Nash Bargaining with Endogenous Outside Options

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Job Market Paper

Abstract

Outside options shape bargaining outcomes, but understanding how they are determined is often challenging, because one’s outside options depend on others’ outside options, which depend, in turn, on others’ outside options, and so on. In this paper, I describe a non-cooperative theory of coalition formation that shows how the Nash bargaining solution uniquely pins down both the sharing rule and the relevant outside options in each potential coalition. This provides a framework to investigate how preference and productivity shocks propagate via outside options. In pairwise matching settings where agents are vertically differentiated by their skills, shocks propagate from the high to the low skill, but not vice versa. Positive assortative matching necessarily arises if and only if skills are complementary, in which case shocks propagate in blocks—in the sense that a shock that propagates from one agent to another one also affects everyone whose skill is in between.

1 Introduction

The Nash bargaining solution is a central concept in economics. It provides a sharing rule in any given coalition as a function of its members’ outside options. Its clean axiomatic

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foundations (Nash 1950) and close connections to strategic bargaining (Binmore 1987) make it theoretically appealing, and its simple functional form makes it convenient in applications.

In many settings of interest, however, agents simultaneously bargain over both which coalitions to form (e.g., which firms employ which workers, which entrepreneurs become partners, which businesses form strategic alliances, etc.) and how to share the resulting gains from trade (e.g., wages, equity shares, etc.), and using the Nash bargaining solution in these settings requires a theory of how the relevant outside options are determined. For example, the outside options of a job candidate when bargaining with a potential employer are often determined by the bargaining outcomes with alternative employers, which depend, in turn, on these employers’ outside options, and so on. Hence, understanding the resulting bargaining outcomes requires a theory that somehow cuts this outside option Gordian knot.

In this paper, I describe a non-cooperative theory of coalition formation that shows how the Nash bargaining solution uniquely pins down both the sharing rule and the relevant outside options in each potential coalition. The key observation that cuts the outside option Gordian knot is that there always exists at least one coalition that is sufficiently productive so that—when bargaining to form this coalition—none of its members has a credible outside option. This allows a recursive characterization of the relevant outside options in each potential coalition. The resulting payoff profile is—in the limit as the bargaining friction vanishes—the only one that is consistent with two principles, roughly stated as follows: Surplus within each coalition is shared according to the Nash bargaining solution, and each agent can justify her outside option in each coalition as resulting from the Nash bargaining solution in another coalition without appealing to her outside option there. In other words, the non-cooperative approach suggests a principle—no circularity in outside options—which combined with the Nash bargaining solution in another coalition without appealing to her outside option there. In other words, the non-cooperative approach suggests a principle—no circularity in outside options—which combined with the Nash bargaining solution uniquely pins down the bargaining outcomes. Figure 1 illustrates a simple example of the negotiation dynamics that can lead to circularity in outside options (which the strategic forces in the non-cooperative game rule out).

This paper provides a framework to investigate how bargaining outcomes are determined in decentralized markets, and is broadly consistent with the view that bargaining is more prominent in the determination of high-skill than low-skill wages. For example, the theory that emerges from this paper predicts that—in two-sided markets where agents are vertically differentiated by their skills—bargaining outcomes are determined from the top down. In

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2 Hall and Krueger (2012) and Brenzel, Gartner, and Schnabel (2014) document a positive correlation between education and wage bargaining in the United States and Germany, respectively.

3 This is reminiscent of—but qualitatively different from—the result in Elliott and Nava (2018) that shows that vertically differentiated markets can endogenously clear from the top down.
Figure 1: An example of the negotiation dynamics between MBA graduates, Wall Street firms and Main Street firms that might lead to the MBA’s outside options being determined in a circular way. When bargaining on Wall Street, MBA’s demand to obtain at least $w$ because they can earn $w$ on Main Street. But the only reason that Main Street pays $w$ is that Wall Street does.

In particular, preference and productivity shocks propagate from the high to the low skill, but not vice versa. As in the canonical marriage model of Becker (1973), agents necessarily match in a positive assortative way if and only if the production function is supermodular. In this case, shocks propagate in blocks, in the sense that a shock that propagates from one agent to another one also affects every agent whose skill is in between.

This paper is related to Binmore, Rubinstein, and Wolinsky (1986), who use a strategic bargaining model in a fixed coalition to investigate how exogenous outside options enter the Nash bargaining solution. The unique subgame-perfect equilibrium of their game predicts that—as the bargaining friction vanishes—the surplus is shared according to the Nash bargaining solution, with the threat points corresponding to the utilities that the agents get in autarky, and the outside options entering only as lower bounds on the payoffs. This is the “outside option principle.” (e.g. Sutton 1986).

For example, consider the situation described in Figure 2, where a recent graduate and an

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4 In many applications, there are different sensible alternatives for both what the relevant outside options are and how they enter the Nash bargaining solution—and different alternatives have qualitatively different implications. For example, the extent to which unemployment is a relevant outside option in wage bargaining determines the effects of unemployment insurance on the labor market—e.g. Pissarides (2000), Krusell et al. (2010), Hagedorn, Karahan, Manovskii, and Mitman (2013) and Chodorow-Reich, Coglianese, and Karabarbouris (2018)—and the ability of macroeconomic models to generate realistic employment fluctuations—e.g. Shimer (2005), Hall and Milgrom (2008), Sorkin (2015), Chodorow-Reich and Karabarbouris (2016), Hall (2017) and Ljungqvist and Sargent (2017).

5 Binmore, Rubinstein, and Wolinsky (1986, Proposition 6) derive the outside option principle in the context of two related models. The bargaining friction incentivizes the agents to reach agreements via their time preferences in one, and via their risk preferences in the other.
employer (both risk neutral) can generate 1 dollar by matching. Suppose that (i) the graduate can sell her labor elsewhere at wage \( w < 1 \), (ii) the employer can hire an equally valuable recent graduate at wage \( w' > w \), and (iii) neither the employer nor the graduate in autarky generate any value. In this case, the outside option principle suggests that the employer hires the graduate at wage \( 1/2 \) (as specified by the Nash bargaining solution with the threat point determined by autarky), unless \( w > 1/2 \) or \( w' < 1/2 \), in which case it suggests that the employer hires the graduate at wage \( w \) or \( w' \), respectively. Intuitively, an agent’s outside option only affects her bargaining power if it is credible, in the sense that her outside option is better than what the Nash bargaining solution would otherwise give her.\(^6\)

Importantly, however, the outside option principle is silent about how the relevant outside options in each coalition are determined. For instance, in the example just described, the wages \( w \) and \( w' \) at which the graduate and the employer, respectively, can match elsewhere are taken as given. But, in many cases, these wages are themselves the result of bargaining with third parties. From this perspective, the contribution of this paper is to describe a non-cooperative theory that shows not only how outside options enter the Nash bargaining solution, but also how the Nash bargaining solution pins down the relevant outside options in each coalition.

In particular, I consider different types of agents that enter a market over time in such a way that there are always agents of each type looking to form a coalition. The model is intended to capture the predominant economic forces in large markets with dynamic entry, where the relevant matching opportunities are roughly constant over time.\(^7\) Examples

\(^{6}\)Binmore, Shaked, and Sutton (1989) provide experimental evidence that is consistent with the outside option principle. More recently, Jäger, Schoefer, Young, and Zweimüller (2018) find that real-world wages are insensitive to sharp increases in unemployment insurance benefits, which suggests that unemployment is not a credible outside option in wage bargaining.

\(^{7}\)In particular, I assume that the surplus of each match is independent of which other matches have formed or will form in the future. The approach is similar in spirit to the one in Rubinstein and Wolinsky (1985) and the subsequent literature studying non-cooperative bargaining in stationary markets.
include thick labor markets where workers and firms arrive over time in search of profitable (potentially many-to-many) matches, and innovation hubs where startups and entrepreneurs cluster to form (potentially multilateral) strategic alliances.

In the model, agents bargain according to a standard protocol (in the spirit of the canonical alternating-offers model of Rubinstein 1982) over both which coalitions to form and how to share the resulting gains from trade. The bargaining friction that incentivizes the agents’ to reach agreements is their fear that an exogenous reason will prevent them from matching in the future. As a result, as in the classical bargaining framework of Nash (1950), risk preferences are essential drivers of the bargaining outcomes.\(^8\)

I show that the model admits an essentially unique stationary subgame-perfect equilibrium, and I characterize which coalitions form and how the resulting gains from trade are shared in this equilibrium. In the limit as the bargaining friction vanishes, the equilibrium sharing rule in each coalition is the one prescribed by the Nash bargaining solution, with the relevant outside options entering as prescribed by the outside option principle, and determined as follows:

Each type’s outside option in any given coalition is her maximum Nash bargaining share in another coalition subject to the others’ outside options.

The main result of this paper is that there is a unique outside option profile that satisfies this property, and that the non-cooperative bargaining model suggests it as a natural point for the agents to settle on when bargaining in a decentralized way. Roughly speaking, non-cooperative bargaining forces each agent to justify her outside option in each coalition without appealing to her own outside option in another coalition, and this prevents outside options from being determined in a circular way. For example, as illustrated in Figure 1, this prevents MBA graduates from justifying a wage \(w\) on Wall Street by arguing that this is what they get on Main Street, while the only reason that Main Street pays them \(w\) is that Wall Street does.

In order to understand how the bargaining outcome is determined, the coalitions that form in equilibrium can be organized into tiers, in such a way that the surplus in each coalition is shared—in the limit as the bargaining friction vanishes—according to the Nash bargaining solution, with the binding outside options determined in higher tiers. This implies that

\(^8\)Analogous results can be derived if one assumes that no agent is ever exogenously forced out of the market but that—instead—the agents are impatient. The Nash bargaining solution then has to be appropriately constructed from agents’ time preferences (see for example Osborne and Rubinstein 1990).
(small) preference and productivity shocks propagate—via outside options—from higher to lower tiers, but not vice versa. This tier structure is behind most of the comparative statics that I describe in this paper. In particular, it reveals that, in pairwise matching settings where preferences are homogeneous and agents are vertically differentiated by their skills, preference and productivity shocks propagate from the high to the low skill, but not vice versa. Analogously, when skills are homogeneous and agents are vertically differentiated by their risk aversion, preference and productivity shocks propagate from the most to the least risk averse, but not vice versa. Intuitively, in any bargaining encounter, the option of bargaining with a less skilled but equally risk averse or a less risk averse but equally skilled agent is not credible. As a result, the preferences and productivities of types that are on the same side of the market but are less skilled (in the former case) or less risk averse (in the latter case) than a given type do not affect her bargaining position.

**Roadmap**

The rest of this paper is organized as follows. I start in section 2 by illustrating the setting and the main result of this paper with a simple example. I then describe the model in section 3 and its essentially-unique stationary subgame-perfect equilibrium in section 4. I illustrate the comparative statics of the resulting theory in section 5, and I further discuss the contribution of this paper to the related literature in section 6. Finally, I conclude in section 7. The formal proofs of most of the results are deferred to Appendix A.

**2 Illustration of the setting and the main result**

In this section, I illustrate the setting and the main result of this paper using the following example. Consider a large city where different agents (in the culinary industry, say) go to in search of business opportunities. For simplicity, assume that there are only four types of agents in this industry: Maîtres, managers, chefs and cooks, all of them risk neutral.

Agents of all types arrive to the city over time (perhaps with the excuse of attending a prestigious culinary school) to find potential partners with whom to start a business venture. For simplicity, assume that each agent can only be part of one such venture (because each feasible venture is a lifelong full-time project, say), and that only bilateral partnerships

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9I am grateful to Rachel Kranton for encouraging me to illustrate the results of this paper along the lines of Hart and Moore’s (1990) gourmet seafare example.
Figure 3: A chef generates 100 dollars when she matches with a maître (by starting a high-end restaurant, say) and 80 dollars when she matches with a manager (by starting the occasional low-end restaurant with great food, say). Similarly, a cook generates 60 dollars when she matches with a maître (by starting the all-too-common high-end restaurant with unimpressive food, say) and 50 dollars when she matches with a manager (by starting a low-end restaurant, say).

between one maître/manager and one chef/cook are feasible. Moreover, assume that the surplus of each match is independent of which other matches form (because each venture is implemented in a different part of the world, say), as illustrated in Figure 3.\(^\text{10}\)

Which ventures form, and how is the resulting surplus shared? How does an increase in the value of the chef-maître partnerships (caused by a global increase in high-end tourism, say), or an improvement in the chefs’ bargaining position (caused by a new technology that allows them to directly fly food to their clients’ doors, say) affect this market?

In order to investigate these questions, I study the equilibrium behavior of these agents when they bargain according to an infinite-horizon protocol in the spirit of the alternating-offers model of Rubinstein (1982): In each period, one of the agents that is in the city hoping to form a business venture is selected uniformly at random to be the proposer. The selected agent can propose a match as well as how to share its surplus. This captures the idea that starting a business venture requires that someone has an idea: Once an agent has an idea, she can propose to implement it with another agent who, in turn, decides whether to join this venture (at the proposed terms of trade) or to wait for better opportunities to arise.

The bargaining friction that incentivizes the agents to reach an agreement is that, after each period, each agent has to leave the city (because of personal reasons, say) with some probability \(q\), preventing her from starting a business. For simplicity, I normalize to zero the surplus that each agent obtains when she has to leave the market before she has created a venture. Hence, when an agent is deciding whether to accept or reject an offer, she has to

\(^{10}\)Food being the most important part of a culinary experience, I assume that the match between an outstanding a chef and a regular manager generates more value than a match between a cook and a maître.
trade off the potential for better opportunities arising in the future (e.g., having a business idea herself), with the risk of having to leave the market before matching.

The unique (stationary subgame-perfect) equilibrium of this game is illustrated in Figure 4. Chefs propose forming business ventures with maîtres, and vice versa. Hence, even if chefs and maîtres can match with managers and cooks, respectively, they effectively bargain over how to share their gains from trade as if they were the only two types in the market. Intuitively, the fact that a chef can always find a maître to bargain with, and vice versa, implies that their surpluses in other matches do not affect their bargaining power. Indeed, since making offers to others is off the equilibrium path, and an agent never benefits from receiving an offer (since equilibrium offers leave the receiver indifferent), in the limit as the bargaining friction \( q \) vanishes, chefs and maîtres equally share their gains from trade. More generally, their terms of trade are as suggested by the Nash bargaining solution, with the threat points given by their payoffs when they are forced out of the city before they can start a business (which are normalized to zero), as prescribed by the outside option principle.

When a manager is the proposer, she offers to match with a chef. In this case, the chef has to trade off the gains from accepting such an offer with the expected gains of waiting to be able to make an offer in the future (at the risk that she might be forced to leave the market before this happens). As a result, when the bargaining friction \( q \) is small enough, the manager has to offer a chef close to 50 dollars (approximately what she gets when proposing to match with a maître) for her to accept. In particular, in the limit as the bargaining friction \( q \) vanishes, chefs and managers share their surplus 50 – 30, as prescribed by the outside option principle.\(^\text{11}\) Similarly, the cooks propose to match with the managers, and—in the limit as

\(^{11}\)As long as the bargaining friction \( q \) is positive, chefs can obtain more from maîtres than from managers
Figure 5: The number associated to an arrow that points from type $i$ to type $j$ is how much $i$ can justify in the $ij$ partnership using the Nash bargaining solution while honoring $j$’s equilibrium (limit) payoff. For example, the chefs’ Nash bargaining share with managers is 40 (the managers’ outside option of 30 does not bind in this case). Chefs can justify a payoff of 50 as being the result of Nash bargaining with maitres while honoring the maitres payoff of 50.

the bargaining friction $q$ vanishes—they share their surplus $20 – 30$, again as prescribed by the outside option principle.

Figure 5 illustrates how, as the bargaining friction vanishes, each type’s payoff is the maximum that she can justify as resulting from the Nash bargaining solution in some match while honoring the others’ payoffs. Remarkably, there is a unique payoff profile that satisfies this property, and the strategic forces in the non-cooperative model suggest it as a natural point to settle on. Informally, the strategic forces in the non-cooperative model require that:

Each type has to be able to justify her outside option in any match as resulting from the Nash bargaining solution in another match without appealing to her own outside option there.

This prevents outside options from being determined in a circular way, and explains why the outcome is uniquely pinned down in equilibrium. For instance, without this requirement, the chefs would be able to justify that their outside option when bargaining with a maître is getting 55 dollars, say, from a manager, by arguing that, when bargaining with a manager, their outside option is getting 55 from a maître. However, the strategic forces in the model prevent both chefs and maîtres to obtain more than what they can justify by their Nash bargaining shares in their match, and this pins down the equilibrium outcome.

In order to understand how bargaining outcomes are determined, the partnerships that...
form in equilibrium can be organized into tiers, with the surplus of each partnership shared—in the limit as the bargaining friction $q$ vanishes—according to the Nash bargaining solution, and the binding outside options determined in higher tiers. Figure 6 illustrates that the chef-maître partnership is in the first tier. The Nash bargaining solution in this partnership pins down the chefs’ binding outside option when bargaining with managers which, in turn, pins down the managers’ binding outside option when bargaining with cooks.

This equilibrium tier structure illustrates how preference and productivity shocks propagate—via outside options—from higher to lower tiers, but not vice versa. For instance, an increase in the surplus of the chef-maître partnership propagates downwards—via the chefs’ and managers’ outside options—to affect everyone. But an increase in the surplus of the cook-manager partnership only affects cooks, who absorb the whole surplus increase.

I now turn to describing how the results illustrated in this section generalize to settings with arbitrarily many types with (potentially) different risk preferences, and where the productive coalitions can be of arbitrary form and size.

### 3 Model: The bargaining game $G$

As emphasized in section 1 above, the model is intended to capture the predominant economic forces in large markets with dynamic entry—where the relevant matching opportunities are roughly constant over time. For simplicity, and following the approach of Rubinstein and Wolinsky (1985) and the subsequent literature on non-cooperative bargaining in stationary markets, I assume that the agents enter the market over time in such a way that there is always one active agent of each type in the market. The model is common knowledge, and

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12 In Elliott and Talamás (2018), we show how these markets can be alternatively modeled by having an exogenous process that determines how agents enter the market over time. While the results of this paper
features perfect information.

3.1 Primitives

There is a finite set $N$ of $n$ different types of agents, and a sequence of agents of each type. Different types of agents can—by matching—produce different amounts of perfectly divisible surplus (e.g., money). For simplicity, I assume that each match containing at least one agent of each type in $C \subseteq N$ produces $y(C) \geq 0$ units of surplus when it forms.$^{13,14}$ For simplicity, I assume that each agent of each type $i$ produces $y(i) > 0$ units of surplus by matching to herself, which ensures that each agent finds it profitable to be in the market.$^{15}$

While surplus is perfectly divisible, the utility generated by each match is—in general—imperfectly transferable, because the agents’ utility functions need not be linear in money. In particular, as in the canonical bargaining setting of Nash (1950), the preferences of each agent of type $i$ are represented by the von-Neumann Morgenstern utility function $u_i$, which is a concave, strictly increasing, and twice-continuously differentiable function of the money that she gets, with $u_i(0)$ normalized to 0 units of surplus.

3.2 Bargaining protocol

Bargaining occurs in discrete periods $t = 1, 2, \ldots$. In each period, the first agent in sequence (yet to leave the market) of each type is active. One active agent is selected uniformly at random to be the proposer. The proposer, of type $i$ say, chooses one coalition $C \subseteq N$ containing $i$, and proposes a split of the corresponding surplus among its members. The active agents of each type in $C - i$ then decide in (a pre-specified) order whether to accept or reject this

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$^{13}$As long as there is an upper bound on how many agents of a given type are productive in a given coalition, the fact that the surplus of each coalition does not depend on whether it contains one or more agents of a given type is without loss of generality, because types can always be defined so that this property holds. For example, suppose that everyone is identical, and that coalitions of one and two agents produce 1 and 2 units of surplus, respectively. This can be captured by letting there be two types of agents, with coalitions consisting of an agent of any one of these types producing 1 unit of surplus, and coalitions containing both these types producing 2 units of surplus.

$^{14}$For expositional clarity, I usually reserve the term “coalition” to refer to a set of types, while I use the term “match” to mean a set of agents (that match).

$^{15}$The quantity $y(i)$ can be interpreted as an exogenous outside option: How much an agent can obtain without the consent of any other agent.
If all of them accept, then they match with the proposer and they, together with
the proposer, leave the market with the agreed shares. Otherwise, they, and the proposer,
wait for the next period, as do all the active agents that are neither proposers nor receivers of
an offer in this period. At the end of each period, each active agent is independently forced
to leave the market with probability \( q > 0 \), in which case she obtains 0 surplus.\(^{17}\)

### 3.3 Histories, strategies and equilibrium

For each period \( t \), let \( h_t \) be a history of the game up to (but not including) period \( t \), which is
a sequence of \( t \) pairs of proposers and coalitions proposed—with corresponding proposals
and responses. There are two types of histories at which some agent must take an action.
First, \((h_t, i)\) consists of \( h_t \) followed by the active agent of type \( i \) being selected to be the
proposer in period \( t \). Second, \((h_t, i \rightarrow C, x, j)\) consists of \((h_t, i)\) followed by the active agent
of type \( i \) proposing that coalition \( C \) shares its surplus according to the profile \( x \) in \( \mathbb{R}^C_{\geq 0} \), and
all the active agents in \( C \) preceding type \( j \) in the response order having accepted.

A strategy \( \sigma_i \) for type \( i \) specifies, for all possible histories \( h_t \), the offer \( \sigma_i(h_t, i) \) that she
makes following the history \( (h_t, i) \) and her response \( \sigma_i(h_t, j \rightarrow C, x, i) \) following history
\( (h_t, j \rightarrow C, x, i) \).\(^{18}\) The strategy profile \( (\sigma_i)_{i \in N} \) is a stationary subgame-perfect equilibrium of the
game \( G \) if it induces a Nash equilibrium in the subgame following every history, and if no
type’s strategy conditions on the history of the game except—in the case of a response—on
the going proposal and on the identity of the proposer. I often refer to a stationary equilib-
rium simply as an equilibrium. I now turn to showing that this game admits an essentially
unique equilibrium, and to investigating which coalitions form and how the resulting sur-
plus is shared in this equilibrium.

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\(^{16}\)The order in which the agents respond is not relevant for the results.

\(^{17}\)Note that an agent of type \( i \) obtains 0 units of surplus when she is forced out of the market, but obtains
\( y(i) > 0 \) when she decides to match with herself while she is in the market. This could be the case, for
example, because the market offers more than just the opportunity to match with others. I assume that these
two quantities are different for simplicity (in order to guarantee that every agent has incentives to be in
the market) but this assumption is not important for the results.

\(^{18}\)I allow for mixed strategies, so \( \sigma_i(h_t, i) \) and \( \sigma_i(h_t, j \rightarrow C, x, i) \) are probability distributions over \( 2^N \times \mathbb{R}^N_{\geq 0} \)
and \( \{\text{Yes, No}\} \), respectively.
4 Essentially unique equilibrium

In this section, I show that the bargaining game $G$ admits an essentially unique equilibrium, and I describe an algorithm that characterizes it. This provides the basis for the main result of this paper, which shows how—in the limit as the bargaining friction $q$ vanishes—the equilibrium payoff profile is—roughly speaking—the only one that is consistent with the following two principles: The sharing rule in each coalition is given by the Nash bargaining solution, and each type obtains the maximum that she can justify in some coalition while honoring the others’ outside options.

I start in subsection 4.1 by deriving the system of equations that determines the equilibrium payoffs. This system formalizes the outside option Gordian knot described in section 1: Each type’s payoff (in a period in which she is not the proposer) is determined by the maximum surplus that she can generate (when she is the proposer) net of the others’ payoffs. But the others’ payoffs depend, in turn, on the others’ payoffs, and so on. The objective of most of the rest of this section is to show that this system admits a unique solution. The strategy to show this is based on the observation that there exists at least one coalition that is sufficiently productive so that—when bargaining in this coalition—none of its members’ outside options bind. For instance, in the example of section 2, the chef-maître partnership is sufficiently productive so that neither chefs nor maîtres outside options are credible when bargaining to form this partnership. As a result, they share the surplus of this partnership equally, and this determines their outside options when bargaining in any other coalition.

I divide the characterization strategy into three steps. First, in subsection 4.2, I describe an auxiliary non-cooperative game of isolated bargaining within each coalition, and I show how the equilibria of these auxiliary games can be used to derive an upper bound on each type’s equilibrium payoff in the game $G$. Second, in subsection 4.3, I show that there exists at least one coalition where this bound is tight for all of its members, which implies that this bound is actually the payoff of all of the members of this coalition in every equilibrium of the game $G$. Third, in subsection 4.4, I leverage these observations to recursively characterize the unique equilibrium payoffs of all the types.

In subsection 4.5, I highlight that the equilibrium payoff profile converges to the unique profile that gives each type the maximum that she can justify in some coalition using the Nash bargaining solution while honoring the others’ payoffs. Finally, in subsection 5.1, I describe how the coalitions that form in equilibrium can be usefully organized into tiers in such a way that preference and productivity shocks propagate via agents’ outside options from higher to lower tiers, but not vice versa. This is the essence of many of the comparative
statics of the theory illustrated in section 5.

4.1 Equilibrium cutoff profile

Proposition 4.1 describes the relevant features of the essentially unique stationary equilibrium of the game $G$. This equilibrium is essentially unique in the sense that each type’s payoff is the same in every equilibrium and, in addition, each type’s proposals and (on-the-equilibrium-path) responses are the same in every equilibrium except in non-generic cases in which one type’s maximum surplus net of the others’ payoffs is achieved in more than one coalition.

Proposition 4.1. Each type $i$ has a cutoff $w_i > 0$ such that, in every stationary subgame-perfect equilibrium of $G$, she always proposes that one of the coalitions $C \ni i$ with the biggest net surplus $y(C) - \sum_{j \in C - i} w_j$ forms, and she offers each of its members $j \neq i$ the amount $w_j$, all of whom accept.

Figure 4 illustrates Proposition 4.1 in the context of the example described in section 2. The cutoff of chefs and maître is 48, the cutoff of managers is 29, and the cutoff of cooks is 19. In this case, each proposer $i$ finds the type $j$ that maximizes the net surplus $y(i, j) - w_j$ and offers the active agent of type $j$ her cutoff $w_j$ (and all such offers are always accepted). In particular, chefs and maîtres make offers to each other, managers make offers to chefs, and cooks make offers to managers.

While I defer the proof of Proposition 4.1 to the appendix, subsection 4.2, subsection 4.3 and subsection 4.4 informally describe the argument behind this proof. The immediate objective here is to derive the system of equations (1) that determines the equilibrium payoffs in the bargaining game $G$. To do this, consider a stationary subgame-perfect equilibrium of this game. For each type $i$, let $w_i$ be the amount that type $i$ is indifferent between accepting and rejecting in any given period. On the equilibrium path, type $j$ accepts every offer that gives her exactly $w_j$ (otherwise, the proposer would have no best response), so the maximum amount that type $i$ can obtain when she is the proposer is the maximum net surplus $y(C) - \sum_{j \in C - i} w_j$ over all coalitions $C \subseteq N$. This quantity is strictly positive, since $y(i) > 0$ for each type $i$, so all the proposers make acceptable offers in equilibrium.\(^\text{20}\)

By definition, each type $i$ is indifferent between obtaining $w_i$ right away, which gives her\(^\text{19}\)All these quantities are approximated to the nearest integer.

\(^\text{20}\)To see this, let $V_i$ and $W_i$ be the expected utility of an agent of type $i$ when she is and is not selected to be the proposer, respectively. We have that $W_i = (1 - q)(pV_i + (1 - p)W_i)$, so $V_i > 0$ implies that $W_i < V_i$. Hence, every agent is strictly worse off by delaying.
utility $u_i(w_i)$, and waiting for the next period, which gives her an expected utility of

$$q u_i(0) + (1 - q) \left[ \frac{1}{n} u_i \left( \max_{C \subseteq N} \left[ y(C) - \sum_{j \in C-i} w_j \right] \right) + \frac{n - 1}{n} u_i(w_i) \right]$$

To see this, note that waiting one period involves a risk of being forced to leave the market (which materializes with probability $q$, and in which case the agent gets zero). In the event that she is not forced to leave at the end of the period, the agent has the opportunity to make a proposal in the next period with probability $1/n$, in which case she obtains $y(C) - \sum_{j \in C-i} w_j$. Otherwise, she either receives an offer that gives her $w_i$ (which she accepts), or she does not receive any offer. In either case, her expected utility is $u_i(w_i)$. Recalling that $u_i(0)$ is normalized to 0, and rearranging terms, gives that the expected utility of an agent when she is not the proposer is determined by her expected utility when she is the proposer:

$$u_i(w_i) = \frac{1 - q}{1 - q + nq} u_i \left( \max_{C \subseteq N} \left[ y(C) - \sum_{j \in C-i} w_j \right] \right)$$

In words, the expected utility of an agent when she is not the proposer is a fraction of her utility when she is, and this fraction converges to 1 as the bargaining friction $q$ converges to 0. In particular, in the absence of bargaining frictions (i.e. when $q = 0$), system (1) becomes

$$w_i = \max_{C \subseteq N} \left[ y(C) - \sum_{j \in C-i} w_j \right]$$

which, in general, has a great multiplicity of solutions.\textsuperscript{21} For instance, in the example described in section 2, one extreme solution to system (2) is the profile that gives chefs and cooks 100 and 60, respectively, and zero to both maîtres and managers. Another extreme solution to system (2) is the profile that gives chefs and cooks zero, and maîtres and managers 100 and 50, respectively.

I now turn to showing that, as long as the bargaining friction $q$ is positive, system (1) has a unique solution, which I refer to as the equilibrium cutoff profile. The limit of this profile as the bargaining friction $q$ vanishes selects a unique profile out of the many possible solutions to system (2). In other words, the unique equilibrium of the non-cooperative game picks one

\textsuperscript{21}Each of the solutions to system (2) can be seen as a profile of competitive equilibrium prices, in the sense that every proposer obtains the maximum possible amount of surplus taking as given the others’ wages. The dynamic entry of agents into the market implies that the market clearing condition does not discipline the prices in this setting.
of the many plausible bargaining outcomes in the frictionless case. Most of the rest of this paper is devoted to describing the properties of this solution, and to using this solution to shed light on the determinants of bargaining outcomes in decentralized markets.

4.2 An upper bound on each type’s equilibrium cutoff

The objective here is to provide an upper bound on the equilibrium cutoff of each type. The idea is the following: When proposing that a coalition forms, its members’ outside options have to be met in order to induce them to accept. Hence, roughly speaking, an agent cannot be made worse off when the others’ outside options deteriorate. This suggests that the equilibrium cutoff of an agent in the hypothetical situation in which she can choose any coalition and prevent its members from making offers to any other coalition is an upper bound on her equilibrium cutoff.\footnote{Preventing agents from accepting offers from other coalitions is not necessary for this exercise, because—given that equilibrium offers leave the respondent indifferent between accepting and rejecting—the source of agents’ bargaining power is what they can obtain when they propose.}

In order to formalize this idea, I consider a family \( \{G_C\}_{C \subseteq N} \) of variants of the bargaining game \( G \) in which only one coalition \( C \subseteq N \) is allowed to form.\footnote{More precisely, for each coalition \( C \subseteq N \), the game \( G_C \) is defined exactly as the bargaining game \( G \), with the following modification: The surplus \( y(D) \) of each coalition \( D \neq C \) is reduced to 0.} The game \( G_C \) is analogous to a multilateral version of the canonical alternating-offers model of Rubinstein (1982), where the one feasible coalition \( C \) is given exogenously, and its members only bargain over how to share the surplus \( y(C) \) of this coalition. Hence—as is the case for this canonical game—each game \( G_C \) has a unique stationary subgame-perfect equilibrium. Moreover, as in Binmore, Rubinstein, and Wolinsky (1986), the equilibrium cutoff profile in this game converges to the Nash bargaining solution in coalition \( C \), with the threat points given by autarky. In other words, the equilibrium cutoff profile in the auxiliary game \( G_C \) converges to the unique profile that solves

\[
\arg\max_{s \in \mathbb{R}^C} \prod_{j \in C} u_j(s_j) \quad \text{subject to the feasibility constraint } y(C) \geq \sum_{j \in C} s_j.
\]

For instance, Figure 7 illustrates each type’s cutoff in each relevant auxiliary game of the example of section 2. Given that the agents’ preferences in this example are homogeneous, isolated bargaining between two agents essentially leads to equal sharing. More precisely, each type’s cutoff in the auxiliary game associated with any given partnership (that she part of) is just below half of its surplus. This reflects the fact that the proposer obtains a
Figure 7: The number associated with a link between type $i$ and type $j$ is approximately their equilibrium cutoff in the auxiliary game in which the surplus of all the other partnerships is artificially set to zero. An arrow from type $i$ to type $j$ indicates that the $ij$ partnership is $i$'s best coalition, and the associated number is her best wage.

...bit more than half of the available gains from trade. But, in the limit as the bargaining friction vanishes, these cutoffs converge to exactly half of the surplus of the corresponding partnership—as prescribed by the Nash bargaining solution.

With the equilibrium payoffs in each auxiliary game at hand, we can formalize the exercise described above: Informally, suppose that we ask each type: “Which coalitions would you be happy choosing if you could pick one coalition $C \subseteq N$ and bargain in isolation with its members (according to the auxiliary game $G_C$)?” Each type’s best coalitions are the ones that she would point to, and her best wage is her equilibrium cutoff in this hypothetical situation. More precisely, I define each type’s best wage to be her maximum equilibrium cutoff—across all coalitions $C \subseteq N$—in the auxiliary game $G_C$, and I say that a coalition $C \subseteq N$ is one of type $i$’s best coalitions if $i$’s equilibrium cutoff in $G_C$ is her best wage. Finally, I say that a coalition is a perfect coalition if it is a best coalition of all of its members.

For instance, the best wage of both chefs and maîtres is 48, and their best coalition is the chef-maître partnership, so this partnership is a perfect coalition. The best wage of managers is 38, and their best coalition is the chef-manager partnership. Finally, the best wage of cooks is 29, and their best coalition is the cook-maître partnership.

Proposition 4.2 below formalizes the intuition that the equilibrium cutoff of each type in the bargaining game $G$ cannot be larger than her best wage. This is especially useful because—as I show in subsection 4.3 below—this bound is always tight for at least one type. In fact, in subsection 4.4, I leverage this observation to recursively pin down everyone’s payoff.

**Proposition 4.2.** Let the profile $w$ in $\mathbb{R}^N$ be a solution to system (1). For each type $i$, $w_i$ is bounded above by $i$’s best wage.

Figure 7 illustrates the hypothetical exercise described above in the context of the example in section 2. For instance, if cooks were able to choose between maîtres and managers, and
bargain with them in isolation, their equilibrium cutoff would be 29 and 24, respectively. Intuitively, 29 must then be an upper bound on the cooks’ equilibrium cutoff, because, in the bargaining game $G$, both managers and maîtres can in fact choose to make offers to chefs as well, which can only improve their bargaining position. Indeed, as illustrated in Figure 4, their equilibrium cutoff is 19.

### 4.3 The upper bound is tight for at least one type

I now describe how the upper bound on each type’s equilibrium cutoff provided by Proposition 4.2 above is tight for at least one type, which provides the basis of the recursive characterization of everyone’s equilibrium cutoffs in subsection 4.4. In particular, the combination of the two results below (Corollary 4.1 and Proposition 4.3) implies that we can always find a coalition that is sufficiently productive so that none of its members’ outside options bind in equilibrium.

On the one hand, Proposition 4.2 above implies that, for each type $i$ in a perfect coalition $C$, her equilibrium cutoff in the auxiliary game $G_C$ is an upper bound on her equilibrium cutoff in the game $G$. On the other hand, as highlighted by Corollary 4.1 below, the fact that this bound holds for all the types in a perfect coalition implies that it is actually a lower bound on their payoffs. Indeed, the fact that the cutoffs in $G_C$ are an upper bound on all of its members’ equilibrium cutoffs in $G$ implies that no one in coalition $C$ has a better outside option when bargaining in the game $G$ than proposing to form coalition $C$. As a result, in every equilibrium of the game $G$, every type in a perfect coalition $C$ can do at least as well as in the auxiliary game $G_C$.

**Corollary 4.1.** Let the profile $w$ in $\mathbb{R}^n$ be a solution to system (1). If type $i$ is in a perfect coalition, then $w_i$ is her best share.

For instance, given that the cutoff of the maîtres is bounded above by 48, in the equilibrium of the game of interest, the chefs can do as well as they can in the auxiliary game in which they can bargain in isolation with the maîtres (because, in this hypothetical case, the cutoff of the maîtres is 48, its upper bound in the game of interest). Hence, the upper bound of 48 on the cutoff of the chefs is tight, and analogously for the upper bound of 48 on the cutoff of the maîtres.

**Corollary 4.1** above is useful mainly for two reasons. First, we can tell whether a coalition is perfect using only the equilibrium cutoff profiles in the auxiliary games $\{G_C\}_{C \subseteq N}$, which, as described above, are familiar and easy to compute. In particular, we do not have to be
able to solve the system (1) in order to identify the perfect coalitions. Second, as highlighted by Proposition 4.3 below, we can always identify at least one perfect coalition. Hence, we can pin down the equilibrium cutoff of a nonempty subset of the types using the auxiliary games \( \{G_C\}_{C \subseteq N} \). In fact, subsection 4.4 describes how we can leverage this observation to pin down everyone’s equilibrium cutoff.

**Proposition 4.3.** There exists at least one perfect coalition.

Proposition 4.3 above follows from the fact that the equilibrium cutoffs in the auxiliary games \( \{G_C\}_{C \subseteq N} \) satisfy the following property: For any two types \( i \) and \( j \) and any two coalitions \( C \) and \( D \) containing both these types, if \( i \)'s cutoff in the auxiliary game \( G_C \) is higher than \( i \)'s cutoff in the auxiliary game \( G_D \), then the same is true for type \( j \) (that is, \( j \)'s cutoff in the auxiliary game \( G_C \) is higher than \( j \)'s cutoff in the auxiliary game \( G_D \)).

To see why this implies the existence of a perfect coalition, note that this precludes the existence of cycles in the network whose nodes are coalitions and whose link from \( C \) to \( C' \) indicates that there is a type in \( C \) whose best wage is strictly higher in \( C' \) than in \( C \). As a result, every path (or sequence of distinct links) in this network must end at a perfect coalition.

Perhaps the best way to gain intuition for Proposition 4.3 is to note that the Nash bargaining shares \( s \) in each coalition \( C \) equalize the ratio \( \frac{u_i(s_i)}{u'_i(s_i)} \) among all of its members. Since each type’s share is increasing in this ratio, the coalition with the highest such ratio is—in the generic case in which each type’s Nash bargaining share is different in different coalitions—a perfect coalition for all small enough bargaining frictions.

### 4.4 Recursive characterization of the equilibrium cutoff profile

The discussion above shows how to pin down the equilibrium cutoffs of a nonempty subset of types from the equilibrium cutoffs in the family of auxiliary games \( \{G_C\}_{C \subseteq N} \). The objective now is to provide an upper bound on the equilibrium cutoff of each type when some of the other types’ cutoffs have already been pinned down. This allows an iterative argument that pins down everyone’s equilibrium cutoffs.

As before, the idea is that a type cannot be hurt when the others’ outside options deteriorate. The difference with the argument above is that, now—in the hypothetical situation

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24 Pycia (2012) coins this property *pairwise-aligned preferences over coalitions*, and shows that the Nash bargaining solution generates pairwise-aligned preferences over coalitions.

25 This is essentially the argument that Pycia (2012) gives to show that the Nash bargaining solution generates pairwise aligned preferences over coalitions.
in which we let the agents bargain in isolation in one coalition—we don’t allow the outside options of the agents whose cutoff has already been defined to deteriorate below their cutoff. In order to formalize this idea, I consider the family of auxiliary games \( \{ G_{C,x} \}_{C \subseteq N, x \in \mathbb{R}^N} \), where \( G_{C,x} \) is a variant of the bargaining game \( G_C \) in which each agent of type \( i \) can choose to leave the market with payoff \( x_i \) any time that she is either the proposer or she rejects any proposal.

The game \( G_{C,x} \) is analogous to a multilateral version of the canonical alternating-offers model of Rubinstein (1982) with exogenous outside options (as considered by Binmore, Rubinstein, and Wolinsky 1986, for example) where both the feasible coalition \( C \) and the outside option profile \( x \) are given exogenously, and its members only bargain over how to share the resulting gains from trade subject to their exogenously given outside options. This game has a unique stationary subgame-perfect equilibrium, and the equilibrium cutoff profile of this game converges to the Nash bargaining solution in coalition \( C \), with the threat points given by autarky, and the outside option profile \( x \) imposing only lower bounds on the payoffs (e.g., Binmore, Rubinstein, and Wolinsky 1986, Proposition 6). In other words, the equilibrium cutoff in the auxiliary bargaining game \( G_{C,x} \) converges to the payoff that solves

\[
\begin{align*}
\arg \max_{s \in \mathbb{R}^C} \prod_{j \in C} u_j(s_j) \quad \text{s.t. } y(C) &\geq \sum_{j \in C} s_j \\
\text{and } s_j &\geq x_j \quad \text{for all } j \in C
\end{align*}
\]

if the coalition \( C \) is feasible (in the sense that \( y(C) \geq \sum_{j \in C} x_j \)) and to the restriction of \( x \) to \( C \) otherwise.

Figure 8 illustrates the equilibrium cutoffs in each of the relevant auxiliary games in the example of section 2, when the outside options are set to \( x_{\text{chef}}^1 = x_{\text{maître}}^1 = 48 \) (and \( x_{\text{cook}}^1 = x_{\text{manager}}^1 = 0 \)). Naturally, the only cutoffs that are different from the case in which everyone’s outside options are zero (the case illustrated in Figure 7) are those associated with partnerships involving either the chefs or the maîtres.

For each payoff profile \( x \) in \( \mathbb{R}^N \), I define each type’s \( x \)-best wage to be her maximum cutoff in the auxiliary game \( G_{C,x} \) across all coalitions \( C \subseteq N \). Similarly, I say that coalition \( C \) is a type’s \( x \)-best coalition if her equilibrium cutoff in \( G_{C,x} \) is her \( x \)-best wage. Finally, I say that a coalition is \( x \)-perfect if it is the \( x \)-best coalition of all of its members.

For example, Figure 8 illustrates that the chef-manager partnership is the managers’ \( x^1 \)-best coalition (and is hence \( x^1 \)-perfect), and that the managers’ \( x^1 \)-best wage is 29. The argument analogous to the one that proves Proposition 4.2 above shows that these \( x^1 \)-best wages are upper bounds on the respective equilibrium cutoffs, and that these bounds are tight in any \( x^1 \)-perfect coalition. In this case, this implies that the \( x^1 \)-best wage of the managers is
Figure 8: The number that is closest to type \( i \) associated with a link between \( i \) and \( j \) is approximately her equilibrium cutoff in the auxiliary game in which the surplus of all the partnerships other than the one between \( i \) and \( j \) is artificially set to zero, and outside options are \( z^{1}_{\text{chef}} = z^{1}_{\text{maître}} = 48 \) and \( z^{1}_{\text{cook}} = z^{1}_{\text{manager}} = 0 \). An arrow from type \( i \) to type \( j \) indicates that the \( ij \) partnership is \( i \)'s \( z \)-best coalition in this case, and the associated number is her \( z \)-best wage.

their equilibrium cutoff. Definition 4.1 below formalizes this argument, which recursively pins down everyone’s cutoffs.

Definition 4.1 (Algorithm \( \mathcal{A} \)). At each step \( s = 1, 2, \ldots \),

1. for each type whose cutoff has been determined in a step \( s' < s \), let \( x^{s}_{i} \) be her cutoff; for all other types \( j \), let \( x^{s}_{j} = 0 \),
2. let the \( x^{s} \)-best wage of each type in an \( x^{s} \)-perfect coalition be her cutoff.

Stop in the first step \( S \) such that \( x^{S}_{i} \neq 0 \) for all types \( i \), and let \( \chi := x^{S} \).

The analog of Proposition 4.3 shows that, at each step \( s \) of algorithm \( \mathcal{A} \), there is always at least one \( x^{s} \)-perfect coalition. Hence, each step identifies the cutoff of at least one type (and this cutoff is positive, since \( y(i) > 0 \) for each type \( i \)). Since there are finitely many types, this guarantees that algorithm \( \mathcal{A} \) ends after finitely many steps. Proposition 4.4 culminates the characterization argument described so far.

Proposition 4.4. The payoff profile \( \chi \) defined by algorithm \( \mathcal{A} \) is the only one that solves system (1).

Figure 9 illustrates the limit equilibrium payoffs in the example of section 2. I now turn to showing that the equilibrium cutoff profile \( \chi \) converges to a special point as the bargaining friction \( q \) converges to zero.
Figure 9: In the limit as the bargaining friction $q$ goes to zero, the cutoffs of chefs, maîtres, managers and cooks converge to 50, 50, 30 and 20, respectively. This is the unique Nash consistent payoff profile in this case.

4.5 Main result

Theorem 1 below highlights that the equilibrium cutoff profile $\chi$ converges—as the bargaining friction $q$ vanishes—to the unique payoff profile that gives each type the maximum that she can justify using the Nash bargaining solution in some coalition subject to the others’ payoffs. I refer to this payoff profile as the Nash consistent payoff profile.

Formally, I say that the payoff profile $x$ in $\mathbb{R}^N$ is Nash consistent if, for each type $i$, $x_i$ is her $x$-Nash best share, defined to be the maximum—across all coalitions $C \subseteq N$—of the amount $z_i$ corresponding to the unique profile $z$ in

$$\arg\max_{s \in \mathbb{R}^C} \prod_{j \in C} u_j(s_j) \text{ s.t. } y(C) \geq \sum_{j \in C} s_j \text{ and } s_j \geq x_j \text{ for all } j \in C - i. \quad (5)$$

when (5) is well defined, and 0 otherwise. In words, type $i$’s $x$-Nash best share is the maximum that she can justify as the result of Nash bargaining in some coalition subject to the others’ outside options—given by $x$ and entering the Nash bargaining solution as bounds on the payoffs, as prescribed by the outside option principle.

Theorem 1. In the limit as the bargaining friction $q$ vanishes, the equilibrium cutoff profile of the bargaining game $G$ converges to the unique Nash-consistent payoff profile.

For instance, Figure 5 illustrates how each type’s limit payoff in the example of section 2 is the maximum (across her two potential partnerships) that she is able to justify using the Nash bargaining solution while honoring the others’ payoffs. Theorem 1 implies that this is the unique payoff profile that satisfies this property—that is, this is the Nash consistent payoff profile in this example. Definition 4.2 formally describes an algorithm that identifies the only Nash-consistent payoff profile. I say that a coalition is $x$-Nash perfect if it it gives the highest $x$-Nash best share across all coalitions to all of its members.

Definition 4.2 (Algorithm $A^*$). At each step $s = 1, 2, \ldots,$
1. for each type whose payoff has been determined in a step \( s' < s \), let \( x_i^{s} \) be her payoff; for all other types \( j \), let \( x_j^{s} = 0 \),

2. let the \( \pi^s \)-Nash best wage of each type in an \( \pi^s \)-Nash perfect coalition be her payoff.

Stop in the first step \( S \) such that \( x_i^S \neq 0 \) for all types \( i \), and let \( \chi^\ast := \pi^S \).

At each step \( s \) of algorithm \( \mathcal{A}^* \), there is always at least one \( \chi^\ast \)-perfect coalition. Hence, each step identifies the payoff of at least one type (and this payoff is positive, since \( y(i) > 0 \) for each type \( i \)). Since there are finitely many types, this guarantees that algorithm \( \mathcal{A}^* \) ends after finitely many steps. Proposition 4.5 highlights that the payoff profile defined by this algorithm is the unique Nash-consistent payoff profile.

**Proposition 4.5.** The payoff profile \( \chi^\ast \) defined by algorithm \( \mathcal{A}^* \) is the unique Nash-consistent payoff profile.

An alternative way to read Theorem 1 is that the equilibrium sharing rule in each coalition converges to the Nash bargaining solution in that coalition, with the binding outside options determined as follows:

Each type’s outside option in any given coalition is the maximum that she can justify using the Nash bargaining solution in some other coalition subject to the others’ outside options.

Indeed, Theorem 1 implies that—when the bargaining friction \( q \) is small enough—in equilibrium each type obtains (close to) her Nash bargaining share in one coalition subject to the others’ outside options. This determines her binding outside option in all the other coalitions that contain her. In all such coalitions, she simply gets her outside option.

For instance, the binding outside option of the chefs when bargaining with the managers is determined by the Nash bargaining solution in the chef-maître partnership (without invoking the chefs’ outside option there). Similarly, the binding outside option of the managers when bargaining with the cooks is determined by the Nash bargaining solution in the chef-manager partnership (without invoking the managers’ outside option there).

## 5 Comparative statics

Throughout this section, I describe how the limit equilibrium payoff profile (i.e. the unique Nash-consistent payoff profile) changes as a function of the primitives of the model. Analogous comparative statics hold for the equilibrium payoffs for any bargaining friction \( q > 0 \).
I refer to each type’s Nash consistent payoff simply as her payoff, and to the coalitions that form in equilibrium for all sufficiently small bargaining frictions simply as the “equilibrium coalitions.”

I start in subsection 5.1 by describing how the equilibrium coalitions can be organized into tiers, in such a way that small shocks propagate via outside options from higher to lower tiers, but not vice versa. In the rest of this section, I describe further comparative statics without restricting attention to shocks that are sufficiently small to keep the equilibrium tier structure unchanged. In subsection 5.2, I describe several comparative statics without imposing further structure on the setting, and, in subsection 5.3, I focus on vertically differentiated markets.

5.1 The equilibrium tier structure

Proposition 4.5 implies that the equilibrium coalitions can be organized into tiers, in such a way that the surplus in each coalition $C \subseteq N$ is shared as predicted by the Nash bargaining solution, with the binding outside options determined in higher tiers. As a result, (small) preference and productivity shocks propagate—via agents’ outside options—from higher to lower tiers, but not vice versa.

In particular, the first-tier coalitions are those coalitions $C \subseteq N$ that are a $\chi^*$-best coalition of all of its members. Note that there is always at least one such coalition. In particular, every Nash-perfect coalition is in the first tier (but coalitions that are not perfect can also be in the first tier). Moreover, none of the types’ outside options in these coalitions bind. In the limit as the bargaining friction $q$ vanishes, the sharing rule in tier 1 coalitions coincides with the Nash bargaining solution, with the threat points determined by autarky. The first-tier types are those that are members of a first-tier coalition. For instance, Figure 10 illustrates an example that has two first tier coalitions, whose members are the first-tier types. The surplus in these coalitions is shared according to the Nash bargaining solution, with the threat points given by autarky and no binding outside options.

Proceeding inductively, after having identified the coalitions in the $\ell$th tier, for $\ell = 1, 2, \ldots, k-1$, a coalition $C \subseteq N$ is in the $k$th tier if and only if (i) it contains at least one $(k-1)$th-tier type and (ii) is a $\chi^*$-best coalition of all its members who are not in the first, second, $\ldots$, or $(k-1)$th tier. If there are no such coalitions, all the coalitions that form in equilibrium are in the tiers above. The $k$th-tier types are those that are in a $k$th-tier coalition and are not in any $\ell$th-tier coalition, for any $\ell < k$. For instance, Figure 10 illustrates an example that has two second-tier coalitions, and three third-tier coalitions. The second-tier types in this case are f, g and
Nash bargaining
\[
\begin{array}{c}
a, b, c \\
d, e
\end{array}
\]
First tier

Nash bargaining subject to first tier’s outside options
\[
\begin{array}{c}
f, g, a \\
h, a, b, c
\end{array}
\]
Second tier

Nash bargaining subject to second tier’s outside options (possibly first tier’s too)
\[
\begin{array}{c}
i, j, c, h \\
k, f \\
l, f
\end{array}
\]
Third tier

Figure 10: Illustration of the equilibrium tier structure. The types are indexed from \(a\) to \(l\). A box containing different types represents an equilibrium coalition containing these types. The index of a type is in bold in the coalition that gives her the highest \(\chi^*\)-Nash best share (this is also the coalition that she proposes to in equilibrium when the bargaining friction is small enough).

\(h\), and the third-tier types are \(i, j, k\) and \(l\). The surplus in second and third-tier coalitions is shared according to the Nash bargaining solution, with the threat points given by autarky and the binding outside options determined in higher tiers.

The equilibrium tier structure implies that a type’s payoff is not affected by preference and productivity shocks that only hit coalitions and types in lower tiers (as long as these shocks are sufficiently small so that they do not change the equilibrium tier structure). Moreover, an increase in the productivity of a \(k\)-th-tier coalition is never beneficial for tier-\(k+1\) types, because it can only increase their partners’ outside options. For instance, in the example illustrated in Figure 10, an increase in the surplus of the coalition \(\{a, b, c\}\) decreases the payoffs of all the second-tier types, because it increases the outside options of the first-tier types in the second-tier coalitions. This leads, in turn, to an increase in the payoffs of the third-tier types \(k\) and \(l\). The effect on the payoffs of the other two third-tier types is ambiguous in this case, because one of the outside options that they have to honor (\(c\)'s) increases, but another one (\(h\)'s) decreases.

As another example, in section 2, an increase in the productivity of the first-tier coalition (the chef-maître partnership) hurts the second-tier type (managers), because it increases the outside options of the first-tier type (chefs) that managers have to honor. In contrast, such an increase is beneficial for third-tier types (cooks). This suggests that, in certain cases,
a positive productivity shock to a tier-$k$ type affects negatively (positively) the tier $k + \ell$ types when $\ell$ is odd (even). In order to formalize this idea, one can construct the coalitional overlap network—whose nodes are all the coalitions that form in equilibrium, and where a link between two coalitions represents the fact that they share some types. Corollary 5.1 highlights how this network can be used to understand how economic shocks that hit one coalition affect the other.

**Corollary 5.1.** If there is only one path in the coalitional overlap network from one coalition to another one, a marginal increase in the surplus of one does not hurt (benefit) the members of the other if the path is even (odd).\(^{26}\)

While the equilibrium tier structure described in this section is useful to illustrate how shocks propagate via agents’ outside options, large shocks will typically affect this structure. Hence, understanding how these shocks propagate requires investigating how they affect the equilibrium tier structure itself. I now turn to discussing several results in this direction, in the form of comparative statics that hold for arbitrarily large shocks.

### 5.2 General comparative statics

I now turn to highlighting that each type’s payoff is increasing in her own productivity and decreasing in her own risk aversion, and to describing how arbitrary preference and productivity shocks to a type affect the payoffs of those types that are either complements or substitutes of this type.

**Corollary 5.2.** A type’s payoff increases when the surplus of any coalition that she is part of increases, or when she becomes less risk averse.\(^ {27}\)

In particular, the payoff that a type can justify using the Nash bargaining solution in some coalition while honoring the others’ payoffs can only increase when a coalition that she is part of becomes more productive, or when she becomes less risk averse. In the case of small shocks, this can be seen directly from the equilibrium tier structure, because the outside options that each type has to honor in the coalition where she can justify her payoff are independent of the surplus of any coalition that she is part of. More generally, Corollary 5.2

---

\(^{26}\)A path of a network is a sequence of distinct links. A path is even (odd) if it contains an even (odd) number of links.

\(^{27}\)I say that a type whose utility function changes from $u$ to $w$ has become more risk averse if there exists a concave function $g$ such that $w = g \circ u$. This preference shock can be interpreted as an exogenous change in a type’s “bargaining power.”
follows from the observation that an increase in the surplus of a coalition that contains a given type \( i \) cannot increase the outside option of any type at the step at which \( i \)'s payoff is determined by algorithm \( A^* \) (Definition 4.2).

While Corollary 5.2 is intuitive enough, it contrasts with well-known theories of coalition formation (e.g., Farrell and Scotchmer 1988 and Pycia 2012) where agents first bargain over which coalitions to form, and then bargain over their terms of trade. For example, Pycia (2012, p. 347) describes how a holdup problem can occur in these models. Somewhat paradoxically, this implies that an agent can be worse off when she becomes more productive or less risk averse. In his own words,

Inflexible sharing of surplus leads to holdup in coalition formation. For instance, consider the setting in which agents share surplus in Nash bargaining with constant bargaining powers. An agent may be better off with a lower rather than a higher bargaining power—other things held equal—when a low bargaining power allows him or her to form a highly productive coalition, while a high bargaining power makes formation of such a productive coalition impossible by lowering the payoffs of its members below their outside options.

Corollary 5.2 above shows that such a holdup problem does not occur in the present setting. However, in contrast to the Nash bargaining solution in a fixed coalition (with exogenous outside options), the payoff of an agent can increase when others become more productive. For instance, as illustrated by the example of section 2, cooks benefit when the surplus of the chef-maître partnership increases. This is because this improves the chefs’ outside options when bargaining with managers which, in turn, deteriorates the managers’ outside options when bargaining with cooks.

Corollary 5.3 below highlights that when two types are perfect complements—in the sense that one is useless without the other—any shock that affects one also affects the other. Moreover, a productivity increase in one has positive spillovers on the other, but an increase in the bargaining power of one has negative spillovers on the other.\(^{28}\)

**Corollary 5.3.** Consider two types that are perfect complements. An increase in the productivity or risk aversion of one strictly increases the payoff of the other.

Intuitively, when two types are perfect complements, their maximum Nash bargaining share—across all coalitions—subject to the others’ outside options is determined in the same

\[^{28}\text{Formally, two types } i \text{ and } i' \text{ are perfect complements if, for any coalition } C \subseteq N - \{i, i'\}, \text{ we have that } y(C) = y(C \cup \{i\}) = y(C \cup \{i'\}) < y(C \cup \{i, i'\}).\]
coalition. As a result, when one of these types becomes more productive, the resulting increase in surplus is shared among these types (and possibly others as well). Similarly, the fact that these two types justify their payoff in the same coalition implies that, when one of these types becomes more risk averse, the amount that the other can justify using Nash bargaining while honoring the others’ payoffs increases.

**Corollary 5.4** below highlights that when two types are perfect substitutes—in the sense that their marginal contribution to any coalition that does not contain either of them is the same—preference shocks propagate from the most risk averse to the least risk averse one, but not vice versa.\(^{29}\)

**Corollary 5.4.** Consider two types that are perfect substitutes, and assume that one is more risk averse than the other (and remains so after the change under consideration). An increase in the risk aversion of the most risk averse one can affect the payoff of the other one, but not vice versa.

Indeed, when two types are perfect substitutes, the payoff of the most risk averse one is determined by algorithm \(A\) first. This means, in particular, that the payoff of the most risk averse type \(i\) is not affected by the risk aversion of the other type \(i'\) (as long as \(i'\) remains less risk averse than \(i\)). But an increase in the risk aversion of the most risk averse type can affect the binding outside options of the equilibrium partners of the other (because both types might match with the same types in equilibrium, for example), and hence affect her payoff.

### 5.3 Vertically differentiated markets

I now turn to deriving further comparative statics in the context of two-sided one-to-one matching markets in which agents are vertically differentiated either by their skill or by their risk aversion. For ease of exposition, I refer to the types on one side as *workers* and the types on the other side as *employers*. Throughout the rest of this section, I assume that each type \(i\) is endowed with a risk aversion parameter \(r_i\) and a skill (or productivity) parameter \(s_i\).

For each worker-employer pair \((i, j)\), the surplus \(y(i, j)\) is strictly increasing in its members’ skills.\(^{30}\) I say that two types \(i\) and \(j\) match if agents of type \(i\) match with agents of type \(j\) in equilibrium.

\(^{29}\)Formally, two types \(i\) and \(i'\) are perfect substitutes if, for any \(C \subseteq N - \{i, i'\}\), we have \(y(C \cup \{i, i'\}) = y(C \cup \{i\}) = y(C \cup \{i'\}) > y(C)\).

\(^{30}\)Formally, let \(i\) and \(i'\) be any two workers and let \(j\) be any employer. On the one hand, we have that \(y(i, j) > y(i', j)\) if and only if \(s_i > s_{i'}\). On the other hand, there exists a concave function \(g\) such that \(u_i = g \circ u_{i'}\) if and only if \(r_i > r_{i'}\).
Figure 11: A two-sided market where agents are vertically differentiated by their skills. Each node corresponds to a type. The nodes on the left column represent one side of the market, and the nodes on the right side the other. The surplus that two types $i$ and $j$ in opposite sides of the market generate is the product of $i$ and $j$. Preferences are homogeneous. A link between node $i$ and node $j$ indicate that $i$ and $j$ match in equilibrium. The number next to a node is her limit equilibrium payoff. An arrow from type $i$ to type $j$ indicates that type $i$ can justify her payoff using the Nash bargaining solution in the $ij$ partnership while honoring $j$’s payoffs.

For example, Figure 11 illustrates a two-sided market where agents are vertically differentiated by their skill. In this example, the sets of types on each side of the market are \{1, 2, 3, 4, 5, 8, 9, 10\} and \{2, 6, 10\}, respectively. The surplus of each partnership is the product of the associated types. Figure 11 illustrates the partnerships that form in equilibrium, and Figure 12 illustrates the corresponding tier structure.

Corollary 5.5 below illustrates how the bargaining outcomes in vertically differentiated markets are determined from the most productive types down (in settings where the agents are vertically differentiated by their productivity but are otherwise identical), and from the most risk averse types down (in settings where agents are vertically differentiated by their risk aversion but are otherwise identical). For instance, in the example illustrated in Figure 11, where preferences are homogeneous and types are vertically differentiated by their skills, algorithm $A^*$ pins down in the first step the payoffs of the types with the highest skill on
Corollary 5.5. When all the types have the same risk aversion, an increase in the productivity of type \( i \) from \( s_i \) to \( s'_i > s_i \) does not affect the payoff of any type \( j \) whose productivity \( s_j \) is strictly higher than \( s'_i \). Similarly, when all the types have the same skill, an increase in the risk aversion of type \( i \) from \( r_i \) to \( r'_i > r_i \) does not affect the payoff of any worker \( j \) whose risk aversion \( r_j \) is strictly higher than \( r'_i \).

For example, in the example illustrated in Figure 11, an increase in the skill of worker 8 to 8 + \( \epsilon \), where \( \epsilon < 2 \), increases the worker 5’s payoff, but it does not affect the payoffs of workers 9 or 10. Intuitively, when all the types have the same risk aversion, algorithm \( A^\ast \) does not determine a worker’s payoff before determining the payoffs of all the more productive workers. This implies that an increase in a type’s productivity does not affect the payoffs of the more productive types. Similarly, when all the types have the same skill, algorithm \( A^\ast \) only determines a worker’s payoff when it has determined the payoffs of all the more risk averse workers. This implies that an increase in a type’s risk aversion does not affect the payoffs of the more risk averse types.

I say that the equilibrium features positive assortative matching if, for any two buyer types \( i \) and \( i' \) and any two seller types \( j \) and \( j' \), with \( s_i > s_{i'} \) and \( s_j > s_{j'} \), if \( i \) matches with \( j' \) in equilibrium, then \( i' \) does not match with \( j \) in equilibrium. Figure 13 illustrates the matching pattern in the example described in section 2 when cooks and maîtres generate 80 units of surplus instead of 60. This matching pattern does not feature positive assortative matching.
Figure 13: Equilibrium matching pattern in the example when the surplus of the cook-maître partnership is 80 instead of 60.

because the most productive cooks (i.e. chefs) match with the lowest productive managers, while the least productive cooks match with the most productive managers (i.e. maîtres). In contrast, the matching pattern of Figure 11 features assortative matching.

Pycia (2012) shows that, when agents’ preferences over coalitions are generated by the Nash bargaining solution (with exogenous outside options), the notion of stability—in the sense of the core—implies that agents match in a positive assortative way with respect to their productivity and their risk aversion. As the example illustrated in Figure 13 illustrates, this is not necessarily the case. In the present setting, where agents bargain simultaneously over both which coalitions to form and how to share the resulting surplus.

Becker (1973) showed that—in the context of one-to-one matching environment with a finite set of agents and transferable utility (the assignment game)—agents match in a positive assortative way for all distribution of types if and only if their skills are complementary—in the sense that the match function is supermodular. Corollary 5.6 is the analogous result in the present setting.

**Corollary 5.6.** The equilibrium features positive assortative matching for all distribution of types if and only if the production function is strictly supermodular.\(^{31}\)

For example, the surplus function in the example illustrated in Figure 11 is strictly supermodular and, as a result, the resulting matching pattern features positive assortative matching. To see the intuition behind Corollary 5.6, consider again the situation illustrated in Figure 13. As argued above, this matching pattern does not feature positive assortative matching, so it must be the case that the production function is not strictly supermodular. Indeed, supermodularity in this case boils down to the following condition

\[
y(\text{Chefs, Maîtres}) + y(\text{Cooks, Managers}) > y(\text{Chefs, Managers}) + y(\text{Cooks, Maîtres}).
\]

---

\(^{31}\)The production function is strictly supermodular if, for any two buyer types \(i\) and \(i'\) and two seller types \(j\) and \(j'\), with \(s_i > s_i'\) and \(s_j > s_{j'}\), we have that \(y(i, j) + y(i', j') > y(i', j) + y(i, j')\).
When this condition is satisfied, we cannot have the matching pattern illustrated in Figure 13, because that matching pattern requires that

\[ y(\text{Cooks, Managers}) < y(\text{Chefs, Managers}) + y(\text{Cooks, Maîtres}) - \overbrace{\text{Sum of all four types payoffs}}^{\text{Sum of Chefs and Maîtres payoffs}} \]

Corollary 5.5 above highlighted that, in settings where types are vertically differentiated by their skill, each type is only affected by shocks that hit more productive types. Corollary 5.7 below highlights further structure on how shocks propagate under positive assortative matching.

**Corollary 5.7.** Suppose that workers and firms match in a positive assortative way. If an increase in the skill of a worker \( i \) from \( s_i \) to \( s'_i \) does not affect the payoff of another worker \( j \) with \( s_j < s_i \), then this increase does not affect the payoff of any worker that is less productive than \( j \) either.

In other words, when the agents match in a positive assortative way, shocks propagate in blocks, in the sense that if a shock that affects worker \( i \) propagates to worker \( i' \), it affects every worker whose skill is in between. For instance, in the example illustrated in Figure 11, an increase in the skill of worker 10 affects the payoff of worker 4. Given that the matching is positive assortative, this means that such increase affects every worker whose skill is between 4 and 10 as well.

Intuitively, if an increase in the skill of a worker \( i \) from \( s_i \) to \( s'_i \) does not affect the payoff of another worker \( j \) with \( s_j < s_i \), it means that there is no path from \( i \) to \( j \) in the matching network (which has a link from one type to another if they match in equilibrium). When the equilibrium matching is positive assortative, this implies that there is no path to \( i \) from any other worker \( k \) less productive than \( j \) either, which implies, in turn, that such an increase cannot affect \( k \)'s payoff either.

The situation described by Corollary 5.7 contrasts with the case in which the production function is strictly submodular.\(^{32}\) Indeed, in this case, all the types that match do so with the most productive type on the other side.\(^ {33}\) As a result, everyone’s payoffs depend on the most

\[^{32}\text{The production function is strictly submodular if, for any two buyers } i \text{ and } i' \text{ and any two sellers } j \text{ and } j', \text{ with } s_i > s_{i'} \text{ and } s_j > s_{j'}, \text{ we have that } y(i, j) + y(i', j') < y(i', j) + y(i, j').}\]

\[^{33}\text{To see this in the context of the example, suppose for contradiction that the surplus function is submodular and that chefs and maîtres match with each other, and so do cooks and managers. Then, the sum of the payoffs of these four types is equal to the sum of the surpluses of these two types of partnerships, which, by submodularity, is smaller than the sum of the surpluses generated by a chef-manager partnership and a cook-maître partnership. In particular, at least one of these partnerships is more profitable than the sum of its members’ payoffs, a contradiction.}\]
productive types. In particular, in this case, an increase in the surplus of the match among the two most productive types makes everyone other than these types worse off. However, an increase in the surplus of any other match that includes the highest type is fully absorbed by the low type of the match. For example, when the equilibrium is as in Figure 13, an increase in the surplus of the chef-manager partnership from 80 to 90 is fully absorbed by the managers.

6 Related literature


These two lines of research focus on two opposite extremes. On the one hand, in order to investigate the strategic forces in a steady state of large dynamic markets, the former typically assumes that the inflow of traders into the market perfectly matches its outflows. On the other hand, in order to investigate the consequences of the endogenous evolution of the set of active traders over time, the latter typically assumes that there are no inflows of traders into the market. Hence, roughly speaking, these two lines of research focus on the likely predominant strategic forces in thick and thin markets, respectively.

This paper is most closely related to the literature on bargaining in stationary markets. The main innovation with respect to this literature is that, in this paper, the agents strategically choose which coalitions to propose, which is an essential feature behind the connection between the predictions of the non-cooperative model and the Nash bargaining solution.35

34 Also related, albeit with a somewhat different spirit, is the large literature that builds on legislative bargaining model of Baron and Ferejohn (1989) (e.g., Eraslan 2002, Eraslan and McLennan 2013 and Eraslan 2016).
35 In Talamàs (2018), I study networked buyer-seller markets using a framework similar to the one in the present paper, and I discuss how allowing the agents to strategically choose whom to make offers to fundamentally alters the determinants of price dispersion in these markets. The characterization in Talamàs (2018) is related to the one of the present paper. There, I provide a simple necessary and sufficient condition for the law of one price to hold in equilibrium (in the limit as bargaining frictions vanish), and I describe an algorithm that decomposes the buyer-seller network into submarkets, from the submarket with the highest limit price down
For example, Nguyen (2015) uses convex programming techniques to characterize the stationary subgame-perfect equilibrium of a non-cooperative bargaining game similar to the one in the present paper, but the equilibrium payoff profile in his framework cannot be understood using the Nash bargaining solution. This contrasts with the present paper, where I show that the (limit) equilibrium payoff profile is the unique profile that satisfies the following property: Each type obtains the maximum that she can justify using the Nash bargaining solution in some coalition while honoring the others’ equilibrium payoffs. As I illustrate in this paper, this provides a tractable theory of coalition formation with sharp comparative statics.

Despite the contrast between the settings under consideration, the strategic forces in the present paper are most related to those that arise in the literature investigating non-cooperative bargaining in non-stationary markets. Indeed, the structure of the equilibrium is similar to that of the no-delay perfect equilibrium characterized in Chatterjee, Dutta, Ray, and Sengupta (1993). Other than the setting, the main difference is that—as in the classical bargaining framework of Nash (1950)—I allow the agents to have arbitrary vN-M utility functions instead of restricting attention to the case of linear utilities. As a consequence, in the limit as the bargaining friction vanishes, in the present setting the coalitional surpluses are shared according to the Nash bargaining solution—instead of according to equal sharing. Moreover, the dynamic entry of the agents into the market implies that the structure of the equilibrium in the present setting is fixed throughout—instead of evolving as different coalitions form—and that the equilibrium that I characterize always exists and is the unique stationary perfect equilibrium—instead of being one of the possible stationary perfect equilibria and existing only under certain conditions. Not surprisingly given the qualitative differences between the settings, however, the predictions of the resulting theories are qualitatively distinct. For example, the endogenous evolution of the market in Chatterjee, Dutta, Ray, and Sengupta (1993) implies that—unlike in the present setting—an agent does not necessarily benefit when she becomes more productive (because her improved outside option can lead others to avoid making offers to her, which can in turn make it more likely that the market will evolve against her).

(or, alternatively, the submarket with the lowest limit price up).

The main difference between the model in Nguyen (2015) and the one that I study in the present paper is that, in the former, coalitions are proposed at random (instead of being selected strategically by the proposer) and, in each period, its members bargain only over whether to form the proposed coalition and, if so, their terms of trade. Another difference is that, in Nguyen (2015), all the types have linear utilities, while I assume instead that—as in the classical framework of Nash (1950)—each type’s preferences can be represented by a concave vN-M utility function.
The similarity between the strategic forces in Chatterjee, Dutta, Ray, and Sengupta (1993) and the present paper might seem surprising given that these two papers differ not only in the setting under study but also in the bargaining protocol. Indeed, they use a rejector-proposes protocol (in which the rejector of a proposal becomes the proposer in the next period) instead of the random-proposer protocol of the present paper. Ray (2007) (see also Compte and Jehiel 2010) discusses how the former protocol gives considerably more bargaining power to the receiver of the offer than the latter, and how this explains the contrasting predictions often obtained under these two protocols. Intuitively, however, the dynamic entry in the present paper implies that agents do not have to consider how the market might evolve after they reject an offer, which implies that the difference between these protocols is much less pronounced—and it actually vanishes with the bargaining frictions.

In the context of convex games,37 Chatterjee, Dutta, Ray, and Sengupta (1993) show that the prediction of the no-delay stationary perfect equilibrium of their coalition formation game converges—as the bargaining frictions vanish—to the egalitarian solution of Dutta and Ray (1989). The similarity between the structure of the equilibrium in their game and the one in the present paper suggests that their results can be generalized beyond the case of linear utilities and can more generally be understood in terms of the Nash bargaining solution. Relatedly, Compte and Jehiel (2010) focus on environments in which the grand coalition generates the highest surplus, and in which only one coalition may form. They show that, if an (asymptotically) efficient stationary equilibrium exists, the corresponding profile of payoffs is the one that maximizes the product of agents’ payoffs among those in the core.38 The analysis of the present paper suggests that this result generalizes to the case in which agents’ have vN-M utility functions, as follows: If an (asymptotically) efficient stationary perfect equilibrium exists, the corresponding profile of payoffs is the unique payoff profile that gives each agent her maximum Nash bargaining share among all the possible coalitions subject to the others’ outside options.

This paper is further related to two other lines of research. First, the idea of building a theory of coalition formation from the Nash bargaining solution goes back at least to Rochford (1984), who defines a symmetrically-pairwise-bargained payoff profile of an assignment game with transferable utility as one that satisfies the following property: Each matched pair shares output according to the Nash bargaining solution—with each agent’s disagreement

37A game is convex if, for any two coalitions $C_1$ and $C_2$, $y(C_1 \cup C_2) \geq y(C_1) + y(C_2) - y(C_1 \cap C_2)$.

38They also provide a necessary and sufficient condition for such an equilibrium to exist. This study contrasts with Krishna and Serrano (1996), who establish a connection between the Nash bargaining solution and a strategic bargaining game in which the grand coalition is the only one that can form.
point being the maximum that she can achieve in any other match (keeping the others’ payoffs fixed). Burguet and Caminal (2018) show that a modification and extension of this idea (in a context in which only one coalition can form) uniquely pins down the agents’ payoffs, and provide strategic foundations for the resulting coalition formation solution concept. While these concepts are similar in spirit to the one described in the present paper, the non-cooperative approach described here suggests that—in the setting of this paper—the outside option principle holds, so outside options do not enter through disagreement points, but act instead as bounds on the range of validity of the Nash bargaining solution.

Second, Collard-Wexler, Gowrisankaran, and Lee (forthcoming) provide strategic foundations for the Nash equilibrium in Nash bargains (Horn and Wolinsky 1988), which is a widely used bargaining solution concept for bilateral oligopoly settings. In contrast to the coalition formation approach of the present paper—in which each agent can be part of at most one coalition—the Nash-in-Nash solution assumes that all the parties trade with each other (i.e. that all possible coalitions form) and derives prices for each bilateral contract as a function of the fundamentals. Ho and Lee (forthcoming) provide a modification of the Nash-in-Nash solution to investigate agents’ incentives to restrict the set of agents with whom they trade.

7 Conclusion

The Nash bargaining solution is a central solution concept in economics. Nash proposed this solution concept using an axiomatic approach. In his own words (Nash, 1953, p. 129),

One states as axioms several properties that it would seem natural for the solution to have and then one discovers that the axioms actually determine the solution uniquely.

My objective in this paper has been to understand how prices and allocations are deter-

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39 Rochford (1984) shows that the set of symmetrically-pairwise-bargained payoff profiles is the intersection of the kernel and the core. Kleinberg and Tardos (2008) refer to such profiles as “balanced outcomes.” Alternative approaches to select a point from the core of the assignment game include Kranton and Minehart (2001) (who focus on an extreme point of the core) and Elliott (2015) (who focuses on different convex combinations of the extreme points of the core).

40 The idea of endogenizing agents’ “threat points” was pursued by Nash (1953) himself (see also Binmore 1987 and Abreu and Pearce 2015) and it is the essence of well-known consistency notions (e.g. Sobolev 1975, Peleg 1986, Hart and Mas-Colell 1989, Serrano and Shimomura 1998).

41 In Agranov, Elliott, and Talamàs (in preparation), we investigate the experimental validity of this prediction.
mined in a thick-market setting in which agents bargain both about which coalitions to form and how the resulting surplus is shared within them. One possible approach to fulfill this objective is to extend Nash’s axioms to this setting, and then to discover what solution comes out of these axioms.

In this paper, I have taken an alternative approach, which leverages the celebrated connection between non-cooperative bargaining and the Nash bargaining solution: I have extended the canonical non-cooperative bargaining model that connects with the Nash bargaining solution (e.g. Binmore, Rubinstein, and Wolinsky 1986) to the setting of interest, and I have let this model suggest how the Nash bargaining solution generalizes to this setting. The payoff profile under the resulting theory of coalition formation is the unique profile that is such that each agent obtains her maximum Nash bargaining share among all the coalitions subject to the others’ outside options. An interesting avenue for future research is to investigate whether Nash’s axioms have natural analogs in the framework of this paper that pin down this solution.

This paper suggests a handful of exciting directions for future research. First, as discussed in section 1, the contribution of this paper can be seen as enriching the outside option principle of Binmore, Rubinstein, and Wolinsky (1986) to determine, not only how outside options enter the Nash bargaining solution, but also how they are determined by the Nash bargaining solution. Binmore, Shaked, and Sutton (1989) and Jäger, Schoefer, Young, and Zweimüller (2018) provide empirical evidence that is consistent with the outside option principle. In Agranov, Elliott, and Talamàs (in preparation), we design and implement a new experimental approach that replicates the relevant strategic forces in stationary markets, and we use it to investigate the empirical relevance of the qualitative predictions that emerge from this theory.

Second, the theory that emerges from this paper opens a door to investigate the agents’ incentives to invest in different skills and relationships—when these investments must be sunk before bargaining takes place. This is the case, for example, in labor markets, in which both employers and employees must make substantial investments before they know who is going to end up matching with whom. In Elliott and Talamàs (2018), we study the extent to which the classical holdup problem (e.g., Williamson 1975, Grossman and Hart 1986, Hart and Moore 1990, Hosios 1990, Acemoglu 1996, 1997, Cole et al. 2001, Elliott 2015) is actually a problem in markets that—as in the present paper—attract traders over time. Leveraging the characterization of the present paper to study how agents’ use their investments to obtain more favorable outcomes in decentralized markets is an exciting avenue of future research.
Third, following most of the related literature, this paper focuses on settings in which each coalition’s productivity is independent of which other coalitions form. In many settings of interest, however, this is counterfactual (e.g., because different coalitions are in competition with each other, or because they provide products that complement each other). The analysis of this paper can be extended to characterize the structure of the coalitions that form in equilibrium in settings with externalities but, in general, these externalities can imply that such equilibria do not exist, or that there is more than one such equilibrium. Ray and Vohra (1999) and Ray (2007) extend the construction of the (no-delay stationary subgame-perfect) equilibria in Chatterjee, Dutta, Ray, and Sengupta (1993) to settings with externalities across coalitions. A similar generalization of the construction of perfect equilibria that I describe in this paper to settings with general externalities across coalitions seems attainable—even if substantially more involved because of the more general preferences considered in this paper.

Finally, an important direction for future research is to investigate the conditions under which the outcome of decentralized bargaining in the thick markets considered in this paper is efficient. If one considers the economy consisting of the set of all the agents that have matched in equilibrium before any given period, the resulting allocation is in the core of this economy, and hence efficient. As a result, the only source of inefficiency in these markets can be the frequency with which agents of different types match in equilibrium. The modeling approach that we develop in Elliott and Talamàs (2018) to study investment incentives is better suited to investigate this question than the one of the present paper (mainly because, in contrast to the present paper, it features exogenous entry). Using a framework along the lines of the one in Elliott and Talamàs (2018), we can investigate the potential sources of inefficiencies in the dynamically-thick markets studied in this paper, thus complementing the large literature investigating the efficiency of decentralized bargaining outcomes in thin markets (e.g., Elliott and Nava 2018).

42 See also de Clippel and Serrano (2008) and Maskin (2016).
A Appendix

A.1 System of Equations that Characterizes the Equilibrium Payoffs

Using the function $f_i$ defined in Definition A.1, system (1) is equivalent to

\begin{equation}
    f_i(w_i) = \max_{C \subseteq I} \left( y(C) - \sum_{j \in C - i} w_j \right)
    \end{equation}

for all $i$ in $N$.

**Definition A.1.** For each type $i$, let the function $f_i : \mathbb{R} \rightarrow \mathbb{R}$ be (implicitly) defined by

$$u_i(w_i) = \lambda u_i(f_i(w_i)).$$

In words, $f_i(w_i)$ is the amount that an agent of type $i$ gets as a proposer that is consistent with her being indifferent between accepting and rejecting the amount $w_i$ (when she is not the proposer). The fact that the utility function $u_i$ is strictly increasing ensures that $f_i$ is well defined and, since $\lambda < 1$, that $f_i(w_i) > w_i$. For example, when the utility function of type $i$ is $u_i(x) = x^{1/\alpha}$, for some $\alpha > 0$, we have that $f_i(x) = x/\lambda^\alpha$, so that, for every value of $x$, the difference between $f_i(x)$ and $x$ increases with the concavity of $u_i$, as measured by $\alpha$. Figure 14 illustrates this definition.

The observation that system (6) characterizes the equilibrium payoff profile $w$ of the bargaining game $G$ is intuitive: When proposing, each type gets the maximum that she can obtain in some coalition given the amount $w_j$ of surplus that she has to offer to each agent of type $j$ in order to induce him to accept. Now the question arises of whether there always exists a solution to system (6), and whether there can be more than one solution to this system. I devote the rest of this section to showing that the answers are yes and no, respectively, and to characterize the resulting unique equilibrium payoff profile.

A.2 Auxiliary Game: Bargaining in a Fixed Coalition with Exogenous Outside Options

In this section, I characterize the equilibrium in an auxiliary game in which (i) only one coalition is productive and (ii) agents have exogenous outside options (Definition A.2). This equilibrium provides the basis for building the equilibrium of the bargaining game $G$.

**Definition A.2.** For any type $\ell$, any coalition $C \subseteq N$ containing type $\ell$, and any payoff profile $x$ in $\mathbb{R}^C$ satisfying the feasibility constraint $\sum_{j \in C - \ell} x_j < y(C)$, the bargaining game
$G(\ell, C, x)$ is defined exactly as the bargaining game $G$ with the following two modifications. First, the surplus $y(D)$ of each coalition $D \neq C$ is reduced to 0. Second, at the end of each period, every active agent of type $j \in C - \ell$ can opt to leave the market and obtain $x_j$ units of surplus.

Consider a stationary subgame-perfect equilibrium of the game $G(\ell, C, x)$. For each type $i$ in $C$, let $v_i$ be the amount that an agent of type $i$ is indifferent between accepting and rejecting in any given period. The argument analogous to the one in subsection A.1 implies that the profile $v$ must satisfy

$$f(\ell)(v) = y(C) - \sum_{j \in C-\ell} v_j, \text{ and}$$

$$f_i(v_i) = \max \left[ f_i(x_i), y(C) - \sum_{j \in C-i} v_j \right] \text{ for all } i \in C - \ell.$$  

The fact that there is only one relevant type of coalition in the bargaining game $G(\ell, C, x)$ substantially simplifies the system that pins down the equilibrium payoffs. Indeed, Lemma A.1 uses a relatively direct argument to show that these payoffs are unique. The proof of Lemma A.1 uses a generalization of standard arguments in the literature (see for example Ray (2007), Chapter 7) to the setting without perfectly transferable utility considered in this paper. Before presenting this result, I derive Equation 8 and Equation 9, which partially characterize the solution of system (7) and will prove useful in the subsequent analysis.

Let the profile $v$ in $\mathbb{R}^C$ be a solution to system (7). The fact that each function $f_i$ is increasing implies that $v_j \geq x_j$ for each type $j$ different than $\ell$, and hence that

$$f(\ell)(v) = y(C) - \sum_{j \in C} v_j \leq y(C) - \sum_{j \in C-\ell} x_j.$$  

Intuitively, Equation 8 says that the amount that an agent of type $\ell$ gets when she is the proposer in the auxiliary game $G(\ell, C, x)$ is bounded above by the available net surplus in the coalition $C$ after giving to each of its members his outside option. Moreover, the fact that,
by definition, \( f_i(v_i) \geq y(C) - \sum_{j \in C - i} v_j \) with equality for all types \( i \) with \( v_i > x_i \), implies that

\[
(9) \quad f_\ell(v_\ell) - v_\ell \leq f_i(v_i) - v_i \quad \text{with equality for all } i \text{ with } v_i > x_i.
\]

In words, Equation 9 says that the difference between the amount that an agent of type \( \ell \) gets when she is the proposer and when she is the receiver in the auxiliary game \( G(\ell, C, x) \) is the same as the difference between the amount that an agent of any other type \( i \) gets when she is the proposer and when she is the receiver, unless type \( i \)’s outside option binds, in which case the former is smaller than the latter.

**Lemma A.1.** There exists a unique profile \( v \) in \( \mathbb{R}^C \) that solves system (7).

**Proof.** Existence follows from Brouwer’s fixed point theorem. To prove uniqueness, suppose for contradiction that there are two profiles \( v, v' \) that solve system (7). Define \( S \) to be the set of all types for which these two solutions differ; that is, \( S := \{ i \in N \mid v_i \neq v'_i \} \). Let \( i \) be one of the elements of the set \( S \) for which \( f_i(v_i) - v_i \) is highest, and suppose without loss of generality that \( f_i(v_i) - v_i \) is an upper bound on \( \{ f_j(v'_j) - v'_j \} \) \( j \in S \). The concavity of the utility function \( u_i \) implies that the function \( f_i(v_i) - v_i \) is increasing in \( v_i \) (see Figure 14), so we also have that \( v_i > v'_i \). Moreover we have that

\[
(10) \quad f_i(v_i) = y(C) - \sum_{j \in C - i} v_j,
\]

since otherwise, \( i \neq \ell \) and \( v_i = x_i \) (see Equation 9), which contradicts the fact that \( v'_i \geq x_i \). In particular, \( f_i(v_i) - v_i = f_\ell(v_\ell) - v_\ell \), so Equation 9 implies

\[
(11) \quad f_j(v_j) - v_j \geq f_i(v_i) - v_i \quad \text{for all } j \in C.
\]

Given the choice of type \( i \), Equation 11 implies that

\[
f_j(v_j) - v_j \geq f_j(v'_j) - v'_j \quad \text{for all } j \in C,
\]

or, using again that \( f_j(v_j) - v_j \) is increasing in \( v_j \), that \( v_j \geq v'_j \). But then, Equation 10 combined with the fact that the function \( f_i \) is increasing and, by definition,

\[
f_i(v'_i) \geq y(C) - \sum_{j \in C - i} v'_j
\]

implies that \( v'_i \geq v_i \), a contradiction. \( \square \)

**Definition A.3.** For any type \( \ell \), any profile \( x \) in \( \mathbb{R}^I \) and any coalition \( C \subseteq N \) satisfying the feasibility condition \( \sum_{j \in C - \ell} x_j \leq y(C) \), type \( \ell \)’s \( x \)-share in \( C \) is the \( \ell \)th element of the profile \( v \) in \( \mathbb{R}^C \) that satisfies system (7).
A.3 Consistency and Equilibrium

I now turn to proving Proposition A.1, which implies that a profile \( w \) in \( \mathbb{R}^N \) is an equilibrium payoff profile in the bargaining game \( G \) if and only it is consistent, as defined in Definition A.5.

Definition A.4. For any type \( \ell \) and any profile \( x \) in \( \mathbb{R}^N \), type \( \ell \)'s \( x \)-best share is her maximum \( x \)-share over all coalitions, and her \( x \)-best coalitions are those coalitions \( C \subseteq N \) for which her \( x \)-share in \( C \) is her \( x \)-best share.

Definition A.5. The payoff profile \( x \) in \( \mathbb{R}^N \) is consistent if, for every type \( i \), \( x_i \) is \( i \)'s \( x \)-best share.

Proposition A.1. A profile \( x \) in \( \mathbb{R}^N \) solves system (6) if and only if it is consistent.

Proof. Let the profile \( x \) in \( \mathbb{R}^N \) be such that, for every type \( i \), \( x_i \) is \( i \)'s \( x \)-best share, and let \( \ell \) be in \( N \). Recalling Equation 8, we have that

\[
f_\ell(x_\ell) \leq \max_{C \subseteq N} \left( y(C) - \sum_{j \in C-\ell} x_j \right),
\]

so we only need to show that there exists a coalition \( C \subseteq N \) with \( f_\ell(x_\ell) \geq y(C) - \sum_{j \in C-\ell} x_j \), but this is satisfied by \( \ell \)'s \( x \)-best coalition. Indeed, let \( C \) be \( \ell \)'s \( x \)-best coalition, and suppose for contradiction that \( f_\ell(x_\ell) < y(C) - \sum_{j \in C-\ell} x_j \). This implies that the profile \( w \) in \( \mathbb{R}^C \) that solves system (7) is such that \( w_\ell = x_\ell \) (by the assumption that \( x_\ell \) is \( \ell \)'s \( x \)-best share), and that \( w_i > x_i \) for some type \( i \) in \( C - \ell \) (otherwise, \( y(C) - \sum_{j \in C-\ell} x_j \) is equal to \( y(C) - \sum_{j \in C-\ell} w_j \), which by definition is itself equal to \( f_\ell(x_\ell) \), a contradiction). Hence, the same profile \( w \) solves system (7) after interchanging the roles of \( i \) and \( \ell \) in this system, a contradiction of the assumption that \( x_i \) is \( i \)'s \( x \)-best share.

In the other direction, suppose that the profile \( x \) in \( \mathbb{R}^N \) solves system (6). Let \( \ell \) in \( N \) and \( C \subseteq N \) be such that \( f_\ell(x_\ell) = y(C) - \sum_{j \in C-\ell} x_j \). First, note that \( \ell \)'s \( x \)-share \( z_\ell \) in any coalition \( D \neq C \) is bounded above by \( x_\ell \), since, using Equation 8,

\[
f_\ell(z_\ell) \leq y(D) - \sum_{j \in D-\ell} x_j \leq y(C) - \sum_{j \in C-\ell} x_j = f_\ell(x_\ell).
\]

Hence, it is enough to show that \( \ell \)'s \( x \)-share in \( C \) is bounded below by \( x_\ell \). Let the profile \( w \) in \( \mathbb{R}^C \) solve system (7). Suppose for contradiction that \( w_\ell < x_\ell \). Then, \( f_\ell(w_\ell) < y(C) - \sum_{j \in C-\ell} x_j \), which implies that \( w_j > x_j \) for some type \( j \) in \( C \). Using Equation 9, the fact that \( f_\ell(w_\ell) - w_\ell \) is increasing in \( w_\ell \), and \( x \) solves system (6), we get

\[
f_j(w_j) - w_j = f_\ell(w_\ell) - w_\ell < f_\ell(x_\ell) - x_\ell = y(C) - \sum_{j \in C} x_j \leq f_j(x_j) - x_j.
\]
which implies that \( w_j < x_j \), a contradiction. 

A.4 Construction of the Unique Consistent Payoff Profile

Proposition A.2 below shows that the algorithm \( A \) defined below (Definition A.6) computes the unique consistent payoff profile. Informally, this algorithm runs as follows: The initial outside option profile is \( x^0 = 0 \). In each step \( k = 1, 2, \ldots \), the outside option profile is \( x^{k-1} \), and each type points to her \( x^{k-1} \)-best coalitions. A coalition that is such that all of its members \( j \) with \( x_j^{k-1} = 0 \) point to it “clears,” and the outside option \( x_j^k \) of each of its members is updated to be her \( x^{k-1} \)-best share in this coalition.

Definition A.6 (Algorithm \( A \)). Let \( x^0 = 0 \in \mathbb{R}^N \) and \( X_0 = \emptyset \). Proceed inductively as follows. In each step \( k = 1, 2, \ldots \):

1. Let \( S_k \) be the set of all types in \( N - X_{k-1} \) that are members of some coalition that is an \( x^{k-1} \)-best coalition of all its members in \( N - X_{k-1} \).

2. For each type \( i \) in \( S_k \), let \( x_i^k \) be her \( x^{k-1} \)-best share; for each other type \( j \), let \( x_j^k = x_j^{k-1} \).

3. Let \( X_k \) be the union of \( X_{k-1} \) and \( S_k \).

End in the first step \( \kappa \) for which \( S_\kappa \) is empty, and let \( \chi := x^\kappa \).

Proposition A.2. The payoff profile \( \chi \) defined by algorithm \( A \) is the unique consistent profile.

Proof. The fact that algorithm \( A \) updates each type’s outside option at most once, and such updates only increase types’ outside options, implies that both the \( \chi \)-best share and the \( \chi \)-best coalitions of each type in \( S_\kappa \) for each \( k \leq \kappa \) are exactly as her \( x^k \)-best share and her \( x^k \)-best coalitions. This implies, in turn, that \( \chi \) is consistent as long as \( X_\kappa = N \). The rest of this proof consists of showing that \( X_\kappa \) is equal to \( N \) and that every consistent profile gives \( \chi_i \) to each type \( i \).

First, I prove by induction in \( k \) that \( S_k \) is empty only if \( X^{k-1} \) is equal to \( N \) (so that \( X_\kappa \) is equal to \( N \)). Let \( k \) be such that \( X_{k-1} \) is a strict subset of \( N \) (this induction hypothesis is vacuously true when \( k = 1 \), so there is no need to prove the base step separately). Denoting, for each coalition \( C \), agent \( i \)'s \( x^{k-1} \)-share in \( C \) by \( x_i^C \), by definition, there exists a number \( \mu_C \) such that \( f_i(x_i^C) - x_i^C = \mu_C \) for every agent \( i \) in \( C - X_{k-1} \). A coalition \( D \) with maximum \( \mu_D \) (among those coalitions \( C \) such that \( C - X_{k-1} \) is nonempty) is an \( x^{k-1} \)-best coalition of all types in \( C - X_{k-1} \), since the fact that \( u_i \) is concave implies that \( f_i(x_i) - x_i \) is increasing in \( x_i \) (see Figure 14).
Second, I prove by induction in \( k \) that, for each \( k \leq \kappa \), every consistent profile gives \( x^k_i \) to each type \( i \) in \( S_k \) (so that \( \chi \) is the only possible consistent profile). Let \( x \) be a consistent payoff profile. Let \( k \) be such that \( x_i \) is equal to \( x_i^{k-1} \) for each agent \( i \) in \( X_{k-1} \) (again, this induction hypothesis is vacuously true when \( k = 1 \), so there is no need to prove the base step separately). Let \( C \) be a coalition that is an \( x^{k-1} \)-best coalition of all its members in \( N - X_{k-1} \). Suppose for contradiction that, for some \( i \) in \( C - X_{k-1} \), \( x_i \) is strictly smaller than \( i \)'s \( x^{k-1} \)-share in \( C \) (the induction hypothesis together with fact that algorithm \( A \) only updates outside options upwards implies that \( x_i \) cannot be strictly bigger than \( i \)'s \( x^{k-1} \)-share in \( C \)). This implies that \( i \)'s \( x \)-share in \( C \) is strictly smaller than \( i \)'s \( x^{k-1} \)-share in \( C \) which implies, in turn, that for some \( j \) in \( C - X_{k-1} \), \( x_j \) is strictly bigger than \( j \)'s \( x^{k-1} \)-share in \( C \) (that is, \( j \)'s \( x^{k-1} \)-best share), which, as just argued, contradicts the induction hypothesis.

\[\square\]

### A.5 The unique Nash consistent payoff profile

**Proposition A.3.** As the bargaining friction \( q \) goes to zero, the unique consistent payoff profile converges to a Nash-consistent payoff profile.

**Proof.** It is enough to show that, for every \( x \) in \( \mathbb{R}^N \) and \( C \subseteq N \) such that \( y(C) \geq \sum_{j \in C - \ell} x_j \), the payoff profile that solves system (7) converges, as \( q \) goes to zero, to the payoff profile that solves

\[
\max_{s \in \mathbb{R}^C} \prod_{j \in C} u_j(s_j) \text{ s.t. } y(C) \geq \sum_{j \in C} s_j \text{ and } s_j \geq x_j \text{ for all } j \in C - \ell.
\]

Indeed, if this is the case, then letting \( x(q) \) be the consistent payoff profile associated with the bargaining friction \( q \), for each type \( i \), \( x_i(q) \) converges to \( i \)'s \( x(q) \)-Nash best share.

Using **Definition A.1**, system (7) can be rewritten as

\[
\begin{align*}
    u_\ell(v_\ell) &= \lambda u_\ell \left( y(C) - \sum_{j \in C - \ell} v_j \right), \text{ and } \\
    u_i(v_i) &= \max \left[ u_i(x_i), \lambda u_i \left( y(C) - \sum_{j \in C - i} v_j \right) \right] \text{ for all } i \in C - \ell.
\end{align*}
\]

Fix a type \( i \) in \( C \), and let \( v'_i \) be such that \( u_i(v'_i) = \lambda u_i \left( y(C) - \sum_{j \in C - i} v_j \right) \). It is enough to show that \((v'_i, v_\ell)\) converges to the maximizer of \( u_\ell(s_\ell)u_i(s_i) \) over the set

\[
S = \left\{ (s_i, s_\ell) \in \mathbb{R}^2 \mid s_i + s_\ell = y(C) - \sum_{j \in C - i - \ell} v_j \right\}.
\]

Indeed, this implies that the solution of system (7) maximizes—in the limit as the bargaining frictions vanish—the product of all the utilities of types in \( C \) whose outside options do not bind in this coalition. In other words, this solution converges to the solution of system (5).
By definition, for each type $j$ and each number $a$, we have that $f_j(a)$ converges to $a$ as $q$ goes to zero. Moreover, we have that
\[
 u_i(f_i(v'_i))u_\ell(v_\ell) = u_i(v'_i)u_\ell(f_i(v_\ell)) = \lambda u_i \left( g(C) - \sum_{j \in C-i} v_j \right) u_\ell \left( g(C) - \sum_{j \in C-\ell} v_j \right)
\]
so $(v'_i, v_\ell)$ indeed converges to the maximizer of $u_\ell(s_\ell)u_i(s_i)$ over the set $S$. Figure 15 illustrates.

I now describe how a reasoning analogous to the one that shows that there exists a unique consistent payoff profile implies that there is also a unique Nash-consistent payoff profile. Proposition A.4 is the analog of Proposition A.2: It shows that algorithm $A^*$ (Definition A.7) computes the unique Nash-consistent payoff profile. This algorithm is analogous to algorithm $A$ using the notion of $x$-Nash-best payoffs and $x$-Nash-best coalitions instead of $x$-best payoffs and $x$-best coalitions.

**Definition A.7 (Algorithm $A^*$).** Let $x^0 = 0 \in \mathbb{R}^N$ and $X_0 = \emptyset$. Proceed inductively as follows. In step $k = 1, 2, \ldots$:

1. Let $S_k$ be the set of all types in $N - X_{k-1}$ that are members of some coalition that is an $x^{k-1}$-Nash best coalition of all its members in $N - X_{k-1}$.
2. For each type $i$ in $S_k$, let $x^k_i$ be her $x^{k-1}$-Nash best share; for each other type $j$, let $x^k_j = x^{k-1}_j$.
3. Let $X_k$ be the union of $X_{k-1}$ and $S_k$.  

Figure 15: Illustration of the proof of Proposition A.3. The graph $U$ corresponds to the set of utility pairs $\{(u_i(s_i), u_\ell(s_\ell)) | s \in S\}$. This figure is the analog of Figure 4.2 in Osborne and Rubinstein (1990).
End in the first step $\kappa$ for which $S_\kappa$ is empty, and let $\chi^* := x^\kappa$.

**Proposition A.4** is analogous to **Proposition A.2**.

**Proposition A.4.** The payoff profile $\chi^*$ defined by algorithm $A^*$ is the unique Nash-consistent profile.

The only part of the proof of **Proposition A.4** that differs from the proof of **Proposition A.2** is the reasoning behind the fact that, for each step $k$ with $X_k \neq N$, $S_k$ is not empty. The argument in this case is analogous to that in Pycia (2012, pages 330-331).\(^43\) Denoting, for each coalition $C$, type $i$’s $x^{k-1}$-Nash share by $x_i^C$, and letting $u_i'$ denote the derivative of the utility function $u_i$, we have that $u_i(x_i^C)/u_i'(x_i^C)$ is the same for every type $i$ in $C-X^{k-1}$; denote by $\mu_C$ this common value. A coalition $C$ with maximum $\mu_C$ is the $x^{k-1}$-best coalition of all its members outside of $X_{k-1}$, since each type’s $x^{k-1}$-Nash share in $C$ is increasing in $\mu_C$.

**Proposition A.5** follows from combining **Proposition A.1**, **Proposition A.3** and **Proposition A.4**, and it culminates the equilibrium characterization of the bargaining game $G$ in the limit as the bargaining frictions vanish.

**Proposition A.5.** As the bargaining friction $q$ goes to 0, the unique profile that solves system (1) converges to the unique Nash-consistent payoff profile $\chi^*$ constructed by algorithm $A^*$.

In words, in the equilibrium of the game $G$, when bargaining frictions are small enough, each agent’s equilibrium payoff is the maximum that she can justify in some coalition using the Nash bargaining solution subject to the others’ equilibrium payoffs.

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\(^{43}\)Pycia (2012) uses this argument to illustrate how there exists a stable coalitional structure when coalitional output is shared according to the Nash bargaining solution (with exogenous outside options).
References


