fact that the product of the sups is always at least as large as the sup of the products (for nonnegative arguments), the second inequality follows from the above calculation and Lemma 5(c), and the second equality comes from a trivial computation. Now, the result immediately above implies that

$$
\begin{aligned}
& \sup_{i,\, \hat{b}_i \in [-d_K, d_K]} \left| E\left[\hat{m}_{iK}\left(\hat{b}_i\right) \mid \hat{b}_i \right] - m_\phi\left(\hat{b}_i\right) \right| \qquad (78) \\
= & \sup_{i,\, \hat{b}_i \in [-d_K, d_K]} \left| \begin{array}{c} \frac{1}{(K-1)} \sum_{j \neq i} \int_{-\infty}^{\infty} w(z)\, m_{ijK}^{C}\left(\hat{b}_i - h_K z | \hat{b}_i\right) dz \\ -m_\phi\left(\hat{b}_i\right) \end{array} \right| \\
= & \sup_{i,\, \hat{b}_i \in [-d_K, d_K]} \left| \int_{-\infty}^{\infty} w(z)\, \bar{m}_{iK}^{C}\left(\hat{b}_i - h_K z | \hat{b}_i\right) dz - m_\phi\left(\hat{b}_i\right) \right| \\
= & \sup_{i,\, \hat{b}_i \in [-d_K, d_K]} \left| \int_{-\infty}^{\infty} w(z) \left[\bar{m}_{iK}^{C}\left(\hat{b}_i - h_K z | \hat{b}_i\right) - m_\phi\left(\hat{b}_i\right) \right] dz \right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sup_{i,\,\hat{b}_i\in[-d_K,d_K]} \int_{-\infty}^{\infty} w(z)\left|\bar{m}_{iK}^{C}\left(\hat{b}_i - h_K z|\hat{b}_i\right) - m_\phi\left(\hat{b}_i\right)\right| dz \\
&\leq \int_{-\infty}^{\infty} w(z) \sup_{i,\,\hat{b}_i\in[-d_K,d_K]} \left|\bar{m}_{iK}^{C}\left(\hat{b}_i - h_K z|\hat{b}_i\right) - m_\phi\left(\hat{b}_i\right)\right| dz \\
&\leq \int_{-\infty}^{\infty} w(z) \sup_{i,\,\hat{b}_i\in[-d_K,d_K]} \left|\bar{m}_{iK}^{C}\left(\hat{b}_i - h_K z|\hat{b}_i\right) - m_\phi\left(\hat{b}_i - h_K z\right)\right| dz \\
&\quad + \int_{-\infty}^{\infty} w(z) \sup_{\hat{b}_i\in[-d_K,d_K]} \left|m_\phi\left(\hat{b}_i - h_K z\right) - m_\phi\left(\hat{b}_i\right)\right| dz \\
&\leq \int_{-\infty}^{\infty} w(z) \sup_{i,x\in[-d_K,d_K],y\in\Re} \left|\bar{m}_{iK}^{C}(y|x) - m_\phi(y)\right| dz \\
&\quad + \int_{-\infty}^{\infty} w(z)\, C_4^* h_K z dz \\
&\leq \int_{-\infty}^{\infty} w(z)\, C_3^* K^{-13/96} \log K dz \; + \; C_4^* h_K \int_{-\infty}^{\infty} z w(z)\, dz \\
&\leq C_3^* K^{-13/96} \log K \; + \; C_5^* h_K
\end{aligned}
$$

where the first equality is by definition, the second equality follows from the linearity of integration, the third equality is due to the fact that $m_\phi(x) = \int_{-\infty}^{\infty} m_\phi(x)\, w(z)\, dz$ as long as w is a probability density, the first inequality is by the convexity of the absolute value function, the second inequality is by the convexity of the sup function, the third inequality follows from adding and subtracting $m_\phi\left(\hat{b}_i - h_K z\right)$ and then applying the triangle inequality (and the convexity of the sup function again), the fourth inequality follows because m_ϕ is uniformly Lipschitz continuous (since ϕ has a uniformly bounded first derivative, and we can safely interchange the derivative and the integral), the fifth inequality follows from the calculation immediately above the current one, and the sixth inequality is due to the fact that the kernel w has a bounded second, and thus first, moment (as well as the fact that it integrates to one, in the case of the first of the two terms).

Finally, consider Term III in display (75) – we know that

$$\sup_{i,\hat{b}_i}\left(\bar{m}_K\left(\hat{b}_i\right)-m_\phi\left(\hat{b}_i\right)\right)^2 \leq CK^{-\frac{1}{2}}\log^2 K \tag{79}$$

from Lemma 3(c) above.

Substituting the bounds (76), (78), and (79) into display (75) yields part (a) of this lemma.

In the case part (b) of this lemma, concerning the derivative of the density, everything is entirely similar, except that the rates slow somewhat, as we use the rates from Lemma 3 which pertain to the derivatives, rather than those which pertain to the densities themselves.

We now have only to demonstrate part (c) of the lemma.

$$\begin{aligned} & \sup_{i,\left|\hat{b}_i\right|\leq d_K} \Pr\left[\left|\breve{l}_i\left(\hat{b}_i\right)\right| > q_K \mid \hat{b}_i\right] \qquad (80)\\ \leq \quad & \sup_{i,\left|\hat{b}_i\right|\leq d_K} \Pr\left[\left|\hat{m}'_{iK}\left(\hat{b}_i\right)\right| > s_K q_K \mid \hat{b}_i\right] \\ \leq \quad & \sup_{i,\left|\hat{b}_i\right|\leq d_K} \left(\frac{1}{s_K q_K}\right)^2 E\left[\left(\hat{m}'_{iK}\left(\hat{b}_i\right)\right)^2 \mid \hat{b}_i\right] \\ \leq \quad & C_0\left(\frac{1}{s_K q_K}\right)^2 \\ \rightarrow \quad & 0 \end{aligned}$$

where the first inequality follows from the definition of $\breve{l} = \frac{\hat{m}'}{\hat{m}+s_K}$ and the nonnegativity of $\hat{m}$, the second inequality is a direct application of Markov's inequality, and the third inequality follows from part (b) of this lemma and the uniform boundedness of $\bar{m}_K$, which itself follows from Lemma 3(c) and the uniform boundedness of m_ϕ, since by Assumption 1 there exists D_2 such that $\sigma_\varepsilon^2 \geq D_2 > 0$. Finally, the convergence to zero is by Assumption 7 and the rate results in display 37. Q.E.D.

Proof of Theorem 3:

Let $\hat{b}_i^{INB} \equiv \hat{b}_i - \sigma_\varepsilon^2 \frac{m'_\phi(\hat{b};\theta)}{m_\phi(\hat{b};\theta)+s_K}$, where s_K is as in Assumption 7. This is the infeasible simple empirical Bayes estimator based on the normal-likelihood marginal $m_\phi(x)$. Now write:

$$
\begin{aligned}
& r_G\left(\hat{b}^{PEB}, f_K\right) - r_G\left(\hat{b}^{NB}, \phi_K\right) && \text{(81a)} \\
= \; & r_G\left(\hat{b}^{PEB}, f_K\right) - r_G\left(\hat{b}^{INB}, f_K\right) && \text{(Term I)} \\
& + r_G\left(\hat{b}^{INB}, f_K\right) - r_G\left(\hat{b}^{INB}, \phi_K\right) && \text{(Term II)} \\
& + r_G\left(\hat{b}^{INB}, \phi_K\right) - r_G\left(\hat{b}^{NB}, \phi_K\right) && \text{(Term III)}
\end{aligned}
$$

We will show that Terms I, II, and III all converge to zero. First consider Term III:

$$
\begin{aligned}
& r_G\left(\hat{b}^{INB}, \phi_K\right) - r_G\left(\hat{b}^{NB}, \phi_K\right) && \text{(82)} \\
= \; & \rho \int\int \frac{1}{K}\sum_{i=1}^{K}\left(\hat{b}_i^{INB}\left(\hat{b}_i\right) - b_i\right)^2 \phi_K\left(\hat{b}-b\right) d\hat{b}\, dG_K\left(b;\theta\right) \\
& - \rho \int\int \frac{1}{K}\sum_{i=1}^{K}\left(\hat{b}_i^{NB}\left(\hat{b}_i\right) - b_i\right)^2 \phi_K\left(\hat{b}-b\right) d\hat{b}\, dG_K\left(b;\theta\right) \\
= \; & \rho \frac{1}{K}\sum_{i=1}^{K}\int\int \left(\hat{b}_i^{INB}\left(\hat{b}_i\right) - b_i\right)^2 \phi\left(\hat{b}_i - b_i\right) d\hat{b}_i\, dG\left(b_i;\theta\right)
\end{aligned}
$$

$$
\begin{aligned}
& -\rho\frac{1}{K}\sum_{i=1}^{K}\int\int\left(\hat{b}_i^{NB}\left(\hat{b}_i\right)-b_i\right)^2\phi\left(\hat{b}_i-b_i\right)d\hat{b}_i dG\left(b_i;\theta\right) \\
= \quad & \rho\int\int\left(\hat{b}_1^{INB}\left(\hat{b}_1\right)-b_1\right)^2\phi\left(\hat{b}_1-b_1\right)d\hat{b}_1 dG\left(b_1;\theta\right) \\
& -\rho\int\int\left(\hat{b}_1^{NB}\left(\hat{b}_1\right)-b_1\right)^2\phi\left(\hat{b}_1-b_1\right)d\hat{b}_1 dG\left(b_1;\theta\right) \\
\rightarrow \quad & 0
\end{aligned}
$$

where the third equality holds because the integrals which make up the summands on lines 3 and 4 above are identical, and depend only on the i^{th} rescaled least-squares coefficient, $\hat{b}_i$. Thus, we could change variables in each of the summand integrals to make them all notationally the same, then simply perform the resulting (trivial) sum to get the equality. The convergence to 0 follows since $\hat{b}_1^{INB} \rightarrow \hat{b}_1^{NB}$ pointwise, as $s_K \rightarrow 0$, so if there is a dominating function for $\left(\hat{b}_1^{INB}\left(\hat{b}_1\right)-b_1\right)^2$ which has a finite integral with respect to the measure $\phi\left(\hat{b}_1-b_1\right)d\hat{b}_1 dG\left(b_1;\theta\right)$, then we may apply the dominated convergence theorem and be finished. But $\left(\hat{b}_1^{INB}\left(\hat{b}_1\right)-b_1\right)^2 \leq 2\left(b_1-\hat{b}_1\right)^2+2\sigma_\varepsilon^4\left(\frac{m'_\phi\left(\hat{b};\theta\right)}{m_\phi\left(\hat{b};\theta\right)+s_K}\right)^2$, the first term of which is integrable by Assumption 1; the second term of this bound is also integrable, by Lemma 4 (since it is less, for each K, than $2\sigma_\varepsilon^4\left(\frac{m'_\phi\left(\hat{b};\theta\right)}{m_\phi\left(\hat{b};\theta\right)}\right)^2$).

The negative of Term II is handled as follows:

$$
\begin{aligned}
& r_G\left(\hat{b}^{INB},\phi_K\right)-r_G\left(\hat{b}^{INB},f_K\right) \qquad (83)\\
= \quad & \rho\int\int\frac{1}{K}\sum_{i=1}^{K}\left(\hat{b}_i^{INB}\left(\hat{b}_i\right)-b_i\right)^2\phi_K\left(\hat{b}-b\right)d\hat{b}dG_K\left(b;\theta\right) \\
& -\rho\int\int\frac{1}{K}\sum_{i=1}^{K}\left(\hat{b}_i^{INB}\left(\hat{b}_i\right)-b_i\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}dG_K\left(b;\theta\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \rho \frac{1}{K} \sum_{i=1}^{K} \int \int \left(\hat{b}_1^{INB} \left(\hat{b}_i \right) - b_i \right)^2 \phi \left(\hat{b}_i - b_i \right) d\hat{b}_i dG \left(b_i; \theta \right) \\
&\quad - \rho \frac{1}{K} \sum_{i=1}^{K} \int \int \left(\hat{b}_1^{INB} \left(\hat{b}_i \right) - b_i \right)^2 f_{iK} \left(\hat{b}_i - b_i \right) d\hat{b}_i dG \left(b_i; \theta \right) \\
&= \rho \int \int \left(\hat{b}_1^{INB} \left(\hat{b}_1 \right) - b_1 \right)^2 \phi \left(\hat{b}_1 - b_1 \right) d\hat{b}_1 dG \left(b_1; \theta \right) \\
&\quad - \rho \int \int \left(\hat{b}_1^{INB} \left(\hat{b}_1 \right) - b_1 \right)^2 \bar{f}_K \left(\hat{b}_1 - b_1 \right) d\hat{b}_1 dG \left(b_1; \theta \right) \\
&\to 0
\end{aligned}
$$

where the first equality is by definition, the second follows from noting that the decision rule $\hat{b}_i^{INB}$, as a function of $\hat{b}_i$, is the same for every i, so that $\hat{b}_i^{INB} = \hat{b}_1^{INB}$ as functions of $\hat{b}_i$, the third equality follows from change of variables and the definition of $\bar{f}_K$, and the convergence to zero comes from: first, the fact that $\bar{f}_K$ converges to ϕ pointwise from Lemma 3, and second, the fact that $\left(\hat{b}_1^{INB} \left(\hat{b}_1 \right) - b_1 \right)^2 \bar{f}_K \leq 2 \left(\hat{b}_1 - b_1 \right)^2 \bar{f}_K + 2\sigma_\varepsilon^4 \left(\frac{m'_\phi\left(\hat{b}_1\right)}{m_\phi\left(\hat{b}_1\right)+s_K} \right)^2 \bar{f}_K$. The first term of this bound is integrable (for every K) by Assumption 1, while the second term is integrable (for every K) by Lemma 4, and both terms have integrals which converge to the integrals of their limits, so we may use the Dominated Convergence Theorem and be finished.

Finally, we show that Term I converges to zero. We will actually work with the negative of Term I.

$$
\begin{aligned}
& r_G \left(\hat{b}^{INB}, f_K \right) - r_G \left(\hat{b}^{PEB}, f_K \right) \qquad (84) \\
&= \rho \int \int \frac{1}{K} \sum_{i=1}^{K} \left(\hat{b}_i^{INB} \left(\hat{b}_i \right) - b_i \right)^2 f_K \left(\hat{b} - b \right) d\hat{b} dG_K \left(b; \theta \right)
\end{aligned}
$$

$$
\begin{aligned}
& - \ \rho \int \int \frac{1}{K} \sum_{i=1}^{K} \left(\hat{b}_i^{PEB} \left(\hat{b}_i \right) - b_i \right)^2 f_K \left(\hat{b} - b \right) d\hat{b} dG_K \left(b; \theta \right) \\
= \quad & \rho \frac{1}{K} \sum_{i=1}^{K} \int \int \left(\left(\hat{b}_i^{INB} \left(\hat{b}_i \right) - b_i \right)^2 - \left(\hat{b}_i^{PEB} \left(\hat{b}_i \right) - b_i \right)^2 \right) \\
& \times f_K \left(\hat{b} - b \right) d\hat{b} dG_K \left(b; \theta \right) \\
= \quad & \rho \frac{1}{K} \sum_{i=1}^{K} \int \int \left[\begin{array}{c} 2 \left(\hat{b}_i^{INB} \left(\hat{b}_i \right) - b_i \right) \\ \times \left(\sigma_\varepsilon^2 \frac{m'_\phi \left(\hat{b}_i; \theta \right)}{m_\phi \left(\hat{b}_i; \theta \right) + s_K} - \hat{\sigma}_\varepsilon^2 \frac{m'_\phi \left(\hat{b}_i; \hat{\theta} \right)}{m_\phi \left(\hat{b}_i; \hat{\theta} \right) + s_K} \right) \\ - \left(\sigma_\varepsilon^2 \frac{m'_\phi \left(\hat{b}_i; \theta \right)}{m_\phi \left(\hat{b}_i; \theta \right) + s_K} - \hat{\sigma}_\varepsilon^2 \frac{m'_\phi \left(\hat{b}_i; \hat{\theta} \right)}{m_\phi \left(\hat{b}_i; \hat{\theta} \right) + s_K} \right)^2 \end{array} \right] \\
& \times f_K \left(\hat{b} - b \right) d\hat{b} dG_K \left(b; \theta \right) \\
= \quad & \rho \frac{1}{K} \sum_{i=1}^{K} \int \int 2 \left(\hat{b}_i^{INB} \left(\hat{b}_i \right) - b_i \right) \\
& \times \left(\sigma_\varepsilon^2 \frac{m'_\phi \left(\hat{b}_i; \theta \right)}{m_\phi \left(\hat{b}_i; \theta \right) + s_K} - \hat{\sigma}_\varepsilon^2 \frac{m'_\phi \left(\hat{b}_i; \hat{\theta} \right)}{m_\phi \left(\hat{b}_i; \hat{\theta} \right) + s_K} \right) \\
& \times f_K \left(\hat{b} - b \right) d\hat{b} dG_K \left(b; \theta \right) \\
& - \rho \frac{1}{K} \sum_{i=1}^{K} \int \int \left(\sigma_\varepsilon^2 \frac{m'_\phi \left(\hat{b}_i; \theta \right)}{m_\phi \left(\hat{b}_i; \theta \right) + s_K} - \hat{\sigma}_\varepsilon^2 \frac{m'_\phi \left(\hat{b}_i; \hat{\theta} \right)}{m_\phi \left(\hat{b}_i; \hat{\theta} \right) + s_K} \right)^2 \\
& \times f_K \left(\hat{b} - b \right) d\hat{b} dG_K \left(b; \theta \right) \\
\leq \quad & \rho \frac{2}{K} \sum_{i=1}^{K} \left\{ \int \int \left(\hat{b}_i^{INB} \left(\hat{b}_i \right) - b_i \right)^2 f_K \left(\hat{b} - b \right) d\hat{b} dG_K \left(b; \theta \right) \right\}^{1/2} \\
& \times \left\{ \begin{array}{c} \int \int \left(\sigma_\varepsilon^2 \frac{m'_\phi \left(\hat{b}_i; \theta \right)}{m_\phi \left(\hat{b}_i; \theta \right) + s_K} - \hat{\sigma}_\varepsilon^2 \frac{m'_\phi \left(\hat{b}_i; \hat{\theta} \right)}{m_\phi \left(\hat{b}_i; \hat{\theta} \right) + s_K} \right)^2 \\ \times f_K \left(\hat{b} - b \right) d\hat{b} dG_K \left(b; \theta \right) \end{array} \right\}^{1/2} \\
& - \rho \frac{1}{K} \sum_{i=1}^{K} \int \int \left(\sigma_\varepsilon^2 \frac{m'_\phi \left(\hat{b}_i; \theta \right)}{m_\phi \left(\hat{b}_i; \theta \right) + s_K} - \hat{\sigma}_\varepsilon^2 \frac{m'_\phi \left(\hat{b}_i; \hat{\theta} \right)}{m_\phi \left(\hat{b}_i; \hat{\theta} \right) + s_K} \right)^2 \\
& \times f_K \left(\hat{b} - b \right) d\hat{b} dG_K \left(b; \theta \right)
\end{aligned}
$$

$$
\begin{aligned}
\leq\quad & 2\rho\left\{\frac{1}{K}\sum_{i=1}^{K}\int\int\left(\hat{b}_i^{INB}\left(\hat{b}_i\right)-b_i\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}dG_K\left(b;\theta\right)\right\}^{1/2} \\
& \times\left\{\begin{array}{c}\frac{1}{K}\sum_{i=1}^{K}\int\int\left(\sigma_\varepsilon^2\frac{m'_\phi\left(\hat{b}_i;\theta\right)}{m_\phi\left(\hat{b}_i;\theta\right)+s_K}-\hat{\sigma}_\varepsilon^2\frac{m'_\phi\left(\hat{b}_i;\hat{\theta}\right)}{m_\phi\left(\hat{b}_i;\hat{\theta}\right)+s_K}\right)^2 \\ \times f_K\left(\hat{b}-b\right)d\hat{b}dG_K\left(b;\theta\right)\end{array}\right\}^{1/2} \\
& -\rho\frac{1}{K}\sum_{i=1}^{K}\int\int\left(\sigma_\varepsilon^2\frac{m'_\phi\left(\hat{b}_i;\theta\right)}{m_\phi\left(\hat{b}_i;\theta\right)+s_K}-\hat{\sigma}_\varepsilon^2\frac{m'_\phi\left(\hat{b}_i;\hat{\theta}\right)}{m_\phi\left(\hat{b}_i;\hat{\theta}\right)+s_K}\right)^2 \\
& \times f_K\left(\hat{b}-b\right)d\hat{b}dG_K\left(b;\theta\right)
\end{aligned}
$$

where the third equality comes from the fact that $a^2-b^2 = 2a\left(a-b\right)-\left(a-b\right)^2$, the first inequality is an application of Hölder's inequality, and the second inequality comes from the Cauchy-Schwartz inequality. The final expression above shows that we need only demonstrate

$$
\begin{aligned}
& \rho\frac{1}{K}\sum_{i=1}^{K}\int\int\left(\begin{array}{c}\sigma_\varepsilon^2\frac{m'_\phi\left(\hat{b}_i;\theta\right)}{m_\phi\left(\hat{b}_i;\theta\right)+s_K} \\ -\hat{\sigma}_\varepsilon^2\frac{m'_\phi\left(\hat{b}_i;\hat{\theta}\right)}{m_\phi\left(\hat{b}_i;\hat{\theta}\right)+s_K}\end{array}\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}dG_K\left(b;\theta\right) \\
\rightarrow\quad & 0
\end{aligned}
\tag{85}
$$

since

$$
\begin{aligned}
& \frac{1}{K}\sum_{i=1}^{K}\int\int\left(\hat{b}_i^{INB}\left(\hat{b}_i\right)-b_i\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}dG_K\left(b;\theta\right) \\
=\quad & r_G\left(\hat{b}^{INB},f_K\right) \\
\rightarrow\quad & r_G\left(\hat{b}^{NB},\phi_K\right)
\end{aligned}
\tag{86}
$$

from the facts that Term II and Term III converge to zero as shown above.

Now,

$$\rho \frac{1}{K} \sum_{i=1}^{K} \int \int \left(\sigma_{\varepsilon}^{2} \frac{m_{\phi}'\left(\hat{b}_{i};\theta\right)}{m_{\phi}\left(\hat{b}_{i};\theta\right)+s_{K}} - \hat{\sigma}_{\varepsilon}^{2} \frac{m_{\phi}'\left(\hat{b}_{i};\hat{\theta}\right)}{m_{\phi}\left(\hat{b}_{i};\hat{\theta}\right)+s_{K}} \right)^{2} f_{K}\left(\hat{b}-b\right) d\hat{b} dG_{K}\left(b;\theta\right) \quad (87)$$

$$\begin{aligned}
\leq \quad & 2\rho \frac{1}{K} \sum_{i=1}^{K} \int \int \left(\hat{\sigma}_{\varepsilon}^{2} - \sigma_{\varepsilon}^{2}\right)^{2} \left(\frac{m_{\phi}'\left(\hat{b}_{i};\hat{\theta}\right)}{m_{\phi}\left(\hat{b}_{i};\hat{\theta}\right)+s_{K}} \right)^{2} f_{K}\left(\hat{b}-b\right) d\hat{b} dG_{K}\left(b;\theta\right) \\
& +2\rho \frac{1}{K} \sum_{i=1}^{K} \int \int \left(\frac{m_{\phi}'\left(\hat{b}_{i};\hat{\theta}\right)}{m_{\phi}\left(\hat{b}_{i};\hat{\theta}\right)+s_{K}} - \frac{m_{\phi}'\left(\hat{b}_{i};\theta\right)}{m_{\phi}\left(\hat{b}_{i};\theta\right)+s_{K}} \right)^{2} f_{K}\left(\hat{b}-b\right) d\hat{b} dG_{K}\left(b;\theta\right) \\
\leq \quad & \frac{C_{1}}{K s_{K}} + 2\rho \frac{1}{K} \sum_{i=1}^{K} \int \int \left(\frac{m_{\phi}'\left(\hat{b}_{i};\hat{\theta}\right)}{m_{\phi}\left(\hat{b}_{i};\hat{\theta}\right)+s_{K}} - \frac{m_{\phi}'\left(\hat{b}_{i};\theta\right)}{m_{\phi}\left(\hat{b}_{i};\theta\right)+s_{K}} \right)^{2} f_{K}\left(\hat{b}-b\right) d\hat{b} dG_{K}\left(b;\theta\right) \\
\leq \quad & \frac{C_{1}}{K s_{K}} \\
& +4\rho \frac{1}{K} \sum_{i=1}^{K} \int \int \left\{ \begin{array}{c} \left(\frac{m_{\phi}'\left(\hat{b}_{i};\hat{\theta}\right)}{m_{\phi}\left(\hat{b}_{i};\hat{\theta}\right)+s_{K}} \right)^{2} \left(\frac{m_{\phi}\left(\hat{b}_{i};\theta\right)-m_{\phi}\left(\hat{b}_{i};\hat{\theta}\right)}{m_{\phi}\left(\hat{b}_{i};\theta\right)+s_{K}} \right)^{2} \\ + \left(\frac{m_{\phi}'\left(\hat{b}_{i};\theta\right)-m_{\phi}'\left(\hat{b}_{i};\hat{\theta}\right)}{m_{\phi}\left(\hat{b}_{i};\theta\right)+s_{K}} \right)^{2} \end{array} \right\} f_{K}\left(\hat{b}-b\right) d\hat{b} dG_{K}\left(b;\theta\right) \\
\leq \quad & \frac{C_{1}}{K s_{K}^{2}} \\
& +4\rho \frac{1}{K} \sum_{i=1}^{K} \int \int \left\{ \begin{array}{c} \left(\frac{m_{\phi}'\left(\hat{b}_{i};\hat{\theta}\right)}{m_{\phi}\left(\hat{b}_{i};\hat{\theta}\right)+s_{K}} \right)^{2} \left(\frac{C_{2}\left\|\theta-\hat{\theta}\right\|}{s_{K}} \right)^{2} \\ + \left(\frac{C_{2}\left\|\theta-\hat{\theta}\right\|}{s_{K}} \right)^{2} \end{array} \right\} f_{K}\left(\hat{b}-b\right) d\hat{b} dG_{K}\left(b;\theta\right) \\
\leq \quad & \frac{C_{1}}{K s_{K}^{2}} \\
& + 4\rho \frac{1}{K s_{K}^{4}} \left[1 + \sup_{x,\theta} \left\{ m_{\phi}'\left(x,\theta\right)^{2} \right\} \right] \int \int K \left\|\hat{\theta}-\theta\right\|^{2} f_{K}\left(\hat{b}-b\right) d\hat{b} dG_{K}\left(b;\theta\right) \\
\leq \quad & \frac{C_{1}}{K s_{K}^{2}} + \frac{C_{3}}{K s_{k}^{4}} \\
\rightarrow \quad & 0
\end{aligned}$$

where the first inequality exploits the fact that $(a+b)^2 \leq 2a^2 + 2b^2$ and uses the trick of adding and subtracting $\sigma_{\varepsilon}^{2} \frac{m_{\phi}'\left(\hat{b}_{i};\hat{\theta}\right)}{m_{\phi}\left(\hat{b}_{i};\hat{\theta}\right)+s_{K}}$, the second inequality comes from Theorem 1 and the fact that $\left(m'_{\phi}\left(\hat{b}_i;\hat{\theta}\right)\right)^2$ is bounded (using the properties of normal convolutions), the third inequality uses the fact that $\left(\frac{a}{b}-\frac{c}{d}\right)^2=\left(\frac{a}{b}\left(\frac{d-b}{d}\right)+\frac{a-c}{d}\right)^2\leq 2\left(\frac{a}{b}\right)^2\left(\frac{d-b}{d}\right)^2+2\left(\frac{a-c}{d}\right)^2$, the fourth inequality uses the Lipschitz continuity (uniform in i and θ) of m' and m, which is implied by the Lipschitz assumption made on $g(b;\theta)$, the fifth inequality follows by rearrangement, and the sixth inequality uses the facts that $\left(m'_{\phi}(x;\theta)\right)^2$ is uniformly bounded, as above, and that, by Assumption 5, $K\left\|\hat{\theta}-\theta\right\|^2$ has uniformly bounded expectation along the sequence. The convergence of the final terms follows from Assumption 7 and display (37).

Please note that the score used in $\hat{b}^{PEB}$ implicitly depends upon $\hat{\sigma}_{\varepsilon}^2$, a dependence which is suppressed above. Since σ_{ε}^2 is bounded above and below by Assumption 1, and $\hat{\sigma}_{\varepsilon}^2$ is $\sqrt{K}$-consistent by Theorem 1, this does not affect our results.

Now we prove part (ii) of the theorem:

We proceed to show that $r_G\left(\hat{b}^{NB},\phi\right)$ is the "asymptotic minimax Bayes risk" in the sense that

$$\lim_{K\to\infty}\inf_{\tilde{b}}\left\{\sup_{f_K} r_G\left(\tilde{b},f_K\right)\right\} \;=\; r_G\left(\hat{b}^{NB},\phi\right) \tag{88}$$

where the supremum is taken over the set of likelihoods satisfying the assumptions of the theorem. This is actually straightforward. First,

$$\lim\inf_{K\to\infty}\inf_{\tilde{b}}\left\{\sup_{f_K} r_G\left(\tilde{b},f_K\right)\right\} \;\geq\; r_G\left(\hat{b}^{NB},\phi\right) \tag{89}$$

follows from $\sup_{f_K} r_G\left(\tilde{b}, f_K\right) \geq r_G\left(\tilde{b}, \phi\right)$ $\forall \tilde{b}, \forall K$, so that $\forall K$,

$$\begin{aligned} \inf_{\tilde{b}}\left\{\sup_{f_K} r_G\left(\tilde{b}, f_K\right)\right\} &\geq \inf_{\tilde{b}}\left\{r_G\left(\tilde{b}, \phi\right)\right\} \\ &= r_G\left(\hat{b}^{NB}, \phi\right). \end{aligned} \tag{90}$$

Further,

$$\inf_{\tilde{b}}\left\{\sup_{f_K} r_G\left(\tilde{b}, f_K\right)\right\} \leq \sup_{f_K} r_G\left(\hat{b}^{INB}, f_K\right) \tag{91}$$

so that

$$\begin{aligned} \lim_{K\to\infty}\sup \inf_{\tilde{b}}\left\{\sup_{f_K} r_G\left(\tilde{b}, f_K\right)\right\} &\leq \lim_{K\to\infty}\sup \sup_{f_K} r_G\left(\hat{b}^{INB}, f_K\right) \\ &= r_G\left(\hat{b}^{NB}, \phi\right). \end{aligned} \tag{92}$$

The last equality is obviously the key to the entire result. It holds because each of the bounding constants in the proof that Terms II and III converge to zero in part (i) depends on the primitive bounding constants in the assumptions. Thus, as long as these primitive bounding constants are fixed, the bounds in the proof of part (i) hold uniformly for f_K satisfying the assumptions, and the final equality above follows. Q.E.D.

Proof of Theorem 4:

Let $\hat{b}_i^{ISEB} = \hat{b}_i - \sigma_\varepsilon^2 \frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)+s_K}$; this is the infeasible simple empirical Bayes estimator based on the true average marginal $\bar{m}_K$. Now write

$$r_G\left(\hat{b}^{NSEB}, f_K\right) - r_G\left(\hat{b}^{NB}, \phi_K\right) \tag{93}$$

$$= r_G\left(\hat{b}^{NSEB}, f_K\right) - r_G\left(\hat{b}^{ISEB}, f_K\right) \quad \text{(Term I)}$$

$$+ r_G\left(\hat{b}^{ISEB}, f_K\right) - r_G\left(\hat{b}^{NB}, \phi_K\right) \quad \text{(Term II)}$$

We will show that both Term I and Term II go to zero. Recall that $r_G\left(\hat{b}^{NB}, \phi_K\right) < \infty$ from Lemma 4. First, let us deal with the easier term, Term II:

$$\begin{aligned}
& r_G\left(\hat{b}^{ISEB}, f_K\right) - r_G\left(\hat{b}^{NB}, \phi_K\right) && (94)\\
= \; & \rho \int\int \frac{1}{K}\sum_{i=1}^{K}\left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right) - b_i\right)^2 f_K\left(\hat{b} - b\right) d\hat{b} dG_K(b) \\
& - \rho \int\int \frac{1}{K}\sum_{i=1}^{K}\left(\hat{b}_i^{NB}\left(\hat{b}_i\right) - b_i\right)^2 \phi_K\left(\hat{b} - b\right) d\hat{b} dG_K(b) \\
= \; & \frac{1}{K}\sum_{i=1}^{K}\rho \int\int \left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right) - b_i\right)^2 f_{iK}\left(\hat{b}_i - b_i\right) d\hat{b}_i dG(b_i) \\
& - \frac{1}{K}\sum_{i=1}^{K}\rho \int\int \left(\hat{b}_i^{NB}\left(\hat{b}_i\right) - b_i\right)^2 \phi\left(\hat{b}_i - b_i\right) d\hat{b}_i dG(b_i) \\
= \; & \frac{1}{K}\sum_{i=1}^{K}\rho \int\int \left(\hat{b}_1^{ISEB}(x) - y\right)^2 f_{iK}(x - y)\, dx dG(y) \\
& - \frac{1}{K}\sum_{i=1}^{K}\rho \int\int \left(\hat{b}_1^{NB}(x) - y\right)^2 \phi(x - y)\, dx dG(y) \\
= \; & \rho \int\int \left(\hat{b}_1^{ISEB}(x) - y\right)^2 \bar{f}_K(x - y)\, dx dG(y) \\
& - \rho \int\int \left(\hat{b}_1^{NB}(x) - y\right)^2 \phi(x - y)\, dx dG(y)
\end{aligned}$$

The second equality comes from the fact that both $\hat{b}_i^{NB}$ and $\hat{b}_i^{ISEB}$ depend only on $\hat{b}_i$ and not on $\hat{b}_{-i}$, while the third equality follows from the fact that the functional form of both $\hat{b}_i^{NB}$ and $\hat{b}_i^{ISEB}$ is the same for each i.

From the bounds in Lemma 3, and the definition of $\hat{b}_i^{ISEB}$, we see that the first integrand in the final line converges to the second pointwise. Thus, if we can produce a dominating function for the first integrand, we may apply the Dominated Convergence Theorem and be finished. But

$$\begin{aligned} & \left(\hat{b}_1^{ISEB}(x) - y\right)^2 \bar{f}_K(x-y) \qquad (95) \\ = & \left((x-y) - \sigma_\varepsilon^2 \frac{\bar{m}'_K(x)}{\bar{m}_K(x) + s_K}\right)^2 \bar{f}_K(x-y) \\ \leq & \; 2(x-y)^2 \bar{f}_K(x-y) + 2\sigma_\varepsilon^4 \left(\frac{\bar{m}'_K(x)}{\bar{m}_K(x) + s_K}\right)^2 \bar{f}_K(x-y) \end{aligned}$$

The first term in this bound is evidently integrable by Assumption 1. The second term is integrable by Lemma 4, so we have shown that Term II converges to zero.

Before dealing with the more challenging Term I, it is worth noting that we will never encounter a "division by zero" problem, because of three facts: 1) the limit density $m_\phi(x)$ is positive everywhere (since it is a convolution of a normal density with the prior); 2) the approximation error given by Lemma 3 is less than s_K by construction (for large enough K, see Assumption 7); 3) the denominator of $\hat{b}_i^{ISEB}$ is thus always greater than m_ϕ, since we take $\bar{m}_K(x) > m_\phi(x) - s_K$ (for large enough K) and add s_K to both sides of the inequality, so that we never encounter difficulties.

Now we shall demonstrate that Term I converges to zero. Our proof is in the spirit of Bickel *et al.* (1993, p. 405 ff) and van der Vaart (1988, p. 169 ff), but we extend their approaches to handle cross-sectional dependence of the $\hat{b}_i$ and to deal with a nonconstant sequence of likelihoods.

Note that, from (2.8) and (2.9) of the body of this paper, we can write $\hat{l}_{iK}(x) = \hat{l}\left(\hat{b}_i\ ;\ \hat{b}_{-i}\right)$, where the function $\hat{l}(\cdot,\cdot)$ does not depend on i. This representation, and a similar one for $\breve{l}$, are adopted here. Although $\hat{\sigma}_\varepsilon^2$ depends on the full data, rather than only $\hat{b}$, this dependence is suppressed for notational convenience; the treatment below does, however, account for the fact that $\hat{\sigma}_\varepsilon^2$ is a function of the full data (see display (100)). Now,

$$
\begin{aligned}
& r_G\left(\hat{b}^{ISEB}, f_K\right) - r_G\left(\hat{b}^{NSEB}, f_K\right) \qquad (96) \\
= \; & \frac{1}{K}\sum_{i=1}^{K}\rho\int\int\left[\left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right) - b_i\right)^2 - \left(\hat{b}_i^{NSEB}\left(\hat{b}\right) - b_i\right)^2\right] f_K\left(\hat{b}-b\right) d\hat{b}dG_K(b) \\
= \; & \frac{1}{K}\sum_{i=1}^{K}\rho\int\int\left[\begin{array}{c}\left(\hat{b}_i - b_i - \sigma_\varepsilon^2\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)+s_K}\right)^2 - \\ \left(\hat{b}_i - b_i - \hat{\sigma}_\varepsilon^2\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)\right)^2\end{array}\right] f_K\left(\hat{b}-b\right) d\hat{b}dG_K(b) \\
= \; & \frac{1}{K}\sum_{i=1}^{K}\rho\int\int\left[\begin{array}{c} -\left(\hat{\sigma}_\varepsilon^2\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right) - \sigma_\varepsilon^2\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)+s_K}\right)^2 + \\ 2\left(\hat{b}_i - b_i - \sigma_\varepsilon^2\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)+s_K}\right)\left(\hat{\sigma}_\varepsilon^2\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right) - \sigma_\varepsilon^2\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)+s_K}\right)\end{array}\right] \\
& \quad \times f_K\left(\hat{b}-b\right) d\hat{b}dG_K(b) \\
= \; & -\frac{1}{K}\sum_{i=1}^{K}\rho\int\int\left(\hat{\sigma}_\varepsilon^2\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right) - \sigma_\varepsilon^2\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 f_K\left(\hat{b}-b\right) d\hat{b}dG_K(b) \\
& + \frac{1}{K}\sum_{i=1}^{K}\rho\int\int 2\left(\hat{b}_i - b_i - \sigma_\varepsilon^2\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)\times \\
& \qquad \left(\hat{\sigma}_\varepsilon^2\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right) - \sigma_\varepsilon^2\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right) f_K\left(\hat{b}-b\right) d\hat{b}dG_K(b) \\
\leq \; & -\frac{1}{K}\sum_{i=1}^{K}\rho\int\int\left(\hat{\sigma}_\varepsilon^2\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right) - \sigma_\varepsilon^2\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 f_K\left(\hat{b}-b\right) d\hat{b}dG_K(b)
\end{aligned}
$$

$$+\frac{2}{K}\sum_{i=1}^{K}\rho\left[\begin{array}{c}\left(\int\int\left(\hat{b}_i-b_i-\sigma_\varepsilon^2\frac{\bar{m}'_K\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)+s_K}\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}dG_K(b)\right)^{\frac{1}{2}}\times\\ \left(\int\int\left(\hat{\sigma}_\varepsilon^2\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\sigma_\varepsilon^2\frac{\bar{m}'_K\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)+s_K}\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}dG_K(b)\right)^{\frac{1}{2}}\end{array}\right]$$

$$\begin{aligned}\leq\quad & -\frac{1}{K}\sum_{i=1}^{K}\rho\int\int\left(\hat{\sigma}_\varepsilon^2\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\sigma_\varepsilon^2\frac{\bar{m}'_K\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}dG_K(b)\\ & +2\left\{\frac{1}{K}\sum_{i=1}^{K}\rho\int\int\left(\hat{b}_i-b_i-\sigma_\varepsilon^2\frac{\bar{m}'_K\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}dG_K(b)\right\}^{\frac{1}{2}}\times\\ & \left\{\frac{1}{K}\sum_{i=1}^{K}\rho\int\int\left(\hat{\sigma}_\varepsilon^2\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\sigma_\varepsilon^2\frac{\bar{m}'_K\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}dG_K(b)\right\}^{\frac{1}{2}}\end{aligned}$$

In the above, the first and second equalities are by definition; the third is a consequence of the fact that $a^2-b^2 = 2a(a-b)-(a-b)^2$; the fourth equality simply breaks out the two terms of the integrand; the first inequality is an application of Hölder's inequality, and the second is an application of the Cauchy-Schwartz inequality.

From the final bound given above, we see that if we can show that

$$\begin{aligned}& \frac{1}{K}\sum_{i=1}^{K}\rho\int\int\left(\hat{\sigma}_\varepsilon^2\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\sigma_\varepsilon^2\frac{\bar{m}'_K\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}dG_K(b)\\ \to\quad & 0\end{aligned}\tag{97}$$

then we will have demonstrated that Term I converges to zero, because

$$\begin{aligned}& \frac{1}{K}\sum_{i=1}^{K}\rho\int\int\left(\hat{b}_i-b_i-\sigma_\varepsilon^2\frac{\bar{m}'_K\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}dG_K(b)\\ =\quad & r_G\left(\hat{b}^{ISEB},f_K\right)\end{aligned}\tag{98}$$

$$
\begin{aligned}
&\to \quad r_G\left(\hat{b}^{NB},\phi_K\right) \\
&< \quad \infty
\end{aligned}
$$

from the fact that Term II converges to zero, and, as noted above, $r_G\left(\hat{b}^{NB},\phi_K\right) < \infty$.

Using $(a+b)^2 \leq 2a^2 + 2b^2$, and adding and subtracting $\sigma_\varepsilon^2 \hat{l}\left(\hat{b}_i\,;\,\hat{b}_{-i}\right)$, we have

$$
\begin{aligned}
& \frac{1}{K}\sum_{i=1}^{K}\rho\int\int\left(\hat{\sigma}_\varepsilon^2\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\sigma_\varepsilon^2\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)+s_K}\right)^2\times \qquad (99)\\
&\qquad f_K\left(\hat{b}-b\right)d\hat{b}dG_K(b)\\
\leq\;& \frac{2}{K}\sum_{i=1}^{K}\rho\int\int\left(\hat{\sigma}_\varepsilon^2-\sigma_\varepsilon^2\right)^2\hat{l}^2\left(\hat{b}_i;\hat{b}_{-i}\right)f_K\left(\hat{b}-b\right)d\hat{b}dG_K(b) \qquad \text{(Term A)}\\
& +\frac{2\sigma_\varepsilon^4}{K}\sum_{i=1}^{K}\rho\int\int\left(\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)+s_K}\right)^2\times \qquad \text{(Term B)}\\
&\qquad f_K\left(\hat{b}-b\right)d\hat{b}dG_K(b)
\end{aligned}
$$

which separates the problem of nonparametric score estimation from the problem of estimating the residual variance. Now, Term A satisfies

$$
\begin{aligned}
& \frac{2}{K}\sum_{i=1}^{K}\rho\int\int\left(\hat{\sigma}_\varepsilon^2-\sigma_\varepsilon^2\right)^2\hat{l}^2\left(\hat{b}_i;\hat{b}_{-i}\right)f_K\left(\hat{b}-b\right)d\hat{b}dG_K(b) \qquad (100)\\
\leq\;& \frac{2\rho q_K^2}{K}\sum_{i=1}^{K}\int\int\left(\hat{\sigma}_\varepsilon^2-\sigma_\varepsilon^2\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}dG_K(b)\\
=\;& 2\rho q_K^2\int\int\left(\hat{\sigma}_\varepsilon^2-\sigma_\varepsilon^2\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}dG_K(b)\\
=\;& 2\rho q_K^2 Var\left[\hat{\sigma}_\varepsilon^2\right]\\
\leq\;& 2\rho q_K^2\frac{C}{K}\\
\to\;& 0
\end{aligned}
$$

where the first inequality comes from the truncation of our estimator $\hat{l}$ of the score, and the second inequality comes from Theorem 1. The convergence of the final bound to zero is by construction: by display (37), $\frac{q_K^2}{K} \rightarrow 0$. Thus, Term A converges to zero.

Now consider Term B. Define $D_i \equiv \left\{\hat{b} : \left|\hat{b}_i\right| \leq \sqrt{\frac{\hat{\sigma}_\varepsilon^2}{128}\log K} \text{ and } \left|\breve{l}\left(\hat{b}_i;\hat{b}_{-i}\right)\right| \leq q_K\right\}$, and let $E_{D_i}\left[(\cdot)\right] \equiv \int_{\hat{b}\in D_i}(\cdot)\, m_K\left(\hat{b}\right)d\hat{b}$ (so that the area of integration is restricted, but in a way which may differ for each i). Now,

$$\frac{2\sigma_\varepsilon^4}{K}\sum_{i=1}^K \rho \int\int \left(\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right) - \frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right) + s_K}\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}dG_K(b) \tag{101}$$

$$= \frac{2\rho\sigma_\varepsilon^4}{K}\sum_{i=1}^K \int \left(\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right) - \frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right) + s_K}\right)^2 m_K\left(\hat{b}\right)d\hat{b}$$

$$= \frac{2\rho\sigma_\varepsilon^4}{K}\sum_{i=1}^K \int E\left[\left(\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right) - \frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right) + s_K}\right)^2 \mid \hat{b}_i\right] m_{iK}\left(\hat{b}_i\right)d\hat{b}_i$$

$$\leq \frac{2\rho\sigma_\varepsilon^4}{K}\sum_{i=1}^K \int_{-\infty}^{\infty} \left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right) + s_K}\right)^2 m_{iK}\left(\hat{b}_i\right) \times \tag{Term Bi}$$

$$\left(\Pr\left(\left|\breve{l}\left(\hat{b}_i;\hat{b}_{-i}\right)\right| > q_K \mid \hat{b}_i\right) + \Pr\left(\left|\hat{b}_i\right| > \sqrt{\frac{\hat{\sigma}_\varepsilon^2}{128}\log K} \mid \hat{b}_i\right)\right)d\hat{b}_i$$

$$+ \frac{2\rho\sigma_\varepsilon^4}{K}\sum_{i=1}^K E_{D_i}\left[\left(\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right) - \frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right) + s_K}\right)^2\right] \tag{Term Bii}$$

where the first equality is by the Tonelli-Fubini Theorem, the second is by definition, and the inequality follows from the truncation of the estimated score function according to the definition of D_i.

Consider Term Bi first.

$$\frac{2\rho\sigma_\varepsilon^4}{K}\sum_{i=1}^{K}\int_{-\infty}^{\infty}\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 m_{iK}\left(\hat{b}_i\right)\times \tag{102}$$

$$\left(\begin{array}{c}\Pr\left(\left|\breve{l}\left(\hat{b}_i;\hat{b}_{-i}\right)\right|>q_K\mid\hat{b}_i\right)+\\ \Pr\left(\left|\hat{b}_i\right|>\sqrt{\frac{\hat{\sigma}_\varepsilon^2}{128}\log K}\mid\hat{b}_i\right)\end{array}\right)d\hat{b}_i$$

$$= \frac{2\rho\sigma_\varepsilon^4}{K}\sum_{i=1}^{K}\int_{-\infty}^{\infty}\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 m_{iK}\left(\hat{b}_i\right)$$

$$\times\Pr\left(\left|\hat{b}_i\right|>\sqrt{\frac{\hat{\sigma}_\varepsilon^2}{128}\log K}\mid\hat{b}_i\right)d\hat{b}_i$$

$$+\frac{2\rho\sigma_\varepsilon^4}{K}\sum_{i=1}^{K}\int_{-\infty}^{\infty}\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 m_{iK}\left(\hat{b}_i\right)\times$$

$$\Pr\left(\left|\breve{l}\left(\hat{b}_i;\hat{b}_{-i}\right)\right|>q_K\mid\hat{b}_i\right)d\hat{b}_i$$

$$\leq \frac{2\rho\sigma_\varepsilon^4}{Ks_K^2}C\sum_{i=1}^{K}\Pr\left(\left|\hat{b}_i\right|>\sqrt{\frac{\hat{\sigma}_\varepsilon^2}{128}\log K}\right)$$

$$+\ 2\rho\sigma_\varepsilon^4\int_{|x|>d_K}\left(\frac{\bar{m}_K'(x)}{\bar{m}_K(x)\ +\ s_K}\right)^2\bar{m}_K(x)\,dx$$

$$+\sup_{i,\left|\hat{b}_i\right|\leq d_K}\Pr\left(\left|\breve{l}\left(\hat{b}_i;\hat{b}_{-i}\right)\right|>q_K\mid\hat{b}_i\right)$$

$$\times 2\rho\sigma_\varepsilon^4\int_{-d_K}^{d_K}\left(\frac{\bar{m}_K'(x)}{\bar{m}_K(x)\ +\ s_K}\right)^2\bar{m}_K(x)\,dx$$

$$\leq \frac{2\rho\sigma_\varepsilon^4}{Ks_K^2}C\sum_{i=1}^{K}\left\{\begin{array}{c}\Pr\left(\left|\hat{b}_i\right|>\sqrt{\frac{\sigma_\varepsilon^2}{256}\log K}\right)\\ +\Pr\left(\hat{\sigma}_\varepsilon^2\leq\frac{1}{2}\sigma_\varepsilon^2\right)\end{array}\right\}$$

$$+\ 2\rho\sigma_\varepsilon^4\int_{|x|>d_K}\left(\frac{\bar{m}_K'(x)}{\bar{m}_K(x)\ +\ s_K}\right)^2\bar{m}_K(x)\,dx$$

$$+\sup_{i,\left|\hat{b}_i\right|\leq d_K}\Pr\left(\left|\breve{l}\left(\hat{b}_i;\hat{b}_{-i}\right)\right|>q_K\mid\hat{b}_i\right)$$

$$
\begin{aligned}
& \times 2\rho\sigma_{\varepsilon}^{4}\int_{-d_K}^{d_K}\left(\frac{\bar{m}_K'(x)}{\bar{m}_K(x)\ +\ s_K}\right)^2 \bar{m}_K(x)\,dx \\
\leq \quad & C_2 s_K^{-2}\left\{\frac{1}{\log K}+\frac{1}{K}\right\} \\
& + 2\rho\sigma_{\varepsilon}^{4}\int_{|x|>d_K}\left(\frac{\bar{m}_K'(x)}{\bar{m}_K(x)\ +\ s_K}\right)^2 \bar{m}_K(x)\,dx \\
& + \sup_{i,\left|\hat{b}_i\right|\leq d_K} \Pr\left(\left|\breve{l}\left(\hat{b}_i;\hat{b}_{-i}\right)\right| > q_K \mid \hat{b}_i\right) \\
& \times 2\rho\sigma_{\varepsilon}^{4}\int_{-d_K}^{d_K}\left(\frac{\bar{m}_K'(x)}{\bar{m}_K(x)\ +\ s_K}\right)^2 \bar{m}_K(x)\,dx
\end{aligned}
$$

The equality is trivial; the first inequality follows from the boundedness of $\left(\bar{m}_K'(x)\right)^2$ (by Lemma 3(c) and the boundedness of $\left(m_{\phi}'(x)\right)^2$), the second inequality follows by $\Pr(A) \leq \Pr(B) + \Pr(C)$ whenever $A \subset (B \cup C)$ (if $\left|\hat{b}_i\right| > \sqrt{\frac{\hat{\sigma}_{\varepsilon}^2}{128}\log K}$, then either $\left|\hat{b}_i\right| > \sqrt{\frac{\sigma_{\varepsilon}^2}{256}\log K}$ or $\hat{\sigma}_{\varepsilon}^2 \leq \frac{1}{2}\sigma_{\varepsilon}^2$, or both), and the third inequality follows from Chebyshev's inequality and the variance bound of Theorem 1 (along with $\sigma_{\varepsilon}^2 > 0$). Of the terms in the final expression, we see immediately that the first term converges to zero by Assumption 7 and the second converges to zero by the uniform integrability of its integrand, which was shown in the course of the proof that Term II converges to zero, and by $d_K \to \infty$. The third term converges to zero by Lemmas 4 and 6(c). Thus, Term Bi converges to zero.

Finally, turn to Term Bii. Consider the i^{th} term of the average which makes up Term Bii. We define $E_{D_i}^{out}[(\cdot)]$ to be $\int_{\hat{b}\in D_i\cap\left\{\hat{\sigma}_{\varepsilon}^2>2\sigma_{\varepsilon}^2\right\}}(\cdot)\,m_K\left(\hat{b}\right)d\hat{b}$ and $E_{D_i}^{cond}\left[(\cdot)\mid\hat{b}_i\right]$ to be $\int_{\hat{b}\in D_i}(\cdot)\,m_K\left(\hat{b}_{-i}\mid\hat{b}_i\right)d\hat{b}_{-i}$ (so that both the probability measure and the area of integration depend on $\hat{b}_i$). Since $\left(\frac{a}{b}-\frac{c}{d}\right)^2 = \left(\frac{a}{b}\left(\frac{d-b}{d}\right)+\frac{a-c}{d}\right)^2 \leq 2\left(\frac{a}{b}\right)^2\left(\frac{d-b}{d}\right)^2 +$

$2\left(\frac{a-c}{d}\right)^2$ We have:

$$
\begin{aligned}
& E_{D_i}\left[\left(\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2\right] && \text{(103a)}\\
\leq\ & 2E_{D_i}^{out}\left[\left(\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)\right)^2\right]+2E_{D_i}^{out}\left[\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2\right]\\
& +\int_{-d_K}^{d_K}E_{D_i}^{cond}\left[\left(\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2|\hat{b}_i\right]m_{iK}\left(\hat{b}_i\right)d\hat{b}_i\\
\leq\ & 2C\left(q_K^2+s_K^{-2}\right)\Pr\left(\hat{\sigma}_\varepsilon^2>2\sigma_\varepsilon^2\right)\\
& +2\int_{-d_K}^{d_K}\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2E_{D_i}^{cond}\left[\left(\frac{\bar{m}_K\left(\hat{b}_i\right)-\hat{m}_{iK}\left(\hat{b}_i\right)}{\hat{m}_{iK}\left(\hat{b}_i\right)+s_K}\right)^2|\hat{b}_i\right]m_{iK}\left(\hat{b}_i\right)d\hat{b}_i\\
& +2\int_{-d_K}^{d_K}E_{D_i}^{cond}\left[\left(\frac{\hat{m}_{iK}'\left(\hat{b}_i\right)-\bar{m}_K'\left(\hat{b}_i\right)}{\hat{m}_{iK}\left(\hat{b}_i\right)+s_K}\right)^2|\hat{b}_i\right]m_{iK}\left(\hat{b}_i\right)d\hat{b}_i\\
\leq\ & 2\left(q_K^2+s_K^{-2}\right)\frac{C_2}{K}\\
& +\frac{2}{s_K^2}\int_{-d_K}^{d_K}\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2\times && \text{(Term Biia)}\\
& \quad E_{D_i}\left[\left(\bar{m}_K\left(\hat{b}_i\right)-\hat{m}_{iK}\left(\hat{b}_i\right)\right)^2|\hat{b}_i\right]m_{iK}\left(\hat{b}_i\right)d\hat{b}_i\\
& +\frac{2}{s_K^2}\int_{-d_K}^{d_K}E_{D_i}\left[\left(\hat{m}_{iK}'\left(\hat{b}_i\right)-\bar{m}_K'\left(\hat{b}_i\right)\right)^2|\hat{b}_i\right]m_{iK}\left(\hat{b}_i\right)d\hat{b}_i. && \text{(Term Biib)}
\end{aligned}
$$

where the first inequality follows from splitting the area of integration and $2a^2+2b^2\geq(a+b)^2$ (and from the definition of d_K), the second inequality is due to the truncation of our score estimator and the boundedness of $\left(\bar{m}_K'(x)\right)^2$ (by Lemma 3(c) and the boundedness of $\left(m_\phi'(x)\right)^2$), and applying the fact that $\left(\frac{a}{b}-\frac{c}{d}\right)^2=\left(\frac{a}{b}\left(\frac{d-b}{d}\right)+\frac{a-c}{d}\right)^2\leq 2\left(\frac{a}{b}\right)^2\left(\frac{d-b}{d}\right)^2+2\left(\frac{a-c}{d}\right)^2$ to the third term in the previous line. The final inequality is by Theorem 1 (for the first term) and is clear for the other two terms. The first term of the final line converges to zero (uniformly in i) due to Assumption 7, so we consider only the remaining two terms.

Term Biia satisfies

$$\begin{aligned} & \frac{2}{s_K^2}\int_{-d_K}^{d_K}\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 \times \\ & \quad E_{D_i}\left[\left(\bar{m}_K\left(\hat{b}_i\right)-\hat{m}_{iK}\left(\hat{b}_i\right)\right)^2|\hat{b}_i\right]m_{iK}\left(\hat{b}_i\right)d\hat{b}_i \\ \leq & \ \frac{2}{s_K^2}C\left(\frac{1}{h_K\left(K-1\right)}+h_K^2+K^{-13/48}\log^2 K\ +\ 2K^{-13/96}h_K\log K\ +\ K^{-1/2}\log^2 K\right) \\ & \times\int_{-\infty}^{\infty}\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 m_{iK}\left(\hat{b}_i\right)d\hat{b}_i \end{aligned} \tag{104}$$

by Lemma 6(a), since omitting the restriction to $\hat{b}\in D_i$ can only make the expectation larger. Also, Term Biib satisfies

$$\begin{aligned} & \frac{2}{s_K^2}\int_{-d_K}^{d_K}E_{D_i}\left[\left(\hat{m}_{iK}'\left(\hat{b}_i\right)-\bar{m}_K'\left(\hat{b}_i\right)\right)^2|\hat{b}_i,\hat{b}\in D_i\right]m_{iK}\left(\hat{b}_i\right)d\hat{b}_i \\ \leq & \ \frac{2}{s_K^2}C\left(\frac{1}{h_K^3\left(K-1\right)}+h_K^2+K^{-5/48}\log^2 K\ +\ 2K^{-5/96}h_K\log K\ +\ K^{-1/4}\log^2 K\right) \end{aligned}$$

by Lemma 6(b), by the same logic. Thus we have that

$$\text{Term Bii} \tag{105}$$

$$
\begin{aligned}
&\leq \quad \frac{4\rho\sigma_\varepsilon^4}{s_K^2}C\left(\begin{array}{c}\left(\begin{array}{c}\frac{1}{h_K(K-1)}+h_K^2+K^{-13/48}\log^2 K\\ +2K^{-13/96}h_K\log K\ +\ K^{-1/2}\log^2 K\end{array}\right)\\ \times\int_{-\infty}^{\infty}\left(\frac{\bar{m}'_K\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)+s_K}\right)^2 m_{iK}\left(\hat{b}_i\right)d\hat{b}_i\\ +\left(\begin{array}{c}\frac{1}{h_K^3(K-1)}+h_K^2+K^{-5/48}\log^2 K\\ +2K^{-5/96}h_K\log K\ +\ K^{-1/4}\log^2 K\end{array}\right)\end{array}\right)\\
&\rightarrow \quad 0
\end{aligned}
$$

by display (37) and Lemma 4 (which applies upon averaging), and we are finished with the proof that $r_G\left(\hat{b}^{NSEB}, f_K\right) - r_G\left(\hat{b}^{NB}, \phi_K\right) \rightarrow 0$.

(ii) The proof of part (ii) exactly parallels the proof of Theorem 3, part (ii), with $\hat{b}^{ISEB}$ replacing $\hat{b}^{INB}$. Q.E.D.

Proof of Theorem 5:

(i) First, we show that permuting $\hat{b}_{-i}$ has no effect on the value of $\tilde{b}_i\left(\hat{b}\right)$ when $\tilde{b} \in \mathcal{B}$. Suppose we define $\mathbf{P}_{-i}$ to be some permutation which leaves the index i fixed but may permute the other indices in an arbitrary way. We will use the same notation for $\mathbf{P}_{-i}$ and for its restriction to $\hat{b}_{-i}$, since context should always make our meaning clear. Then, for any $i = 1, \ldots, K$,

$$
\begin{aligned}
\tilde{b}_i\left(\mathbf{P}_{-i}\hat{b}\right) &= \left(\mathbf{P}_{-i}\tilde{b}\right)_i\left(\hat{b}\right) \qquad (106)\\
&= \tilde{b}_i\left(\hat{b}\right)
\end{aligned}
$$

where the first line follows directly from $\tilde{b}\left(\mathbf{P}\hat{b}\right) = \mathbf{P}\left(\tilde{b}\left(\hat{b}\right)\right)$ for any permutation $\mathbf{P}$ (the definition of equivariance) and the second line is by the definition of $\mathbf{P}_{-i}$. Since this is true for any such permutation $\mathbf{P}_{-i}$, we see that $\tilde{b}_i\left(\hat{b}\right)$ depends only on the values, not on the ordering, of the coordinates $\hat{b}_{-i}$. Thus, letting $\hat{G}_{K,-i}$ be the empirical c.d.f. of the (realized values) $\hat{b}_{-i}$, we may write

$$\tilde{b}_i\left(\hat{b}\right) = \tilde{b}_i\left(\hat{b}_i\ ;\ \hat{G}_{K,-i}\right) \tag{107}$$

for any $i = 1, \ldots, K$.

Further, we have, for any $i = 1, \ldots, K$,

$$\begin{aligned} \tilde{b}_i\left(\hat{b}\right) &= \tilde{b}_i\left(\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_{i-1}, \hat{b}_i, \hat{b}_{i+1}, \ldots, \hat{b}_K\right) \\ &= \tilde{b}_1\left(\hat{b}_i, \hat{b}_2, \ldots, \hat{b}_{i-1}, \hat{b}_1, \hat{b}_{i+1}, \ldots, \hat{b}_K\right) \\ &= \tilde{b}_1\left(\hat{b}_i\ ;\ \hat{G}_{K,-i}\right) \end{aligned} \tag{108}$$

where the first line is simply a rewriting for clarity, the second line follows by considering the permutation which transposes the indices i and 1 and applying the definition of equivariance, and the final equality is due to the fact shown in equation (107), setting $i = 1$ and noting that the equation shows that the argument treated specially is the one in the first coordinate position, which, here, is $\hat{b}_i$.

Now consider the i^{th} component of the risk function when $\tilde{b} \in \mathcal{B}$,

$$\begin{aligned} R_i\left(b,\ \tilde{b};\ \phi_K\right) &= E\left[\left(\tilde{b}_i\left(\hat{b}\right) - b_i\right)^2 \mid b\right] \\ &= E\left[\left(\tilde{b}_1\left(\hat{b}_i\ ;\ \hat{G}_{K,-i}\right) - b_i\right)^2 \mid b\right] \end{aligned} \tag{109}$$

where the first equality is by definition (with the conditioning on b stated for clarity), and the second equality is due to the fact that $\tilde{b}_i\left(\hat{b}\right) = \tilde{b}_1\left(\hat{b}_i\ ;\ \hat{G}_{K,-i}\right)$ as shown above. Note that $\hat{b}_i$ and $\hat{G}_{K,-i}$ are independent since, under ϕ_K, $\hat{b}_i$ and $\hat{b}_{-i}$ are independent, and $\hat{G}_{K,-i}$ is a function of $\hat{b}_{-i}$. Moreover, the only part of b which $\hat{b}_i$ depends on is b_i, and the only part of b that $\hat{G}_{K,-i}$ depends on is $\tilde{G}_{K,-i}$, the empirical c.d.f. of the elements of b_{-i} (observe that the fact that the $\hat{b}_j$ are i.i.d. is crucial in making this hold). Using these facts and the above, we may write

$$R_i\left(b,\ \tilde{b};\ \phi_K\right) \ = \ E\left[\left(\tilde{b}_1\left(\hat{b}_i\ ;\ \hat{G}_{K,-i}\right) - b_i\right)^2 \mid b_i, \tilde{G}_{K,-i}\right]. \tag{110}$$

But this has the same functional form, as a function of b_i and $\tilde{G}_{K,-i}$, for every $i = 1, \ldots, K$, since the estimator is, for each i, the same function of $\hat{b}_i$ and $\hat{G}_{K,-i}$. Therefore we have that

$$\begin{aligned} R_i\left(b,\ \tilde{b};\ \phi_K\right) &= E\left[\left(\tilde{b}_1\left(\hat{b}_i\ ;\ \hat{G}_{K,-i}\right) - b_i\right)^2 \mid b_i, \tilde{G}_{K,-i}\right] \\ &\equiv q\left(b_i\ ;\ \tilde{G}_{K,-i}\right) \\ &= q\left(b_i\ ;\ \tilde{G}_K\right) \end{aligned} \tag{111}$$

where $\tilde{G}_K$ is the empirical c.d.f. of the b_i (the last line is trivial) and we suppress the dependence on the estimator and the likelihood. This yields, for $\tilde{b} \in \mathcal{B}$,

$$\begin{aligned} R\left(b,\ \tilde{b};\ \phi_K\right) &= \rho\frac{1}{K}\sum_{i=1}^{K} R_i\left(b,\ \tilde{b};\ \phi_K\right) \\ &= \rho\frac{1}{K}\sum_{i=1}^{K} q\left(b_i\ ;\ \tilde{G}_K\right) \end{aligned} \tag{112}$$

$$= \rho \int_{-\infty}^{\infty} q\left(z\ ;\ \tilde{G}_K\right) d\tilde{G}_K(z)$$

$$= \rho \int_{-\infty}^{\infty} E\left[\left(\tilde{b}_1\left(\hat{b}_{ind(z)}, \hat{G}_{K,-ind(z)}\right) - z\right)^2 \mid z, \tilde{G}_{K,-i}\right] d\tilde{G}_K(z)$$

where z is simply a variable of integration, and the last line follows by definition, where $\hat{b}_{ind(z)}$ is just the coordinate of the least-squares coefficient vector corresponding to the coordinate of b whose value z takes on. Now, it is clear that the form of the estimator $\tilde{b}_1$ above is restricted in the way in which it may depend on $\hat{b}$. Since removing the restriction and seeking the best decision rule in an unrestricted way can only lower the risk, we have

$$\inf_{\tilde{b}\in\mathcal{B}} R\left(b,\ \tilde{b};\ \phi_K\right) \geq \inf_{b^*\left(\hat{b}\right)} \left\{\rho \int_{-\infty}^{\infty} E\left[\left(b^*\left(\hat{b}\right) - z\right)^2 \mid z, \tilde{G}_{K,-i}\right] d\tilde{G}_K(z)\right\} \tag{113}$$

and by standard Bayesian calculation, we obtain that the vector of coordinates

$$\hat{b}_i^{NB}\left(\hat{b}, \tilde{G}_K\right) = \frac{\int_{-\infty}^{\infty} z\phi\left(\hat{b}_i - z\right) d\tilde{G}_K(z)}{\int_{-\infty}^{\infty} \phi\left(\hat{b}_i - z\right) d\tilde{G}_K(z)} \tag{114}$$

is the solution to the problem

$$\arg \inf_{b^*\left(\hat{b}\right)} \left\{\rho \int_{-\infty}^{\infty} E\left[\left(b^*\left(\hat{b}\right) - z\right)^2 \mid z, \tilde{G}_{K,-i}\right] d\tilde{G}_K(z)\right\} \tag{115}$$

so that we have

$$\inf_{\tilde{b}\in\mathcal{B}} R\left(b,\ \tilde{b};\ \phi_K\right) \geq R\left(b,\ \hat{b}^{NB};\ \phi_K\right) = r_{\tilde{G}_K}\left(\hat{b}^{NB}, \phi\right). \tag{116}$$

An important point to observe is that $\hat{b}^{NB}\left(\hat{b}, \tilde{G}_K\right)$ is not an estimator: it is infeasible, since it depends on information (that is, the empirical c.d.f. of the b_i) which is not part of the data.

(ii) We prove that $\left|R\left(b,\ \hat{b}^{NSEB};\ f_K\right) - R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right)\right| \to 0$, then simply set $f_K = \phi_K$ and observe that our result is uniform in $\|b\|_2 \leq M$. Let $\hat{b}_i^{ISEB} = \hat{b}_i - \sigma_\varepsilon^2 \frac{\bar{m}'_K(\hat{b}_i)}{\bar{m}_K(\hat{b}_i)+s_K}$; this is the infeasible simple empirical Bayes estimator based on the average marginal

$$\bar{m}_K(x) = \frac{1}{K}\sum_{j=1}^{K} \bar{f}_K(x - b_j)$$

(with respect to the empirical c. d. f. of the true b_j). Likewise, we have the modified definitions

$$\begin{aligned} m_\phi(x) &= \frac{1}{K}\sum_{j=1}^{K} \phi(x - b_j) \\ m_{iK}(x) &= \frac{1}{K}\sum_{j=1}^{K} f_{iK}(x - b_j). \end{aligned}$$

With these definitions, and imposing $\|b\|_2 \leq M$ uniformly along the K sequence, we note that Lemmas 1 through 6 hold uniformly along the K sequence. This is a critical conclusion, and is used throughout the following. The reason we may make this observation is that the only feature of the prior used in the lemmas is that the variance is bounded, and our assumption that $\|b\|_2 \leq M$ implies that the (empirical) variance of the b_i is uniformly bounded along the K sequence.

Now write

$$R\left(b,\ \hat{b}^{NSEB};\ f_K\right) - R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right) \tag{117}$$

$$= R\left(b,\ \hat{b}^{NSEB};\ f_K\right) - R\left(b,\ \hat{b}^{ISEB};\ f_K\right) \tag{Term I}$$

$$+R\left(b,\ \hat{b}^{ISEB};\ f_K\right) - R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right) \tag{Term II}$$

We will show that both Term I and Term II go to zero. Note that $\sup_{\|b\|_2 \leq M} \limsup_{K \to \infty} R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right) < \infty$ (for any $M < \infty$, where $\|b\|^2 \equiv \frac{1}{K}\sum_{i=1}^{K} b_i^2$) from reasoning exactly similar to that of Lemma 4. First, let us deal with the easier term, Term II:

$$
\begin{aligned}
& \left| R\left(b,\ \hat{b}^{ISEB};\ f_K\right) - R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right) \right| \qquad (118) \\
= & \left| \begin{array}{c} \rho \int \frac{1}{K} \sum_{i=1}^{K} \left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right) - b_i\right)^2 f_K\left(\hat{b} - b\right) d\hat{b} \\ - \rho \int \frac{1}{K} \sum_{i=1}^{K} \left(\hat{b}^{NB}_{\tilde{G}_K,i}\left(\hat{b}_i\right) - b_i\right)^2 \phi_K\left(\hat{b} - b\right) d\hat{b} \end{array} \right| \\
= & \left| \begin{array}{c} \frac{1}{K} \sum_{i=1}^{K} \rho \int \left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right) - b_i\right)^2 f_{iK}\left(\hat{b}_i - b_i\right) d\hat{b}_i \\ - \frac{1}{K} \sum_{i=1}^{K} \rho \int \left(\hat{b}^{NB}_{\tilde{G}_K,i}\left(\hat{b}_i\right) - b_i\right)^2 \phi\left(\hat{b}_i - b_i\right) d\hat{b}_i \end{array} \right| \\
\leq & \rho \frac{1}{K} \sum_{i=1}^{K} \int \left| \left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right) - b_i\right)^2 - \left(\hat{b}^{NB}_{\tilde{G}_K,i}\left(\hat{b}_i\right) - b_i\right)^2 \right| \\
& \times \phi\left(\hat{b}_i - b_i\right) d\hat{b}_i \\
& + \rho \frac{1}{K} \sum_{i=1}^{K} \int \left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right) - b_i\right)^2 \\
& \times \left| f_{iK}\left(\hat{b}_i - b_i\right) - \phi\left(\hat{b}_i - b_i\right) \right| d\hat{b}_i \\
\leq & \rho \frac{1}{K} \sum_{i=1}^{K} \int \left| \begin{array}{c} \left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right) - \hat{b}^{NB}_{\tilde{G}_K,i}\left(\hat{b}_i\right)\right)^2 \\ +2\left(\hat{b}^{NB}_{\tilde{G}_K,i}\left(\hat{b}_i\right) - b_i\right) \\ \times \left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right) - \hat{b}^{NB}_{\tilde{G}_K,i}\left(\hat{b}_i\right)\right) \end{array} \right| \phi\left(\hat{b}_i - b_i\right) d\hat{b}_i \\
& + \rho \frac{1}{K} \sum_{i=1}^{K} \int_{-z_K}^{z_K} \left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right) - b_i\right)^2 C K^{-1/4} \log K d\hat{b}_i \\
& + \rho \frac{1}{K} \sum_{i=1}^{K} \int_{\left|\hat{b}_i\right| > z_K} \left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right) - b_i\right)^2 \\
& \times \left(f_{iK}\left(\hat{b}_i - b_i\right) + \phi\left(\hat{b}_i - b_i\right)\right) d\hat{b}_i
\end{aligned}
$$

$$
\begin{aligned}
\leq\quad & \rho\frac{1}{K}\sum_{i=1}^{K}\int\left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right)-\hat{b}_{\tilde{G}_K,i}^{NB}\left(\hat{b}_i\right)\right)^2\phi\left(\hat{b}_i-b_i\right)d\hat{b}_i \\
& +2\rho\frac{1}{K}\sum_{i=1}^{K}\left(\int\left(\hat{b}_{\tilde{G}_K,i}^{NB}\left(\hat{b}_i\right)-b_i\right)^2\phi\left(\hat{b}_i-b_i\right)d\hat{b}_i\right)^{1/2} \\
& \times\left(\int\left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right)-\hat{b}_{\tilde{G}_K,i}^{NB}\left(\hat{b}_i\right)\right)^2\phi\left(\hat{b}_i-b_i\right)d\hat{b}_i\right)^{1/2} \\
& +\rho\frac{1}{K}\sum_{i=1}^{K}\int_{-z_K}^{z_K}\left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right)-b_i\right)^2CK^{-1/4}\log K d\hat{b}_i \\
& +\rho\frac{1}{K}\sum_{i=1}^{K}\int_{\left|\hat{b}_i\right|>z_K}\left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right)-b_i\right)^2 \\
& \times\left(f_{iK}\left(\hat{b}_i-b_i\right)+\phi\left(\hat{b}_i-b_i\right)\right)d\hat{b}_i \\
\leq\quad & \rho\frac{1}{K}\sum_{i=1}^{K}\int\left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right)-\hat{b}_{\tilde{G}_K,i}^{NB}\left(\hat{b}_i\right)\right)^2\phi\left(\hat{b}_i-b_i\right)d\hat{b}_i \\
& +2\rho\left(\frac{1}{K}\sum_{i=1}^{K}\int\left(\hat{b}_{\tilde{G}_K,i}^{NB}\left(\hat{b}_i\right)-b_i\right)^2\phi\left(\hat{b}_i-b_i\right)d\hat{b}_i\right)^{1/2} \\
& \times\left(\frac{1}{K}\sum_{i=1}^{K}\int\left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right)-\hat{b}_{\tilde{G}_K,i}^{NB}\left(\hat{b}_i\right)\right)^2\phi\left(\hat{b}_i-b_i\right)d\hat{b}_i\right)^{1/2} \\
& +\rho\frac{1}{K}\sum_{i=1}^{K}\int_{-z_K}^{z_K}\left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right)-b_i\right)^2CK^{-1/4}\log K d\hat{b}_i \\
& +\rho\frac{1}{K}\sum_{i=1}^{K}\int_{\left|\hat{b}_i\right|>z_K}\left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right)-b_i\right)^2 \\
& \times\left(f_{iK}\left(\hat{b}_i-b_i\right)+\phi\left(\hat{b}_i-b_i\right)\right)d\hat{b}_i
\end{aligned}
$$

$$
\begin{aligned}
\leq\quad & \rho\frac{\sigma_\varepsilon^4}{K}\sum_{i=1}^{K}\int\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}-\frac{m_\phi'\left(\hat{b}_i\right)}{m_\phi\left(\hat{b}_i\right)}\right)^2\phi\left(\hat{b}_i-b_i\right)db_i \\
& +2\rho\left(\frac{1}{K}\sum_{i=1}^{K}\int\left(\hat{b}_{\tilde{G}_K,i}^{NB}\left(\hat{b}_i\right)-b_i\right)^2\phi\left(\hat{b}_i-b_i\right)d\hat{b}_i\right)^{1/2} \\
& \times\left(\frac{\sigma_\varepsilon^4}{K}\sum_{i=1}^{K}\int\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}-\frac{m_\phi'\left(\hat{b}_i\right)}{m_\phi\left(\hat{b}_i\right)}\right)^2\phi\left(\hat{b}_i-b_i\right)d\hat{b}_i\right)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&+\rho\frac{2}{K}\sum_{i=1}^{K}\int_{-z_K}^{z_K}\left(\hat{b}_i-b_i\right)^2 CK^{-1/4}\log K d\hat{b}_i\\
&+\rho\frac{2\sigma_\varepsilon^4}{K}\sum_{i=1}^{K}\int_{-z_K}^{z_K}\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\;+\;s_K}\right)^2 CK^{-1/4}\log K d\hat{b}_i\\
&+\rho\frac{2\sigma_\varepsilon^4}{K}\sum_{i=1}^{K}\int_{\left|\hat{b}_i\right|>z_K}\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\;+\;s_K}\right)^2\\
&\times\left(f_{iK}\left(\hat{b}_i-b_i\right)+\phi\left(\hat{b}_i-b_i\right)\right)d\hat{b}_i\\
&+\rho\frac{2\sigma_\varepsilon^4}{K}\sum_{i=1}^{K}\int_{\left|\hat{b}_i\right|>z_K}\left(\hat{b}_i-b_i\right)^2\\
&\times\left(f_{iK}\left(\hat{b}_i-b_i\right)+\phi\left(\hat{b}_i-b_i\right)\right)d\hat{b}_i
\end{aligned}
$$

The second equality comes from the fact that both $\hat{b}^{NB}_{\tilde{G}_K,i}$ and $\hat{b}^{ISEB}_i$ depend only on $\hat{b}_i$ and not on $\hat{b}_{-i}$, the first inequality is trivial, the second inequality follows from $a^2-b^2=(a-b)^2+2b(a-b)$ applied to the integrand of the first term of the preceding expression, splitting up the area of integration of the second term of the preceding expression, and noting that $\left|f_{iK}\left(\hat{b}_i-b_i\right)-\phi\left(\hat{b}_i-b_i\right)\right|\le f_{iK}\left(\hat{b}_i-b_i\right)+\phi\left(\hat{b}_i-b_i\right)$, the third inequality follows from applying the triangle inequality and then Hölder's inequality to the first term of the preceding display, the fourth inequality follows from applying the Cauchy-Schwartz inequality to the second term of the preceding display, the fifth inequality follows from rewriting the first two terms of the preceding display and using $(a+b)^2\le 2a^2+2b^2$ on the last two terms of the preceding display, and the last expression converges to zero by the Dominated Convergence Theorem through Lemma 4 (for the first two terms), and $z_K=s_K^{-2}$ (for the remaining terms) along with Chebyshev's inequality and the observation that $\left(\frac{\bar{m}'_K(\hat{b}_i)}{\bar{m}_K(\hat{b}_i)+s_K}\right)^2 \leq C s_K^{-2}$ for sufficiently large K by Lemma 3(e), the boundedness of m'_ϕ, and the nonnegativity of $\bar{m}_K$.

Now we shall demonstrate that Term I converges to zero. Our proof is in the spirit of Bickel *et al.* (1993, p. 405 ff) and van der Vaart (1988, p. 169 ff), but we extend their approaches to handle cross-sectional dependence of the $\hat{b}_i$ and to deal with a nonconstant sequence of likelihoods.

Note that, from (2.8) and (2.9) of the body of this paper, we can write $\hat{l}_{iK}(x) = \hat{l}\left(\hat{b}_i\,;\,\hat{b}_{-i}\right)$, where the function $\hat{l}(\cdot,\cdot)$ does not depend on i. This representation, and a similar representation for $\breve{l}$, are adopted here. Although $\hat{\sigma}_\varepsilon^2$ depends on the full data, rather than only $\hat{b}$, this dependence is suppressed for notational convenience; the treatment below does, however, account for the fact that $\hat{\sigma}_\varepsilon^2$ is a function of the full data (see display (123)). Now,

$$
\begin{aligned}
& R\left(b,\ \hat{b}^{ISEB};\ f_K\right) - R\left(b,\ \hat{b}^{NSEB};\ f_K\right) \qquad (119) \\
= \ & \frac{1}{K}\sum_{i=1}^{K}\rho\int\left[\left(\hat{b}_i^{ISEB}\left(\hat{b}_i\right)-b_i\right)^2-\left(\hat{b}_i^{NSEB}\left(\hat{b}\right)-b_i\right)^2\right] f_K\left(\hat{b}-b\right)d\hat{b} \\
= \ & \frac{1}{K}\sum_{i=1}^{K}\rho\int\left[\begin{array}{c}\left(\hat{b}_i-b_i-\sigma_\varepsilon^2\frac{\bar{m}'_K(\hat{b}_i)}{\bar{m}_K(\hat{b}_i)+s_K}\right)^2- \\ \left(\hat{b}_i-b_i-\hat{\sigma}_\varepsilon^2\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)\right)^2\end{array}\right] f_K\left(\hat{b}-b\right)d\hat{b} \\
= \ & \frac{1}{K}\sum_{i=1}^{K}\rho\int\left[\begin{array}{c}-\left(\hat{\sigma}_\varepsilon^2\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\sigma_\varepsilon^2\frac{\bar{m}'_K(\hat{b}_i)}{\bar{m}_K(\hat{b}_i)+s_K}\right)^2+ \\ 2\left(\hat{b}_i-b_i-\sigma_\varepsilon^2\frac{\bar{m}'_K(\hat{b}_i)}{\bar{m}_K(\hat{b}_i)+s_K}\right)\left(\hat{\sigma}_\varepsilon^2\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\sigma_\varepsilon^2\frac{\bar{m}'_K(\hat{b}_i)}{\bar{m}_K(\hat{b}_i)+s_K}\right)\end{array}\right]
\end{aligned}
$$

$$\times f_K\left(\hat{b}-b\right)d\hat{b}$$

$$= -\frac{1}{K}\sum_{i=1}^{K}\rho\int\left(\hat{\sigma}_{\varepsilon}^{2}\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\sigma_{\varepsilon}^{2}\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}$$

$$+\frac{1}{K}\sum_{i=1}^{K}\rho\int 2\left(\hat{b}_i-b_i-\sigma_{\varepsilon}^{2}\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)\times$$

$$\left(\hat{\sigma}_{\varepsilon}^{2}\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\sigma_{\varepsilon}^{2}\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)f_K\left(\hat{b}-b\right)d\hat{b}$$

$$\leq -\frac{1}{K}\sum_{i=1}^{K}\rho\int\left(\hat{\sigma}_{\varepsilon}^{2}\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\sigma_{\varepsilon}^{2}\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}$$

$$+\frac{2}{K}\sum_{i=1}^{K}\rho\left[\begin{array}{c}\left(\int\left(\hat{b}_i-b_i-\sigma_{\varepsilon}^{2}\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)+s_K}\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}\right)^{\frac{1}{2}}\times\\ \left(\int\left(\hat{\sigma}_{\varepsilon}^{2}\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\sigma_{\varepsilon}^{2}\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)+s_K}\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}\right)^{\frac{1}{2}}\end{array}\right]$$

$$\leq -\frac{1}{K}\sum_{i=1}^{K}\rho\int\left(\hat{\sigma}_{\varepsilon}^{2}\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\sigma_{\varepsilon}^{2}\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}$$

$$+2\left\{\frac{1}{K}\sum_{i=1}^{K}\rho\int\left(\hat{b}_i-b_i-\sigma_{\varepsilon}^{2}\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}\right\}^{\frac{1}{2}}\times$$

$$\left\{\frac{1}{K}\sum_{i=1}^{K}\rho\int\left(\hat{\sigma}_{\varepsilon}^{2}\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\sigma_{\varepsilon}^{2}\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 f_K\left(\hat{b}-b\right)d\hat{b}\right\}^{\frac{1}{2}}$$

In the above, the first and second equalities are by definition; the third is a consequence of the fact that $a^2-b^2\ =\ 2a\left(a-b\right)-\left(a-b\right)^2$; the fourth equality simply breaks out the two terms of the integrand; the first inequality is an application of Hölder's inequality, and the second is an application of the Cauchy-Schwartz inequality.

From the final bound given above, we see that if we can show that

$$\frac{1}{K}\sum_{i=1}^{K}\rho\int\left(\hat{\sigma}_{\varepsilon}^{2}\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\sigma_{\varepsilon}^{2}\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 f_K\left(\hat{b}-b\right)d\hat{b} \quad (120)$$

$$\to \quad 0$$

then we will have demonstrated that Term I converges to zero, because

$$\frac{1}{K}\sum_{i=1}^{K}\rho\int\left(\hat{b}_i-b_i-\sigma_{\varepsilon}^{2}\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 f_K\left(\hat{b}-b\right)d\hat{b} \quad (121)$$

$$= \quad R\left(b,\ \hat{b}^{ISEB};\ f_K\right)$$

$$< \quad \infty \text{ uniformly along the } K,\ T \text{ sequence}$$

from the fact that Term II $= R\left(b,\ \hat{b}^{ISEB};\ f_K\right) - R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right)$ converges to zero, and, as noted above, $\sup_{\|b\|\leq M}\limsup_{K\to\infty} R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right)\ <\ \infty$.

Using $(a+b)^2\ \leq\ 2a^2\ +\ 2b^2$, and adding and subtracting $\sigma_{\varepsilon}^{2}\hat{l}\left(\hat{b}_i\ ;\ \hat{b}_{-i}\right)$, we have

$$\frac{1}{K}\sum_{i=1}^{K}\rho\int\left(\hat{\sigma}_{\varepsilon}^{2}\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\sigma_{\varepsilon}^{2}\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2\times \quad (122)$$

$$f_K\left(\hat{b}-b\right)d\hat{b}$$

$$\leq \quad \frac{2}{K}\sum_{i=1}^{K}\rho\int\left(\hat{\sigma}_{\varepsilon}^{2}-\sigma_{\varepsilon}^{2}\right)^2\hat{l}^2\left(\hat{b}_i;\hat{b}_{-i}\right)f_K\left(\hat{b}-b\right)d\hat{b} \quad \text{(Term A)}$$

$$+\frac{2\sigma_{\varepsilon}^{4}}{K}\sum_{i=1}^{K}\rho\int\left(\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2\times \quad \text{(Term B)}$$

$$f_K\left(\hat{b}-b\right)d\hat{b}$$

which separates the problem of nonparametric score estimation from the problem of estimating the residual variance. Now, Term A satisfies

$$
\begin{aligned}
& \frac{2}{K}\sum_{i=1}^{K}\rho\int\left(\hat{\sigma}_{\varepsilon}^{2}-\sigma_{\varepsilon}^{2}\right)^{2}\hat{l}^{2}\left(\hat{b}_{i};\hat{b}_{-i}\right)f_{K}\left(\hat{b}-b\right)d\hat{b} \\
\leq\ & \frac{2\rho q_{K}^{2}}{K}\sum_{i=1}^{K}\int\left(\hat{\sigma}_{\varepsilon}^{2}-\sigma_{\varepsilon}^{2}\right)^{2}f_{K}\left(\hat{b}-b\right)d\hat{b} \\
=\ & 2\rho q_{K}^{2}\int\left(\hat{\sigma}_{\varepsilon}^{2}-\sigma_{\varepsilon}^{2}\right)^{2}f_{K}\left(\hat{b}-b\right)d\hat{b} \\
=\ & 2\rho q_{K}^{2}Var\left[\hat{\sigma}_{\varepsilon}^{2}\right] \\
\leq\ & 2\rho q_{K}^{2}\frac{C}{K} \\
\rightarrow\ & 0
\end{aligned}
\tag{123}
$$

where the first inequality comes from the truncation of our estimator $\hat{l}$ of the score, and the second inequality comes from Theorem 1. The convergence of the final bound to zero is by construction: by display (37), $\frac{q_K^2}{K} \rightarrow 0$. Thus, Term A converges to zero.

Now consider Term B. Define $D_i \equiv \left\{\hat{b} : \left|\hat{b}_i\right| \leq \sqrt{\frac{\hat{\sigma}_{\varepsilon}^2}{128}\log K} \text{ and } \left|\breve{l}\left(\hat{b}_i;\hat{b}_{-i}\right)\right| \leq q_K\right\}$, and let $E_{D_i}\left[(\cdot)\right] \equiv \int_{\hat{b}\in D_i}(\cdot)\, f_K\left(\hat{b}-b\right)d\hat{b}$ (so that the area of integration is restricted, but in a way which may differ for each i). Now,

$$
\begin{aligned}
& \frac{2\sigma_{\varepsilon}^{4}}{K}\sum_{i=1}^{K}\rho\int\left(\hat{l}\left(\hat{b}_{i};\hat{b}_{-i}\right)-\frac{\bar{m}_{K}'\left(\hat{b}_{i}\right)}{\bar{m}_{K}\left(\hat{b}_{i}\right)\ +\ s_{K}}\right)^{2}f_{K}\left(\hat{b}-b\right)d\hat{b} \\
=\ & \frac{2\rho\sigma_{\varepsilon}^{4}}{K}\sum_{i=1}^{K}\int E\left[\left(\hat{l}\left(\hat{b}_{i};\hat{b}_{-i}\right)-\frac{\bar{m}_{K}'\left(\hat{b}_{i}\right)}{\bar{m}_{K}\left(\hat{b}_{i}\right)\ +\ s_{K}}\right)^{2}\mid\hat{b}_{i},b\right]f_{iK}\left(\hat{b}_{i}-b_{i}\right)d\hat{b}_{i}
\end{aligned}
\tag{124}
$$

$$\leq \frac{2\rho\sigma_\varepsilon^4}{K}\sum_{i=1}^{K}\int_{-\infty}^{\infty}\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 f_{iK}\left(\hat{b}_i-b_i\right)\times \tag{Term Bi}$$

$$\left(\Pr\left(\left|\breve{l}\left(\hat{b}_i;\hat{b}_{-i}\right)\right|>q_K\mid\hat{b}_i,b\right)+\Pr\left(\left|\hat{b}_i\right|>\sqrt{\frac{\hat{\sigma}_\varepsilon^2}{128}\log K}\mid\hat{b}_i,b\right)\right)d\hat{b}_i$$

$$+\frac{2\rho\sigma_\varepsilon^4}{K}\sum_{i=1}^{K}E_{D_i}\left[\left(\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2\mid b\right] \tag{Term Bii}$$

where the first equality is by definition, and the inequality follows from the truncation of the estimated score function according to the definition of D_i.

Consider Term Bi first.

$$\frac{2\rho\sigma_\varepsilon^4}{K}\sum_{i=1}^{K}\int_{-\infty}^{\infty}\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 f_{iK}\left(\hat{b}_i-b_i\right)\times \tag{125}$$

$$\left(\begin{array}{c}\Pr\left(\left|\breve{l}\left(\hat{b}_i;\hat{b}_{-i}\right)\right|>q_K\mid\hat{b}_i,b\right)+\\ \Pr\left(\left|\hat{b}_i\right|>\sqrt{\frac{\hat{\sigma}_\varepsilon^2}{128}\log K}\mid\hat{b}_i,b\right)\end{array}\right)d\hat{b}_i$$

$$=\frac{2\rho\sigma_\varepsilon^4}{K}\sum_{i=1}^{K}\int_{-\infty}^{\infty}\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 f_{iK}\left(\hat{b}_i-b_i\right)$$

$$\times\Pr\left(\left|\hat{b}_i\right|>\sqrt{\frac{\hat{\sigma}_\varepsilon^2}{128}\log K}\mid\hat{b}_i,b\right)d\hat{b}_i$$

$$+\frac{2\rho\sigma_\varepsilon^4}{K}\sum_{i=1}^{K}\int_{-\infty}^{\infty}\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2 f_{iK}\left(\hat{b}_i-b_i\right)\times$$

$$\Pr\left(\left|\breve{l}\left(\hat{b}_i;\hat{b}_{-i}\right)\right|>q_K\mid\hat{b}_i,b\right)d\hat{b}_i$$

$$\leq\frac{2\rho\sigma_\varepsilon^4}{Ks_K^2}C\sum_{i=1}^{K}\Pr\left(\left|\hat{b}_i\right|>\sqrt{\frac{\hat{\sigma}_\varepsilon^2}{128}\log K}\mid b\right)$$

$$+\frac{2\rho\sigma_\varepsilon^4}{K}\sum_{i=1}^{K}\int_{|x|>d_K}\left(\frac{\bar{m}_K'(x)}{\bar{m}_K(x)\ +\ s_K}\right)^2 f_{iK}(x-b_i)\,dx$$

$$+\sup_{i,\left|\hat{b}_i\right|\leq d_K}\Pr\left(\left|\breve{l}\left(\hat{b}_i;\hat{b}_{-i}\right)\right|>q_K\mid\hat{b}_i,b\right)$$

$$
\begin{aligned}
& \times \frac{2\rho\sigma_\varepsilon^4}{K} \sum_{i=1}^{K} \int_{-d_K}^{d_K} \left(\frac{\bar{m}_K'(x)}{\bar{m}_K(x) \;+\; s_K} \right)^2 f_{iK}(x - b_i)\, dx \\
\leq \quad & \frac{2\rho\sigma_\varepsilon^4}{K s_K^2} C \sum_{i=1}^{K} \left\{ \begin{array}{c} \Pr\left(\left| \hat{b}_i \right| > \sqrt{\frac{\sigma_\varepsilon^2}{256} \log K} \right) \\ + \Pr\left(\hat{\sigma}_\varepsilon^2 > \frac{1}{2}\sigma_\varepsilon^2 \right) \end{array} \right\} \\
& + \frac{2\rho\sigma_\varepsilon^4}{K} \sum_{i=1}^{K} \int_{|x|>d_K} \left(\frac{\bar{m}_K'(x)}{\bar{m}_K(x) \;+\; s_K} \right)^2 f_{iK}(x - b_i)\, dx \\
& + \sup_{i, \left| \hat{b}_i \right| \leq d_K} \Pr\left(\left| \breve{l}\left(\hat{b}_i; \hat{b}_{-i} \right) \right| > q_K \mid \hat{b}_i, b \right) \\
& \times \frac{2\rho\sigma_\varepsilon^4}{K} \sum_{i=1}^{K} \int_{-d_K}^{d_K} \left(\frac{\bar{m}_K'(x)}{\bar{m}_K(x) \;+\; s_K} \right)^2 f_{iK}(x - b_i)\, dx \\
\leq \quad & C_2 s_K^{-2} \left\{ \frac{1}{\log K} + \frac{1}{K} \right\} \\
& + \frac{2\rho\sigma_\varepsilon^4}{K} \sum_{i=1}^{K} \int_{|x|>d_K} \left(\frac{\bar{m}_K'(x)}{\bar{m}_K(x) \;+\; s_K} \right)^2 f_{iK}(x - b_i)\, dx \\
& + \sup_{i, \left| \hat{b}_i \right| \leq d_K} \Pr\left(\left| \breve{l}\left(\hat{b}_i; \hat{b}_{-i} \right) \right| > q_K \mid \hat{b}_i, b \right) \\
& \times \frac{2\rho\sigma_\varepsilon^4}{K} \sum_{i=1}^{K} \int_{-d_K}^{d_K} \left(\frac{\bar{m}_K'(x)}{\bar{m}_K(x) \;+\; s_K} \right)^2 f_{iK}(x - b_i)\, dx
\end{aligned}
$$

The equality is trivial; the first inequality follows from the boundedness of $(\bar{m}_K'(x))^2$ (by Lemma 3(c) and the boundedness of $\left(m_\phi'(x)\right)^2$), the second inequality follows by $\Pr(A) \leq \Pr(B) + \Pr(C)$ whenever $A \subset (B \cap C)$, and the third inequality follows from Chebyshev's inequality and the variance bound of Theorem 1 (along with $\sigma_\varepsilon^2 > 0$). Of the terms in the final expression, we see immediately that the first term converges to zero by Assumption 7 and the second converges to zero by the uniform integrability of its integrand, which was shown in the course of the proof that Term II converges to zero, and by $d_K \to \infty$. The third term converges to zero by Lemmas 4 and 6(c).

Thus, Term Bi converges to zero.

Finally, turn to Term Bii. Consider the i^{th} term of the average which makes up Term Bii. We define $E_{D_i}^{out}\left[(\cdot)\ \mid b\right]$ to be $\int_{\hat{b}\in D_i\cap\left\{\hat{\sigma}_\varepsilon^2>2\sigma_\varepsilon^2\right\}}(\cdot)\, f_K\left(\hat{b}\mid b\right)d\hat{b}$ and $E_{D_i}^{cond}\left[(\cdot)\ \mid \hat{b}_i,\ b\right]$ to be $\int_{\hat{b}\in D_i}(\cdot)\, f_K\left(\hat{b}_{-i}\mid \hat{b}_i, b\right)d\hat{b}_{-i}$ (so that both the probability measure and the area of integration depend on $\hat{b}_i$). Since $\left(\frac{a}{b}-\frac{c}{d}\right)^2=\left(\frac{a}{b}\left(\frac{d-b}{d}\right)+\frac{a-c}{d}\right)^2\leq 2\left(\frac{a}{b}\right)^2\left(\frac{d-b}{d}\right)^2+2\left(\frac{a-c}{d}\right)^2$ We have:

$$E_{D_i}\left[\left(\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2\ \mid b\right] \tag{126a}$$

$$\begin{aligned}
\leq\ & 2E_{D_i}^{out}\left[\left(\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)\right)^2\ \mid b\right]+2E_{D_i}^{out}\left[\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2\ \mid b\right]\\
&+\int_{-d_K}^{d_K}E_{D_i}^{cond}\left[\left(\hat{l}\left(\hat{b}_i;\hat{b}_{-i}\right)-\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2|\hat{b}_i,b\right]f_{iK}\left(\hat{b}_i-b_i\right)d\hat{b}_i\\
\leq\ & 2C\left(q_K^2+s_K^{-2}\right)\Pr\left(\hat{\sigma}_\varepsilon^2>2\sigma_\varepsilon^2\right)\\
&+2\int_{-d_K}^{d_K}\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2E_{D_i}^{cond}\left[\left(\frac{\bar{m}_K\left(\hat{b}_i\right)-\hat{m}_{iK}\left(\hat{b}_i\right)}{\hat{m}_{iK}\left(\hat{b}_i\right)+s_K}\right)^2|\hat{b}_i,b\right]f_{iK}\left(\hat{b}_i-b_i\right)d\hat{b}_i\\
&+2\int_{-d_K}^{d_K}E_{D_i}^{cond}\left[\left(\frac{\hat{m}_{iK}'\left(\hat{b}_i\right)-\bar{m}_K'\left(\hat{b}_i\right)}{\hat{m}_{iK}\left(\hat{b}_i\right)+s_K}\right)^2|\hat{b}_i,b\right]f_{iK}\left(\hat{b}_i-b_i\right)d\hat{b}_i\\
\leq\ & 2\left(q_K^2+s_K^{-2}\right)\frac{C_2}{K}
\end{aligned}$$

$$+\frac{2}{s_K^2}\int_{-d_K}^{d_K}\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\ +\ s_K}\right)^2\times \tag{Term Biia}$$

$$E_{D_i}\left[\left(\bar{m}_K\left(\hat{b}_i\right)-\hat{m}_{iK}\left(\hat{b}_i\right)\right)^2|\hat{b}_i,b\right]f_{iK}\left(\hat{b}_i-b_i\right)d\hat{b}_i$$

$$+\frac{2}{s_K^2}\int_{-d_K}^{d_K}E_{D_i}\left[\left(\hat{m}_{iK}'\left(\hat{b}_i\right)-\bar{m}_K'\left(\hat{b}_i\right)\right)^2|\hat{b}_i,b\right]f_{iK}\left(\hat{b}_i-b_i\right)d\hat{b}_i. \tag{Term Biib}$$

where the first inequality follows from splitting the area of integration and $2a^2+2b^2 \geq (a+b)^2$ (and from the definition of d_K), the second inequality is due to the truncation of our score estimator and the boundedness of $(\bar{m}'_K(x))^2$ (by Lemma 3(c) and the boundedness of $\left(m'_\phi(x)\right)^2$), and applying the fact that $\left(\frac{a}{b}-\frac{c}{d}\right)^2 = \left(\frac{a}{b}\left(\frac{d-b}{d}\right)+\frac{a-c}{d}\right)^2 \leq 2\left(\frac{a}{b}\right)^2\left(\frac{d-b}{d}\right)^2 + 2\left(\frac{a-c}{d}\right)^2$ to the third term in the previous line. The final inequality is by Theorem 1 (for the first term) and is clear for the other two terms. The first term of the final line converges to zero (uniformly in i) due to Assumption 7, so we consider only the remaining two terms.

Term Biia satisfies

$$
\begin{aligned}
&\frac{2}{s_K^2}\int_{-d_K}^{d_K}\left(\frac{\bar{m}'_K\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\;+\;s_K}\right)^2 \times \\
&\quad E_{D_i}\left[\left(\bar{m}_K\left(\hat{b}_i\right)-\hat{m}_{iK}\left(\hat{b}_i\right)\right)^2|\hat{b}_i,b\right] f_{iK}\left(\hat{b}_i-b_i\right)d\hat{b}_i \\
\leq\;& \frac{2}{s_K^2}C\left(\frac{1}{h_K(K-1)}+h_K^2+K^{-13/48}\log^2 K \;+\; 2K^{-13/96}h_K\log K \;+\; K^{-1/2}\log^2 K\right) \\
&\times\int_{-\infty}^{\infty}\left(\frac{\bar{m}'_K\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)\;+\;s_K}\right)^2 f_{iK}\left(\hat{b}_i-b_i\right)d\hat{b}_i
\end{aligned}
\tag{127}
$$

by Lemma 6(a), since omitting the restriction to $\hat{b}\in D_i$ can only make the expectation larger, and conditioning on any b such that $\|b\|^2 \leq M < \infty$ will not change the results of that lemma, as can be easily verified. Also, Term Biib satisfies

$$
\begin{aligned}
&\frac{2}{s_K^2}\int_{-d_K}^{d_K} E_{D_i}\left[\left(\hat{m}'_{iK}\left(\hat{b}_i\right)-\bar{m}'_K\left(\hat{b}_i\right)\right)^2|\hat{b}_i,\hat{b}\in D_i,b\right] f_{iK}\left(\hat{b}_i-b_i\right)d\hat{b}_i \\
\leq\;& \frac{2}{s_K^2}C\left(\frac{1}{h_K^3(K-1)}+h_K^2+K^{-5/48}\log^2 K \;+\; 2K^{-5/96}h_K\log K \;+\; K^{-1/4}\log^2 K\right)
\end{aligned}
$$

by Lemma 6(b), by the same logic. Thus we have that

$$\begin{aligned}
&\text{Term Bii} && (128)\\
\leq\quad & \frac{4\rho\sigma_\varepsilon^4}{s_K^2}C\left(\begin{array}{c}\left(\begin{array}{c}\frac{1}{h_K(K-1)}+h_K^2+K^{-13/48}\log^2 K\\ +\,2K^{-13/96}h_K\log K\;+\;K^{-1/2}\log^2 K\end{array}\right)\\ \times\int_{-\infty}^{\infty}\left(\frac{\bar{m}_K'\left(\hat{b}_i\right)}{\bar{m}_K\left(\hat{b}_i\right)+s_K}\right)^2 f_{iK}\left(\hat{b}_i-b_i\right)d\hat{b}_i\\ +\left(\begin{array}{c}\frac{1}{h_K^3(K-1)}+h_K^2+K^{-5/48}\log^2 K\\ +\,2K^{-5/96}h_K\log K\;+\;K^{-1/4}\log^2 K\end{array}\right)\end{array}\right)\\
\to\quad & 0
\end{aligned}$$

by display (37) and Lemma 4 (after averaging), and we are finished with the proof that $R\left(b,\ \hat{b}^{NSEB};\ f_K\right)-R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right)\ \to\ 0$.

Now, to complete the proof that $\lim_{K\to\infty}\sup_{\|b\|_2\,\leq\,M}\left|R\left(b,\ \hat{b}^{NSEB};\ \phi_K\right)-R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right)\right|=0$, we simply note that we never used any property of the vector b other than a bound on the empirical variance of the b_i. Over the set $\{b\ :\ \|b\|_2\ \leq\ M\}$ this bound is uniform, so our results hold uniformly over this set. Thus, we simply substitute ϕ_K in for f_K in the result derived above.

Further, we observe that

$$\begin{aligned}
&\lim\sup_{K\to\infty}\sup_{\|b\|_2\,\leq\,M}\left\{R\left(b,\ \hat{b}^{NSEB};\ \phi_K\right)-\inf_{\tilde{b}\in\mathcal{B}}\left\{R\left(b,\ \tilde{b};\ \phi_K\right)\right\}\right\} && (129)\\
=\quad & \lim\sup_{K\to\infty}\sup_{\|b\|_2\,\leq\,M}\left\{\begin{array}{c}R\left(b,\ \hat{b}^{NSEB};\ \phi_K\right)-R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right)+\\ R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right)-\inf_{\tilde{b}\in\mathcal{B}}\left\{R\left(b,\ \tilde{b};\ \phi_K\right)\right\}\end{array}\right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \limsup_{K\to\infty} \sup_{\|b\|_2 \leq M} \left\{ R\left(b,\ \hat{b}^{NSEB};\ \phi_K\right) - R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right) \right\} \\
&\quad + \limsup_{K\to\infty} \sup_{\|b\|_2 \leq M} \left\{ R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right) - \inf_{\tilde{b}\in\mathcal{B}} \left\{ R\left(b,\ \tilde{b};\ \phi_K\right) \right\} \right\} \\
&\leq \limsup_{K\to\infty} \sup_{\|b\|_2 \leq M} \left| R\left(b,\ \hat{b}^{NSEB};\ \phi_K\right) - R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right) \right| \\
&\quad + \limsup_{K\to\infty} \sup_{\|b\|_2 \leq M} \left\{ R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right) - \inf_{\tilde{b}\in\mathcal{B}} \left\{ R\left(b,\ \tilde{b};\ \phi_K\right) \right\} \right\} \\
&\leq \limsup_{K\to\infty} \sup_{\|b\|_2 \leq M} \left\{ R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right) - \inf_{\tilde{b}\in\mathcal{B}} \left\{ R\left(b,\ \tilde{b};\ \phi_K\right) \right\} \right\} \\
&\leq 0
\end{aligned}
$$

where the second line follows from adding and subtracting $R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right)$, the third line follows from the fact that $\limsup_{K\to\infty} \sup_x \{h_K(x) + g_K(x)\} \leq \limsup_{K\to\infty} \sup_x \{h_K(x)\} + \limsup_{K\to\infty} \sup_x \{g_K(x)\}$, the fourth line follows from noting that $|x| \geq x$, the fifth line follows from $\lim_{K\to\infty} \sup_{\|b\|_2 \leq M} \left| R\left(b,\ \hat{b}^{NSEB};\ \phi_K\right) - R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right) \right| = 0$, and the final line follows from $R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right) \leq \inf_{\tilde{b}\in\mathcal{B}} \left\{ R\left(b,\ \tilde{b};\ \phi_K\right) \right\} \forall K$ and $\forall b$, as shown in part (i) of this proof.

To bound the limsup of the sup of the absolute value, we must also consider the liminf of the inf. To do so, observe that $\hat{b}^{NSEB}$ is an equivariant estimator, so that $R\left(b,\ \hat{b}^{NSEB};\ \phi_K\right) \geq \inf_{\tilde{b}\in\mathcal{B}} \left\{ R\left(b,\ \tilde{b};\ \phi_K\right) \right\} \forall K$ and $\forall b$. Note that this domination of $\hat{b}^{NSEB}$ may be by a different equivariant estimator at each b. This immediately yields

$$
-\liminf_{K\to\infty} \inf_{\|b\|_2 \leq M} \left\{ R\left(b,\ \hat{b}^{NSEB};\ \phi_K\right) - \inf_{\tilde{b}\in\mathcal{B}} \left\{ R\left(b,\ \tilde{b};\ \phi_K\right) \right\} \right\} \leq 0. \tag{130}
$$

which, together with our result above on the $\limsup_{K\to\infty}\sup_{\|b\|_2\,\le\,M}$, implies that

$$\begin{aligned}
&\limsup_{K\to\infty}\sup_{\|b\|_2\,\le\,M}\left|R\left(b,\ \hat{b}^{NSEB};\ \phi_K\right)-\inf_{\tilde{b}\in\mathcal{B}}\left\{R\left(b,\ \tilde{b};\ \phi_K\right)\right\}\right| \qquad (131)\\
\le\ & \max\left\{\begin{array}{c} -\liminf_{K\to\infty}\inf_{\|b\|_2\,\le\,M}\left\{R\left(b,\ \hat{b}^{NSEB};\ \phi_K\right)-\inf_{\tilde{b}\in\mathcal{B}}\left\{R\left(b,\ \tilde{b};\ \phi_K\right)\right\}\right\}, \\ \limsup_{K\to\infty}\sup_{\|b\|_2\,\le\,M}\left\{R\left(b,\ \hat{b}^{NSEB};\ \phi_K\right)-\inf_{\tilde{b}\in\mathcal{B}}\left\{R\left(b,\ \tilde{b};\ \phi_K\right)\right\}\right\}\end{array}\right\}\\
\le\ & \max\{0,0\}\\
\le\ & 0
\end{aligned}$$

where the first inequality follows from the fact that $|x|=\max\{x,-x\}$ and the fact that we can interchange the maximum operation with the sup and limsup operations. The second inequality is by the bounds demonstrated above, and the final inequality is trivial. Now, since the limsup and sup operations are being performed on a nonnegative sequence, the result must be nonnegative. But by nonnegativity of the sequence again, the liminf of the sup must be nonnegative. Thus we obtain the desired conclusion:

$$\limsup_{K\to\infty}\sup_{\|b\|_2\,\le\,M}\left|R\left(b,\ \hat{b}^{NSEB};\ \phi_K\right)-\inf_{\tilde{b}\in\mathcal{B}}\left\{R\left(b,\ \tilde{b};\ \phi_K\right)\right\}\right|=0 \qquad (132)$$

implies

$$\lim_{K\to\infty}\sup_{\|b\|_2\,\le\,M}\left|R\left(b,\ \hat{b}^{NSEB};\ \phi_K\right)-\inf_{\tilde{b}\in\mathcal{B}}\left\{R\left(b,\ \tilde{b};\ \phi_K\right)\right\}\right|=0.$$

(iii) The proof of part (iii) follows from:

$$\lim_{K\to\infty}\left\{\sup_{\|b\|_2\le M}\left\{\sup_{f_K}\left|R\left(b,\ \hat{b}^{NSEB};\ f_K\right)-R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right)\right|\right\}\right\}\ =\ 0 \qquad (133)$$

which is due to a procedure identical to that used in part (ii) Theorem 3, noting simply that the results there are uniform in the prior over a set of priors with the same variance bound. This uniformity is due to the fact that the only feature of the prior that is used is the variance of the prior or, in this frequentist case, the empirical variance of the b_i. Then we apply reasoning identical to that of the final portion of part (ii) above to conclude that

$$\lim_{K\to\infty} \sup_{\|b\|_2 \leq M} \sup_{f_K} \left| R\left(b,\ \hat{b}^{NSEB};\ f_K\right) - \inf_{\tilde{b}\in\mathcal{B}} \left\{ R\left(b,\ \tilde{b};\ \phi_K\right)\right\} \right| = 0 \tag{134}$$

as desired.

Q.E.D.

Proof of Theorem 6: As in the proof of Theorem 5, we note that, if we restrict $\|b\|_2 \leq M$ uniformly along the K sequence, we have that Lemmas 1 through 6 hold uniformly along the K sequence. But if Lemma 4 holds uniformly along the K sequence, then we get that, if $\tilde{G}_K \Longrightarrow G$, $\forall b$ s.t. $\|b\|_2 \leq M$ uniformly along the K sequence,

$$\lim_{K\to\infty} \left| R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right) - r_G\left(\hat{b}^{NB}, \phi\right)\right| = 0 \tag{135}$$

because

$$\begin{aligned} & R\left(b,\ \hat{b}^{NB}_{\tilde{G}_K};\ \phi_K\right) && (136)\\ = \quad & \rho \int_{-\infty}^{\infty} \left(\hat{b}^{NB}_{\tilde{G}_K,1}\left(\hat{b}_1\right) - b_1\right)^2 d\tilde{G}_K\left(b_1\right) \end{aligned}$$

$$= \quad \rho \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\hat{b}_1 - \sigma_\varepsilon^2 \frac{m'_\phi\left(\hat{b}_1\right)}{m_\phi\left(\hat{b}_1\right)} - b_1 \right)^2 \phi\left(\hat{b}_1 - b_1\right) d\tilde{G}_K\left(b_1\right)$$

so if there is an uniformly integrable (with respect to $d\tilde{G}_K\left(b_1\right) \forall K$ and $dG\left(b_1\right)$) dominating function of b_1 for $\int_{-\infty}^{\infty} \left(\hat{b}_1 - \sigma_\varepsilon^2 \frac{m'_\phi\left(\hat{b}_1\right)}{m_\phi\left(\hat{b}_1\right)} - b_1 \right)^2 \phi\left(\hat{b}_1 - b_1\right) d\hat{b}_1$, then we have the desired result by weak convergence (see, e. g., Billingsley (1968)). Now, $\left(\hat{b}_1 - \sigma_\varepsilon^2 \frac{m'_\phi\left(\hat{b}_1\right)}{m_\phi\left(\hat{b}_1\right)} - b_1 \right)^2 \leq 2\left(\hat{b}_1 - b_1\right)^2 + 2\sigma_\varepsilon^4 \left(\frac{m'_\phi\left(\hat{b}_1\right)}{m_\phi\left(\hat{b}_1\right)}\right)^2$, so that $\int_{-\infty}^{\infty} \left[2\left(\hat{b}_1 - b_1\right)^2 + 2\sigma_\varepsilon^4 \left(\frac{m'_\phi\left(\hat{b}_1\right)}{m_\phi\left(\hat{b}_1\right)}\right)^2 \right] \phi\left(\hat{b}_1 - b_1\right) d\hat{b}_1$ is a dominating function. This function is uniformly integrable w. r. t. $d\tilde{G}_K\left(b_1\right) \forall K$ and $dG\left(b_1\right)$ by, for the first term, the fact that $\sigma_\varepsilon^2 < \infty$, and, for the second term, Lemma 4. Note that to apply Lemma 4, we must recognize that the second term doesn't depend on b_1, so that we may integrate with respect to $d\tilde{G}_K\left(b_1\right)$ or $dG\left(b_1\right)$ first, obtaining $\int_{-\infty}^{\infty} \frac{\left(m'_\phi\left(\hat{b}_1\right)\right)^2}{m_\phi\left(\hat{b}_1\right)} d\hat{b}_1$, which is uniformly bounded along the K sequence by Lemma 4 and the observation at the beginning of this proof, i. e., that Lemma 4 holds uniformly along the K sequence because $\|b\|_2 \leq M$ uniformly along the K sequence.

Q. E. D.

1 Appendix B: Berry-Esseen Theorems for Densities and Their Derivatives

This appendix provides Berry-Esseen-type theorems for densities and derivatives of densities for univariate and bivariate random variables. These theorems are referred to below as local limit results.

The presentation proceeds in two steps. First, the local limit results are proven assuming that a Berry-Esseen theorem (for c.d.f.'s) and a smoothness condition hold. Because Berry-Esseen theorems hold under a variety of primitive conditions, this provides general conditions under which the local limit results hold. Second, it is shown that a (multivariate) Berry-Esseen theorem does in fact hold for averages of strongly mixing random variables satisfying certain moment and mixing-rate conditions. This theorem is an adaptation of Tikhomirov's (1980) univariate result.

All theorems, lemmas, and equation numbers herein refer to this appendix only; this appendix is self-contained.

2 Local Limit Result

Let $\eta_1, \eta_2, \ldots$ be a sequence of m-dimensional random variables with mean zero and finite second moments. Without loss of generality, let $\lim_{n\to\infty} E\left[\left(\sqrt{\frac{1}{n}}\sum_{i=1}^n \eta_i\right)\left(\sqrt{\frac{1}{n}}\sum_{i=1}^n \eta_i\right)'\right] = I_m$. Let J_n denote the distribution function of $\sqrt{\frac{1}{n}}\sum_{i=1}^n \eta_i$ and let Φ denote the m-

variate standard normal distribution function. The local limit results will be proven under the following two conditions.

Condition A: The random variables $\eta_1, \eta_2, \ldots$ have conditional characteristic functions $\psi_1, \psi_2, \ldots$ (so that ψ_i is the characteristic function of the distribution of η_i conditional on $\eta_j,\ 1 \leq j < i$) with the property that

$$\begin{aligned} \exists\, \alpha &> 0,\ C_0 < \infty,\ M_0 < \infty\ \ s.t. \\ \sup_i |\psi_i(t)| &\leq M_0 |t|^{-\alpha}\quad \forall\, |t| \geq C_0. \end{aligned} \tag{1}$$

Condition A is weaker than requiring, for instance, that the conditional densities of the η_i be uniformly bounded, or even that any of them be bounded, though it does rule out discreteness.

Condition B: A Berry-Esseen theorem holds for $\eta_1, \eta_2, \ldots$, that is,

$$\exists\, \beta > 0,\ \mu < \infty,\ M_1 < \infty\ \ s.t.\ \sup_{z \in R^m} |J_n(z) - \Phi(z)| \leq M_1 n^{-\beta} \log^{\mu}(n) \tag{2}$$

Typical Berry-Esseen theorems specify a particular constant M_1 and have $\beta = \frac{1}{2}$ and $\mu = 0$ (c. f. Feller (1971), F. Götze (1991), Hall and Heyde (1980)). However, the local limit results here do not rely on these specific values, and so are proven for the more general statement 2.

Lemma 1 (Cramér 1937) *Suppose $t, \zeta, \chi \in \Re^m$. If $\psi(t)$ is a characteristic function such that $|\psi(t)| \leq \nu < 1$ for all $|t| \geq M$, then we have for $|t| < M$*

$$|\psi(t)| \leq 1 - (1-\nu)^2 \frac{|t|^2}{8M^2} \tag{3}$$

Proof of Lemma 1: A terse proof for $m = 1$ is to be found on page 26 of Cramér (1937). However, it is given in an expanded form here for the reader's convenience.

Recall first that for scalars A and B, $\cos A \leq \frac{3}{4} + \frac{1}{4}\cos(2A)$ (since $\frac{1}{4}\cos(2A) - \cos(A) + \frac{3}{4} = \frac{1}{2}\cos^2(A) - \cos(A) + \frac{1}{2} = \frac{1}{2}(\cos(A) - 1)^2 \geq 0$) and $\sin(A - B) = \sin(A)\cos(B) - \sin(B)\cos(A)$.

$$\begin{aligned}
|\psi(t)|^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{it'(\zeta-\chi)} dJ(\zeta)\, dJ(\chi) \qquad (4) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i\left[\sin(t'\zeta)\cos(t'\chi) - \sin(t'\chi)\cos(t'\zeta)\right] + \\
&\quad \left[\cos(t'(\zeta - \chi))\right] dJ(\zeta)\, dJ(\chi) \\
&= i\left[\begin{array}{c} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin(t'\zeta)\cos(t'\chi)\, dJ(\zeta)\, dJ(\chi) - \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin(t'\chi)\cos(t'\zeta)\, dJ(\zeta)\, dJ(\chi) \end{array}\right] \\
&\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos(t'(\zeta-\chi))\, dJ(\zeta)\, dJ(\chi) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos(t'(\zeta-\chi))\, dJ(\zeta)\, dJ(\chi) \\
&\leq \frac{3}{4} + \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos(2t'(\zeta-\chi))\, dJ(\zeta)\, dJ(\chi) \\
&= \frac{3}{4} + \frac{1}{4}\left\{\begin{array}{c} i\left[\begin{array}{c} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin(2t'\zeta)\cos(2t'\chi)\, dJ(\zeta)\, dJ(\chi) - \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin(2t'\chi)\cos(2t'\zeta)\, dJ(\zeta)\, dJ(\chi) \end{array}\right] + \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos(2t'(\zeta-\chi))\, dJ'(\zeta)\, dJ(\chi) \end{array}\right\} \\
&= \frac{3}{4} + \frac{1}{4}|\psi(2t)|^2
\end{aligned}$$

where the first equality holds because the square of a complex number's modulus equals its product (in complex multiplication) with its complex conjugate, the second equality holds by the trigonometric identity recalled above, the third is merely a rearrangement, and the fourth follows from the fact that the two integrals in the imaginary part are equal, so their difference is zero (we may simply relabel the variables). The inequality comes from the trigonometric inequality noted above, and the last two equalities are simply the first four "in reverse." Thus, we have that for $|t| \in \left[\frac{M}{2}, M\right)$

$$|\psi(t)|^2 \leq 1 - \frac{1}{4}(1-\nu)^2 \tag{5}$$

and we may repeat this argument to show that for $|t| \in \left[\frac{M}{2^q}, \frac{M}{2^{q-1}}\right)$ (for any integer $q \geq 1$)

$$|\psi(t)|^2 \leq 1 - \left(\frac{1}{4}\right)^q (1-\nu)^2 < 1 - (1-\nu)^2 \frac{|t|^2}{4M^2} \tag{6}$$

so that, in the same region,

$$|\psi(t)| \leq 1 - (1-\nu)^2 \frac{|t|^2}{8M^2} \tag{7}$$

as can be seen by squaring the righthand side of the inequality immediately above and comparing it to the rightmost quantity of the inequality in the previous display. Now, q is arbitrary, so the desired conclusion must hold for $|t| \in (0, M)$. But we know that $\psi(0) = 1$, so we are finished. Q.E.D.

Theorem 2 (Univariate Local Limit Theorem) *Suppose that Conditions A and B hold, and let $m = 1$. Then $\forall\ d \in N$, $\exists\ B(d) < \infty$ and $n_0(d) < \infty$ such that,*

$\forall n > n_0(d)$,

$$\sup_{z\in R}\left|j_n^{(d)}(z) - \phi^{(d)}(z)\right| \;\leq\; B(d)\, n^{-\beta\,/\,2^{d+1}}\,[\log(n)]^{\mu\,/\,2^{d+1}}$$

where $j_n^{(d)}$ *is the* d^{th} *derivative,* $d = 0, 1, 2, \ldots$*, of the density of* $\frac{\eta_1+\eta_2+\cdots+\eta_n}{\sqrt{n}}$ *(this density will be shown to exist for sufficiently large* n *as part of the proof), and* $\phi^{(d)}$ *is the* d^{th} *derivative of the standard normal density.*

Proof of Theorem 2: We will proceed by induction on the order d of the derivative to be taken. First we will prove that the result holds for $d = 0$. Note that if the first derivative of j_n is uniformly bounded for all n greater than some given n_0, then j_n will clearly satisfy a Lipschitz condition uniformly beyond n_0, that is, we will have $\sup_{z,w\in\Re}\sup_{n\geq n_0}|j_n(z) - j_n(w)| \;\leq\; B_0|z-w|$ for some $B_0 < \infty$. Clearly, ϕ satisfies such a smoothness condition (all its derivatives exist and are bounded). Let $B_1 < \infty$ be the Lipschitz constant for ϕ. Now, if the Lipschitz constant holds for j_n,

$$\begin{aligned} sup_{z\in R}|j_n(z) - \phi(z)| \;&\leq\; \sup_{z\in R}\left|j_n(z) - \frac{\int_z^{z+r} j_n(w)\,dw}{r}\right| + \\ &\quad \sup_{z\in R}\frac{1}{r}\left|\int_z^{z+r} j_n(w)\,dw - \int_z^{z+r}\phi(w)\,dw\right| + \\ &\quad \sup_{z\in R}\left|\phi(z) - \frac{\int_z^{z+r}\phi(w)\,dw}{r}\right| \\ &\leq\; \sup_{z\in R}|j_n(z) - j_n(c_f)| + \\ &\quad \sup_{z\in R}\frac{1}{r}\left|\int_z^{z+r} j_n(w)\,dw - \int_z^{z+r}\phi(w)\,dw\right| + \\ &\quad \sup_{z\in R}|\phi(z) - \phi(c_\phi)| \end{aligned} \tag{8}$$

$$
\begin{aligned}
&\leq \quad B_0 \left| z - c_f \right| \ + \\
&\quad\ \frac{1}{r} \sup_{z \in \Re} \left| J_n \left(z + r \right) - J_n \left(z \right) - \Phi \left(z + r \right) + \Phi \left(z \right) \right| \ + \\
&\quad\ B_1 \left| z - c_\phi \right| \\
&\leq \quad 2 \left(B_0 + B_1 \right) r \ + \ \frac{2}{r} \sup_{z \in \Re} \left| J_n \left(z \right) - \Phi \left(z \right) \right| \\
&\leq \quad \left(B_0 + B_1 \right) r \ + \ \frac{2 M_1}{r n^\beta} \log^\mu \left(n \right) \\
&\leq \quad 2 \left[2^{1/2} \left(B_0 + B_1 \right)^{1/2} M_1^{1/2} \right] n^{-\beta/2} \log^{\mu/2} \left(n \right)
\end{aligned}
$$

where $c_f, c_\phi \in [z, z + r]$ by the mean value theorem. The second term in the fifth inequality uses Condition B, and the final inequality follows by setting $r = \sqrt{\frac{2M_1}{B_0 + B_1}} n^{-\beta/2} \log^{\mu/2}(n)$, which is certainly permitted, since r is arbitrary.

Thus the lemma is proven with $B(0) = 2\left[2^{1/2}(B_0 + B_1)^{1/2} M_1^{1/2}\right]$ if we can show that the Lipschitz condition holds.

This is where Cramér's (1937) lemma is useful: consider general $d \in N$. By the Fourier inversion theorem,

$$
\begin{aligned}
\left| j_n^{(d)} \left(z \right) \right| \quad &\leq \quad \int_{-\infty}^{\infty} \left| t \right|^d \left\{ \prod_{i=1}^n \left| \psi_i \left(\frac{t}{\sqrt{n}} \right) \right| \right\} dt \qquad (9) \\
&\leq \quad \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} \left| t \right|^d \left\{ \prod_{i=1}^n \left| \psi_i \left(\frac{t}{\sqrt{n}} \right) \right| \right\} dt \ + \ \int_{\delta\sqrt{n}}^{\infty} \left| t \right|^d \left\{ \prod_{i=1}^n \left| \psi_i \left(\frac{t}{\sqrt{n}} \right) \right| \right\} dt \\
&\quad + \int_{-\infty}^{-\delta\sqrt{n}} \left| t \right|^d \left\{ \prod_{i=1}^n \left| \psi_i \left(\frac{t}{\sqrt{n}} \right) \right| \right\} dt.
\end{aligned}
$$

Now choose $\delta = \max\left\{(2M_0)^{1/\alpha}, C_0\right\}$, where the parameters refer to Condition A. Then, by Condition A, $\forall t > \delta$, $\sup_i |\psi_i(t)| \leq \frac{1}{2} \equiv \nu$. We can now apply Cramér's lemma to obtain $\sup_i |\psi_i(t)| \leq 1 - (1-\nu)^2 \frac{|t|^2}{8\delta^2} \; \forall \; |t| < \delta$. Thus, we have,

$$\begin{aligned}
\sup_{n \geq n_0} \left| j_n^{(d)}(z) \right| \quad \leq \quad & \sup_{n \geq n_0} \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} |t|^d \left(1 - (1-\nu)^2 \frac{t^2 / n}{8\delta^2} \right)^n dt \; + \\
& \sup_{n \geq n_0} \nu^{n-m} \int_{\delta\sqrt{n}}^{\infty} |t|^d \left(\frac{M_0}{|t / \sqrt{n}|^\alpha} \right)^m dt \; + \\
& \sup_{n \geq n_0} \nu^{n-m} \int_{-\infty}^{-\delta\sqrt{n}} |t|^d \left(\frac{M_0}{|t / \sqrt{n}|^\alpha} \right)^m dt \\
\leq \quad & \sup_{n \geq n_0} \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} |t|^d \exp\left(-Ct^2 \right) dt \; + \\
& 2 \sup_{n \geq n_0} \nu^{n-m} M_0^m n^{\alpha m / 2} \int_{-\infty}^{-\delta\sqrt{n}} |t|^{d - \alpha m} dt \\
\leq \quad & \sup_{n \geq n_0} \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} |t| \exp\left(-Ct^2 \right) dt \; + \\
& 2 \sup_{n \geq n_0} M_0^{\lceil (d+2) / \alpha \rceil} \nu^{n - \lceil (d+2) / \alpha \rceil} n^{(d+2) / 2} \int_{\delta}^{\infty} |t|^{-2} dt \\
\leq \quad & M_2(d) < \infty
\end{aligned} \tag{10}$$

where the second inequality follows by setting $C = \frac{(1-\nu)^2}{8\delta^2}$ and observing that $\exp(-Ct^2) = \sup_n \left(1 - \frac{Ct^2}{n}\right)^n$, the third inequality follows by choosing $m = \left\lceil \frac{d+2}{\alpha} \right\rceil$, where $\lceil \cdot \rceil$ is the least greater (or equal) integer function. The desired Lipschitz condition is now verified. We have only to set $d = 0$ to prove the existence and boundedness of the density itself for sufficiently large n in an identical fashion.

We have now proven the $d = 0$ case. To prove higher-d cases, we now simply substitute $j_n^{(d)}$ for j_n and $\phi^{(d)}$ for ϕ in (8) and, following the steps in (8), we obtain

$$\begin{aligned}
& \sup_{z \in \Re} \left| j_n^{(d)}(z) - \phi^{(d)}(z) \right| \\
\leq \quad & (B_0(d) + B_1(d)) \, r \; + \; \frac{2}{r} \sup_{z \in \Re} \left| j_n^{(d-1)}(z) - \phi^{(d-1)}(z) \right|
\end{aligned} \tag{11}$$

where the Lipschitz condition holds as a consequence of (10). It is readily verified that the bound in the statement of the theorem satisfies the recursion (11), and since we have shown the bound for $d = 0$, we are finished. Q. E. D.

Theorem 3 (Bivariate Local Limit Theorem) *Suppose that Conditions A and B hold with* $m = 2$*. Let* $j_n^{(0)}(z,w)$ *denote the density of* $\sqrt{\frac{1}{n}}\sum_{i=1}^{n}\eta_i$*, and let* $\phi(z,w)$ *denote the bivariate standard normal density. Then* $\forall\ d \in N$, $\exists\ B(d) < \infty$ *such that*

$$\sup_{z\in R, w\in R}\left|j_n^{(d)}(z,w) - \phi^{(d)}(z,w)\right| \leq B(d)\, n^{-\beta\,/\,\left(3\times 2^d\right)}\left[\log(n)\right]^{\mu\,/\,\left(3\times 2^d\right)}$$

where $j_n^{(d)}$ *is the* d^{th} *derivative,* $d = 0, 1, 2, \ldots,$ *of* $j_n^{(0)}(z,w)$ *with respect to* z *(this density will be shown to exist for sufficiently large* n *as part of the proof) and* $\phi^{(d)}$ *is the* d^{th} *derivative of* $\phi(z,w)$ *with respect to* z*.*

Proof of Theorem 3: We will proceed by induction on the order d of the derivative to be taken. First we will prove that the result holds for $d = 0$. Note that if the first derivative of j_n with respect to z is uniformly bounded for all n greater than some given n_0, then j_n will clearly satisfy a Lipschitz condition in z uniformly beyond n_0, that is, we will have $\sup_{z,w,u,v\in\Re}\sup_{n\geq n_0}\left|j_n(z,w) - j_n(u,v)\right| \leq B_0\left\|\begin{pmatrix} z \\ w \end{pmatrix} - \begin{pmatrix} u \\ v \end{pmatrix}\right\|$ and for some $B_0 < \infty$. Now, if such a uniform Lipschitz condition holds, we can easily show that, since ϕ clearly satisfies such a smoothness condition (all its derivatives exist and are bounded),

$$
\begin{aligned}
& sup_{z,w\in R}\left|j_n(z,w)-\phi(z,w)\right| && (12)\\
\leq\; & \sup_{z,w\in\Re}\left|j_n(z,w)-\frac{\int_z^{z+r}\int_w^{w+r}j_n(u,v)\,dudv}{r^2}\right| + \\
& \sup_{z,w\in\Re}\frac{1}{r^2}\left|\int_z^{z+r}\int_w^{w+r}j_n(u,v)\,dudv-\int_z^{z+r}\int_w^{w+r}\phi(u,v)\,dudv\right| + \\
& \sup_{z,w\in\Re}\left|\phi(z,w)-\frac{\int_z^{z+r}\int_w^{w+r}\phi(u,v)\,dudv}{r^2}\right| \\
\leq\; & \sup_{z,w\in\Re}\left|j_n(z,w)-j_n(c_f,d_f)\right| + \\
& \sup_{z,w\in\Re}\frac{1}{r^2}\left|\int_z^{z+r}\int_w^{w+r}j_n(u,v)\,dudv-\int_z^{z+r}\int_w^{w+r}\phi(u,v)\,dudv\right| + \\
& \sup_{z,w\in\Re}\left|\phi(z,w)-\phi(c_\phi,d_\phi)\right| \\
\leq\; & B_0\left|\begin{pmatrix} z\\ w\end{pmatrix}-\begin{pmatrix} c_f\\ d_f\end{pmatrix}\right| + \frac{2}{r^2}\sup_{z,w\in\Re}\left|J_n(z,w)-\Phi(z,w)\right| + \\
& B_1\left|\begin{pmatrix} z\\ w\end{pmatrix}-\begin{pmatrix} c_\phi\\ d_\phi\end{pmatrix}\right| \\
\leq\; & (B_0+B_1)\,r + \frac{2M_1}{r^2n^\beta}\log^\mu(n) \\
\leq\; & \left[\left(4^{1/3}+4^{1/6}\right)(B_0+B_1)^{2/3}M_1^{1/3}\right]n^{-\beta/3}\log^{\mu/3}(n)
\end{aligned}
$$

where $c_f, c_\phi \in [z, z+r]$ and $d_f, d_\phi \in [w, w+r]$ by the mean value theorem. The second term in the fourth inequality uses Condition B, and the final inequality follows by setting $r = \sqrt[3]{\frac{4M_1}{B_0+B_1}}n^{-\beta/3}\log^{\mu/3}(n)$, which is certainly permitted, since r is arbitrary.

Thus the lemma is proven with $B(0) = \left[\left(4^{1/3} + 4^{1/6}\right)(B_0 + B_1)^{2/3} M_1^{1/3}\right]$ if we can show that the Lipschitz condition holds.

This is where Cramér's (1937) lemma is useful: consider general $d \in N$. By the Fourier inversion theorem,

$$\begin{aligned} \left|j_n^{(d)}(z,w)\right| &\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |t_1|^d \left\{\prod_{i=1}^{n}\left|\psi_i\left(\frac{t}{\sqrt{n}}\right)\right|\right\} dt_1 dt_2 \\ &\leq \int_{B_{\delta\sqrt{n}}} |t_1|^d \left\{\prod_{i=1}^{n}\left|\psi_i\left(\frac{t}{\sqrt{n}}\right)\right|\right\} dt + \int_{\Re^2 - B_{\delta\sqrt{n}}} |t|^d \left\{\prod_{i=1}^{n}\left|\psi_i\left(\frac{t}{\sqrt{n}}\right)\right|\right\} dt \end{aligned} \tag{13}$$

where $B_{\delta\sqrt{n}} \equiv \{t : |t| \leq \delta\sqrt{n}\}$, and, by choosing $\delta = \max\left\{(2M_0)^{1/\alpha}, C_0\right\}$, we will have $\nu = \frac{1}{2}$ such that $\sup_i |\psi_i(t)| \leq \nu\ \forall\ |t| \geq \delta$. But then Cramér's lemma proves that we have $\sup_i |\psi_i(t)| \leq 1 - (1-\nu)^2 \frac{|t|^2}{8\delta^2}\ \forall\ |t| < \delta$. Thus, we have

$$\begin{aligned} & \sup_{n\geq n_0} \left|j_n^{(d)}(z,w)\right| \\ \leq\ & \sup_{n\geq n_0} \int_{B_{\delta\sqrt{n}}} |t_1|^d \left(1 - (1-\nu)^2 \frac{|t|^2 / n}{8\delta^2}\right)^n dt\ + \\ & \sup_{n\geq n_0} \nu^{n-m} \int_{\Re^2 - B_{\delta\sqrt{n}}} |t_1|^d \left(\frac{M_0}{|t / \sqrt{n}|^{\alpha}}\right)^m dt \\ \leq\ & \sup_{n\geq n_0} \int_{B_{\delta\sqrt{n}}} |t_1|^d \exp\left(-C|t|^2\right) dt\ + \\ & \sup_{n\geq n_0} M_0^m \nu^{n-m} n^{\alpha m / 2} \int_{\Re^2 - B_{\delta\sqrt{n}}} |t_1|^d |t|^{-\alpha m} dt \\ \leq\ & \sup_{n\geq n_0} \int_{B_{\delta\sqrt{n}}} |t_1|^d \exp\left(-C|t|^2\right) dt\ + \\ & \sup_{n\geq n_0} \dot{M}_0^{\lceil (d+4) / \alpha \rceil} \nu^{n-\lceil (d+4) / \alpha \rceil} n^{(d+4) / 2} \int_{\Re^2 - B_{\delta\sqrt{n}}} |t_1|^d |t|^{-(d+4)} dt \\ \leq\ & \sup_{n\geq n_0} \int_{\Re^2} |t_1|^d \exp\left(-C|t|^2\right) dt\ + \\ & \sup_{n\geq n_0} M_0^{\lceil (d+4) / \alpha \rceil} \nu^{n-\lceil (d+4) / \alpha \rceil} n^{(d+4) / 2} \int_{\Re^2 - B_{\delta\sqrt{n}}} |t|^{-4} dt \end{aligned} \tag{14}$$

$$
\begin{aligned}
&\leq \sup_{n\geq n_0} \int_{\Re^2} |t_1| \exp\left(-C|t|^2\right) dt \quad + \\
&\quad \sup_{n\geq n_0} M_0^{\lceil (d+4)\ /\ \alpha \rceil} \nu^{n-\lceil (d+4)\ /\ \alpha\rceil} n^{(d+4)\ /\ 2} \int_{\Re^2 - B_{\delta\sqrt{n}}} r^{-3} dr d\theta \\
&\leq M_2 + \sup_{n\geq n_0} M_0^{\lceil (d+4)\ /\ \alpha \rceil} \nu^{n-\lceil (d+4)\ /\ \alpha\rceil} n^{(d+2)\ /\ 2} 2\pi \int_{\delta}^{\infty} r^{-3} dr \\
&\leq M_3 < \infty
\end{aligned}
$$

where the second inequality follows by setting $C = \frac{(1-\nu)^2}{8\delta^2}$ and noting that $\exp\left(-C|t|^2\right) = \sup_n \left(1 - \frac{(1-\nu)^2|t|^2}{8\delta^2 n}\right)^n$, and the third inequality follows by setting $m = \left\lceil \frac{d+4}{\alpha} \right\rceil$, where $\lceil\cdot\rceil$ is the least integer greater function. The fourth inequality is due to the fact that $|t_1|^2 = ((1\ 0\ \cdots\ 0)\,t)^2 \leq |t|^2$ by the Cauchy-Schwartz inequality, so that $|t_1| \leq |t|$, and the fifth inequality follows upon a change of variables into polar coordinates. So the d^{th} derivative of j_n w. r. t. z is bounded. An identical proof in which $|t_2|$ is substituted for $|t_1|$ shows that the d^{th} derivative of j_n w. r. t. w is bounded. Together, these two bounds produce the Lipschitz condition. We have only to set $d = 0$ in the proof above to prove the existence and boundedness of the density itself for sufficiently large n.

We have now proven the $d = 0$ case. To prove higher-d cases, we now simply substitute $j_n^{(d)}$ for j_n and $\phi^{(d)}$ for ϕ in (12), and, only slightly modifying the steps in (12), we obtain

$$
\begin{aligned}
&sup_{z,w\in R} \left| j_n^{(d)}(z,w) - \phi^{(d)}(z,w) \right| \qquad (15) \\
&\leq \sup_{z,w\in\Re} \left| j_n^{(d)}(z,w) - \frac{\int_z^{z+r} j_n^{(d)}(u,w)\, du}{r} \right| +
\end{aligned}
$$

$$
\begin{aligned}
& \sup_{z,w\in\Re} \frac{1}{r} \left| \int_z^{z+r} j_n^{(d)}(u,w)\, du - \int_z^{z+r} \phi^{(d)}(u,w)\, du \right| + \\
& \sup_{z,w\in\Re} \left| \phi(z,w) - \frac{\int_z^{z+r} \phi^{(d)}(u,w)\, du}{r} \right| \\
\leq \quad & (B_0(d) + B_1(d))\, r \; + \; \frac{2}{r} \sup_{z,w\in\Re} \left| j_n^{(d-1)}(z,w) - \phi^{(d-1)}(z,w) \right| .
\end{aligned}
$$

Note that the second term of the upper bound is, as a function of r, only an inverse, rather than an inverse squared, because we have taken only a first partial derivative, rather than a cross or second partial. It is readily verified that the bound in the statement of the theorem satisfies the recursion (15), and since we have shown the bound for $d = 0$, we are finished. Q. E. D.

3 Multivariate Berry-Esseen Theorem Under Strong Mixing

We now provide a multivariate Berry-Esseen theorem that applies to sequences $\{\eta_i\}_{i=1}^{\infty}$ of random variables in $\Re^m$ which satisfy a strong mixing condition and a moment condition.

Definition 1 *A sequence of random variables $\eta_1, \eta_2, \ldots$ will be said to be strongly mixing with coefficients $\alpha(n)$ if*

$$
\alpha(n) = \sup_{k, A\in\mathcal{F}_1^k, B\in\mathcal{F}_{k+n}^{\infty}} |P(AB) - P(A)\,P(B)| \tag{16}
$$

where $\mathcal{F}_a^b$ is the σ-algebra generated by η_j, $j \in \{a, a+1, \ldots b\}$.

Condition C. Let $\eta_1, \eta_2, \ldots$ be a sequence of $\Re^m$-valued random variables with $E[\eta_i] = 0$, $\sup_i E\left[|\eta_i|^{4+\gamma}\right] < \infty$ for some $\gamma > 0$, and $\alpha(n) \leq M_3 e^{-\beta n}$ for some $M_3 < \infty$ and some $\beta > 0$.

Theorem 4 provides a m-variate Berry-Esseen theorem which holds under Condition C. The results of this theorem satisfy Condition B. Thus the local limit results given above hold under, in particular, Conditions A and C.

This section concludes with the statement of this theorem, which is minor modification of a result of Tikhomirov (1980).

Theorem 4 *Suppose Condition C is satisfied. Then there is a constant C_2 depending only on m, β, γ, M_3 such that*

$$\sup_{z \in \Re^m} |J_n(z) - \Phi(z)| \leq C_2 n^{-1/2} \log n \tag{17}$$

where $J_n(z) = \Pr(S_{n,1} \leq z_1, S_{n,2} \leq z_2, \ldots, S_{n,m} \leq z_m)$, *with* $S_n = H_n^{-1} \sum_{i=1}^n \eta_i$, *where* $H_n \equiv$ *Cholesky*$\left\{E\left[\left(\sum_{i=1}^n \eta_i\right)\left(\sum_{i=1}^n \eta_i\right)'\right]\right\}$ *(the Cholesky factor of the given expectation).*

Proof: Consider Tikhomirov's (1980) Theorem 4. Although this is a univariate result, we see that minor modifications make it applicable to $\Re^m$. Namely, the ODE that Tikhomirov derives for the characteristic function becomes a system of ODEs, and the solution, naturally, becomes a multivariate characteristic function. However, the structure of his proof remains exactly the same; his lemmas transfer naturally to the multivariate case.

Appendix C: Description of Time Series Data Used in the Empirical Analysis

This appendix lists the time series used to construct the forecasts discussed in section 4. The format is: series number; series mnemonic; data span used; transformation code; and brief series description. The transformation codes are: 1 = no transformation; 2 = first difference; 4 = logarithm; 5 = first difference of logarithms; 6 = second difference of logarithms. An asterisk after the date denotes a series that was included in the unbalanced panel but not the balanced panel, either because of missing data or because of gross outliers which were treated as missing data. The series were either taken directly from the DRI-McGraw Hill Basic Economics database, in which case the original mnemonics are used, or they were produced by authors' calculations based on data from that database, in which case the authors calculations and original DRI/McGraw series mnemonics are provided. The following abbreviations appear in the data definitions: SA = seasonally adjusted; NSA = not seasonally adjusted; SAAR = seasonally adjusted at an annual rate; FRB = Federal Reserve Board; AC = Authors calculations

Real output and income

ip	1959:01-1998:12	5	industrial production: total index (1992=100,sa)
ipp	1959:01-1998:12	5	industrial production: products, total (1992=100,sa)
ipf	1959:01-1998:12	5	industrial production: final products (1992=100,sa)
ipc	1959:01-1998:12	5	industrial production: consumer goods (1992=100,sa)
ipcd	1959:01-1998:12	5	industrial production: durable consumer goods (1992=100,sa)
ipcn	1959:01-1998:12	5	industrial production: nondurable condsumer goods (1992=100,sa)
ipe	1959:01-1998:12	5	industrial production: business equipment (1992=100,sa)
ipi	1959:01-1998:12	5	industrial production: intermediate products (1992=100,sa)
ipm	1959:01-1998:12	5	industrial production: materials (1992=100,sa)
ipmd	1959:01-1998:12	5	industrial production: durable goods materials (1992=100,sa)
ipmnd	1959:01-1998:12	5	industrial production: nondurable goods materials (1992=100,sa)
ipmfg	1959:01-1998:12	5	industrial production: manufacturing (1992=100,sa)
ipd	1959:01-1998:12	5	industrial production: durable manufacturing (1992=100,sa)
ipn	1959:01-1998:12	5	industrial production: nondurable manufacturing (1992=100,sa)
ipmin	1959:01-1998:12	5	industrial production: mining (1992=100,sa)
iput	1959:01-1998:12	5	industrial production: utilities (1992-=100,sa)
ipxmca	1959:01-1998:12	1	capacity util rate: manufacturing,total(% of capacity,sa)(frb)
pmi	1959:01-1998:12	1	purchasing managers' index (sa)
pmp	1959:01-1998:12	1	NAPM production index (percent)
gmyxpq	1959:01-1998:12	5	personal income less transfer payments (chained) (#51) (bil 92$,saar)

Employment and hours

lhel	1959:01-1998:12	5	index of help-wanted advertising in newspapers (1967=100;sa)
lhelx	1959:01-1998:12	4	employment: ratio; help-wanted ads:no. unemployed clf
lhem	1959:01-1998:12	5	civilian labor force: employed, total (thous.,sa)
lhnag	1959:01-1998:12	5	civilian labor force: employed, nonagric.industries (thous.,sa)
lhur	1959:01-1998:12	1	unemployment rate: all workers, 16 years & over (%,sa)
lhu680	1959:01-1998:12	1	unemploy.by duration: average(mean)duration in weeks (sa)
lhu5	1959:01-1998:12	1	unemploy.by duration: persons unempl.less than 5 wks (thous.,sa)
lhu14	1959:01-1998:12	1	unemploy.by duration: persons unempl.5 to 14 wks (thous.,sa)
lhu15	1959:01-1998:12	1	unemploy.by duration: persons unempl.15 wks + (thous.,sa)
lhu26	1959:01-1998:12	1	unemploy.by duration: persons unempl.15 to 26 wks (thous.,sa)
lpnag	1959:01-1998:12	5	employees on nonag. payrolls: total (thous.,sa)
lp	1959:01-1998:12	5	employees on nonag payrolls: total, private (thous,sa)
lpgd	1959:01-1998:12	5	employees on nonag. payrolls: goods-producing (thous.,sa)
lpcc	1959:01-1998:12	5	employees on nonag. payrolls: contract construction (thous.,sa)

lpem	1959:01-1998:12	5	employees on nonag. payrolls: manufacturing (thous.,sa)
lped	1959:01-1998:12	5	employees on nonag. payrolls: durable goods (thous.,sa)
lpen	1959:01-1998:12	5	employees on nonag. payrolls: nondurable goods (thous.,sa)
lpsp	1959:01-1998:12	5	employees on nonag. payrolls: service-producing (thous.,sa)
lpt	1959:01-1998:12	5	employees on nonag. payrolls: wholesale & retail trade (thous.,sa)
lpfr	1959:01-1998:12	5	employees on nonag. payrolls: finance,insur.&real estate (thous.,sa
lps	1959:01-1998:12	5	employees on nonag. payrolls: services (thous.,sa)
lpgov	1959:01-1998:12	5	employees on nonag. payrolls: government (thous.,sa)
lphrm	1959:01-1998:12	1	avg. weekly hrs. of production wkrs.: manufacturing (sa)
lpmosa	1959:01-1998:12	1	avg. weekly hrs. of prod. wkrs.: mfg.,overtime hrs. (sa)
pmemp	1959:01-1998:12	1	NAPM employment index (percent)

Real retail, manufacturing and trade sales

msmtq	1959:01-1998:12	5	manufact. & trade: total (mil of chained 1992 dollars)(sa)
msmq	1959:01-1998:12	5	manufact. & trade:manufacturing;total(mil of chained 1992 dollars)(sa)
msdq	1959:01-1998:12	5	manufact. & trade:mfg; durable goods (mil of chained 1992 dollars)(sa)
msnq	1959:01-1998:12	5	manufact. & trade:mfg;nondurable goods (mil of chained 1992 dollars)(sa)
wtq	1959:01-1998:12	5	merchant wholesalers: total (mil of chained 1992 dollars)(sa)
wtdq	1959:01-1998:12	5	merchant wholesalers:drble goods total (mil of chained 1992 dollars)(sa)
wtnq	1959:01-1998:12	5	merchant wholesalers:nondurable goods (mil of chained 1992 dollars)(sa)
rtq	1959:01-1998:12	5	retail trade: total (mil of chained 1992 dollars)(sa)
rtnq	1959:01-1998:12	5	retail trade:nondurable goods (mil of 1992 dollars)(sa)

Consumption

gmcq	1959:01-1998:12	5	personal consumption expend (chained)-total (bil 92$,saar)
gmcdq	1959:01-1998:12	5	personal consumption expend (chained)-total durables (bil 92$,saar)
gmcnq	1959:01-1998:12	5	personal consumption expend (chained)-nondurables (bil 92$,saar)
gmcsq	1959:01-1998:12	5	personal consumption expend (chained)-services (bil 92$,saar)
gmcanq	1959:01-1998:12	5	personal cons expend (chained)-new cars (bil 92$,saar)

Housing starts and sales

hsfr	1959:01-1998:12	4	housing starts:nonfarm(1947-58);total farm&nonfarm(1959-)(thous.,sa
hsne	1959:01-1998:12	4	housing starts:northeast (thous.u.)s.a.
hsmw	1959:01-1998:12	4	housing starts:midwest(thous.u.)s.a.
hssou	1959:01-1998:12	4	housing starts:south (thous.u.)s.a.
hswst	1959:01-1998:12	4	housing starts:west (thous.u.)s.a.
hsbr	1959:01-1998:12	4	housing authorized: total new priv housing units (thous.,saar)
hmob	1959:01-1998:12	4	mobile homes: manufacturers' shipments (thous.of units,saar)
hsbne	1959:01-1998:12	4	houses authorized by build. permits:northeast(thou.u.)s.a
hsbmw	1959:01-1998:12	4	houses authorized by build. permits:midwest(thou.u.)s.a
hsbsou	1959:01-1998:12	4	houses authorized by build. permits:south(thou.u.)s.a
hsbwst	1959:01-1998:12	4	houses authorized by build. permits:west(thou.u.)s.a

Real inventories and inventory-sales ratios

ivmtq	1959:01-1998:12	5	manufacturing & trade inventories:total (mil of chained 1992)(sa)
ivmfgq	1959:01-1998:12	5	inventories, business, mfg (mil of chained 1992 dollars, sa)
ivmfdq	1959:01-1998:12	5	inventories, business durables (mil of chained 1992 dollars, sa)
ivmfnq	1959:01-1998:12	5	inventories, business, nondurables (mil of chained 1992 dollars, sa)
ivwrq	1959:01-1998:12	5	manufact. & trade inv: merchant wholesalers (mil of chained 1992 dollars)(sa)
ivrrq	1959:01-1998:12	5	manufacturing & trade inv:retail trade (mil of chained 1992 dollars)(sa)
ivsrq	1959:01-1998:12	2	ratio for mfg & trade: inventory/sales (chained 1992 dollars, sa)
ivsrmq	1959:01-1998:12	2	ratio for mfg & trade:mfg;inventory/sales (87$)(s.a.)
ivsrwq	1959:01-1998:12	2	ratio for mfg & trade:wholesaler;inventory/sales(87$)(s.a.)
ivsrrq	1959:01-1998:12	2	ratio for mfg & trade:retail trade;inventory/sales(87$)(s.a.)
pmnv	1959:01-1998:12	1	napm inventories index (percent)

Orders and unfilled orders

pmno	1959:01-1998:12	1	napm new orders index (percent)
pmdel	1959:01-1998:12	1	napm vendor deliveries index (percent)
mocmq	1959:01-1998:12	5	new orders (net)-consumer goods & materials, 1992 dollars (bci)
mdoq	1959:01-1998:12	5	new orders, durable goods industries, 1992 dollars (bci)
msondq	1959:01-1998:12	5	new orders, nondefense capital goods, in 1992 dollars (bci)
mo	1959:01-1998:12	5	mfg new orders: all manufacturing industries, total (mil$,sa)
mowu	1959:01-1998:12	5	mfg new orders: mfg industries with unfilled orders(mil$,sa)
mdo	1959:01-1998:12	5	mfg new orders: durable goods industries, total (mil$,sa)
mduwu	1959:01-1998:12	5	mfg new orders:durable goods indust with unfilled orders(mil$,sa)
mno	1959:01-1998:12	5	mfg new orders: nondurable goods industries, total (mil$,sa)
mnou	1959:01-1998:12	5	mfg new orders: nondurable gds ind.with unfilled orders(mil$,sa)
mu	1959:01-1998:12	5	mfg unfilled orders: all manufacturing industries, total (mil$,sa)
mdu	1959:01-1998:12	5	mfg unfilled orders: durable goods industries, total (mil$,sa)
mnu	1959:01-1998:12	5	mfg unfilled orders: nondurable goods industries, total (mil$,sa)
mpcon	1959:01-1998:12	5	contracts & orders for plant & equipment (bil$,sa)
mpconq	1959:01-1998:12	5	contracts & orders for plant & equipment in 1992 dollars (bci)

Stock prices

fsncom	1959:01-1998:12	5	NYSE common stock price index: composite (12/31/65=50)
fspcom	1959:01-1998:12	5	S&P's common stock price index: composite (1941-43=10)
fspin	1959:01-1998:12	5	S&P's common stock price index: industrials (1941-43=10)
fspcap	1959:01-1998:12	5	S&P's common stock price index: capital goods (1941-43=10)
fsput	1959:01-1998:12	5	S&P's common stock price index: utilities (1941-43=10)
fsdxp	1959:01-1998:12	1	S&P's composite common stock: dividend yield (% per annum)
fspxe	1959:01-1998:12	1	S&P's composite common stock: price-earnings ratio (%,nsa)

Exchange rates

exrus	1959:01-1998:12	5	United States effective exchange rate (merm)(index no.)
ex*rG*er	1959:01-1998:12	5	foreign exchange rate: Germany (deutsche mark per U.S.$)
exrsw	1959:01-1998:12	5	foreign exchange rate: Switzerland (swiss franc per U.S.$)
exrjan	1959:01-1998:12	5	foreign exchange rate: Japan (yen per U.S.$)
exrcan	1959:01-1998:12	5	foreign exchange rate: Canada (canadian $ per U.S.$)

Interest rates

fygt5	1959:01-1998:12	2	interest rate: U.S.treasury const maturities,5-yr.(% per ann,nsa)
fygt10	1959:01-1998:12	2	interest rate: U.S.treasury const maturities,10-yr.(% per ann,nsa)
fyaaac	1959:01-1998:12	2	bond yield: moody's aaa corporate (% per annum)
fybaac	1959:01-1998:12	2	bond yield: moody's baa corporate (% per annum)
fyfha	1959:01-1998:12	2	secondary market yields on fha mortgages (% per annum)
sfycp	1959:01-1998:12	1	spread fycp - fyff
sfygm3	1959:01-1998:12	1	spread fygm3 - fyff
sfygm6	1959:01-1998:12	1	spread fygm6 - fyff
sfygt1	1959:01-1998:12	1	spread fygt1 - fyff
sfygt5	1959:01-1998:12	1	spread fygt5 - fyff
sfygt10	1959:01-1998:12	1	spread fygt10 - fyff
sfyaaac	1959:01-1998:12	1	spread fyaaac - fyff
sfybaac	1959:01-1998:12	1	spread fybaac - fyff
sfyfha	1959:01-1998:12	1	spread fyfha - fyff

Money and credit quantity aggregates

fm1	1959:01-1998:12	6	money stock: m1(curr,trav.cks,dem dep,other ck'able dep)(bil$,sa)
fm2	1959:01-1998:12	6	money stock:m2(m1+o'nite rps,euro$,g/p&b/d mmmfs&sav&sm time dep(bil$,nsa)
fm3	1959:01-1998:12	6	money stock: m3(m2+lg time dep,term rp's&inst only mmmfs)(bil$,sa)

fm2dq	1959:01-1998:12	5	money supply-m2 in 1992 dollars (bci)
fmfba	1959:01-1998:12	6	monetary base, adj for reserve requirement changes(mil$,sa)
fmrra	1959:01-1998:12	6	depository inst reserves:total,adj for reserve req chgs(mil$,sa)
fmrnbc	1959:01-1998:12	6	depository inst reserves:nonborrow+ext cr,adj res req cgs(mil$,sa)

Price indexes

pmcp	1959:01-1998:12	1	napm commodity prices index (percent)
pwfsa	1959:01-1998:12	6	producer price index: finished goods (82=100,sa)
pwfcsa	1959:01-1998:12	6	producer price index:finished consumer goods (82=100,sa)
psm99q	1959:01-1998:12	6	index of sensitive materials prices (1990=100)(bci-99a)
punew	1959:01-1998:12	6	cpi-u: all items (82-84=100,sa)
pu83	1959:01-1998:12	6	cpi-u: apparel & upkeep (82-84=100,sa)
pu84	1959:01-1998:12	6	cpi-u: transportation (82-84=100,sa)
pu85	1959:01-1998:12	6	cpi-u: medical care (82-84=100,sa)
puc	1959:01-1998:12	6	cpi-u: commodities (82-84=100,sa)
pucd	1959:01-1998:12	6	cpi-u: durables (82-84=100,sa)
pus	1959:01-1998:12	6	cpi-u: services (82-84=100,sa)
puxf	1959:01-1998:12	6	cpi-u: all items less food (82-84=100,sa)
puxhs	1959:01-1998:12	6	cpi-u: all items less shelter (82-84=100,sa)
puxm	1959:01-1998:12	6	cpi-u: all items less midical care (82-84=100,sa)
gmdc	1959:01-1998:12	6	pce,impl pr defl:pce (1987=100)
gmdcd	1959:01-1998:12	6	pce,impl pr defl:pce; durables (1987=100)
gmdcn	1959:01-1998:12	6	pce,impl pr defl:pce; nondurables (1987=100)
gmdcs	1959:01-1998:12	6	pce,impl pr defl:pce; services (1987=100)

Average hourly earnings

lehcc	1959:01-1998:12	6	avg hr earnings of constr wkrs: construction ($,sa)
lehm	1959:01-1998:12	6	avg hr earnings of prod wkrs: manufacturing ($,sa)

Miscellaneous

hhsntn	1959:01-1998:12	1	u. of mich. index of consumer expectations(bcd-83)