

Testing the Drift-Diffusion Model

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1 Introduction

The *drift diffusion model* (DDM) is a model of sequential sampling with diffusion (Brownian) signals, where the decision maker accumulates evidence until the process hits a stopping boundary, and then stops and chooses the alternative that corresponds to that boundary. This model has been widely used in psychology, neuroeconomics, and neuroscience to explain the observed patterns of choice and response times in a range of binary choice decision problems. One class of papers study “perception tasks” with an objectively correct answer (e.g. “are more of the dots on the screen moving left or moving right?”); here the drift of the process is related to which choice is objectively correct [Ratcliff and McKoon \(2008\)](#); [Shadlen and Kiani \(2013\)](#). The other class of papers study “consumption tasks” such as “which of these snacks would you rather eat?”; here the drift is related to the relative appeal of the alternatives ([Clithero and Rangel, 2013](#); [Fehr and Rangel, 2011](#); [Krajbich, Armel, and Rangel, 2010](#); [Krajbich, Bartling, Hare, and Fehr, 2015](#); [Krajbich, Lu, Camerer, and Rangel, 2012](#); [Krajbich and Rangel, 2011](#); [Milosavljevic, Malmaud, Huth, Koch, and Rangel, 2010a](#); [Reutskaja, Nagel, Camerer, and Rangel, 2011](#); [Roe, Busemeyer, and Townsend, 2001](#)).

The simplest version of the DDM assumes that the stopping boundaries are constant over time [Edwards \(1965\)](#); [Ratcliff \(1978\)](#); [Stone \(1960\)](#); [Wald \(1947\)](#). More recently a number of papers use non-constant boundaries to better fit the data, and in particular the observed correlation between response times and choice accuracy, i.e., that correct responses are faster than error responses [Drugowitsch, Moreno-Bote, Churchland, Shadlen, and Pouget \(2012\)](#); [Fu-](#)

denberg, Strack, and Strzalecki (2018); Luce (1986); Milosavljevic, Malmaud, Huth, Koch, and Rangel (2010b).

Constant stopping boundaries is the optimal solution for perception tasks where the volatility of the signals and the flow cost of sampling are both constant, and the prior belief is that the drift of the diffusion has only two possible values, depending on which decision is correct. Even with constant volatility and costs, non-constant boundaries are optimal for other priors. Fudenberg, Strack, and Strzalecki (2018) characterize the optimal boundaries for the consumption task: the decision maker is uncertain about the utility of each choice, with independent normal priors on the value of each option. Drugowitsch, Moreno-Bote, Churchland, Shadlen, and Pouget (2012) show how to computationally derive the optimal boundaries for the perception task: the signal coherence varies from trial to trial, so some decision problems are harder than others.

This paper provides a statistical test for DDM's with general boundaries. We first prove a characterization theorem: we find a condition on choice probabilities that is satisfied if and only if the choice probabilities are generated by some DDM. Moreover, we show that the drift and the boundary are uniquely identified. We then use our condition to nonparametrically estimate the drift and the boundary and construct a test statistic based on finite samples.

Recent related work on DDM includes Drugowitsch, Moreno-Bote, Churchland, Shadlen, and Pouget (2012) who conducted a Bayesian estimation of a collapsing boundary model and Fudenberg, Strack, and Strzalecki (2018) who conducted a maximum likelihood estimation. Hawkins, Forstmann, Wagenmakers, Ratcliff, and Brown (2015) estimate collapsing boundaries in a parametric class, allowing for a random nondecision time at the start. Chiong, Shum, Webb, and Chen (2018) estimate a version of DDM with constant boundaries but random starting point of the signal accumulation process; Ratcliff (2002) estimates a similar model where other parameters are made random. Baldassi, Cerreia-Vioglio, Maccheroni, and Marinacci (2018) partially characterize DDM with constant boundary.¹

Other work on DDM-like models includes the decision field theory of Busemeyer and Johnson

¹They ignore the issue of correlation between response times choices by looking only at marginal distributions, which makes their conditions necessary but not sufficient.

(2004); Busemeyer and Townsend (1992, 1993) allows the signal process to be mean-reverting. Alós-Ferrer, Fehr, and Netzer (2018) and Echenique and Saito (2017) study models where response time is a deterministic function of the utility difference. Che and Mierendorff (2016); Hebert and Woodford (2016); Liang and Mu (2019); Liang, Mu, and Syrgkanis (2019); Woodford (2014); Zhong (2019) study dynamic costly optimal information acquisition.

2 The Stochastic Choice Function

Let X be the universe of alternatives (actions) and $T = \mathbb{R}_+$ be time. For every pair of objects $\{x, y\}$ the analyst observes pairwise stochastic choices and decision times. In the limit as the sample size grows large, the analyst will have access to the joint distribution over which object is chosen and at which time a choice is made. We denote by $F^{xy}(t)$ the probability that the agent makes a choice by time t , and let $p^{xy}(t)$ the probability that the agent picks x conditional on stopping at time t . Throughout, we restrict attention to cases where F has full support and no atoms at time 0, so that $F(0) = 0$, and we assume that F is strictly increasing with $\lim_{t \rightarrow \infty} F(t) = 1$. These restrictions imply the agent never stops immediately, that there is a positive probability of stopping in every time interval, and that the agent always eventually stops. We call (p^{xy}, F^{xy}) , the *stochastic choice function*.

An immediate restriction on the stochastic choice function is that the choices of the agent are unaffected by which object we consider to be the first and which object we consider to be the second. This is formally equivalent to

$$p^{xy}(t) \equiv 1 - p^{yx}(t) \text{ for all } t \text{ and } F^{xy} \equiv F^{yx} \text{ for all } x, y \in X.$$

Without loss of generality we only consider stochastic choice functions which satisfy this restriction. We also assume that each option is chosen with positive probability $0 < p^{xy}(t) < 1$ for all t .

Given (p^{xy}, F^{xy}) we define the choice imbalance at each time t to be

$$I^{xy}(t) := p^{xy}(t) \log \left(\frac{p^{xy}(t)}{1 - p^{xy}(t)} \right) + (1 - p^{xy}(t)) \log \left(\frac{1 - p^{xy}(t)}{p^{xy}(t)} \right).$$

This is the Kullback-Leibler divergence (or relative entropy) between the Binomial distribution of the agent's time t choice $P(t) = (p(t), 1 - p(t))$ and the permuted choice distribution $Q(t) = (1 - p(t), p(t))$. As the Kullback-Leibler divergence is a statistical measure of the similarity between distributions $I(t)$ captures the imbalance of the agent's choice at time t . Note that $I = 0$ means that both choices are equally likely; $I = \infty$ when p equals 0 or 1, and that I is symmetric about 0.5. We define \bar{I}^{xy} to be the average choice imbalance,

$$\bar{I}^{xy} := \int_0^\infty I^{xy}(t) dF^{xy}(t),$$

and we define \bar{T}^{xy} to be the average decision time,

$$\bar{T}^{xy} := \int_0^\infty t dF^{xy}(t),$$

and define \bar{p}^{xy} to be the average choice probability,

$$\bar{p}^{xy} := \int_0^\infty p^{xy}(t) dF^{xy}(t),$$

and assume that all of these integrals exist. Finally, we relabel objects as needed so that the first object is chosen weakly more often, i.e. $\bar{p}^{xy} \geq 0.5$ for all x, y .

3 DDM representation

The drift diffusion model (DDM) is commonly used to explain the stochastic choice data in neuroscience and psychology. The two main ingredients of a DDM are the stimulus process Z_t and a time-dependent stopping boundary $b(t)$. In the DDM representation, the stimulus

process Z_t is a Brownian motion with drift δ and volatility α :

$$Z_t = \delta t + \alpha B_t, \tag{1}$$

where B_t is a standard Brownian motion, so in particular $Z_0 = 0$. Define the hitting time τ

$$\tau = \inf\{t \geq 0 : |Z_t| \geq b(t)\}, \tag{2}$$

i.e., the first time the absolute value of the process Z_t hits the boundary b . Let $F^*(t; \delta, b, \alpha) := \mathbb{P}[\tau \leq t]$ be the distribution of the stopping time τ . Likewise, let $p^*(t; \delta, b, \alpha)$ be the conditional choice probability induced by (1) and (2) and a decision rule that chooses x if $Z_\tau = b(\tau)$ and y if $Z_\tau = -b(\tau)$.

Our goal in this paper is to determine which data is consistent with a DDM representation, and when it is, when the representation is unique. When the drift $\delta = 0$, each alternative will be chosen half of the time regardless of the shape of the boundary, so we will exclude this case going forward.

The original formulation of the DDM was for “perception tasks” where the drift δ is either $+1$ or -1 depending on which decision is correct; more generally there can be a distinct drift δ^{xy} for each pair x, y . In consumption-choice problems (otherwise known as value-based problems, see, e.g., Milosavljevic, Malmaud, Huth, Koch, and Rangel (2010b)) it is natural to assume that the net drift δ^{xy} is the difference between two signals, an x -signal with drift $u(x)$ equal to the utility of x and a y -signal with drift $u(y)$ equal to the utility of y , so that $\delta^{xy} = u(x) - u(y)$. This imposes some consistency conditions that we discuss below.

Definition 1 (DDM Representation). Stochastic choice data $(p^{xy}, F^{xy})_{x,y \in X}$ has a DDM representation if there exists a utility function $u : X \rightarrow \mathbb{R}$, a volatility parameter $\alpha > 0$ as well as a boundary $b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for all $x, y \in X$ and $t \in \mathbb{R}$

$$p^{xy}(t) = p^*\left(t, u(x) - u(y), b, \alpha\right)$$

and $F^{xy}(t) = F^*\left(t, u(x) - u(y), b, \alpha\right).$

Note that this definition requires that the data from all of the menus $\{x, y\}$ is generated with the *same* boundary function b . This corresponds to cases where the agent treats each decision problem as a random draw from a fixed environment.² We are interested in characterizing which stochastic choice functions admits a DDM representation. The following result follows immediately from rescaling δ and b .

Lemma 1. *If a stochastic choice function exhibits a DDM representation for some α , then it also exhibits a DDM representation for $\alpha = 1$.*

We will thus without loss of generality only consider the DDM model where we normalized $\alpha = 1$. We write $p^*(t, \delta, b)$ and $F^*(t, \delta, b)$ as short-hands for $p^*(t, \delta, b, 1)$ and $F^*(t, \delta, b, 1)$.

4 Characterization

Given a stochastic choice function (p^{xy}, F^{xy}) , define the *revealed drift*

$$\tilde{\delta}^{xy} := \sqrt{\frac{\bar{I}^{xy}}{2\bar{T}^{xy}}}. \quad (3)$$

When the revealed drift is non zero, we define the *revealed boundary* as

$$\tilde{b}^{xy}(t) := \frac{\ln p^{xy}(t) - \ln(1 - p^{xy}(t))}{2\tilde{\delta}^{xy}}. \quad (4)$$

The revealed drift is high for a pair x, y whenever the agent either makes very imbalanced choices or decides quickly, and low for choices that are slow and close to 50-50. Over time the boundary at time t follows the log-odds ratio of the agent's choice at time t which is zero whenever the agent's choice is balanced and increases in the imbalance of the agent's choice. The revealed boundary is smaller for pairs with a larger revealed drift. In the knife-edge case when the revealed drift is 0 the revealed boundary is not defined and our results do not apply.

Theorem 1 below says that if the true data generating process is a DDM, then the revealed drift and boundary will exactly match the true parameters. Moreover, Theorem 1 allows us to

²In an optimal stopping model, the shape of the boundary is determined by the agent's prior over these draws.

test whether the true data generating process is indeed a DDM.

4.1 Characterization for a fixed pair

Our first result characterizes the DDM for a fixed pair $x, y \in X$.

Theorem 1. *For a fixed pair x, y with $\tilde{\delta}^{xy} \neq 0$ the stochastic choice function (p^{xy}, F^{xy}) admits a DDM representation if and only if for all $t \geq 0$*

$$F^{xy}(t) = F^*(t; \tilde{\delta}^{xy}, \tilde{b}^{xy})$$

If such a representation exists it is unique (up to the choice of α) and given by $\tilde{\delta}^{xy}, \tilde{b}^{xy}$.

Thus, the stochastic choice function (p^{xy}, F^{xy}) is consistent with DDM whenever the observed distribution of stopping times F^{xy} equals to the distribution of hitting times generated by the revealed drift $\tilde{\delta}^{xy}$ and revealed boundary \tilde{b}^{xy} . Theorem 1 shows that the revealed drift and boundary are the unique candidate for a DDM representation. It thus allows us to identify the parameters of the DDM model directly from choice data. This permits the model to be calibrated to the data without computing the likelihood function, which requires computationally costly Monte-Carlo simulations. More substantially, as Theorem 1 connects the primitives of the model directly to data it allows us to better understand their economic meaning. The drift in the DDM model is a measure of how imbalanced and quick the agent's choices are and the shape of the boundary follows the imbalance of the agent's choices over time. We hope that this interpretation makes the empirical content of the parameters of DDM model more transparent and the model thus more useful.

Note that this theorem shows that the distribution of stopping times contains additional information that is not captured by the mean. For example, choice data where $p^{xy}(t)$ and \bar{T}^{xy} are any 2 given constants is only consistent with one possible distribution of stopping times F^{xy} . However a test based only on the mean choice probability and mean stopping time will accept any model that matches those two numbers, and in particular regardless of F^{xy} the data is consistent with a constant stopping boundary. (See Baldassi, Cerreia-Vioglio, Maccheroni, and Marinacci (2018)).

4.2 Characterization for menus of pairs

Our next result extends the characterization to all pairs $x, y \in X$.

Theorem 2. *The stochastic choice function $(\{p^{xy}\}, \{F^{xy}\}_{x,y \in X})$ has a DDM representation iff*

- (i) $F^{xy}(t) = F^*(t; \tilde{\delta}^{xy}, \tilde{b}^{xy})$ for all $t \geq 0$,
- (ii) $\tilde{b}^{xy}(t) = \tilde{b}^{xz}(t)$ for all $x, y, z \in X$ and all $t \geq 0$.
- (iii) $\tilde{\delta}^{xy} + \tilde{\delta}^{yz} = \tilde{\delta}^{xyz}$ for all $x, y, z \in X$,

Thus, in addition to satisfying the condition from Theorem 1 pairwise, we have two additional consistency conditions imposed across pairs. Condition (ii) follows from our assumption that the agent uses the same stopping boundary in every menu. Condition (iii) comes from the assumption that the drift in a given menu depends on the difference of utilities, that is $\delta^{xy} = u(x) - u(y)$.³

5 An Econometric Test for a Fixed Pair of Alternatives

The idea for the test is based on Theorem 1, which requires that the observed distribution of stopping times matches the distribution induced by the revealed boundary \tilde{b} and drift \tilde{d} . We first describe a nonparametric estimator of \tilde{b} and \tilde{d} based on a finite data set. Next, we show how to test the distribution matching condition. This test could be extended to multiple-alternatives settings along the lines of Theorem 2, but we do not do so here.

5.1 Estimation of drift and boundary

Suppose that we have a fixed pair $x, y \in X$. Define

$$\gamma_\tau := \begin{cases} 1, & \text{when choice } x \text{ is made,} \\ 0, & \text{when choice } y \text{ is made.} \end{cases}$$

³The proof of the theorem follows from Theorem 1 and the Sincov functional equation, see, e.g., [Aczél \(1966\)](#).

Each data point consists of the time τ_i at which the choice is made and the choice γ_i made at time τ_i .

Assumption 1. The data $(\tau_1, \gamma_1), \dots, (\tau_n, \gamma_n)$ are i.i.d.

The unknown features of the DDM model are the drift δ and the boundary $b(t)$. We use estimators based on equations (3) and (4) that identify the revealed drift and boundary. Both of them depend on the choice probability, so we first give an estimator of that. Here $p^{xy}(t) := \Pr(\gamma_i = 1 | \tau_i = t)$ is the probability of choice x conditional on the choice being made at t .

The nonparametric estimator we construct is a spline regression: that is, a least squares regression of γ_i on approximating functions of τ_i . For simplicity, we use a linear probability estimator of $p^{xy}(t)$.⁴

We first transform τ_i to the unit interval.⁵ For this purpose let $G(t)$ be a CDF of a positive random variable with PDF that is positive on $(0, \infty)$. Consider

$$G_i = G(\tau_i).$$

Because G_i lies in the unit interval we can use standard series estimation to estimate $p^{xy}(t)$.

We consider regression spline estimation of $p^{xy}(t)$. For this purpose let

$$q^K(G) = (q_{1K}(G), \dots, q_{KK}(G))'$$

be a B -spline vector, say for evenly spaced knots on $(0, 1)$. Let $\hat{\beta}$ be OLS coefficients from regressing γ_i on $q_i^K = q^K(G_i)$. The choice probability estimator we consider is

$$\hat{p}(t) := q^K(G(t))' \hat{\beta}, \quad \hat{\beta} := \left[\sum_{i=1}^n q_i^K q_i^{K'} \right]^{-1} \sum_{i=1}^n q_i^K \gamma_i.$$

We give conditions for this estimator to be consistent and have other important large sample

⁴We reserve consideration of other estimators of the choice probability to future work, including logit or probit with a series approximation inside the logit or probit CDF.

⁵In DDM models where $b(t)$ does not reach zero, there is no uniform bound on realized decision times τ_i . Because τ_i is the conditioning variable (i.e. regressor) in the choice probability, it is important to allow for an unbounded regressor.

properties in Assumptions 2 and 3 to follow.

We can estimate the drift δ by plugging in $\hat{p}(t)$ for $p^{xy}(t)$ in formula (3) and replacing expectations with sample averages. Let

$$\hat{I}(t) := \hat{p}(t) \ln \left[\frac{\hat{p}(t)}{1 - \hat{p}(t)} \right] + [1 - \hat{p}(t)] \ln \left[\frac{1 - \hat{p}(t)}{\hat{p}(t)} \right],$$

$$\bar{I} := \frac{1}{n} \sum_{i=1}^n \hat{I}(\tau_i), \quad \bar{\tau} := \frac{1}{n} \sum_{i=1}^n \tau_i.$$

The estimator of δ is then

$$\hat{\delta} := \sqrt{\frac{\bar{I}}{\bar{\tau}}}.$$

The estimator of the boundary $b(t)$ is obtained by plugging in $\hat{\delta}$ and $\hat{p}(t)$ in the expression of equation (4), giving

$$\hat{b}(t) := \frac{1}{\hat{\delta}} \ln \left[\frac{\hat{p}(t)}{1 - \hat{p}(t)} \right].$$

5.2 Testing

We now have to test whether the observed distribution of stopping times matches the one induced by the revealed drift and boundary. We do this by comparing sample moments of functions of the decision time with estimators of the moments that predicted by the model. To describe such a test let $m_J(\tau) = (m_{1J}(\tau), \dots, m_{JJ}(\tau))'$ be a vector of functions of τ . Examples include indicator functions for intervals and B-splines in $G(\tau)$. The sample average vector will be $\bar{m} = \sum_{i=1}^n m_J(\tau_i)/n$.⁶ We use simulation to obtain model prediction. To describe the simulated predictions, let $\{B_t^1, \dots, B_t^S\}$ be S independent copies of Brownian motion and

$$\hat{\tau}_s = \inf\{t \geq 0 : |\hat{\delta}t + B_t^s| \geq \hat{b}(t)\}.$$

⁶The Kolmogorov–Smirnov test uses indicator functions but instead of the the average of m it takes the supremum. The Cramer–von Mises test takes the sum of squares. We look at the average of m because the target cdf we are comparing with is not fixed, but involves estimates of the boundary and drift, see Newey (1994).

A moment vector predicted by the model would be $\hat{m}_S = \sum_{s=1}^S m_J(\hat{\tau}_s)/S$. A test of the model can be based on comparing \bar{m} and \hat{m} . Let \hat{V} be a consistent estimator of the asymptotic variance of $\sqrt{n}(\bar{m} - \hat{m}_S)$ when the model is correctly specified, as we will describe below. A test statistic can be formed as

$$\hat{A} := n(\bar{m} - \hat{m}_S)' \hat{V}^{-1} (\bar{m} - \hat{m}_S).$$

The model would be rejected if \hat{A} exceeds the critical value of a $\chi^2(J)$ distribution. If J is allowed to grow with n and the $m_J(\tau)$ is allowed to grow in dimension and richness as n grows then this approach will test all the restrictions implied by DDM as n grows. In Appendix A we describe the construction of \hat{V} .

In formulating conditions for the asymptotic distribution of this test we will let $m_{jJ}(\tau)$, ($j = 1, \dots, J$) be indicator functions for disjoint intervals. Let $\tau_{jJ} = G^{-1}(j/(J+1))$, ($j = 0, \dots, J$), $\tau_{J+1,J} = \infty$. Consider

$$m_{jJ}(t) = \sqrt{J+1} \cdot \mathbb{1}(\tau_{j,J} \leq t < \tau_{j+1,J}), \quad (j = 1, \dots, J).$$

The test based on these functions will be based on comparing empirical probabilities of intervals with those predicted by the model. The normalization of multiplying by $\sqrt{J+1}$ is convenient in making the second moment of these functions of the same magnitude for different values of J . Note that we have left out the indicator for the interval $(0, 1/(J+1))$. We have done this to account for the fact that the estimator the drift parameter uses some information about τ_i , so that we are not able to test all of the implications of the DDM for the distribution of τ_i . As usual we can only test overidentifying restrictions.

We derive results under the following conditions:

Assumption 2. The pdf of $G(\tau_i)$ is bounded and bounded away from zero.

This assumption is equivalent to the ratio of the pdf of τ_i to $dG(t)/dt$ being bounded and bounded away from zero. It is straightforward to weaken this condition to allow it to only hold on compact, connected interval that is a subset of $(0, 1)$, if we assume the $b(t)$ is constant on known intervals near 0 and where τ is large.

We also make a smoothness assumption on the boundary function.

Assumption 3. $b(G^{-1}(g))$ is bounded and $s \geq 1$ times differentiable with bounded derivatives on $g \in [0, 1]$ and the $q_{kK}(G)$, $k = 1, \dots, K$ are b-splines of order $s - 1$.

This condition requires that the derivatives of $b(t)$ go to zero in the tails of the distribution of τ_i as fast as the pdf of $G(t)$ does. We also require that the drift parameter be nonzero.

Assumption 4. $\delta \neq 0$.

We need to add other conditions about the smoothness of CDF of τ_i as a function of the drift δ and the boundary and about rates of growth of J and K . They involve much notation, so we state them in Assumption 5 in Appendix C.

We can now state the following result on the limiting distribution of \hat{A} .

Theorem 3. *Suppose that Assumptions 2, 3, 4 and Assumption 5 in Appendix C are satisfied. Then for the $1 - \alpha$ quantile $c(\alpha, J)$ of a chi-square distribution with J degrees of freedom*

$$\Pr\left(\hat{A} \geq c(\alpha, J)\right) \rightarrow \alpha.$$

A Proofs from Section 4

A.1 Proof of Lemma 1

Dividing (1) by α and observing that $\inf\{t \geq 0 : |Z_t| \geq b(t)\} = \inf\{t \geq 0 : |\frac{Z_t}{\alpha}| \geq \frac{b(t)}{\alpha}\}$ yields that $p^*(\delta(x, y), b, \alpha) = p^*\left(\frac{1}{\alpha}\delta(x, y), \frac{b}{\alpha}, \alpha\right)$ and thus the result. \square

A.2 Proof of Theorem 1

(1) We first show that these conditions are necessary for (p, F) to admit a DDM representation for a given pair $\{x, y\}$.

By equation (4) in Fudenberg, Strack, and Strzalecki (2018) we have $\frac{p^{xy}(t)}{1-p^{xy}(t)} = \exp(2\delta(x, y)b(t))$.

Thus, we have that

$$b(t) = \frac{1}{2\delta(x, y)} \log \left(\frac{p^{xy}(t)}{1-p^{xy}(t)} \right). \quad (5)$$

This proves (4). By the definition of τ in equation (2) we have $Z_\tau = \text{sgn}(Z_\tau)b(\tau)$. By (1), $Z_\tau = \delta(x, y)\tau + B_\tau$. Combining these two equations and taking expectations, it follows from Doob's optional sampling theorem that

$$\delta(x, y) \mathbb{E}^{xy} [\tau] = \mathbb{E}^{xy} [\text{sgn}(Z_\tau)b(\tau)] \quad (6)$$

Plugging (5) into (6) yields

$$\delta(x, y) \mathbb{E}^{xy} [\tau] = \mathbb{E}^{xy} \left[\text{sgn}(Z_\tau) \frac{1}{2\delta(x, y)} \log \left(\frac{p^x(\tau)}{1-p^{xy}(\tau)} \right) \right]$$

Dividing by $\mathbb{E}^{xy} [\tau]$ and multiplying by $2\delta(x, y)$ yields

$$\begin{aligned} 2\delta(x, y)^2 &= \frac{\mathbb{E}^{xy} \left[\text{sgn}(Z_\tau) \log \left(\frac{p^x(\tau)}{1-p^{xy}(\tau)} \right) \right]}{\mathbb{E}^{xy} [\tau]} = \frac{\mathbb{E}^{xy} \left[[\mathbf{1}_{Z_\tau > 0} - \mathbf{1}_{Z_\tau < 0}] \log \left(\frac{p^x(\tau)}{1-p^{xy}(\tau)} \right) \right]}{\mathbb{E}^{xy} [\tau]} \\ &= \frac{\mathbb{E}^{xy} \left[\int_0^\infty \mathbf{1}_{\tau=t} [\mathbf{1}_{Z_\tau > 0} - \mathbf{1}_{Z_\tau < 0}] \log \left(\frac{p^{xy}(t)}{1-p^{xy}(t)} \right) dt \right]}{\int_0^\infty t dF^{xy}(t)} \\ &= \frac{\mathbb{E}^{xy} \left[\int_0^\infty \mathbf{1}_{\tau=t} \mathbb{E}^{xy} \left[[\mathbf{1}_{Z_\tau > 0} - \mathbf{1}_{Z_\tau < 0}] \log \left(\frac{p^{xy}(t)}{1-p^{xy}(t)} \right) \mid \tau = t \right] dt \right]}{\int_0^\infty t dF^{xy}(t)} \\ &= \frac{\mathbb{E}^{xy} \left[\int_0^\infty \mathbf{1}_{\tau=t} [\mathbb{E}^{xy} [\mathbf{1}_{Z_\tau > 0} \mid \tau = t] - \mathbb{E}^{xy} [\mathbf{1}_{Z_\tau < 0} \mid \tau = t]] \log \left(\frac{p^{xy}(t)}{1-p^{xy}(t)} \right) dt \right]}{\int_0^\infty t dF^{xy}(t)} \\ &= \frac{\mathbb{E}^{xy} \left[\int_0^\infty \mathbf{1}_{\tau=t} [p^{xy}(t) - (1 - p^{xy}(t))] \log \left(\frac{p^{xy}(t)}{1-p^{xy}(t)} \right) dt \right]}{\int_0^\infty t dF^{xy}(t)} \\ &= \frac{\int_0^\infty [p^{xy}(t) - (1 - p^{xy}(t))] \log \left(\frac{p^{xy}(t)}{1-p^{xy}(t)} \right) dF^{xy}(t)}{\int_0^\infty t dF^{xy}(t)} \\ &= \frac{\int_0^\infty [2p^{xy}(t) - 1] \log \left(\frac{p^{xy}(t)}{1-p^{xy}(t)} \right) dF^{xy}(t)}{\int_0^\infty t dF^{xy}(t)}. \end{aligned}$$

This proves (3). Finally, we know that $\delta > 0$ if and only if the probability with which the

first object is chosen $\mathbb{P}^{xy}[Z_\tau > 0] = \int_0^\infty p^{xy}(t) dF^{xy}(t)$ is greater $\frac{1}{2}$ which yields the result.

To show sufficiency, consider the DDM model with parameters $(\tilde{\delta}^{xy}, \tilde{b}^{xy})$ given by (3–4). It follows that F^{xy} equals the distribution over stopping times in the DDM model with boundary \tilde{b}^{xy} and drift $\tilde{\delta}^{xy}$. Finally, we will show that this DDM model also generates the correct conditional stopping probabilities p^{xy} . By equation (4) in Fudenberg, Strack, and Strzalecki (2018), the conditional probability of stopping in the DDM model \tilde{p}^{xy} satisfies

$$\frac{\tilde{p}^{xy}(t)}{1 - \tilde{p}^{xy}(t)} = \exp\left(2\tilde{\delta}(x, y) \tilde{b}^{xy}(t)\right) = \frac{p^{xy}(t)}{1 - p^{xy}(t)},$$

which completes the proof as we have argued that each stochastic choice function is uniquely identified by the associated pair (p, F) . \square

B Construction of \hat{V}

To construct \hat{V} we use the fact that there are three asymptotically independent sources of variation in $\bar{m} - \hat{m}$. These sources are the variation in τ_i , the variation in $\hat{\beta}$, and the variation from simulation. The variation in τ_i affects both \bar{m} and $\hat{\delta}$ and the variation in $\hat{\delta}$ has an effect through \hat{m} . Generally \hat{m} will not be differentiable in $\hat{\delta}$ so we use a difference quotient to estimate the derivative of \hat{m} with respect to δ . To describe how this source of variation can be estimated let

$$\tau_s(\delta, \beta) = \inf\{t \geq 0 : |\delta t + B_t^s| \geq \frac{1}{\delta} \ln \left[\frac{q^K(G(t))' \beta}{1 - q^K(G(t))' \beta} \right]\}, \quad \hat{m}(\delta, \beta) = \frac{1}{S} \sum_{s=1}^S m_J(\tau_s(\delta, \beta)).$$

denote one simulation $\tau_s(\delta, \beta)$ of τ_s when δ is the true drift and $q^K(G(t))' \beta$ the true $p(t) = p^{xy}(t)$ and $\hat{m}(\delta, \beta)$ denote the average over S simulations. Let

$$\hat{M}_\delta = \frac{\hat{m}(\hat{\delta} + \Delta, \hat{\beta}) - \hat{m}(\hat{\delta} - \Delta, \hat{\beta})}{2\Delta}$$

be the difference quotient that serves as an estimator of the derivative of the the expectation of the model moments with respect to the drift. Then

$$\hat{\psi}_{i1} = m_J(\tau_i) - \bar{m} - \hat{M}_\delta \frac{1}{2\hat{\delta}\bar{\tau}} [\hat{I}(\tau_i) - \bar{I} - \hat{\delta}^2\{\tau_i - \bar{\tau}\}]$$

will estimate the influence of τ_i on the difference of moments coming from the effect of τ_i on the sample moments as well as on $\hat{\delta}$. An estimator of the variance of the moment differences due to variation in τ_i is then

$$\hat{V}_1 = \frac{1}{n} \sum_{i=1}^n \hat{\psi}_{i1} \hat{\psi}'_{i1}.$$

To estimate the component of the variance due to $\hat{\beta}$ we use

$$\hat{M}_k = \frac{\hat{m}(\hat{\delta}, \hat{\beta} + e_k \Delta) - \hat{m}(\hat{\delta}, \hat{\beta} - e_k \Delta)}{2\Delta}, \quad \hat{M}_\beta = [\hat{M}_1, \dots, \hat{M}_K].$$

to estimate the derivative of $E[m_J(\tau_s(\delta, \beta))]$ with respect to β at $\hat{\delta}$ and $\hat{\beta}$, where e_k is the k^{th} unit vector. Let $\hat{p}_i = \hat{p}(\tau_i)$ and $d(p) = d \ln[p/(1-p)]/dp = p^{-1}(1-p)^{-1}$. Accounting also for the effect of β on $\hat{\delta}$, an estimator of the Jacobian of $E[m_J(\tau_s(\delta, \beta))]$ with respect to β is

$$\hat{D}_\beta = \hat{M}_\delta \frac{1}{2\hat{\delta}\bar{\tau}n} \sum_{i=1}^n d(\hat{p}_i) q_i^{K'} + \hat{M}_\beta.$$

The variation in $\bar{m} - \hat{m}$ due to $\hat{\beta}$ can then be estimated by

$$\hat{V}_2 = \hat{D}_\beta \hat{\Sigma}^{-1} \left[\frac{1}{n} \sum_{i=1}^n q_i^K q_i^{K'} (\gamma_i - \hat{p}_i)^2 \right] \hat{\Sigma}^{-1} \hat{D}'_\beta, \quad \hat{\Sigma} = \frac{1}{n} \sum_{i=1}^n q_i^K q_i^{K'}.$$

This is a delta method estimator of the asymptotic variance of $E[m_J(\tau_s(\delta, \beta))]$ due to the $\hat{\beta}$ in the nonparametric estimator $\hat{p}(t)$. As in [Newey \(1994\)](#), it is formed by treating \hat{m} as depending on the vector of parameters $\hat{\beta}$ and applying the delta method as if K were fixed and not growing with the sample size.

The variation due to simulation is easy to estimate as $\hat{V}_3 = (n/S^2) \sum_{s=1}^S [m_J(\hat{\tau}_s) - \hat{m}] [m_J(\hat{\tau}_s) - \hat{m}]'$. In the theory we assume that the number of simulations is large enough so that we can replace

this \hat{V}_3 by zero without affecting the results. Computing \hat{V}_3 in practice may still be a good idea check whether the number of simulations is large enough to make \hat{V}_3 negligible.

The estimators of the variance from independent sources of variation can then be combined into an asymptotic variance estimator for $\sqrt{n}[\bar{m} - \hat{m}_S]$ as

$$\hat{V} = \hat{V}_1 + \hat{V}_2 + \hat{V}_3.$$

We give conditions in Theorem 3 sufficient for the chi-squared approximation to the distribution of \hat{A} to be correct for n , J , and S growing and Δ shrinking in specific ways.

C Smoothness Conditions for the CDF of τ_i .

To obtain the limiting distribution of the test statistic we make use of smoothness conditions for the CDF of τ_i as $F(t|\delta, b)$ as a function of the drift δ and boundary $b(\cdot)$. The three key primitive regularity conditions that will be useful involve a Frechet derivative $D(\tilde{\delta} - \delta, \tilde{b} - b; \delta, b, t)$ of $F(t|\delta, b)$ with respect to δ and b . We collect these conditions in the following assumption. Let $\varepsilon_{pn} = \sqrt{n^{-1}K \ln(K)/n} + K^{-s}$.

Assumption 5. For $|\tilde{b}| = \sup_t |\tilde{b}(t)|$ there is $C > 0$ not depending on δ, b, t such that

a)

$$|F(t|\tilde{\delta}, \tilde{b}) - F(t|\delta, b) + D(\tilde{\delta} - \delta, \tilde{b} - b; \delta, b, t)| \leq C(|\tilde{\delta} - \delta|^2 + |\tilde{b} - b|^2);$$

b) for each t there is a constant D_{0t}^δ and function $\alpha_{0t}(t)$ such that $|\alpha_{0t}(\tau_i)| \leq C$, $|D_{0t}^\delta| \leq C$, $|d^s \alpha_{0t}(t)/dt^s| \leq C$ for s equal to the order of the spline plus 1, and

$$D(\tilde{\delta} - \delta, \tilde{b} - b; \delta, b, t) = D_{0t}^\delta(\tilde{\delta} - \delta) + E[\alpha_{0t}(\tau_i)\{\tilde{b}(\tau_i) - b(\tau_i)\}];$$

c)

$$|D(\delta, b; \tilde{\delta}, \tilde{b}, t) - D(\delta, b; \delta_0, b_0, t)| \leq C(|\delta| + |b|)(|\tilde{\delta} - \delta_0| + |\tilde{b} - b_0|).$$

d) there is $C > 0$ such that for $\psi_{i\delta x} = I(\tau_i) - E[I(\tau_i)] - \delta^2\{\tau_i - E[\tau_i]\}$ and all J ,

$$(J + 1)E[1(\tau_i < 1/(J + 1))\psi_{i\delta x}^2] \geq C.$$

e) Each of the following converge to zero: $\sqrt{n}J\varepsilon_{pn}^2$, nJ^3/S , $J^{7/2}K/(\sqrt{S}\Delta)$, $J^{7/2}K\Delta$, $J^{7/2}K^{3/2}\varepsilon_{pn}$, $J^{5/2}K^{-s_\alpha}$

Part a) is Frechet differentiability of the CDF of τ_i in the drift and boundary, b) is implied by mean square continuity of the derivative and the Riesz representation Theorem, and c) is continuity of the functional derivative D in δ and b . The test statistic will continue to be asymptotically chi-squared for a stronger norm for b under corresponding stronger rate conditions for J , K , and Δ .

D Proofs from Section 5

We will use two Lemmas on the asymptotic behavior of quadratic forms to prove the properties of the test statistic. For the first Lemma let h_i be a $J \times 1$ vector of random variables with $E[h_i] = 0$ and h_1, \dots, h_n i.i.d. Let

$$\Omega = E[h_i h_i'], \quad \bar{h} = \frac{1}{n} \sum_i h_i.$$

Consider \hat{h} that is approximately equal to \bar{h} in the sense that $\hat{h} - \bar{h}$ is small. Also consider an estimator $\hat{\Omega}$ of Ω and let $\|A\| = \sqrt{\text{tr}(A'A)}$ be the L_2 norm on matrices.

Lemma 2. *If i) $\lambda_{\min}(\Omega) \geq c > 0$, ii) $J^{-1/2}\sqrt{n}\text{tr}(\Omega)^{1/2}\|\hat{h} - \bar{h}\| \xrightarrow{p} 0$, iii) $J^{-1/2}\text{tr}(\Omega)\|\hat{\Omega} - \Omega\| \xrightarrow{p} 0$, and iv) $E[(h_i' h_i)^2]/nJ \rightarrow 0$ then for the $1 - \alpha$ quantile $c(\alpha, J)$ of a chi-square distribution with J degrees of freedom*

$$\Pr\left(n\hat{h}'\hat{\Omega}^{-1}\hat{h} \geq c(\alpha, J)\right) \rightarrow \alpha.$$

Proof: By i) we have $\lambda_{\min}(\Omega) \geq c$, so that $J^{-1/2}\text{tr}(\Omega)^{1/2} \geq c$. Then iii) implies $\|\hat{\Omega} - \Omega\| \xrightarrow{p} 0$ and hence w.p.a.1,

$$\lambda_{\min}(\hat{\Omega}) \geq c.$$

Since this event occurs w.p.a.1 we can assume it is true henceforth. Define

$$T_1 = n'\hat{h} \left(\hat{\Omega}^{-1} - \Omega^{-1} \right) \hat{h}, \quad T_2 = n \left[\hat{h}'\Omega^{-1}\hat{h} - \bar{h}'\Omega^{-1}\bar{h} \right]$$

Note that $E[n \|\bar{h}\|^2] = nE[\bar{h}'\bar{h}] = \text{tr}(\Omega)$. Then by the Markov inequality we have

$$\sqrt{n} \|\bar{h}\| = O_p(\text{tr}(\Omega)^{1/2}).$$

Also by ii) $\sqrt{n} \|\hat{h} - \bar{h}\| \leq CJ^{-1/2}\text{tr}(\Omega)^{1/2}\sqrt{n} \|\hat{h} - \bar{h}\| \xrightarrow{p} 0$. Then by the triangle inequality

$$\sqrt{n} \|\hat{h}\| \leq \sqrt{n} \|\bar{h}\| + \sqrt{n} \|\hat{h} - \bar{h}\| = O_p(\text{tr}(\Omega)^{1/2}).$$

It therefore follows that

$$\begin{aligned} |T_1| &= \left| n\hat{h}'\hat{\Omega}^{-1} \left(\Omega - \hat{\Omega} \right) \Omega^{-1}\hat{h} \right| \leq \left\| \sqrt{n}\hat{h}'\hat{\Omega}^{-1} \right\| \left\| \hat{\Omega} - \Omega \right\| \left\| \sqrt{n}\hat{h}'\Omega^{-1} \right\| \leq cn \left\| \hat{h} \right\|^2 \left\| \hat{\Omega} - \Omega \right\| \\ &= O_p(\text{tr}(\Omega)) \left\| \hat{\Omega} - \Omega \right\| = o_p(J^{1/2}). \end{aligned}$$

Similarly we have

$$\begin{aligned} |T_2| &= n \left| \left(\hat{h} - \bar{h} \right)' \Omega^{-1}\hat{h} + \bar{h}'\Omega^{-1} \left(\hat{h} - \bar{h} \right) \right| \leq n \left(\left\| \hat{h} - \bar{h} \right\| \left(\left\| \hat{h} \right\| + \left\| \bar{h} \right\| \right) \right) \\ &= O_p(\text{tr}(\Omega)^{1/2}\sqrt{n} \left\| \hat{h} - \bar{h} \right\|) = o_p(J^{1/2}). \end{aligned}$$

It then follows by the triangle inequality that

$$n'\hat{h}\hat{\Omega}^{-1}\hat{h} - n\bar{h}\Omega^{-1}\bar{h} = T_1 + T_2 = o_p(J^{1/2}).$$

In addition, by iv) and Lemma A.15 of [Newey and Windmeijer \(2009\)](#),

$$\frac{n\bar{h}'\Omega^{-1}\bar{h} - J}{\sqrt{2J}} \xrightarrow{d} N(0, 1).$$

Also, by standard results for the chi-squared distribution, as $J \rightarrow \infty$ we have $(c(\alpha, J) - J)/\sqrt{2J}$

converges to the $1 - \alpha$ quantile of a $N(0, 1)$. Hence

$$\Pr(n\bar{h}'\Omega^{-1}\bar{h} \geq c(\alpha, J)) = \Pr\left(\frac{n\bar{h}'\Omega^{-1}\bar{h} - J}{\sqrt{2J}} \geq \frac{c(\alpha, J) - J}{\sqrt{2J}}\right) \rightarrow \alpha.$$

The conclusion then follows by the Slutsky Lemma. Q.E.D.

The next Lemma gives a rate of growth for the number of simulation draws to ensure that the limiting distribution of the test statistic based on \hat{m}_S is the same as that based on $\hat{m} = \int m\left(\tau_s\left(\hat{\delta}, \hat{\beta}\right)\right) dF(s)$.

Let h_s be simulated moments. Then we have

Lemma 3. *If $\max_{1 \leq j \leq J} \sup_{\tau > 0} |m_{jJ}(\tau)| \leq C\sqrt{J}$ and $nJtr(\Omega)/S \rightarrow 0$ then*

$$J^{-1/2}\sqrt{ntr(\Omega)^{1/2}} \|\hat{m}_S - \hat{m}\| \xrightarrow{p} 0,$$

Proof: Let $Z = ((\gamma_1, \tau_1), \dots, (\gamma_n, \tau_n))$ denote the data. Note that by definition, $E[\hat{m}_S|Z] = \hat{m}$. Then for any constant ℓ

$$\lim Prob(\|\hat{m}_S - \hat{m}\| > \ell) = E[\Pr(\|\hat{m}_S - \hat{m}\| > \ell | Z)].$$

By the Markov inequality

$$\begin{aligned} \Pr(\|\hat{m}_S - \hat{m}\| > \ell | Z) &= \Pr(\|\hat{m}_S - \hat{m}\|^2 > \ell^2 | Z) \leq E\left[\sum_{j=1}^J (\hat{m}_{Sj} - \hat{m}_j)^2 | Z\right] / \ell^2 \\ &\leq \frac{1}{S} \sum_{j=1}^J E\left[\hat{m}_j\left(\tau_s\left(\hat{\delta}, \hat{\beta}\right)\right)^2 | Z\right] / \ell^2 \leq \frac{C^2 J^2}{S\ell^2}. \end{aligned}$$

By iterated expectations we then have

$$\Pr(\|\hat{m}_S - \hat{m}\| > \ell) \leq \frac{C^2 J^2}{S\ell^2}.$$

Let $\ell = J^{1/2} \text{tr}(\Omega)^{-1/2} n^{-1/2} \varepsilon$. Then

$$\begin{aligned} \Pr\left(J^{-1/2} \text{tr}(\Omega)^{1/2} \sqrt{n} \|\hat{m}_s - \hat{m}\| \geq \varepsilon\right) &= \Pr(\|\hat{m}_s - \hat{m}\| \geq \ell) \leq C^2 J^2 [S J \text{tr}(\Omega)^{-1} n^{-1} \varepsilon^2]^{-1} \\ &= \frac{J^2 \text{tr}(\Omega) n}{S J \varepsilon^2} = \frac{n J \text{tr}(\Omega)}{S} \frac{1}{\varepsilon^2} \longrightarrow 0. \end{aligned}$$

Q.E.D.

We next give a uniform convergence rate for $\hat{p}(t)$. For notational simplicity we let $p(t) := p^{xy}(t)$.

Lemma 4: If Assumptions 2 and 3 are satisfied then

$$\sup_t |\hat{p}(t) - p(t)| = O_p\left(\sqrt{\frac{K \ln(K)}{n}} + K^{-s}\right).$$

Proof: Follows from Theorem 4.3 and Comments 4.5 and 4.6 of [Belloni, Chernozhukov, Chetverikov, and Kato \(2015\)](#). *Q.E.D.*

We next give an asymptotic expansion for $\hat{\delta}$. Define

$$\begin{aligned} I(p) &= p \ln\left(\frac{p}{1-p}\right) + (1-p) \ln\left(\frac{1-p}{p}\right) = (1-2p) \ln\left(\frac{1-p}{p}\right), \\ \psi_i^\delta &= \frac{1}{2E[\tau_i]\delta} \left\{ I(p_i) - I_0 + I_p(p_i)(\gamma_i - p_i) - \delta^2(\tau_i - E[\tau_i]) \right\}. \end{aligned}$$

Lemma 5: If Assumptions 2 and 3 are satisfied and $\sqrt{n}\varepsilon_{pn}^2 \longrightarrow 0$ then

$$\hat{\delta} - \delta = \frac{1}{n} \sum_i \psi_i^\delta + O_p(\varepsilon_{pn}^2) = \frac{1}{n} \sum_i \psi_i^\delta + o_p(1/\sqrt{n}) = O_p(1/\sqrt{n}).$$

Proof: Equation (4) and Assumption 3 imply that $p(t)$ is bounded away from zero and one. It then follows from Lemma 4 that with probability approaching one (w.p.a.1) there is $\varepsilon > 0$ with $\varepsilon \leq \hat{p}(t) \leq 1 - \varepsilon$. It is straightforward to check that $I(p)$ is twice continuously differentiable in $p \in (0, 1)$ with first and second derivatives that are bounded when p is bounded away from zero

and one. It then follows by an expansion and Lemma 4 that

$$I(\hat{p}_i) = I(p_i) + I_p(p_i)(\hat{p}_i - p_i) + \hat{R}_i, \left| \hat{R}_i \right| \leq C|\hat{p}_i - p_i|^2.$$

Therefore we have

$$\hat{I} = \frac{1}{n} \sum_i I(\hat{p}_i) = \frac{1}{n} \sum_i [I(p_i) + I_p(p_i)(\hat{p}_i - p_i)] + \hat{R}, \quad \hat{R} = O_p(\varepsilon_{pn}^2).$$

Define

$$\begin{aligned} \Gamma &= (\gamma_1, \dots, \gamma_n)', \quad P = (p_1, \dots, p_n)', \quad Q = [q^K(G_1), \dots, q^K(G_n)]', \quad I_p = (I_p(p_1), \dots, I_p(p_n)), \\ H &= I - Q(Q'Q)^{-1}Q'. \end{aligned}$$

Note that derivatives of $I_p(p)$ to any order are bounded on $[\varepsilon, 1 - \varepsilon]$, so that by the fact that the approximation rate of a general s differentiable function by a b-spline of at least order $s - 1$ is K^{-s} we have

$$\frac{1}{n} P'HP = O(K^{-2s}), \quad \frac{1}{n} I_p'HI_p = O(K^{-2s}).$$

Note also that

$$\frac{1}{n} \sum_i I_p(p_i)(\hat{p}_i - p_i) - \frac{1}{n} \sum_i I_p(p_i)(\gamma_i - p_i) = -\frac{1}{n} I_p'HI\Gamma$$

Furthermore,

$$E[-\frac{1}{n} I_p'HI\Gamma | \tau_1, \dots, \tau_n] = -\frac{1}{n} I_p'HP = O(K^{-2s}), \quad \text{Var}(-\frac{1}{n} I_p'HI\Gamma | \tau_1, \dots, \tau_n) \leq \frac{1}{n^2} I_p'HI_p = O(\frac{K^{-2s}}{n}).$$

Then by $2K^{-s}/\sqrt{n} \leq 1/n + K^{-2s} \leq \varepsilon_{pn}^2$ it follows that

$$\frac{1}{n} \sum_i I_p(p_i)(\hat{p}_i - p_i) - \frac{1}{n} \sum_i I_p(p_i)(\gamma_i - p_i) = O_p(\frac{K^{-s}}{\sqrt{n}} + K^{-2s}) = O_p(\varepsilon_{pn}^2).$$

Then by the triangle inequality

$$\hat{I} = \frac{1}{n} \sum_i I(\hat{p}_i) = \frac{1}{n} \sum_i [I(p_i) + I_p(p_i)(\gamma_i - p_i)] + O_p(\varepsilon_{pn}^2).$$

Note that for $\delta(I, \tau) = \sqrt{I/\tau}$,

$$\frac{\partial \delta(I, \tau)}{\partial I} = \frac{1}{2\delta(I, \tau)\tau}, \quad \frac{\partial \delta(I, \tau)}{\partial \tau} = -\frac{\delta(I, \tau)}{2\tau}.$$

The conclusion then follows by the usual delta method argument. *Q.E.D.*

Next for any $\alpha(\tau)$ define

$$\psi_i^\alpha = -\delta^{-1} \{E[\alpha(\tau_i)b(\tau_i)]\psi_i^\delta + \frac{\alpha(\tau_i)}{p(\tau_i)[1-p(\tau_i)]}(\gamma_i - p_i)\}.$$

The next result gives a rate of convergence for the boundary estimator $\hat{b}(t)$ and a uniform expansion for a mean square continuous linear functional of $\hat{b}(t)$

Lemma 6: If there is a constant C such that $\alpha(G^{-1}(g))$ is continuously differentiable of order s with $|d\alpha(G^{-1}(g))/dg| \leq C$ on $[0, 1]$, then $\sup_t |\hat{b}(t) - b(t)| = O_p(\varepsilon_{pn})$ and

$$\int \alpha(\tau) \{\hat{b}(\tau) - b(\tau)\} F_0(d\tau) = \frac{1}{n} \sum_i \psi_i^\alpha + O_p(\varepsilon_{np}^2),$$

uniformly in α .

Proof: Note that for $b(\delta, p) = \delta^{-1} \ln(p/[1-p])$,

$$\frac{\partial b(\delta, p)}{\partial \delta} = \frac{-b(\delta, p)}{\delta}, \quad \frac{\partial b(\delta, p)}{\partial p} = \frac{1}{\delta p(1-p)}.$$

Then by Lemma 5, a delta method argument similar to that used in the proof of Lemma 5, and $\hat{\delta} = \delta + O_p(1/\sqrt{n})$ we have

$$\hat{b}(t) = b(t) - b(t) \frac{[\hat{\delta} - \delta]}{\delta} + \frac{1}{\delta p(t)[1-p(t)]} [\hat{p}(t) - p(t)] + \hat{R}(t), \quad \sup_t |\hat{R}(t)| = O_p(\varepsilon_{pn}^2).$$

The first conclusion then follows by $b(t)$ bounded, which implies $p(t)$ is bounded away from zero and one, and by Lemma 5. To show the second conclusion note that for any bounded $a(t)$ it follows by the proof of Corollary 10 of [Ichimura and Newey \(2018\)](#) that

$$\int a(\tau)[\hat{p}(\tau) - p(\tau)]F_0(d\tau) = \frac{1}{n} \sum_i a(\tau_i)[\gamma_i - p_i] + O_p(\varepsilon_{pn}^2),$$

uniformly in $a(\tau)$ with uniformly bounded derivatives to order s . Let $a(\tau) = \alpha(\tau)/\{\delta p(t)[1 - p(t)]\}$. By plugging in the above expansion for $\hat{b}(t)$ and using boundedness of $\alpha(\tau)$ we obtain

$$\begin{aligned} & \int \alpha(\tau)\{\hat{b}(\tau) - b(\tau)\}F_0(d\tau) \\ &= -\delta^{-1}\{E[\alpha(\tau_i)b(\tau_i)](\hat{\delta} - \delta) + \int \alpha(\tau)[\hat{p}(\tau) - p(\tau)]F_0(d\tau) + \int \alpha(\tau)\hat{R}(\tau)F_0(d\tau)\}. \\ &= \frac{1}{n} \sum_i \psi_i^\alpha + O_p(\varepsilon_{np}^2) + \int \alpha(\tau)\hat{R}(\tau)F_0(d\tau) = \frac{1}{n} \sum_i \psi_i^\alpha + O_p(\varepsilon_{np}^2). \text{ Q.E.D..} \end{aligned}$$

Proof of Theorem 4: We first show that conditions i)-iv) of Lemma 2 are satisfied. Let

$$\begin{aligned} h_{ji} &= m_{ji} - E[m_{ji}] + M_{\delta j} \psi_i^\tau + \alpha_{j0}(\tau_i)(\gamma_i - p_i), \\ \psi_i^\tau &= \frac{1}{2\delta E[\tau_i]} \{I(p_i) - I_0 - \delta^2(\tau_i - E[\tau_i])\}, \\ M_{\delta j} &= \sqrt{J}(D_{0\tau_{j+1}}^\delta - D_{0\tau_j}^\delta - \delta^{-1}E[\{\alpha_{0,\tau_{j+1}}(\tau_i) - \alpha_{0,\tau_j}(\tau_i)\}b(\tau_i)]) \\ \alpha_{j0}(\tau_i) &= M_{\delta j} \frac{1}{2E[\tau_i]\delta} I_p(p_i) + \frac{\sqrt{J}[\alpha_{0,\tau_{j+1}}(\tau_i) - \alpha_{0,\tau_j}(\tau_i)]}{\delta p_i[1 - p_i]}. \end{aligned}$$

Also let

$$\begin{aligned} h_i &= (h_{i1}, \dots, h_{iJ})' = m_i - E[m_i] + M_\delta \psi_i^\tau + \alpha_0(\tau_i)(\gamma_i - p_i), \\ M_\delta &= (M_{\delta 1}, \dots, M_{\delta J})', \quad \alpha_0(\tau) = (\alpha_{10}(\tau), \dots, \alpha_{J0}(\tau))', \\ \Omega &= E[h_i h_i'], \quad V_1 = \text{Var}(m_i + M_\delta \psi_i^\tau), \quad V_2 = E[\alpha_0(\tau_i) \alpha_0(\tau_i)' \text{Var}(\gamma_i | \tau_i)]. \end{aligned}$$

Note that $\Omega = V_1 + V_2$ by $E[\gamma_i | \tau_i] = p(\tau_i)$.

To show condition i) of Lemma 2 it suffices to show that $\lambda_{\min}(V_1) \geq C$, which we now

proceed to show. Let

$$\tilde{m}_i = (\sqrt{J+1}\psi_i^\tau, m_i')'.$$

It follows in a straightforward way from Assumption 5 d) that

$$\lambda_{\min}(E[\tilde{m}_i\tilde{m}_i']) \geq C.$$

Also, for $B = [M_\delta, I]$ we have

$$V_1 = BE[\tilde{m}_i\tilde{m}_i']B'.$$

Therefore for any conformable vector λ with $\lambda'\lambda = 1$,

$$\lambda'V_1\lambda = \frac{\lambda'BE[\tilde{m}_i\tilde{m}_i']B'\lambda}{\lambda'BB'\lambda}\lambda'BB'\lambda \geq C\lambda'BB'\lambda \geq C\lambda_{\min}(BB') \geq C\lambda_{\min}(I) = C.$$

We next show that condition ii) of the Lemma 2 is satisfied. Recall that

$$m_{jJ}(t) = \sqrt{J}1(\tau_{j,J} \leq t < \tau_{j+1,J}), \quad (j = 1, \dots, J).$$

Then taking expectations over the simulation,

$$\begin{aligned} E[m_{jS}(\delta, b)] &= \bar{m}_j(\delta, b) = \int m_{jJ}(\tau_s(\delta, b))F_s(ds) \\ &= \sqrt{J}[F(\tau_{j+1,J}|\delta, b) - F(\tau_{j,J}|\delta, b)], \quad (j = 1, \dots, J). \end{aligned}$$

From Assumption 5 let

$$\hat{D}_j(\tilde{\delta}, \tilde{b}) = D(\tilde{\delta}, \tilde{b}; \hat{\delta}, \hat{b}, \tau_j), \quad D_j(\tilde{\delta}, \tilde{b}) = D(\tilde{\delta}, \tilde{b}; \delta, b, \tau_j).$$

By Assumption 5 a) and Lemma 5,

$$\begin{aligned} \bar{m}_j(\hat{\delta}, \hat{b}) - \bar{m}_j(\delta, b) &= \sqrt{J}[D_{j+1}(\hat{\delta} - \delta, \hat{b} - b) - D_j(\hat{\delta} - \delta, \hat{b} - b)] + \hat{R}_j, \\ |\hat{R}_j| &\leq \sqrt{J}2C[(\hat{\delta} - \delta)^2 + \sup_t |\hat{b}(t) - b(t)|^2] = O_p(\sqrt{J}\varepsilon_{pn}^2), \end{aligned}$$

uniformly in j . By Assumption 5 b) and Lemmas 5 and 6,

$$\begin{aligned}
& \sqrt{J}[D_{j+1}(\hat{\delta} - \delta, \hat{b} - b) - D_j(\hat{\delta} - \delta, \hat{b} - b)] \\
&= \sqrt{J}[(D_{0\tau_{j+1}}^\delta - D_{0\tau_j}^\delta)(\hat{\delta} - \delta) + \int \{\alpha_{0,\tau_{j+1}}(\tau) - \alpha_{0,\tau_j}(\tau)\} \{\hat{b}(\tau) - b(\tau)\} F_0(d\tau)] \\
&= \sqrt{J}[(D_{0\tau_{j+1}}^\delta - D_{0\tau_j}^\delta) \{ \frac{1}{n} \sum_i \psi_i^\delta + O_p(\varepsilon_{pn}^2) \}] \\
&\quad - \sqrt{J} \delta^{-1} E[\{\alpha_{0,\tau_{j+1}}(\tau_i) - \alpha_{0,\tau_j}(\tau_i)\} b(\tau_i)] \left(\frac{1}{n} \sum_i \psi_i^\delta \right) \\
&\quad + \sqrt{J} \frac{1}{n} \sum_i \frac{[\alpha_{0,\tau_{j+1}}(\tau_i) - \alpha_{0,\tau_j}(\tau_i)]}{\delta p_i [1 - p_i]} (\gamma_i - p_i) + \sqrt{J} O_p(\varepsilon_{pn}^2) \\
&= \frac{1}{n} \sum_i h_{ji} + O_p(\sqrt{J} \varepsilon_{pn}^2)
\end{aligned}$$

Then by $tr(\Omega)^{1/2} = O(J)$ we have

$$J^{-1/2} \sqrt{n} tr(\Omega)^{1/2} \|\hat{h} - \bar{h}\| \leq C J^{1/2} \sqrt{n} \|\hat{h} - \bar{h}\| \leq C \sqrt{n} \sqrt{J} O_p(\sqrt{J} \varepsilon_{pn}^2).$$

Hypothesis ii) of Lemma 2 then follows by $\sqrt{n} J \varepsilon_{pn}^2 \rightarrow 0$, and by Lemma 3 and $n J^3 / S \rightarrow 0$.

Next we verify hypothesis iii) of Lemma 2. Note that

$$\hat{M}_{\delta j} = \frac{\hat{m}_j(\hat{\delta} + \Delta, \hat{\beta}) - \hat{m}_j(\hat{\delta} - \Delta, \hat{\beta})}{2\Delta}$$

Let $\bar{m}_j(\delta, \beta) = \int m_j(\tau_s(\delta, \beta)) F(ds)$ and

$$\bar{M}_{\delta j} = \frac{\bar{m}_j(\hat{\delta} + \Delta, \hat{\beta}) - \bar{m}_j(\hat{\delta} - \Delta, \hat{\beta})}{2\Delta}.$$

By the simulations i.i.d. given $\hat{\delta}, \hat{\beta}$ and $m_{jJ}(\tau) \leq C\sqrt{J}$,

$$E \left[\left(\hat{M}_{\delta j} - \bar{M}_{\delta j} \right)^2 \mid \hat{\delta}, \hat{\beta} \right] \leq \frac{CJ}{S\Delta^2}.$$

Then for $\bar{M}_\delta = (\bar{M}_{\delta_1}, \dots, \bar{M}_{\delta_J})'$ the Markov inequality gives

$$E \left[\left\| \hat{M}_\delta - \bar{M}_\delta \right\|^2 \right] \leq \frac{CJ^2}{S\Delta^2}, \quad \left\| \hat{M}_\delta - \bar{M}_\delta \right\| = O_p \left(\frac{J}{\sqrt{S}\Delta} \right).$$

Note that replacing $\hat{\delta}$ with $\hat{\delta} + \Delta$ in the boundary estimator \hat{b} gives $[\hat{\delta}/(\hat{\delta} + \Delta)]\hat{b}$ and replacing $\hat{\delta}$ with $\hat{\delta} - \Delta$ gives $[\hat{\delta}/(\hat{\delta} - \Delta)]\hat{b}$. Also,

$$\frac{\hat{\delta}}{\hat{\delta} + \Delta} - 1 = \frac{-\Delta}{\hat{\delta} + \Delta}, \quad \frac{\hat{\delta}}{\hat{\delta} - \Delta} - 1 = \frac{\Delta}{\hat{\delta} - \Delta}$$

Let $\hat{D}_j(\delta, b) = D(\delta, b; \hat{\delta}, \hat{b}, j)$ and $D_j(\delta, b) = D(\delta, b; \delta_0, b_0, j)$ for true values δ_0 and b_0 . Then by Assumption 5 a),

$$\begin{aligned} \bar{M}_{\delta_j} &= \frac{\bar{m}_j(\hat{\delta} + \Delta, \hat{\beta}) - \bar{m}_j(\hat{\delta}, \hat{\beta}) - [\bar{m}_j(\hat{\delta} - \Delta, \hat{\beta}) - \bar{m}_j(\hat{\delta}, \hat{\beta})]}{2\Delta} \\ &= \frac{\sqrt{J}[\hat{D}_{j+1}(\Delta, \frac{-\Delta}{\hat{\delta} + \Delta}\hat{b}) - \hat{D}_{j+1}(-\Delta, \frac{\Delta}{\hat{\delta} - \Delta}\hat{b})]}{2\Delta} - \frac{\sqrt{J}[\hat{D}_j(\Delta, \frac{-\Delta}{\hat{\delta} + \Delta}\hat{b}) - \hat{D}_j(-\Delta, \frac{\Delta}{\hat{\delta} - \Delta}\hat{b})]}{2\Delta} + \hat{R}_j \\ |\hat{R}_j| &\leq C\sqrt{J}\Delta^{-1}(\Delta^2 + \left| \frac{\Delta}{\hat{\delta} + \Delta}\hat{b} \right|^2 + \left| \frac{\Delta}{\hat{\delta} - \Delta}\hat{b} \right|^2) \leq C\sqrt{J}\Delta(1 + |\hat{b}|^2). \end{aligned}$$

We also have

$$\begin{aligned} \sqrt{J}\frac{1}{\Delta}\hat{D}_{j+1}(\Delta, \frac{-\Delta}{\hat{\delta} + \Delta}\hat{b}) &= \sqrt{J}\hat{D}_{j+1}(1, \frac{-1}{\hat{\delta} + \Delta}\hat{b}), \\ \sqrt{J}|\hat{D}_{j+1}(1, \frac{-1}{\hat{\delta} + \Delta}\hat{b}) - D_{j+1}(1, \frac{-1}{\hat{\delta} + \Delta}\hat{b})| &\leq C\sqrt{J}\left| \frac{\hat{b}}{\hat{\delta} + \Delta} \right| (|\hat{\delta} - \delta| + |\hat{b} - b|) \leq C\sqrt{J}O_p(\varepsilon_{pn}). \end{aligned}$$

Also,

$$\begin{aligned} &\left| \sqrt{J} \left[D_{j+1}(1, \frac{-1}{\hat{\delta} + \Delta}\hat{b}) - D_{0\tau_{j+1}}^\delta + \frac{1}{\delta} \int \alpha_{0, \tau_{j+1}}(\tau) b(\tau) F_0(d\tau) \right] \right| \\ &\leq C\sqrt{J}(|\hat{\delta} - \delta| + |\hat{b} - b|) = \sqrt{J}O_p(\varepsilon_{pn}). \end{aligned}$$

Applying an analogous set of inequalities to other terms and collecting remainders gives

$$|\bar{M}_{\delta j} - M_{\delta j}| \leq C\sqrt{J}(\Delta + O_p(\varepsilon_{pn})).$$

Combining results and stacking over j then give

$$\|\hat{M}_\delta - M_\delta\| = O_p(J(\frac{1}{\sqrt{S}\Delta} + \Delta + \varepsilon_{pn})).$$

Next, for $\hat{\psi}_i^\tau = (2\hat{\delta}\bar{\tau})^{-1} [\hat{I}(\tau_i) - \bar{I} - \hat{\delta}^2\{\tau_i - \bar{\tau}\}]$ it follows straightforwardly that

$$\frac{1}{n} \sum_{i=1}^n (\hat{\psi}_i^\tau - \psi_i^\tau)^2 = O_p(\varepsilon_{pn}^2).$$

Let $\tilde{V}_1 = n^{-1} \sum_{i=1}^n \psi_{1i} \psi'_{1i}$ and $\psi_{1i} = m_i - E[m_i] + M_\delta \psi_i^\tau$. Note that

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \|\hat{\psi}_{1i} - \psi_{1i}\|^2 &\leq \|\bar{m} - E[m_i]\|^2 + \|\hat{M}_\delta - M_\delta\|^2 \frac{1}{n} \sum_{i=1}^n \|\hat{\psi}_i^\tau\|^2 + \|M_\delta\|^2 \frac{1}{n} \sum_{i=1}^n (\hat{\psi}_{1i} - \psi_{1i})^2 \\ &= O_p(\frac{J^2}{n}) + O_p(J^2(\frac{1}{\sqrt{S}\Delta} + \Delta + \varepsilon_{pn})^2) + O_p(J^2\varepsilon_{pn}^2) \\ &= O_p(J^2(\frac{1}{\sqrt{S}\Delta} + \Delta + \varepsilon_{pn})^2). \end{aligned}$$

Then by the Cauchy-Schwartz and triangle inequalities,

$$\begin{aligned} \|\hat{V}_1 - \tilde{V}_1\| &\leq \frac{1}{n} \sum_{i=1}^n \|\hat{\psi}_{1i} - \psi_{1i}\|^2 + \sqrt{\frac{1}{n} \sum_{i=1}^n \|\hat{\psi}_{1i} - \psi_{1i}\|^2} \sqrt{\frac{1}{n} \sum_{i=1}^n \|\psi_{1i}\|^2} \\ &= O_p(J^2(\frac{1}{\sqrt{S}\Delta} + \Delta + \varepsilon_{pn})). \end{aligned}$$

It follows similarly that $\|\tilde{V}_1 - V_1\| = O_p(J^{3/2}/\sqrt{n})$, so by the triangle inequality,

$$\|\hat{V}_1 - V_1\| = O_p(J^2(\frac{1}{\sqrt{S}\Delta} + \Delta + \varepsilon_{pn})).$$

Next we derive a convergence rate for $\|\hat{V}_2 - V_2\|$. Let

$$\begin{aligned} D_\beta &= E[\alpha_0(\tau_i)q_i^{K'}], \quad \Sigma = E[q_i^K q_i^{K'}], \quad \alpha_K(\tau_i) = D_\beta \Sigma^{-1} q_i^K, \\ \Lambda &= E[q_i^K q_i^{K'}(\gamma_i - p_i)^2], \quad \bar{V}_2 = D_\beta \Sigma^{-1} \Lambda \Sigma^{-1} D_\beta' = E[\alpha_K(\tau_i) \alpha_K(\tau_i)' (\gamma_i - p_i)^2]. \end{aligned}$$

Note that by Assumption 5 b) and standard approximation properties of splines

$$E[\{(\alpha_{0j}(\tau_i) - \alpha_{Kj}(\tau_i))(\gamma_i - p_i)\}^2] \leq CE[\{\alpha_{0j}(\tau_i) - \alpha_{Kj}(\tau_i)\}^2] \leq CK^{-2s_\alpha},$$

for a constant C that does not depend on j . Then we have

$$\begin{aligned} \|\bar{V}_2 - V_2\|^2 &= \sum_{j,\ell=1}^J \{E[\alpha_{Kj}(\tau_i) \alpha_{K\ell}(\tau_i) (\gamma_i - p_i)^2] - E[\alpha_{0j}(\tau_i) \alpha_{0\ell}(\tau_i) (\gamma_i - p_i)^2]\}^2 \\ &= \sum_{j,\ell=1}^J \{E[\{\alpha_{Kj}(\tau_i) - \alpha_{0j}(\tau_i)\} \alpha_{K\ell}(\tau_i) (\gamma_i - p_i)^2] + E[\alpha_{0j}(\tau_i) \{\alpha_{K\ell}(\tau_i) - \alpha_{0\ell}(\tau_i)\} (\gamma_i - p_i)^2]\}^2 \\ &\leq C \sum_{j,\ell=1}^J \{\sqrt{E[\{\alpha_{Kj}(\tau_i) - \alpha_{0j}(\tau_i)\}^2]} \sqrt{E[\alpha_{K\ell}(\tau_i)^2]} \\ &\quad + \sqrt{E[\{\alpha_{K\ell}(\tau_i) - \alpha_{0\ell}(\tau_i)\}^2]} \sqrt{E[\alpha_{0j}(\tau_i)^2]}\}^2 \\ &\leq C \left(\sum_{j=1}^J E[\{\alpha_{Kj}(\tau_i) - \alpha_{0j}(\tau_i)\}^2] \right) \left(\sum_{j=1}^J \{E[\alpha_{0j}(\tau_i)^2] + E[\alpha_{K\ell}(\tau_i)^2]\} \right) \leq CJ^2 K^{-2s_\alpha}. \end{aligned}$$

Taking square roots we have

$$\|\bar{V}_2 - V_2\| \leq CJK^{-s_\alpha}.$$

Define

$$\bar{M}_{\beta jk} = \frac{\bar{m}_j(\hat{\delta}, \hat{\beta} + e_k \Delta) - \bar{m}_j(\hat{\delta}, \hat{\beta} - e_k \Delta)}{2\Delta}.$$

It follows similarly to $\|\hat{M}_\delta - \bar{M}_\delta\| = \|\hat{M}_\delta - \bar{M}_\delta\| = O_p(J/\sqrt{S}\Delta)$ that

$$\|\hat{M}_\beta - \bar{M}_\beta\| = O_p(J\sqrt{K}/\sqrt{S}\Delta).$$

Next, let $\hat{p}_{\Delta k}(t) = \hat{p}(t) + \Delta q_{kK}(G(t))$ and $\hat{b}_{\Delta k}(t) = \hat{\delta}^{-1} \ln(\hat{p}_{\Delta k}(t)/[1 - \hat{p}_{\Delta k}(t)])$. By $\Delta\sqrt{K} \rightarrow 0$

and $\sup_{G \in [0,1]} |q_{kK}(G)| \leq C\sqrt{K}$ it follows that $\sup_t \Delta q_{kK}(G(t)) \rightarrow 0$. Then w.p.a.1 we have

$$\hat{b}_{\Delta k}(t) = \hat{b}(t) + \frac{\Delta q_{kK}(G(t))}{\hat{\delta}\hat{p}(t)[1-\hat{p}(t)]} + \hat{R}_k(t, \Delta), \quad \left| \hat{R}_k(t, \Delta) \right| \leq C\Delta^2 K.$$

Then we have

$$\begin{aligned} \bar{M}_{\beta jk} &= \frac{\bar{m}_j(\hat{\delta}, \hat{\beta} + e_k \Delta) - \bar{m}_j(\hat{\delta}, \hat{\beta}) - [\bar{m}_j(\hat{\delta}, \hat{\beta} - e_k \Delta) - \bar{m}_j(\hat{\delta}, \hat{\beta})]}{2\Delta} \\ &= \frac{\sqrt{J}[\hat{D}_{j+1}(0, \hat{b}_{\Delta k} - \hat{b}) - \hat{D}_{j+1}(0, \hat{b}_{-\Delta k} - \hat{b})]}{2\Delta} \\ &\quad - \frac{\sqrt{J}[\hat{D}_j(0, \hat{b}_{\Delta k} - \hat{b}) - \hat{D}_j(0, \hat{b}_{-\Delta k} - \hat{b})]}{2\Delta} + \hat{R}_{jk} \\ \left| \hat{R}_{jk} \right| &\leq C\sqrt{J}\Delta^{-1} \left(\left| \hat{b}_{\Delta k} - \hat{b} \right|^2 + \left| \hat{b}_{-\Delta k} - \hat{b} \right|^2 \right) \leq C\sqrt{J}\Delta K. \end{aligned}$$

We also have

$$\begin{aligned} \sqrt{J} \frac{1}{\Delta} \hat{D}_{j+1}(0, \hat{b}_{\Delta k} - \hat{b}) &= \sqrt{J} \hat{D}_{j+1}(0, \frac{\hat{b}_{\Delta k} - \hat{b}}{\Delta}), \\ \sqrt{J} \left| \hat{D}_{j+1}(0, \frac{\hat{b}_{\Delta k} - \hat{b}}{\Delta}) - D_{j+1}(0, \frac{\hat{b}_{\Delta k} - \hat{b}}{\Delta}) \right| &\leq C\sqrt{J} \left| \frac{\hat{b}_{\Delta k} - \hat{b}}{\Delta} \right| (|\hat{\delta} - \delta| + |\hat{b} - b|) \leq C\sqrt{J}\sqrt{K}O_p(\varepsilon_{pn}). \end{aligned}$$

In addition

$$\begin{aligned} \sqrt{J} D_{j+1}(0, \frac{\hat{b}_{\Delta k} - \hat{b}}{\Delta}; \delta, b, \tau_{j+1}) &= \sqrt{J} D(0, \frac{q_{kK}(G(\cdot))}{\hat{\delta}\hat{p}(\cdot)[1-\hat{p}(\cdot)]}; \delta, b, \tau_{j+1}) + \sqrt{J}\Delta D(0, \hat{R}_k(\cdot, \Delta); \delta, b, \tau_{j+1}) \\ &= \sqrt{J} D(0, \frac{q_{kK}(G(\cdot))}{\delta p(\cdot)[1-p(\cdot)]}; \delta, b, \tau_{j+1}) + \hat{R}_{jk}, \\ \left| \hat{R}_{jk} \right| &\leq \sqrt{J}\sqrt{K}O_p(\varepsilon_{pn}) + \sqrt{J}K\Delta. \end{aligned}$$

Combining terms we have

$$\left\| \hat{M}_\beta - M_\beta \right\| = O_p(J\sqrt{K}/\sqrt{S}\Delta + JK\varepsilon_{pn} + JK^{3/2}\Delta)$$

Next, we have

$$\begin{aligned}
& \left\| \hat{M}_\delta \frac{1}{2\hat{\delta}\bar{\tau}n} \sum_{i=1}^n I_p(\hat{p}_i) q_i^{K'} - M_\delta \frac{1}{2\delta E[\tau_i]} E[I_p(p_i) q_i^{K'}] \right\| \\
& \leq \left\| \hat{M}_\delta - M_\delta \right\| \frac{1}{2\hat{\delta}\bar{\tau}} \left(\frac{1}{n} \sum_{i=1}^n I_p(\hat{p}_i)^2 \right)^{1/2} \left(\frac{1}{n} \sum_{i=1}^n q_i^{K'} q_i^K \right)^{1/2} \\
& \quad + \left\| M_\delta \right\| \left\| \frac{1}{2\hat{\delta}\bar{\tau}n} \sum_{i=1}^n I_p(\hat{p}_i) q_i^{K'} - \frac{1}{2\delta E[\tau_i]} E[I_p(p_i) q_i^{K'}] \right\| \\
& = O_p(J\sqrt{K}(\frac{1}{\sqrt{S}\Delta} + \Delta + \varepsilon_{pn})) + O_p(JK\varepsilon_{pn}) = O_p(J\sqrt{K}(\frac{1}{\sqrt{S}\Delta} + \Delta + \sqrt{K}\varepsilon_{pn})).
\end{aligned}$$

Combining terms we then have

$$\left\| \hat{D}_\beta - D_\beta \right\| = O_p(J\sqrt{K}/\sqrt{S}\Delta + JK\varepsilon_{pn} + JK^{3/2}\Delta).$$

Next, for $\hat{\pi} = \hat{\Sigma}^{-1}\hat{D}_\beta$ and $\pi = \Sigma^{-1}D_\beta$ note that $\hat{V}_2 = \hat{\pi}'\hat{\Lambda}\hat{\pi}$ and $\bar{V}_2 = \pi'\Lambda\pi$. Also we have

$$\hat{V}_2 - \bar{V}_2 = (\hat{\pi} - \pi)'\hat{\Lambda}(\hat{\pi} - \pi) + 2\pi'\hat{\Lambda}(\hat{\pi} - \pi) + \pi'(\hat{\Lambda} - \Lambda)\pi.$$

By the law of large number for symmetric matrices, $\left\| \hat{\Sigma} - \Sigma \right\|_{op} = O_p(\sqrt{n^{-1}K \ln K}) = o_p(1)$, where $\|\cdot\|_{op}$ denotes the operator norm on symmetric matrices. Then by the eigenvalues of Σ bounded and bounded away from zero, $\lambda_{\max}(\hat{\Sigma}) = O_p(1)$ and $1/\lambda_{\min}(\hat{\Sigma}) = O_p(1)$. Let $\tilde{\Lambda} = \frac{1}{n} \sum_i q_i^K q_i^{K'} (\gamma_i - p_i)^2$. Note that

$$\begin{aligned}
\hat{\Lambda} - \tilde{\Lambda} &= \frac{1}{n} \sum_i q_i^K q_i^{K'} [(\gamma_i - \hat{p}_i)^2 - (\gamma_i - p_i)^2] \leq \frac{1}{n} \sum_i q_i^K q_i^{K'} |(\gamma_i - \hat{p}_i)^2 - (\gamma_i - p_i)^2| \\
&\leq C\hat{\Sigma} \max_i |\hat{p}_i - p_i| = \hat{\Sigma} O_p(\varepsilon_{pn}), \quad \hat{\Lambda} - \tilde{\Lambda} \geq -C\hat{\Sigma} O_p(\varepsilon_{pn}).
\end{aligned}$$

Also by the law of large numbers for symmetric matrices $\left\| \tilde{\Lambda} - \Lambda \right\|_{op} = O_p(\sqrt{n^{-1}K \ln K})$. Therefore by the triangle inequality,

$$\left\| \hat{\Lambda} - \Lambda \right\|_{op} = O_p(\varepsilon_{pn}).$$

It follows that $\lambda_{\max}(\hat{\Lambda}) = O_p(1)$, $1/\lambda_{\min}(\hat{\Lambda}) = O_p(1)$, and for $\hat{\Upsilon} = \hat{\Lambda} - \Lambda$,

$$\|\hat{\Upsilon}\| = \sqrt{\text{tr}(\hat{\Upsilon}^2)} \leq C\sqrt{J} \|\hat{\Lambda} - \Lambda\|_{op} = O_p(\sqrt{J}\varepsilon_{pn}).$$

Similarly we have $\|\hat{\Sigma} - \Sigma\| = O_p(K\sqrt{\ln(K)/n})$. We also have $\|D_\beta\| \leq CJ\sqrt{K}$. Then it follows that for $\varepsilon_{Dn} = J\sqrt{K}/\sqrt{S}\Delta + JK\varepsilon_{pn} + JK^{3/2}\Delta$

$$\|\hat{\pi} - \pi\| \leq \left\| (\hat{D}_\beta - D_\beta)' \hat{\Sigma}^{-1} \right\| + \left\| D_\beta' \hat{\Sigma}^{-1} (\Sigma - \hat{\Sigma}) \Sigma^{-1} \right\| \leq O_p(\varepsilon_{Dn}) + O_p(JK\sqrt{\ln(K)/n}) = O_p(\varepsilon_{Dn}).$$

It then follows by the triangle inequality that

$$\begin{aligned} \|\hat{V}_2 - \bar{V}_2\| &\leq O_p(1)(\|\hat{\pi} - \pi\|^2 + \|\pi\| \|\hat{\pi} - \pi\| + \|\pi\|^2 \|\hat{\Lambda} - \Lambda\|) \\ &= O_p(J\sqrt{K}\varepsilon_{Dn} + J^2K^2\sqrt{\ln(K)/n}) = O_p(J^2K/\sqrt{S}\Delta + J^2K^{3/2}\varepsilon_{pn} + J^2K\Delta). \end{aligned}$$

By the triangle inequality we then have

$$\|\hat{\Omega} - \Omega\| = O_p(J^2K/\sqrt{S}\Delta + J^2K\Delta + J^2K^{3/2}\varepsilon_{pn} + JK^{-s_\alpha})$$

It then follows that Assumption iii) is satisfied by Assumption 5 e).

Finally, for Assumption iv) of Lemma A2, note that

$$(h_i' h_i)^2 = \left(\sum_{j=1}^J h_{ij}^2 \right)^2 = \sum_{j=1}^J \sum_{k=1}^K h_{ij}^2 h_{ik}^2 \leq CJ \sum_{j=1}^J h_{ij}^4 \leq CJ^4,$$

so that

$$E \left[(h_i' h_i)^2 \right] / nJ \leq CJ^3/n \longrightarrow 0.$$

Therefore condition iv) is satisfied. Q.E.D.

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