Temporal Resolution of Uncertainty and Recursive Models of Ambiguity Aversion

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Preference for Earlier Resolution of Uncertainty

- instrumental value of information Spence Zeckhauser (1972)
- intrinsic value of information Kreps Porteus (1978)
- hidden actions Ergin Sarver (2012)
Aversion to Persistence/Long Term Risk

Duffie Epstein (1992)
Ambiguity

Kenes (1921), Ellsberg (1961)
Ambiguity

Kenes (1921), Ellsberg (1961)
Ambiguity

50 B
50 R

? B
? R

Kenes (1921), Ellsberg (1961)
Ambiguity

\[
\begin{array}{c|c}
50 & B \\
\hline
50 & R
\end{array}
\quad
\begin{array}{c|c}
? & B \\
\hline
? & R
\end{array}
\]

Kenes (1921), Ellsberg (1961)
Ambiguity

\[
\begin{array}{c|c}
50 & B \\
50 & R \\
\end{array}
\quad \quad
\begin{array}{c|c}
? & B \\
? & R \\
\end{array}
\]

B

II

R

Kenes (1921), Ellsberg (1961)
Ambiguity

Kenes (1921), Ellsberg (1961)
Ambiguity

Kenes (1921), Ellsberg (1961)
Ambiguity

\[
\begin{array}{c|c}
50 & B \\
50 & R \\
\end{array}
\quad
\begin{array}{c|c}
? & B \\
? & R \\
\end{array}
\]

B > B
II > II
R > R

Kenes (1921), Ellsberg (1961)
The way we choose to model ambiguity aversion will impact:

- preference for earlier resolution of uncertainty
- aversion to long-run risk,

so the model ties these dimensions of preference together.
Similar to the modeling tradeoff in Epstein Zin (1989), where

- risk aversion
- intertemporal elasticity of substitution
- preference for earlier resolution of uncertainty and aversion to long-run risk

are tied together
Preferences

\[ V_t = u(c_t) + \beta E(V_{t+1}) \quad \text{Discounted Expected Utility} \]
Preferences

\[ V_t = u(c_t) + \beta E(V_{t+1}) \]  
Discounted Expected Utility

\[ V_t = W(u(c_t), E(V_{t+1})) \]  
Kreps–Porteus, Epstein–Zin
Preferences

\[ V_t = u(c_t) + \beta E(V_{t+1}) \]  
Discounted Expected Utility

\[ V_t = W(u(c_t), E(V_{t+1})) \]  
Kreps–Porteus, Epstein–Zin

\[ V_t = u(c_t) + \beta I(V_{t+1}) \]  
Discounted Ambiguity Aversion
Preferences

\[ V_t = u(c_t) + \beta E(V_{t+1}) \]  

Discounted Expected Utility

this \succsim \text{ is indifferent to timing and to long run risks}
Preferences

\[ V_t = W(u(c_t), E(V_{t+1})) \]

Kreps–Porteus, Epstein–Zin

this \( \succeq \) is \( \begin{cases} \text{PERU} \\ \text{PLRU} \\ \text{IERU} \end{cases} \) iff \( W(u, \cdot) \) is \( \begin{cases} \text{convex} \\ \text{concave} \\ \text{linear} \end{cases} \)
Preferences

\[ V_t = u(c_t) + \beta I(V_{t+1}) \]

Discounted Ambiguity Aversion

- here the operator \( I \) replaces expectation
- captures ambiguity aversion
- question: how does PERU depend on \( I \)
Setting
Setting

\( \mathcal{T} = \{0, 1, \ldots, T\} \) — discrete time, \( T < \infty \)

(\( S, \Sigma \)) — shocks, measurable space

\( \Omega = S^{\mathcal{T}} \) — states of nature

\( X \) — consequences, convex subset of a real vector space

\( h = (h_0, h_1, \ldots, h_T) \) — consumption plan, \( h_t : S^t \to X \)
**Consumption Plan**

\[ h_0 \]

\[ h_1(s^1) \quad h_2(s^2) \]

\[ h_1(s^1) \quad h_2(s^2) \quad h_2(s^2) \]

\[ h_2(s^2) \]
IID uncertainty with EU

\[ V_t(s^t, h) = u(h_t(s^t)) + \beta \int_S V_{t+1}((s^t, s_{t+1}), h) \, dp(s_{t+1} | s^t) \]
IID uncertainty with EU

\[ V_t(s^t, h) = u(h_t(s^t)) + \beta \int_S V_{t+1}((s^t, s_{t+1}), h) \, dp(s_{t+1} | s^t) \]

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IID uncertainty with EU

\[ V_t(s^t, h) = u(h_t(s^t)) + \beta \int_{s} V_{t+1}((s^t, s_{t+1}), h) \, dp(s_{t+1} | s^t) \]

\[ V_t(s^t, h) = u(h_t(s^t)) + \beta \int_{s} V_{t+1}((s^t, s_{t+1}), h) \, dp(s_{t+1}) \]

- what is IID is the underlying state process \( s^t \)
IID uncertainty with EU

\[ V_t(s^t, h) = u(h_t(s^t)) + \beta \int_S V_{t+1}((s^t, s_{t+1}), h) \, dp(s_{t+1} | s^t) \]

\[ V_t(s^t, h) = u(h_t(s^t)) + \beta \int_S V_{t+1}((s^t, s_{t+1}), h) \, dp(s_{t+1}) \]

- what is IID is the underlying state process \( s^t \)
- consumption can have correlation
generalize beyond EU

\[ V_t(s^t, h) = u(h_t(s^t)) + \beta \int_S V_{t+1}((s^t, s_{t+1}), h) \, dp(s_{t+1}) \]
generalize beyond EU

\[ V_t(s^t, h) = u(h_t(s^t)) + \beta \int_{s} V_{t+1}((s^t, s_{t+1}), h) \, dp(s_{t+1}) \]

\[ V_t(s^t, h) = u(h_t(s^t)) + \beta I\left(V_{t+1}((s^t, \cdot), h)\right) \]

where \( I : \mathbb{R}^S \rightarrow \mathbb{R} \)
generalize beyond EU

\[ V_t(s^t, h) = u(h_t(s^t)) + \beta \int_S V_{t+1}((s^t, s_{t+1}), h) \, dp(s_{t+1}) \]

\[ V_t(s^t, h) = u(h_t(s^t)) + \beta I(V_{t+1}((s^t, \cdot), h)) \]

where \[ I : \mathbb{R}^S \rightarrow \mathbb{R} \]

I constant over time—IID Ambiguity (Epstein and Schneider, 2003)
Uncertainty Averse Preferences

$I : \mathbb{R}^S \to \mathbb{R}$ is:

- continuous (supnorm)
- monotonic \((\forall s \in S \; \xi(s) \geq \zeta(s) \Rightarrow I(\xi) \geq I(\zeta))\)
- normalized \((\forall r \in \mathbb{R} \; I(r) = r)\)
- quasiconcave
  \(\Rightarrow\) Uncertainty Aversion (Schmeidler, 1989)

Axiomatic foundations: Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011)
special cases

1. **Maxmin expected utility:** 
   \[ I(\xi) = \min_{p \in C} \int \xi \, dp \]

2. **Variational:** 
   \[ I(\xi) = \min_{p \in \Delta(\Sigma)} \left[ \int \xi \, dp + c(p) \right] \]

3. **Multiplier:** 
   \[ I(\xi) = \min_{p \in \Delta(\Sigma)} \left[ \int \xi \, dp + \theta R(p \| q) \right] \]

4. **Confidence:** 
   \[ I(\xi) = \min\{p \in \Delta(\Sigma) | \phi(p) \geq \alpha\} \left[ \frac{1}{\phi(p)} \int \xi \, dp \right] \]

5. **Second order expected utility:** 
   \[ I(\xi) = \phi^{-1} \left( \int \phi(\xi) \, dp \right) \]

6. **Smooth ambiguity:** 
   \[ I(\xi) = \phi^{-1} \left( \int_{\Delta(\Sigma)} \phi(\int \xi \, dp) \, d\mu(p) \right) \]
special cases

relations between them

- $\text{MEU} \subset \text{Variational}$
- $\text{MEU} \subset \text{Confidence}$
- $\text{Variational} \cap \text{Confidence} = \text{MEU}$
- $\text{Multiplier} \subset \text{Variational}$
- $\text{SOEU} \cap \text{Variational} = \text{Multiplier}$
- $\text{SOEU} \subset \text{KMM}$
two key properties

• \( I(\xi + k) = I(\xi) + k \) for all \( \xi \in \mathbb{R}^S, k \in \mathbb{R} \)
  - shift invariance, Constant Absolute Ambiguity Aversion (Variational)

• \( I(\beta \xi) = \beta I(\xi) \) for all \( \xi \in \mathbb{R}^S, \beta \in (0, 1) \)
  - scale invariance, Constant Relative Ambiguity Aversion (Confidence)

• so MEU has both
Discounted Uncertainty Averse Preferences

define by backward induction

\[ V_T(s^T, h) := u(h_T(s^T)) \]
\[ V_t(s^t, h) := u(h_t(s^t)) + \beta I\left(V_{t+1}((s^t, \cdot), h)\right); \quad t = 0, \ldots, T-1 \]

so Dynamically Consistent:

- Sarin and Wakker (1998)
- Epstein and Schneider (2003)
- Marinacci, Maccheroni, and Rustichini (2006)
- Klibanoff, Marinacci, and Muhkerhi (2009)

(without a fixed filtration, well known problems with DC)
Applications to macroeconomics and finance


Results
Main Message

What we assume about I will have impact on Preference for Earlier Resolution of Uncertainty.
ambiguity
averse
preferences
ambiguity
averse
preferences

timing
indifference
**Theorem 1.** A family of discounted uncertainty averse preferences satisfies indifference to timing of resolution of uncertainty if and only if $I(\xi) = \min_{p \in C} \int \xi \, dp$. 
\textit{Proof}

\begin{itemize}
  \item indifference to timing $\Rightarrow$ shift-invariance \textbf{and} scale-invariance
  \item shift-invariance and scale-invariance $\Rightarrow$ MEU
\end{itemize}
Proof

\( V(\begin{array}{c}
  x \\
  x \\
  z \\
  y \\
  y \\
  x \\
  z \\
\end{array}) \)
Proof

\[ V \left( \begin{array}{c}
  x \\
  y \\
  z \\
\end{array} \right) = x + \beta I \left( \begin{array}{c}
  y \\
  z \\
\end{array} \right) \]
Proof

\[ V \left( \begin{array}{c}
    y \\
    x \\
    x \\
    z
\end{array} \right) = x + \beta I \left( \begin{array}{c}
    y \\
    x + \beta I \\
    x + \beta I \\
    z
\end{array} \right) \]

\[ = x + \beta \left( x + \beta I \left( \begin{array}{c}
    y \\
    z
\end{array} \right) \right) \]
Proof
Proof

\[ V \begin{pmatrix} x \\ x \\ z \end{pmatrix} = x + \beta I \begin{pmatrix} x + \beta I \begin{pmatrix} y \\ y \end{pmatrix} \\ x + \beta I \begin{pmatrix} z \\ z \end{pmatrix} \end{pmatrix} \]
Proof

\[ V \left( \begin{array}{c}
  x \\
  y \\
  z \\
\end{array} \right) = x + \beta I \left( \begin{array}{c}
  x + \beta I \left( \begin{array}{c}
    y \\
  \end{array} \right) \\
  x + \beta I \left( \begin{array}{c}
    z \\
  \end{array} \right) \\
\end{array} \right) \]

= \[ x + \beta I \left( \begin{array}{c}
  x + \beta y \\
  x + \beta z \\
\end{array} \right) \]
Proof

\[
x + \beta (x + \beta I) (y) (z)
\]

\[
x + \beta I (x + \beta y) (x + \beta z)
\]
Proof

\[ x + \beta I \begin{pmatrix} y \\ z \end{pmatrix} \]

\[ I \begin{pmatrix} x + \beta y \\ x + \beta z \end{pmatrix} \]
Proof

we have:

$$\exists \beta \in (0,1) \ \forall x \in \mathbb{R} \ \forall \xi \in \mathbb{R}^s \ \ x + \beta I(\xi) = I(x + \beta \xi)$$
Proof

we have:

$$\exists \beta \in (0,1) \forall x \in \mathbb{R} \forall \xi \in \mathbb{R}^s \quad x + \beta I(\xi) = I(x + \beta \xi)$$

need to show:

$$\forall \beta \in (0,1) \forall x \in \mathbb{R} \forall \xi \in \mathbb{R}^s \quad x + \beta I(\xi) = I(x + \beta \xi)$$

(details in the paper) \qed
• in some sense this argument could be used to axiomatize the recursive multiple priors model

• a related paper by Kochov (2012) axiomatizes MEU using a strong version of Stationarity, which has a flavor of IERU
Comparison to Risk

• Chew Epstein (1989) show IERU ⇒ EU

• Grant Kajii Polak (2000) show (rank-dependent or betweenness) + PERU ⇒ EU

• so dispensing with objective probability makes more room
preference for earlier resolution

MEU

timing indifference
**Theorem 2.** A family of discounted variational preferences, 
\[ I(\xi) = \min_{p \in \Delta(\Sigma)} \int \xi \, dp + c(p), \] 
always satisfies preference toward earlier resolution of uncertainty.
VARIATIONAL PREFERENCES

preference for earlier resolution

timing indifference

MEU
Theorem 3. A family of discounted confidence preferences,
\[ I(\xi) = \min_{p \in \Delta(\Sigma) \mid \varphi(p) \geq \alpha} \frac{1}{\varphi(p)} \int \xi \, dp, \]
displays a preference for earlier resolution of uncertainty if and only if
\[ I(\xi) = \min_{p \in C} \int \xi \, dp. \]
Theorem 4. A family of discounted second order expected utility preferences \( I(\xi) = \phi^{-1}\left( \int \phi(\xi) \, dp \right) \) with \( \phi \) concave, strictly increasing and twice differentiable displays a preference for earlier resolution of uncertainty iff Condition 1 holds.

Condition 1. There exists a real number \( A > 0 \) such that

\[ -\frac{\phi''(x)}{\phi'(x)} \in [\beta A, A] \text{ for all } x \in \mathbb{R} \]

(this condition means that the curvature of \( \phi \) doesn’t vary too much)
Theorem 5. A family of discounted smooth ambiguity preferences
\[ I(\xi) = \phi^{-1}\left( \int_{\Delta(\Sigma)} \phi\left( \int \xi \, dp \right) \, d\mu(p) \right) \]
with \( \phi \) concave, strictly increasing and twice differentiable displays a preference for earlier resolution of uncertainty if Condition 1 holds

Only if under additional assumption on the support of \( \mu \).
Persistence
Theorem 6. A family of discounted variational preferences, \[ I(\xi) = \min_{p \in \Delta(\Sigma)} \int \xi \, dp + c(p), \] always satisfies preference for iid.

A family of discounted confidence preferences, \[ I(\xi) = \min\{p \in \Delta(\Sigma) | \varphi(p) \geq \alpha\} \frac{1}{\varphi(p)} \int \xi \, dp, \] always satisfies preference for iid.

In both cases, indifference to iid is satisfied if and only if \[ I(\xi) = \min_{p \in C} \int \xi \, dp. \]

(I do not know how to extend this result to all of \( I \))
Aggregator
Aggregator

\[ V_t(s^t, h) = u(h_t(s^t)) + \beta l(V_{t+1}((s^t, \cdot), h)) \]
Aggregator

\[ V_t(s^t, h) = u(h_t(s^t)) + \beta \mathcal{I}(V_{t+1}((s^t, \cdot), h)) \]

\[ V_t(s^t, h) = \mathcal{W}\left(h_t(s^t), \mathcal{I}(V_{t+1}((s^t, \cdot), h))\right) \]

where \( \mathcal{W} : X \times \mathbb{R} \rightarrow \mathbb{R} \)
Recursive Uncertainty Averse Preferences

\[ V_T(s^T, h) = \nu(h_T(s^T)) \]
\[ V_t(s^t, h) = W\left(h_t(s^t), I\left(V_{t+1}(\langle s^t, \cdot \rangle, h)\right)\right); \ t = 0, \ldots, T - 1 \]
Question

$$(I^{MEU}, W^{disc})$$
Question

$$(I, W^{disc})$$

$$(I^{MEU}, W^{disc})$$
Question

\[(I, W^{disc})\]

\[(I^{MEU}, W^{disc})\]

\[(I^{MEU}, W)\]
Are they the same?
Question

\( (I, W^{\text{disc}}) \)

\( (I^{\text{MEU}}, W^{\text{disc}}) \)

Same iff \( I(\xi) = \min_{p \in C} \phi^{-1} \left( \int \phi(\xi) \, dp \right) \)

\( (I^{\text{MEU}}, W) \)
relation between the two models

• both $W$ and $I$ induce PERU

• but save for the case above, they are different $\sim$

• do they fit the data in a different way?

• $W$—workhorse model of macrofinance (Epstein–Zin)

• what does $I$ add?
recent work: Epstein, Farhi, Strzalecki (2014)

• suppose you are endowed with a consumption process $h_t$

• for what $\pi \in (0, 1)$ are you indifferent between

\[ [h_t, \text{gradual resolution}] \sim [(1 - \pi)h_t, \text{early resolution}] \]

• $\pi \in (20\%, 40\%)$ for workhorse models in finance using Epstein–Zin preferences (Bansal and Yaron 2004; Barro, 2009)

• how high is this number for models of ambiguity?
Conclusion:
interdependence of ambiguity and timing

MEU—only case of indifference

Questions:
theoretical: is this it? can we disentangle more?

empirical: how to measure this?
Thank you


