Fair Stable Sets of Simple Games

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June 4, 2017

Abstract

Simple games are abstract representations of voting systems and other group-decision procedures. A *stable set*—or von Neumann-Morgenstern solution—of a simple game represents a "standard of behavior" that satisfies certain internal and external stability properties. *Compound simple games* are built out of *component* games, which are, in turn, "players" of a *quotient* game. I describe a method to construct *fair*—or symmetry-preserving—stable sets of compound simple games from fair stable sets of their quotient and components. This method is closely related to the composition theorem of Shapley (1963c), and contributes to the answer of a question that he formulated: What is the set $\mathcal{G}$ of simple games that admit a fair stable set? In particular, this method shows that the set $\mathcal{G}$ includes all simple games whose *factors*—or quotients in their "unique factorization" of Shapley (1967)—are in $\mathcal{G}$, and suggests a path to characterize $\mathcal{G}$.

JEL classification: `C71`

Keywords: `fair stable set; simple game; compound simple game, symmetry; aggregation.`

*Department of Economics, Harvard University. Email: talamas@fas.harvard.edu. I dedicate this article to the memory of Lloyd Stowell Shapley, whose work I deeply admire. I thank Benjamin Golub for encouraging me to write this article; his detailed feedback has greatly improved it. I also thank Aubrey Clark, Jerry Green, Rajiv Vohra and the participants of Harvard's *Games and Markets* and *Contracts and Organizations* seminars for useful comments. Finally, I thank the editor and an anonymous referee for useful suggestions. All errors are my own.

1 Introduction

Simple games—in which every coalition of players either wins or loses—represent political structures in which "power" is the fundamental driving force. For example, a legislature in which any two of three parties have enough parliamentary seats to form a new government can be represented by the three-player simple majority game.[1]

A *fair stable set* of a simple game is a set of distributions of power among the players that (i) satisfies certain internal and external stability properties, and (ii) does not discriminate among players based on their names. For example, the unique fair stable set of the three-player simple majority game consists of the three possible outcomes in which power is divided equally among two of the three players; intuitively, this stable set reflects that—when all coalitions obtain the same surplus from forming government—the governing coalition is vulnerable when one of its parties receives less than the other.[2]

In 1978, Lloyd Shapley asked which simple games admit a fair stable set. Rabie (1985) showed that the answer is not "all," but—to the best of my knowledge—the set of simple games that admit a fair stable set has not been characterized.[3]

In this article I describe a method to construct fair stable sets of *compound simple games* from fair stable sets of their *quotient* and *components*, and I discuss how this method contributes to the characterization of the set of simple games that admit a fair stable set.

Compound simple games are built out of component games, which are, in turn, "players" of a quotient game. An example of a compound simple game is the multimillion-person game "US Presidential Election"—whose quotient is a weighted majority game (the Electoral College), and whose components are symmetric majority games of assorted sizes (the electorates of the 50 states and the District of Columbia). *Sums of games*—whose

[1]See Taylor and Zwicker (1999) for an excellent exposition of the theory of simple games.

[2]Stable sets were first studied by von Neumann and Morgenstern (1944), who—among many other things—showed that all simple games have a stable set. In fact, they showed that many interesting games admit multiple stable sets, which lead to the advancement of several refinements of the theory. The fairness requirement is one such refinement; see also Shapley (1952), Luce and Raiffa (1957), Vickrey (1959), Harsanyi (1974), Roth (1976), Muto (1980), Greenberg (1990), Bogomolnaia and Jackson (2002), Béal et al. (2008), Mauleon et al. (2011), Jordan and Obadia (2015), Ray and Vohra (2015) and Dutta and Vohra (forthcoming).

[3]Shapley raised this question during the Fourth International Workshop in Game Theory; see Lucas (1978) and Rabie (1985). Rabie's result is analogous to those of Stearns (1964) and Lucas (1968), who showed that not all coalitional games with non-transferable and transferable utility, respectively, have a stable set (see Lucas (1992) for an illuminating review of these and other results in the theory of stable sets).

winning coalitions are exactly those that win in at least one of these games—and *products of games*—whose winning coalitions are exactly those that win in all of these games—are particular classes of compound games. For example, the game "US Congress" can be represented as the product of two simple majority games (the Senate and the House of Representatives). A game is *prime* if it does not have any non-trivial compound representation.[4]

The main result of this article is closely related the composition theorem of Shapley (1963c). To emphasize this connection, I quote his verbal description of his result (Shapley, 1963c, page 269) adding words in italics that transform it into a description of my result:[5]

> If one first divides the proceeds of a *regular*[6] [compound simple] game among the components, in accordance with a *fair* [stable set] of the quotient, and then subdivides them among the players of each component according to a scaled-down *fair* [stable set] of that component, *using isomorphic fair stable sets for isomorphic components,* then the resulting set of imputations is a *fair* [stable set] of the compound.

This new composition result combined with Shapley's (1967) "unique factorization" theorem[7] reveals that the set $\mathcal{G}$ of simple games that admit a fair stable set includes all simple games whose *factors*—or prime quotients in their unique factorization—are in $\mathcal{G}$. In other words, a game that does not admit a fair stable set has at least one factor that does not admit a fair stable set. It is an open question whether the converse holds; if it does, the composition result presented in this article implies that characterizing the set of prime games in $\mathcal{G}$ is equivalent to characterizing the set $\mathcal{G}$.

The rest of this article is organized as follows. In section 2 I provide background material: The definitions of simple game, compound simple game, committee and stable set,

[4]Compound simple games and their stable sets were first studied by Shapley (1963a, 1963b, 1963c, 1967). See Shapley (1962) for an illuminating introduction to this theory.

[5]In the Rand Mimeo version of this study (Shapley, 1963d), this sentence can be found in pages 3 and 4. Stable sets were originally called "solutions." In this quote, I have replaced "solution" for "[stable set]" to reflect the modern terminology.

[6]A compound simple game is *regular* if either its quotient is prime, or it is a sum of products that are not themselves sums, or it is a product of games which are not themselves products.

[7]This theorem shows how every simple game can be uniquely decomposed into a hierarchical arrangement of compound simple games that use only prime quotients and the operations of sums and products.

and Shapley's unique factorization and composition theorems. In section 3 I define what it means for a stable set to be *fair*, and I state and prove the main result of this article: A composition theorem that shows how to construct fair stable sets of compound simple games from fair stable sets of their quotient and components. I conclude in section 4 by discussing the implications of this result for the characterization of the set of simple games that admit a fair stable set, and for the problem of aggregation in the theory of fair stable sets.

2 Preliminaries

In this section I review the two fundamental results of Lloyd Shapley that this article builds on. In subsection 2.1 I review the definitions of simple game, compound simple game, dual of a game and committee of a game, and I present the unique factorization theorem of Shapley (1967). In subsection 2.2 I review the definition of stable set of a game and the composition theorem presented in Shapley (1963c).

2.1 Compound Simple Games

2.1.1 Simple Games

Let P be the set of all players that might ever come under consideration. A *(simple) game* $\mathcal{W}$ is a collection of subsets of P (the winning coalitions) that includes the grand coalition P, excludes the empty coalition, and is *monotonic*—in the sense that every superset of a winning coalition is also winning.

The monotonicity of a game $\mathcal{W}$ implies that it can be identified with the set $\mathcal{W}^m$ that contains only its *minimal winning coalitions*. For example, $\{ab, ac\}$ represents the game in which only the coalitions that contain both player a and one of players b and c win, and $\{ab, ac, bc\}$ represents the three-person simple majority game.[8] A player is said to be a *dummy* of a game if it is not in any of the minimal winning coalitions of the game.[9]

[8]For brevity I write ab for $\{a, b\}$, etc.

[9]Abusing terminology slightly, I often refer to the non-dummy players of a game as its *players*.

2.1.2 Compound Simple Games

Throughout this article, let there be given m *non-overlapping component games* $\mathcal{W}_i$, $i = 1, 2, \ldots m$, together with an m-person *quotient game* $\mathcal{W}$, whose players are identified with the integers $1, 2, \ldots, m$.[10]

For each set $S \subset P$, let $K(S)$ be the set $\{i \mid S \in \mathcal{W}_i\}$; intuitively, the set $K(S)$ consists of the set of all components that S wins.

Definition 2.1. The *compound simple game* $\mathcal{W}[\mathcal{W}_1, \ldots, \mathcal{W}_m]$ is defined by the following condition:

$$S \in \mathcal{W}[\mathcal{W}_1, \ldots, \mathcal{W}_m] \text{ if and only if } K(S) \in \mathcal{W}.$$

Thus, a coalition wins in the compound if and only if it wins enough of the components to make up a winning coalition of the quotient. The game $\mathcal{W}[\mathcal{W}_1, \ldots, \mathcal{W}_m]$ is a *compound representation* of game $\mathcal{M}$ if it satisfies

$$S \in \mathcal{M} \text{ if and only if } S \in \mathcal{W}[\mathcal{W}_1, \ldots, \mathcal{W}_m].$$

For example, the game $\{ab, ac\}$ can be represented as the compound game whose quotient is a two-player unanimity game, and whose components are the games $\{a\}$ and $\{b, c\}$. In contrast, the three-player majority game is *prime*, in the sense that it does not have a non-trivial compound representation.

2.1.3 Sums and Products

Quotients having the maximum and minimum possible number of winning coalitions play a central part in the theory of compound simple games; it is useful to represent them as operations on games, as follows.[11] The *sum of $m \geq 2$ non-overlapping games*

$$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \cdots \oplus \mathcal{W}_m \tag{1}$$

is the compound game $\mathcal{S}_m[\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_m]$ where the minimal winning coalitions of the quotient $\mathcal{S}_m$ are all the singleton subsets of $\{1, 2, \ldots, m\}$. That is, a coalition wins in the sum of games whenever it contains a winning contingent from at least one of these games.

[10] Non-overlapping in the sense that their non-dummy player sets do not overlap; that is, for any $i \neq j$, the union of $\mathcal{W}_i^m$ is disjoint from the union of $\mathcal{W}_j^m$.

[11] The quotient of a sum of games has the maximum possible number of winning coalitions (all nonempty coalitions win) and the quotient of the product has the minimum number of winning coalitions (only the coalition containing all non-dummy players wins).

Similarly, the *product of $m \geq 2$ non-overlapping games*

$$\mathcal{W}_1 \otimes \mathcal{W}_2 \otimes \cdots \otimes \mathcal{W}_m \tag{2}$$

is the compound game $\mathcal{P}_m[\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_m]$ where the only minimal winning coalition of the quotient $\mathcal{P}_m$ is $\{1, 2, \ldots, m\}$. That is, a coalition wins in the product of games whenever it contains winning contingents from all of them.

2.1.4 Regular Compound Representations

The main result of this article concerns *regular* compound representations, defined as follows.

Definition 2.2. The compound representation $\mathcal{W}[\mathcal{W}_1, \ldots, \mathcal{W}_m]$ of a simple game is *regular* if either its quotient $\mathcal{W}$ is prime, or its quotient $\mathcal{W}$ is a sum and none of its components $\mathcal{W}_i$ is a sum, or its quotient $\mathcal{W}$ is a product and none of its components $\mathcal{W}_i$ is a product.

For example, the compound representation $\mathcal{S}_3[a, b, c]$ of the game $\{a, b, c\}$ is regular, but the compound representation $\mathcal{S}_2[a \oplus b, c]$ of the same game is not.[12]

2.1.5 Dual Games

The following duality between sums and products of games is useful to prove the main result of this paper. The *dual* $\mathcal{M}^*$ of a game $\mathcal{M}$ is the set of all coalitions B that *block* in $\mathcal{M}$; that is, the set of all coalitions B whose complement $P - B$ does not win in $\mathcal{M}$. For example, the dual of the sum of games (1) is $\mathcal{W}_1^* \otimes \mathcal{W}_2^* \otimes \cdots \otimes \mathcal{W}_m^*$, and the dual of the product of games (2) is $\mathcal{W}_1^* \oplus \mathcal{W}_2^* \oplus \cdots \oplus \mathcal{W}_m^*$.

2.1.6 Committees

Shapley's unique factorization theorem (Theorem 2.1 below) describes how a simple game can be decomposed into a hierarchial arrangement of *committees*, defined as follows.

Definition 2.3. A *committee* of a game $\mathcal{M}$ is another game $\mathcal{M}_C$ (with non-dummy player set C) which is related to the first as follows: For every coalition S such that

$$S \cup C \in \mathcal{M} \text{ and } S - C \notin \mathcal{M},$$

[12] For brevity, in compound representations I denote the one-player component $\{a\}$ by a, etc.

we have

$$S \in \mathcal{M} \text{ if and only if } S \cap C \in \mathcal{M}_C.$$

A committee of a game $\mathcal{M}$ is *proper* if it is not the committee of the whole set of its non-dummy players or a committee that consists of only one individual. For example, denoting the three player majority game by $\mathcal{M}_3$, the game $\mathcal{M}_3[b,c,d]$ is a proper committee of the game $\mathcal{M}_3[a,\mathcal{M}_3[b,c,d],e]$. Prime games are exactly those that do not possess proper committees (Shapley, 1967, Section 7).[13]

2.1.7 Shapley's Unique Factorization Theorem

The fundamental result of Shapley (1967) is that—just like every natural number can be uniquely expressed as the product of prime numbers—every simple game has a unique compound representation that only uses prime quotients and the operations of sums and products.

Theorem 2.1 (Shapley, 1967, Theorem 8)**.** *Every simple game has a compound representation that uses nothing but prime quotients and the associative operations $\oplus$ and $\otimes$ and that is unique except for the arbitrariness in the ordering of the players.*[14]

Continuing the analogy with the natural numbers, the *factors* of a simple game are its quotients in the above compound representation.

2.2 Stable Sets of Compound Simple Games

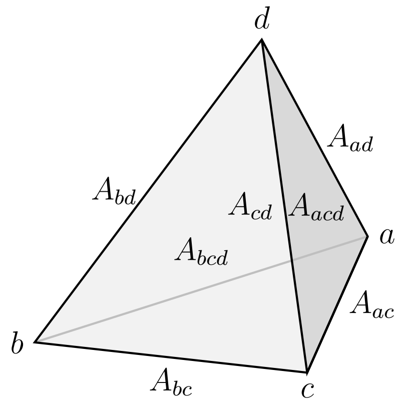
2.2.1 Stable Sets of Simple Games

For any set of players Q, the set of *imputations* A_Q is the simplex of real nonnegative vectors x with $x_j = 0$ for any $j \notin Q$, and whose entries sum to one. Geometrically, the

[13]The games that have only two non-dummy players are an exception: even though they do not have any proper committee, they are not regarded to be prime; see Shapley (1967, page 5).

[14]Shapley's statement adds an additional exception regarding "the disposition of dummy players." This is because he defines a simple game to be a finite set N (the players) and a set of subsets of N (the winning coalitions), so in order to define a compound simple game, he needs to specify to which component each dummy player belongs. In contrast, I identify a simple game with the set of its minimal winning coalitions, so I do not need to assign dummy players to components.

Figure 1: A geometric representation of several imputation simplices.



simplex of imputations A_Q is the set of convex combinations of the vectors that divide a unit of surplus among the players in Q. Figure 1 illustrates some imputation simplices.[15]

Fix a simple game $\mathcal{M}$. An imputation x *dominates another imputation* y *via the coalition* S if $S \in \mathcal{M}$ and if each of the players in S gets strictly more payoff in x than in y. For example, in the game $\{ab, ac, bc\}$, the imputation that gives one half of the payoff to each of the players a and b dominates the imputation that gives all the payoff to player c via the coalition ab. An imputation x *dominates another imputation* y if x dominates y via some coalition.

Definition 2.4. Given a simple game $\mathcal{M}$, a set X of imputations is (i) *internally stable* if no imputation in X is dominated by any imputation in X, (ii) *externally stable* if each imputation that is not in X is dominated by some imputation in X, and (iii) *stable* if it is internally and externally stable.

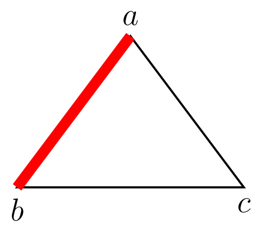
Figure 2 depicts a set of imputations that constitutes a stable set of both $\{ab, ac, bc\}$ and $\{ab, ac\}$. Stable sets are the classical *solutions* of von Neumann and Morgenstern (1944).

2.2.2 Shapley's Composition Theorem

The main contribution of Shapley (1963c) is the description of a method to construct a stable set of a compound simple game from stable sets of its quotient and components. I now formally describe this method. For each $i = 1, 2, \ldots m$, let X_i be a stable set of the game $\mathcal{W}_i$, and let χ be a stable set of the quotient game $\mathcal{W}$.

[15] In the figures, I write a, b, c, d for A_a, A_b, A_c and A_d respectively.

Figure 2: The imputation simplex A_{ab} is a stable set of both $\{ab, ac, bc\}$ and $\{ab, ac\}$.



Definition 2.5. The *compound set* $\chi[X_1, \ldots, X_m]$ is the set of all imputations of the form

$$x = \sum_{i=1}^{m} \alpha_i x_i,\ \alpha \in \chi,\ x_i \in X_i,\ i = 1, 2, \ldots, m.$$

Theorem 2.2 (Shapley, 1963c, Part II, Section 2, Theorem 1)**.** *The compound set* $\chi[X_1, \ldots, X_m]$ *is a stable set of the compound game* $\mathcal{W}[\mathcal{W}_1, \ldots, \mathcal{W}_m]$.

For an illustration of Theorem 2.2, consider the game $\{ab, ac\}$. This game can be represented as a compound game, whose quotient is the two-player unanimity game, and whose components are $\{a\}$ and $\{b, c\}$. A stable set of its quotient consists of all imputations that share the payoff among its two players, and A_a and A_b are stable sets of its components $\{a\}$ and $\{b, c\}$, respectively. Shapley's composition theorem then says that the set A_{ab} (illustrated in Figure 2) is a stable set of the game $\{ab, ac\}$. Similarly, since any singleton set that consists of an imputation that shares the payoff arbitrarily between players b and c is a stable set of the game $\{b, c\}$, any straight line in A_{abc} from A_a to any point in A_{bc} is a stable set of the game $\{ab, ac\}$; the right diagram of Figure 3 illustrates another stable set of this game that can be constructed in this way.[16]

3 Fair Stable Sets of Compound Simple Games

Fair stable sets are those that do not discriminate among players based on their names. Not all stable sets are fair; for example, the stable set of the three-player majority game depicted in Figure 2 is not fair, since it discriminates among its three non-dummy players (who play exactly the same role in this game). In subsection 3.1 I formally describe what it means for a stable set to be *fair*, and in subsection 3.2 I provide the main result of this

[16] Shapley (1963c) also presents a generalization of his composition theorem that shows that the requirement that such a line be straight is not necessary.

article: A composition theorem that shows how to construct fair stable sets of compound simple games from fair stable sets of its quotient and components.

3.1 Fair Stable Sets of Simple Games

A permutation π of the set of players P acts on an imputation by permuting its indices—for example,[17] $(ab)A_b = A_a$—and it acts on a set of imputations by acting on each of the imputations of this set—for example, $(ab)A_{bc} = A_{ac}$.

Definition 3.1. A permutation π of the player set P is an *isomorphism between an imputation set X and an imputation set Y* if $\pi X = Y$.

An imputation set X is said to be *isomorphic* to an imputation set Y if there exists an isomorphism between X and Y. For example, the permutation (bc) is an isomorphism between A_{ab} and A_{ac}. Isomorphisms between an imputation set X and itself are *symmetries* of X. For example, the permutation (ab) is a symmetry of A_{abc}.

A permutation π of the set of players P acts on a game $\mathcal{M}$ by permuting the players in each of the coalitions in $\mathcal{M}$. For example, $(ac)\{ab, ac\} = \{cb, ca\}$.

Definition 3.2. A permutation π of the player set P is an *isomorphism between a game $\mathcal{M}_1$ and a game $\mathcal{M}_2$* if $\pi\mathcal{M}_1 = \mathcal{M}_2$, or, equivalently, $\pi\mathcal{M}_1^m = \mathcal{M}_2^m$.

A game $\mathcal{M}_1$ is said to be *isomorphic* to a game $\mathcal{M}_2$ if there is an isomorphism between $\mathcal{M}_1$ and $\mathcal{M}_2$. For example, the permutation (ac) is an isomorphism between the game $\{ab, ac\}$ and the game $\{cb, ca\}$. Isomorphisms between a game $\mathcal{M}$ and itself are *symmetries* of $\mathcal{M}$. For example, the permutation (bc) is a symmetry of the game $\{ab, ac\}$, and every permutation of the players is a symmetry of the three-player majority game.

Definition 3.3. A stable set X of a game $\mathcal{M}$ is *fair* if every symmetry of $\mathcal{M}$ is also a symmetry of X.

The stable set of the games $\{ab, ac, bc\}$ and $\{ab, ac\}$ illustrated in Figure 2 is not fair, since it is not invariant under the permutation (bc) (which is a symmetry of both these games). The left and right diagrams in Figure 3 illustrate the unique fair stable set of the game $\{ab, ac, bc\}$ and $\{ab, ac\}$, respectively. The fair stable set of $\{ab, ac, bc\}$ is a set of

[17] I denote by $(a_1a_2a_3 \ldots a_n)$ the permutation that maps a_1 to a_2, a_2 to a_3, $\ldots$, a_{n-1} to a_n, a_n to a_1, and every other player to herself.

Figure 3: The fair stable sets of $\{ab, ac, bc\}$ (left) and $\{ab, ac\}$ (right).

three imputations; in each of these imputations, two players divide the payoff equally, leaving the remaining player with zero payoff. The fair stable set of $\{ab, ac\}$ is the set of all imputations in the simplex A_{abc} that give the same payoff to both players b and c.

Since the isomorphism relation between games is an *equivalence relation*, we can partition every set of games into *isomorphic classes*. For example, the set of games $\{\{ab, ac\}, \{cb, ca\}\}$ has only one isomorphic class (itself), but the set of games $\{\{ab, ac\}, \{ab, ac, bc\}\}$ has two isomorphic classes ($\{ab, ac\}$, and $\{ab, ac, bc\}$).

3.2 A New Composition Theorem

Let χ be a fair stable set of the quotient $\mathcal{W}$. For each isomorphic class $\mathcal{C}$ of components, pick a representative $\mathcal{W}_i$, let X_i be a fair stable set of $\mathcal{W}_i$ and, for each component $\mathcal{W}_j$ in $\mathcal{C}$, let $X_j = \mu X_i$, where μ is any isomorphism between $\mathcal{W}_i$ and $\mathcal{W}_j$.[18]

Theorem 3.1. *If the compound representation $\mathcal{W}[\mathcal{W}_1, \ldots, \mathcal{W}_m]$ is regular then the compound set $\chi[X_1, \ldots, X_m]$ is a fair stable set of the compound game $\mathcal{W}[\mathcal{W}_1, \ldots, \mathcal{W}_m]$.*

The reason why we need to construct the fair stable sets of the components so that X_i is isomorphic to X_j when $\mathcal{W}_i$ is isomorphic to $\mathcal{W}_j$ is that some games admit multiple fair stable sets:[19] In order to make sure that the composition process respects the symmetry of the game, we need to choose "the same" fair stable set in any two isomorphic components.[20]

[18]Note that this construction does not depend on the isomorphism μ chosen because, for every two isomorphisms μ_1 and μ_2 between $\mathcal{W}_i$ and $\mathcal{W}_j$ there is a symmetry σ of $\mathcal{W}_i$ such that $\mu_1\sigma = \mu_2$ (namely $\sigma := \mu_1^{-1}\mu_2$). Hence, since X_i is a fair stable set, $\mu_2 X_i = \mu_1 \sigma X_i = \mu_1 X_i$.

[19]For example, both A_{bc} and the union of all imputations that give $1/2$ to c and share the other $1/2$ between a and b and all imputations that give $1/2$ to b and share the other $1/2$ between c and d are fair stable sets of the game $\{ab, bc, cd\}$.

[20]In fact, it is enough to require the use of isomorphic fair stable sets for isomorphic components $\mathcal{W}_i$ and $\mathcal{W}_j$ for which there is a symmetry of the quotient $\mathcal{W}$ that maps $\mathcal{W}_i$ to $\mathcal{W}_j$.

The condition that the compound representation be regular is to avoid situations like the one described in Example 3.2.1 below, in which the compound representation "hides" certain symmetries of the game by having a component $\mathcal{W}_i$ that is isomorphic to a component of another component $\mathcal{W}_j$.

Example 3.2.1. The following example shows that the conclusion of Theorem 3.1 is not generally true without the requirement that the compound representation be regular. Consider the compound simple game $\mathcal{S}_2[a \oplus b, c]$, where $\mathcal{S}_2$ is the sum of components 1 and 2. This compound representation is not regular, since both its quotient and one of its components are sums of games.

Both $\mathcal{S}_2$ and $a \oplus b$ have a unique fair stable set, which consists of the imputation that divides the unit payoff equally among their two players; denote them by η and Y_1, respectively. Similarly, the game c has a unique fair stable set, which consists of the imputation that gives the unit payoff to player c; denote it by Y_2.

The compound set $\eta[Y_1, Y_2]$ consists of the singleton set containing the imputation that gives $1/4$ to each of players a and b, and $1/2$ to player c. But this is not a fair stable set of the game; for example, (ac) is a symmetry of the game but not a symmetry of $\eta[Y_1, Y_2]$.

3.3 Proof of Theorem 3.1

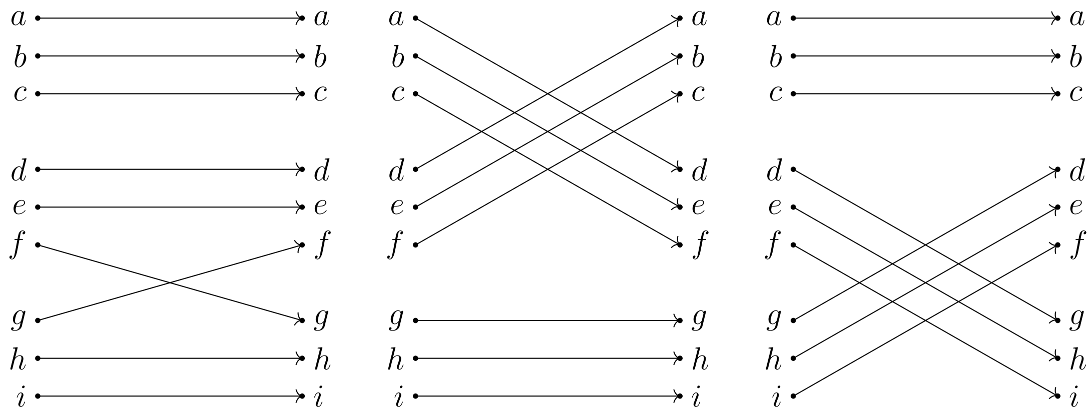
Let π be a symmetry of the compound game in regular form $\mathcal{W}[\mathcal{W}_1, \ldots, \mathcal{W}_m]$. Given Theorem 2.2, it is enough to show that π is a symmetry of the compound set $\chi[X_1, \ldots, X_m]$. This follows from Proposition 3.5, Proposition 3.6 and Proposition 3.7 below, which I now outline.

3.3.1 Outline of the Proof

The first step (Proposition 3.5) is the most subtle one: I show that for every component $\mathcal{W}_i$, there exists a component $\mathcal{W}_j$ such that the map π is an isomorphism between $\mathcal{W}_i$ and $\mathcal{W}_j$. When a map has this property, I say that it is *compatible with the compound representation* $\mathcal{W}[\mathcal{W}_1, \ldots, \mathcal{W}_m]$. The requirement that the compound representation be regular is used in this first step. Indeed, a symmetry of a game need not be compatible with its non-regular compound representations; for example, (ac) is a symmetry of the game in Example 3.2.1, but it is not an isomorphism between any of its components.

A corollary of the first step is that π naturally defines a permutation π^* of the players

Figure 4: The maps π_L, π_M and π_R of Example 3.3.1.



(a) The map π_L (b) The map π_M (c) The map π_R

of the quotient. The second step (Proposition 3.6) is to show that π^* is a symmetry of the quotient. The third and final step (Proposition 3.7) is to show that any permutation of the players with the two properties above is a symmetry of the compound set $\chi[X_1, \ldots, X_m]$.

3.3.2 Intuition for the Proof

Before proving the three steps of the proof, I present Example 3.3.1 to help build intuition for why the statement proved in each step holds.

Example 3.3.1. Consider the nine-player game with regular compound representation

$$\mathcal{M}[\mathcal{M}_3, \mathcal{M}_3, \mathcal{M}_3], \tag{3}$$

where $\mathcal{M}$ denotes the three-player game $\{\{1,2\},\{1,3\}\}$ and $\mathcal{M}_3$ denotes the three-player majority game. Let the set of non-dummy players in the first, second, and third component be $\{a,b,c\}$, $\{d,e,f\}$ and $\{g,h,i\}$ respectively.

The two sets of players $\{d,e,f\}$ and $\{g,h,i\}$ play the same role, in the sense that any two players in one of these sets combined with any two players in $\{a,b,c\}$ win. In fact, the set of all minimal winning coalitions of this compound game are exactly the set of all four-player coalitions just described.

To gain intuition for the first step of the proof of Theorem 3.1, consider the maps π_L, π_M and π_R depicted in Figure 4. The map π_L is not compatible with the compound representation (3), since it maps players d and e to one component and player f to a different component. To see why π_L is not a symmetry of the compound game, note that it maps the winning coalition $\{a, b, e, f\}$ to the losing coalition $\{a, b, e, g\}$. In contrast, the maps π_M and π_R are compatible with the compound representation (3).

To gain intuition for the second step of the proof of Theorem 3.1, note that since the maps π_M and π_R are compatible with the compound representation (3), they define the following two maps on the players of $\mathcal{M}$: The map π_M^* interchanges players 1 and 2 and keeps 3 fixed (so it is not a symmetry of the quotient $\mathcal{M}$), and the map π_R^* interchanges players 2 and 3 and keeps 1 fixed (so it is a symmetry of the quotient $\mathcal{W}$). To see why π_M is not a symmetry of the compound game (3), note that it maps the winning coalition $\{a, b, g, h\}$ to the losing coalition $\{d, e, g, h\}$.

To gain intuition for the third step of the proof of Theorem 3.1, note that the map π_R (which is compatible with the compound representation and defines a map on the quotient that is a symmetry of the quotient) is a symmetry of the compound set $\eta[Y_1, Y_2, Y_3]$, where η and $\{Y_i\}_{i=1,2,3}$ denote the unique fair stable sets of the quotient and components of the compound game (3), respectively; this compound set consists of the set of all imputations that give $0 \leq x \leq 1/2$ units of surplus to each of two players in component 1, and $1/4 - x/2$ to each of two players in each of the other two components.

3.3.3 Terminology

Abusing terminology slightly, I often denote the component $\mathcal{W}_i$ by its index i, I refer to the intersection of a coalition A of players with the non-dummy player set of a given component i as the *intersection of coalition A with component i*, and I say that *coalition A intersects with component i* when this intersection is not empty.

3.3.4 Three Auxiliary Results

In this subsection I present three auxiliary results that facilitate the proof of Proposition 3.5. On the one hand, Lemma 3.2 is useful to reduce the number of cases to be considered in the proof of Proposition 3.5.[21] On the other hand, Lemma 3.3 and Lemma 3.4 give useful information about the map π; both of these results are weaker than the statement that π is compatible with the compound representation $\chi[X_1, \ldots, X_m]$, but they are useful to establish this fact.

[21] In particular, it is because of this result that *Case 2* in the proof of Proposition 3.5 follows from *Case 1*.

Lemma 3.2. *A permutation is an isomorphism between two games if and only if it is an isomorphism between their duals.*

Proof. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two games and let μ be a permutation such that $\mu\mathcal{M}_1 = \mathcal{M}_2$. Let B be a blocking coalition of $\mathcal{M}_1$; that is, suppose that $P - B \notin \mathcal{M}_1$. Then $\mu(P - B) \notin \mathcal{M}_2$, so $P - \mu(B) \notin \mathcal{M}_2$; that is, $\mu(B)$ is also a blocking coalition of $\mathcal{M}_2$. Since μ is one to one, this implies that $\mu\mathcal{M}_1^* = \mathcal{M}_2^*$. The converse follows from the fact that every game is the dual of its dual. □

Lemma 3.3 holds irrespective of whether the compound representation is regular.

Lemma 3.3. *Let A and B be two minimal winning coalitions of a given component. If $\pi(A)$ intersects with component j but $\pi(B)$ does not, then the intersection of $\pi(A)$ with j is a minimal winning coalition of j.*

Proof. Let A and B be two minimal winning coalitions of a given component, and suppose that $\pi(A)$ intersects with component j but $\pi(B)$ does not. Let C be a minimal winning coalition (of the compound) that includes A (such a coalition C can be found because i is not a dummy of the quotient). The intersection of $\pi(C)$ with j is not empty, since it is the union of the intersection of $\pi(C - A)$ with j and the intersection of $\pi(A)$ with j. This intersection is in fact a minimal winning coalition of j, since $\pi(C)$ is minimal winning coalition of the compound. Hence, it is enough to show that the intersection of $\pi(C - A)$ with j is empty.

Note that—since $\pi(B)$ does not intersect with j—the intersection of $\pi(C - A)$ with j is equal to the intersection of $\pi((C - A) \cup B)$ with j. Suppose for contradiction that this intersection is not empty. Then, since $(C - A) \cup B$ is a minimal winning coalition of the compound, this intersection must in fact be a minimal winning coalition of j, which contradicts the fact that it is a strict subset of the intersection of $\pi(C)$ with j (which is itself a minimal winning coalition of j). □

Lemma 3.4 is only relevant for compound representations whose quotient is prime.

Lemma 3.4. *If the quotient is prime, there is a unique component u with the property that, for all minimal winning coalitions A of a given component, the image of A under π intersects with u.*

Proof. On the one hand, suppose for contradiction that there is no such component u. Then there are two minimal winning coalitions A and B of component i, and two components j and k, such that $\pi(A)$ intersects with j and not with k, and $\pi(B)$ intersects with

k and not with j. Indeed, if this was not the case, then, for every two minimal winning coalitions C and D of i, either $\pi(C)$ would intersect only with a subset of those components that $\pi(D)$ intersects with, or vice versa. But this would imply that there is a minimal winning coalition C of i such that, for every minimal winning coalition D of i, $\pi(D)$ intersects with all the components that $\pi(C)$ intersects with (so any component that $\pi(C)$ intersects with would serve as u).

I show that the sum of j and k is a committee of the quotient, which is a contradiction of the assumption that the latter is prime. Let S be a set (of components) such that $S \cup \{j, k\}$ wins and $S - \{j, k\}$ does not win in the quotient. By Lemma 3.3, the intersection of $\pi(A)$ with j is a minimal winning coalition of j, and the intersection of $\pi(B)$ with k is a minimal winning coalition of k. This means that no minimal winning coalition C (of the compound) that intersects with j can intersect with k (and vice versa);[22] that is, that S wins in the quotient if and only if it contains either l or k.

On the other hand, suppose for contradiction that there are (at least) two different components u_1 and u_2 with the property that, for all minimal winning coalitions A in i, the image of A under π intersects with both u_1 and u_2. I show that the product of u_1 and u_2 is a committee of the quotient, which is again a contradiction of the assumption that the quotient is prime.

Let S be a set (of components) such that $S \cup \{u_1, u_2\}$ wins and $S - \{u_1, u_2\}$ does not win in the quotient. Every minimal winning coalition C (of the compound) that contains a minimal winning coalition C_1 of u_1 also contains a minimal winning coalition of u_2 (and vice versa).[23] So S wins in the quotient if and only if S contains both u_1 and u_2. $\square$

3.3.5 Step 1 of the Proof

Proposition 3.5. *The map π is compatible with the representation $\mathcal{W}[\mathcal{W}_1, \ldots, \mathcal{W}_m]$.*

[22] To see this, let C be a minimal winning coalition (of the compound) that contains a minimal winning coalition of j and that intersects with k. Letting J and K denote the non-dummy player sets of j and k, the coalition $H = (C - J - K) \cup \pi(B) \cup \pi(A)$ is minimal winning of the compound, so the image of H under π^{-1} is also a minimal winning coalition of the compound. But this cannot be, since this image contains $A \cup B$, which is a strict superset of any minimal winning coalition.

[23] This is because, since π^{-1} is a symmetry of the compound, and the image of C under π^{-1} intersects with i, this intersection is in fact a minimal winning coalition of i. Hence, by the definition of u_2, $\pi(\pi^{-1}(C)) = C$ intersects with u_2, so this intersection is in fact a minimal winning coalition of u_2 (since $\pi(C)$ is a minimal winning coalition of the compound).

Proof. Fix an arbitrary component i. It is enough to show that there exists a component j such that the image of every minimal winning coalition of i under π intersects only with j.[24] Since the compound representation $\mathcal{W}[\mathcal{W}_1, \ldots, \mathcal{W}_m]$ is regular, the following three cases are exhaustive.

Case 1: The quotient $\mathcal{W}$ is a sum and the component $\mathcal{W}_i$ is not a sum: Let A and B be two minimal winning coalitions of i. Since the set of minimal winning coalitions of the compound consists of the union of the set of minimal winning coalitions of each component, we can assume without loss of generality that π maps A to a minimal winning coalition of some component j. Since component i is not a sum, we cannot partition its set of non-dummy players into two nonempty sets in such a way that there is no minimal winning coalition that intersects both of them. In other words, there is a set $\{A_1 = A, A_2, \ldots, A_l = B\}$ of minimal winning coalitions of i with the property that, for all $t = 1, 2, \ldots, l-1$, the coalition A_t intersects with the coalition A_{t+1}. Since π maps A_1 to a minimal winning coalition of j, and A_1 overlaps with A_2, π maps A_2 to a minimal winning coalition of j as well.[25] Iterating on this observation, we conclude that π maps B to a minimal winning coalition of j.

Case 2: The quotient $\mathcal{W}$ is a product and the component $\mathcal{W}_i$ is not a product: Since the dual of the product of games is the sum of their duals (see subsubsection 2.1.5), this case follows from the combination of *Case 1* and Lemma 3.2.

Case 3: The quotient $\mathcal{W}$ is prime: Let u be the unique component with the property that, for all minimal winning coalitions A of component i, the image of A under π intersects with u (see Lemma 3.4). Also, let T be the union of all components j for which there is a minimal winning coalition A of i such that $\pi(A)$ intersects with j. I prove that T is both a committee and a strict subset of all components. This implies that T is a singleton (that is, $T = \{u\}$) since the fact that the quotient is prime implies that T cannot be a proper committee.

First, I prove that T is a committee. For each component $j \in T - \{u\}$, let A_j be a

[24] Indeed, the same logic then proves that the image of every minimal winning coalition of j under π^{-1} intersects only with i, so π in fact maps every minimal winning coalition of i to a minimal winning coalition of j, and vice versa.

[25] Indeed, since A_2 is a minimal winning coalition of a component (and hence the compound), $\pi(A_2)$ is also a minimal winning coalition of the compound (and hence of some component). Since A_2 intersects with A_1, and $\pi(A_1)$ is a minimal winning coalition of j, $\pi(A_2)$ intersects with j (and is therefore a minimal winning coalition of j as well).

minimal winning coalition of j whose image under π^{-1} is a minimal winning coalition of i (we can find such coalitions by Lemma 3.3), and let A_u be a minimal winning coalition in u whose image under π^{-1} intersects with i.

Suppose for contradiction that there exist two sets (of components) S_1, S_2 and a subset Q of T such that both $S_1 \cup T$ and $S_2 \cup T$ win, both $S_1 - T$ and $S_2 - T$ lose, and $(S_1 - T) \cup Q$ wins but $(S_2 - T) \cup Q$ loses in the quotient.

Note that u must be an element of Q, because the image under π^{-1} of a coalition (of players) that does not intersect with u but intersects with $\pi(A)$, for some minimal winning coalition A of i, cannot be a winning coalition of the compound (since the image under π of every winning coalition of the compound that intersects with i intersects with u).

Let B_1 and C be two coalitions (of players) that contain at most one minimal winning coalition from each of the components in $S_1 - T$ and in $\{A_j\}_{j \in Q}$, respectively, and such that $B_1 \cup C$ is a minimal winning coalition of the compound (we can do this, because the union of one minimal winning coalition in each component in $(S_1 - T) \cup \{A_j\}_{j \in Q}$ wins in the compound). Note that the image of C under π^{-1} intersects with i, and hence wins in i.

Let B_2 be a coalition (of players) that contains one minimal winning coalition from each of the components in $S_2 - T$. Since the union of $S_2 - T$ with Q does not win in the quotient, $B_2 \cup C$ does not win in the compound. But since the union of S_2 with T wins in the quotient, the union of $B_2 \cup C$ with the union D of the coalitions $\{A_j\}_{j \in T-Q}$ wins in the compound. This contradicts the fact that—since the image of D under π^{-1} only intersects component i, and the image of C under π^{-1} wins in i—the image of $B_2 \cup C \cup D$ under π^{-1} wins in exactly the same components as does the image of $B_2 \cup C$ under π^{-1}.

Second, I prove that T is a strict subset of the set of all components. For contradiction, suppose otherwise. Let A be a minimal winning coalition (of the compound) that does not win i (such a coalition can be found because the quotient is prime, and hence it is not the product of i and some other game), let j be a component that $\pi(A)$ wins, let B_j denote the intersection of $\pi(A)$ with j, and let A_j be a minimal winning coalition of j whose image under π^{-1} intersects with i (we can find such A_j because of the assumption that T is the set of all components). Then the image of $(\pi(A) - B_j) \cup A_j$ under π^{-1} is a minimal winning coalition of the compound, and it intersects with component i; hence, this intersection is a minimal winning coalition of i. But, since $\pi(\pi^{-1}(A)) = A$ does not intersect i, this means that the image of A_j under π^{-1} is a minimal winning coalition of

i, which is only possible if j is equal to u (since the image of every minimal winning coalition in i under π intersects with u). Hence, the only component that $\pi(A)$ wins is u. This means that we can decompose the quotient as the sum of u and another game, a contradiction of the assumption that the quotient is prime. □

3.3.6 Step 2 of the Proof

Given Proposition 3.5, we can define, for every symmetry μ of the compound game, the map μ^* from the set of components to itself such that $\mu^*(i) = j$ if $\mu\mathcal{W}_i = \mathcal{W}_j$.

Proposition 3.6. *The map π^* is a symmetry of the quotient $\mathcal{W}$.*

Proof. Let S be a minimal winning coalition of the quotient. Since π^* is one-to-one, it is enough to show that the image of S under π^* is a minimal winning coalition of the quotient. For contradiction, suppose otherwise. Let the coalition A contain exactly one minimal winning coalition of each of the components with index in S. Then A is a minimal winning coalition of the compound, but π maps it to a non-minimal winning coalition of the compound, a contradiction. □

3.3.7 Step 3 of the Proof

Proposition 3.7. *If a permutation μ is compatible with the compound representation $\mathcal{W}[\mathcal{W}_1, \ldots, \mathcal{W}_m]$ and μ^* is a symmetry of $\mathcal{W}$, then μ is a symmetry of the compound set $\chi[X_1, \ldots, X_m]$.*

Proof. Let μ be a permutation that is compatible with the compound $\mathcal{W}[\mathcal{W}_1, \ldots, \mathcal{W}_m]$, and that is such that μ^* is a symmetry of the quotient $\mathcal{W}$. Let x be in the compound set $\chi[X_1, \ldots, X_m]$. Since μ is one-to-one, it is enough to show that $\mu(x)$ is also in this compound set. By definition,

$$x = \sum_{i=1}^{m} \alpha_i x_i, \text{ and } \mu(x) = \sum_{i=1}^{m} \alpha_i \mu(x_i).$$

for some $\alpha \in \chi$ and, for each $i = 1, \ldots, m$, $x_i \in X_i$. Since μ is an isomorphism between the components i and $\pi^*(i)$, by construction we have that μX_i is equal to $X_{\mu^*(i)}$. Also, since χ is a fair stable set of $\mathcal{W}$ and μ^* is a symmetry of $\mathcal{W}$, there exists $\beta \in \chi$ such that $\beta_{\mu^*(i)} = \alpha_i$ for all $i = 1, 2, \ldots, m$ (namely, $\beta := \mu^*\alpha$). Hence, we can write $\mu(x)$ as

$$\mu(x) = \sum_{i=1}^{m} \beta_{\mu^*(i)} \mu(x_i),$$

where $\beta \in \chi$ and, for $i = 1, \ldots, m$, $\mu(x_i) \in X_{\mu^*(i)}$; that is, $\mu(x)$ is in the compound $\chi[X_1, \ldots, X_m]$. □

4 Conclusion

Lloyd Shapley made fundamental contributions to the theory of simple games. In particular, he was the first to define and study compound simple games. One of the reasons he thought compound simple games are interesting is that they allow us to study the problem of aggregation of players in game theory. In his own words (Shapley, 1963c, pages 4-5):

> An important question in the application of n-person game theory is the extent to which it is permissible to treat firms, committees, political parties, labor unions, nations, etc., as though they were individual players. Behind every game model played by such aggregates, there lies another, more detailed model: a compound game of which the original is the quotient. Given any solution concept, it is legitimate to ask how well it stands up under the aggregation—or disaggregation—of its players. How sensitive are its theoretical predictions to the detail adopted in constructing the model?

Shapley's (1963c) composition theorem shows that the *stable sets* proposed by von Neumann and Morgenstern (1944) stand up well under the disaggregation of their players: A stable set of the gross model (the quotient), with details added at the component level, becomes a stable set of the refined model (the compound game).

In this article, I have shown that *fair stable sets*—that is, stable sets that do not discriminate among players based on their names—stand up well under the disaggregation of their players in a similar manner. This result can also be used to shed light on a question that Lloyd Shapley asked in 1978 and that remains open to this day: What is the set of simple games that admit a fair stable set? The composition theorem presented in this article implies that a game that does not admit a fair stable set must have a *factor*—or prime quotient in its unique factorization (Shapley, 1967)—that does not admit a fair stable set.

This raises several natural questions that I leave for future research. For example: Is there any game that admits a fair stable set some of whose factors do not admit a fair stable set? Or: What is the set of prime games that admit a fair stable set? Answers to these questions might provide the key to the characterization of the set $\mathcal{G}$ of simple games that admit a fair stable set. In particular, the composition result presented in this article implies that if the answer to the first question is negative, answering the second question would be equivalent to characterizing the set $\mathcal{G}$.

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