Stochastic Choice Theory

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Introduction

What is this Book About?

This book is about models used in:

- decision theory,
- behavioral economics,
- discrete choice econometrics.

Such models have a variety of applications in game theory, industrial organization, labor economics, marketing, behavioral, and experimental economics. The manuscript offers a systematic introduction to the field, building up from scratch without any prior knowledge requirements. It surveys recent developments and brings the reader all the way to the frontier of research. It is addressed primarily to PhD and advanced Masters students in economics.

Math

I wanted to keep this book relatively short, so I had to make some compromises. To focus on the conceptual description of the theory and the directions of its development, I made math as simple as I could. Whenever things get dense, I refer the reader to the original sources. If you get interested in this topic, please be mindful that this monograph will not build much “muscle.” If you find a decision theory class near you, take it!

At the same time, I wanted to state the main results of the theory. For the purpose of this book, I introduced the concept of Theorem†. The dagger means that additional technical details or definitions are needed because not all terms are properly defined. Fully formal statements are contained in the original source, which a serious reader will want to consult. You should not use any daggers in your job market paper!
Part 1

Static Choice
Chapter 1

Random Utility

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1.1. The Analyst and the Agent

The main character in this book is the analyst. She is a researcher, an econometrician, an experimental economist, etc. The analyst has access to data about the behavior of an agent (or a population of agents) summarized by a stochastic choice function $\rho$. The analyst wants to understand $\rho$ to predict the agent’s behavior in a new situation, e.g., forecast demand for a new product. In addition, if the analyst is benevolent she will want to be able to measure the agent’s welfare.\footnote{I will refer to the analyst as “she,” or sometimes “us.” I use “their” for the agent(s).}

This book focuses on non-strategic situations, where the data are for example the occupational choices in a population or response frequencies in a laboratory or field experiment. Our agents don’t play games with each other or with the analyst.\footnote{Of course, strategic interactions are prevalent in economics, but it’s worthwhile to first see how much we can understand about individual behavior.} We assume that the analyst is passively studying the agent. The analyst’s decisions (which will be unmodeled here) may ultimately impact the agent as new products get introduced or new contract or mechanisms get designed, new policies get made; our agents are not strategic enough to take this into account.\footnote{In fact, the situation of the mechanism designer is similar to the situation of our analyst: she has some information about behavior in various situations, $\rho$, and picks a situation (mechanism) to induce the agent to behave in a desired way.}

Many analysts model the agent as a utility-maximizing creature and make various other “rationality” assumptions. Each model puts some restrictions on the class of behaviors that are allowed. We will try to understand these restrictions and the ways the various classes connect to each other.

Understanding the relationships between models is interesting in its own right, but can also serve some practical purposes. For instance, the analyst often has to pick a particular model and it’s good to know what the possible tradeoffs between these models are.

1.2. Deterministic Choice

We start with deterministic choice because this will be the basis for much of what is to come in this book. This will also establish notation used throughout.

Let $X$ be the set of all possible alternatives that our agent might be facing. Typical elements are denoted $x, y \in X$ and may stand for things like: brand choices, employment status, number of children, market entry decisions, or choosing which perceptual stimulus is stronger.

In introductory microeconomics and consumer theory $X$ is typically an infinite set of consumption bundles ($X = \mathbb{R}^n_+$, where $n$ is the number of goods) and the agent is choosing how much of each good to consume (the budget set is a polytope given by the price vector). The analysis then quickly assumes
1.2. Deterministic Choice

differentiability and convexity and characterizes optimality by first order conditions. In discrete choice theory the analysis is somewhat different: the budget set, or the _menu_ is finite, and the optimality conditions are a set of inequalities instead of equalities. We allow $X$ to be infinite, but the menu will always be finite.

The analyst observes the agent’s choices in multiple choice situations. The data of the analyst is a _choice function_ that says what the agent does in each situation. We will treat the choice function as observable to the analyst—we will assume that she can collect this data by observing how people behave in real life, or by designing a lab or field experiment.

In decision theory and consumer theory a choice situation is typically summarized by the _menu_ (a subset of $X$) the agent is choosing from (for example the actual menu at the restaurant, or the set of insurance plans an employer offers, or the budget set in consumer theory).

Let $\mathcal{A}$ be the collection of _menus_—all nonempty and finite subsets of $X$, with typical elements $A, B, C$. A single-valued _choice function_ is a mapping $\chi : \mathcal{A} \to X$ such that $\chi(A) \in A$.

The “revealed preference” exercise of Samuelson (1938), seeks to rationalize such observations by preference maximization and to uncover the preference relation from the observed data.

A binary relation $\succeq$ on $X$ is a _preference_ if it is:

- complete ($x \succeq y$ or $y \succeq x$ for all $x, y \in X$)
- transitive ($x \succeq y$ and $y \succeq z$ implies $x \succeq z$ for all $x, y, z \in X$).

Moreover, the relation is a _strict preference_ if it is also satisfies the following property:

- $x \succ y$ and $y \succ x$ implies $x = y$ for all $x, y \in X$.

The last requirement (called _antisymmetry_) means that the agent is never indifferent between two distinct options.

We say that a strict preference $\succ$ _represents_ $\chi$ whenever, for each $A \in \mathcal{A}$, $\chi(A)$ is the highest ranked element of $A$ according to $\succ$. The key here is that the agent maximizes _the same_ preference on $X$ irrespectively of which menu they are facing. If we are allowed to tailor the preference to the menu, we can explain every possible choice function and our model is not falsifiable (so there is no way of testing if it’s true).

Many $\chi$’s cannot be represented by a single $\succeq$; they are sometimes called “irrational,” “behavioral,” or “boundedly rational.”

---

\(^4\) We will talk about the multi-valued choice correspondences shortly.
1.2. Deterministic Choice

The key test for deterministic preference maximization is known under many names, such as Sen’s $\alpha$ condition, Arrow’s IIA, Chernoff’s condition, or Contraction.\(^5\)

**Axiom 1.1** (Sen’s $\alpha$). If $x \in A \subseteq B$, then $x = \chi(B)$ implies $x = \chi(A)$.

This axiom says that if alternative $x$ beats all things in a menu, it must also beat all things in a subset of the menu. Thus, the axiom imposes consistency conditions on choices from various menus.

**Proposition 1.2.** A choice function $\chi$ satisfies Sen’s $\alpha$ if and only if there exists a strict preference relation that represents it. Moreover, $\succeq$ is unique.

A simple proof is for example in Osborne and Rubinstein (2020). The assumption that $A$ contains all menus can be relaxed as long as it contains all pairs and triples.

There are many situations in which Sen’s $\alpha$ is violated; an early example comes from Luce and Raiffa (1957).

**Example 1.3** (Frog Legs). Suppose that there are two restaurants, one with menu $A = \{\text{steak tartare, chicken}\}$ and the other $B = \{\text{steak tartare, chicken, frog legs}\}$.

The agent’s utility function depends on the meal and also on how good the chef is.

<table>
<thead>
<tr>
<th></th>
<th>$U(\text{steak tartare})$</th>
<th>$U(\text{chicken})$</th>
<th>$U(\text{frog legs})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>good chef</td>
<td>10</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>bad chef</td>
<td>0</td>
<td>5</td>
<td>0</td>
</tr>
</tbody>
</table>

As Luce and Raiffa (1957) argue, the very existence of frog legs on the menu could persuade the agent that the chef is good, because only good chefs would attempt to prepare frog legs. If that is so, the agent chooses steak tartare from $B$ and chicken from $A$, thereby violating Sen’s $\alpha$. The reason why this example “works” is that different menus are associated with different information, which lead to different expected utilities. So in a sense, this is a “rational” violation of Sen’s $\alpha$. In Chapter 2 we will discuss some “behavioral” violations. ▲

Decision theorists like results like Proposition 1.2 that provide an exact translation between two languages:

- what is observable (the choice function $\chi$)
- what is a mathematical *representation* (the preference $\succeq$).

Building such a bridge helps us understand the connections between the two ways of describing choice. It also offers a test of “rationality:” if our agent violates Sen’s $\alpha$, then they cannot be maximizing a complete and transitive preference.

To deal with indifferences, economists often consider a multi-valued *choice correspondence* $\chi : A \rightarrow A$ such that $\chi(A) \subseteq A$. The idea behind multi-valued

choice is that from any given menu the agent sometimes chooses one alternative and sometimes another. The analyst records both of these choices and interprets this as indifference. For choice correspondences, an additional condition, known as Sen’s $\beta$, is needed to characterize preference maximization. Conditions $\alpha$ and $\beta$ combined are called WARP (weak axiom of revealed preferences). For details see Chapter 2 of Kreps (1988) and Chapter 1 of Mas-Colell, Whinston, Green, et al. (1995). We will not deal with choice correspondences. The theory of stochastic choice provides an alternative way of modeling the situation where the agent makes different choices from the same menu.

So far, we have two languages: the observables (choice function or correspondence $\chi$) and the representation (preference relation $\succeq$). To make the math easier, we often use yet another language for representations—the utility functions. This way we can use familiar tools from optimization theory, such as first- and second-order conditions, Hamilton-Jacobi-Bellman equations, etc.

A preference $\succeq$ is represented by a utility function $U : X \rightarrow \mathbb{R}$ whenever

$$x \succeq y \text{ if and only if } U(x) \geq U(y).$$

We will interchangeably write $U : X \rightarrow \mathbb{R}$ and $U \in \mathbb{R}^X$ for the same object. Recall that if $X$ and $Y$ are sets then $Y^X$ is the collection of all functions from $X$ to $Y$.

A utility representation always exists when $X$ is finite or countable. A classic counterexample when $X$ is uncountable are lexicographic preferences. Since we often have to deal with uncountable $X$, e.g., consumption bundles (as in price theory) or lotteries (Chapter 4), typically continuity is assumed to get a representation.\(^6\) There is a multitude of such representations: any monotone transform $\phi(U)$ represents the same preference as $U$, see Proposition A.1.1 in the Appendix.

This is called ordinal uniqueness. This implies that the “curvature” of $U$ does not have a behavioral meaning because we can always take $\phi$ concave or convex.

1.3. Stochastic Choice

So far, we assumed that whenever the analyst observed the agent alternating their choices, she ignored the frequency of such choices treating them as indifferent. This means that a person who chooses $x$ 99% of the time and another person who chooses $y$ 99% of the time are classified as the same type.

In this book we will take the choice frequencies seriously and try to extract information from them. To do this, we need to enrich the set of observables: for each menu $A$ and item $x \in A$ let $\rho(x, A)$ be the frequency with which a

\(^6\)For the finite and countable cases, see Propositions 3.2 and 3.3 of Kreps (1988). For uncountable $X$ see Theorems 3.5 and 3.7 of Kreps (1988) or Chapter 9 of Ok (2014).
choice of \( x \) from \( A \) was observed.\(^7\) In reality we will have a finite sample of \( n \) observations, but we will think of \( \rho(x, A) \) as the limiting frequency as \( n \to \infty. \)\(^8\) A stochastic choice function (s.c.f.) collects these frequencies as a function of the menu.

For any finite set \( Z \) let \( \Delta(Z) \) denote the set of probability distributions over \( Z, \) i.e., functions \( p : Z \to [0, 1] \) such that \( \sum_{z \in Z} p(z) = 1. \) For each menu \( A \) the values of \( \rho(\cdot, A) \) form a probability distribution over \( A, \) so we can think of the s.c.f. as a map that takes a menu \( A \) and maps it into \( \Delta(A). \)

**Definition 1.4.** A stochastic choice function is a mapping 
\[
\rho : A \to \Delta(X)
\]
such that \( \sum_{x \in A} \rho(x, A) = 1 \) for all \( A \in \mathcal{A}. \)

Sometimes, not all menus are observed, in which case the domain of \( \rho \) is smaller. For example, in experiments often there are just binary menus. On the other hand, in econometrics, the menu is fixed but what varies are the attributes of these alternatives. The first three parts of the book focus on choice set variation and the last part of the book focuses on attribute variation. However, the difference is not clear cut, for example for lotteries, each alternative is characterized by a vector of attributes (probabilities of each payoff).

If our analyst is observing a single individual who faces the problem repeatedly (as it happens in some within-subject experiments); then \( \rho(x, A) \) is the fraction of times the agent chose \( x \) from \( A. \) Stochastic choice functions can also capture population-level data. For example, McFadden (1974) studied transportation choices of the Bay area population. In this situation \( \rho(x, A) \) is the fraction of the population choosing \( x \) from \( A. \) In such applications choice has two sources of stochastic variation: individual randomness (how much choice varies if a given person is sampled over and over again) and heterogeneity of preferences (how much choice varies across people).

While it’s easy to imagine that preference heterogeneity leads to non-trivial choice frequencies in the aggregate data, it’s less obvious why the choices of a single individual should be stochastic. Yet, stochastic choice is routinely observed. First and foremost, in the perceptual domain, in the context of discrimination between stimuli (Fechner, 1860; Thurstone, 1927). The following example discusses perception of weight, but similar experiments are used in the study of other senses: vision, hearing, touch, etc.\(^9\) To simplify notation for binary menus we will write \( \rho(x, y) := \rho(x, \{x, y\}) \) when \( x \neq y, \) agreeing that \( \rho(x, x) = 0.5. \)

---

\(^7\)The recent paper of Ok and Tserenjigmid (2020) compares the choice-correspondence approach to the choice-frequency approach.

\(^8\)This assumption is routinely made in econometrics for the purpose of estimation and identification of parameters. We will talk about this more in Chapter 2.

\(^9\)For a good introduction on this topic, see Gescheider (1997).
1.3. Stochastic Choice

Example 1.5 (Perception Task). Let $X = \mathbb{R}_+$ be a collection of weights (the weights all look the same, or the experimental subject’s view is obstructed). The subject is facing a series of binary menus $A_i := \{x_i, y_i\}$, $i = 1, \ldots, n$, where $x_i, y_i$ are drawn i.i.d. from some distribution $\pi \in \Delta(X)$. The subject is tasked with picking which object is heavier: there is a positive payoff for a correct guess and zero for incorrect. The analyst records the subject’s choice over many i.i.d. trials. In the limit, we get $\rho(x, y)$.

It is interesting to examine $\rho(x, y)$ as a function. If we fix $y$ as the reference weight and $x$ is varied, then we get what is called a psychometric function; in fact we have a family of psychometric functions indexed by $y$.

Numerous experiments in psychology and psychophysics can be summarized by the following stylized facts (see, Woodrow (1933) and Gescheider (1997)). First, psychometrics functions are typically S-shaped (otherwise known as sigmoid). This means that if $x$ is close to $y$ it is hard for the subject to discriminate between them and accuracy is low. As $|x - y|$ grows, the accuracy improves. It is typical to use the cdf of the Normal distribution $\Phi$ to model psychometric functions.

![Figure 1.1. An S-shaped psychometric function.](image)

Another stylized fact is diminishing sensitivity: a given weight difference $|x - y|$ may be big enough for the subject to notice when both $x$ and $y$ are small, but not big enough when they are both big.\(^\text{10}\) Diminishing sensitivity has

\(^{10}\)This is related to the Weber-Fechner law. The law was originally formulated in the language of just noticeable differences, which assumes that psychometric functions are step functions: discrimination is either perfect or absent. It goes against our first stylized fact (smooth discrimination). One way to state the Weber-Fechner law in the world of S-shaped psychometric functions is to say that the interquartile range (IQR) depicted in Figure 1.1 is an increasing function of $y$.\)
been incorporated into many psychological theories, such as Prospect Theory (Kahneman and Tversky, 1979).

Yet another stylized fact is payoff-monotonicity, which says that the error rate diminishes if the payoff for guessing correctly increases.

The final stylized fact is frequency-dependence, which says that $\rho(x, y)$ depends on the distribution $\pi$ of weights across trials. Intuitively, the agent gets attuned to the variation, so that the same weight difference can be perceptible if all weights in the experiment are in some small range, but will go unnoticed if the weights vary a lot from trial to trial. Notice that frequency-dependence implies that we should more accurately be talking about $\rho^\pi(x, y)$, where $\pi$ is fixed in a given batch of trials and the analyst runs several batches each with a different $\pi$.

While it may not be surprising that perception of physical stimuli is random, there is a body of experimental evidence showing that economic choices are random as well. Mosteller and Nogee (1951) show that choices between lotteries show substantial switching. Whether trials are separated by days (Tversky, 1969; Hey and Orme, 1994) or minutes (Camerer, 1989; Ballinger and Wilcox, 1997; Agranov and Ortoleva, 2017; Agranov, Healy, and Nielsen, 2020) the fraction of choice reversals is substantial: between 20% and 30%. This is true even in questions that offer dominated options.

We will now discuss various reasons why individual choice could fluctuate. Each of them corresponds to a particular representation of $\rho$.

1.4. Representations

The easiest case is population heterogeneity. For example, in the Hotelling (1929) model, consumers’ or voters’ blisspoints are distributed along a line. More generally, we are given a probability distribution over utility functions that specifies the frequency of each utility in the population. This is called a random utility representation and our formal analysis of stochastic choice will begin with it. Each individual’s utility function is deterministic, but choices appear random to the analyst as she only observes aggregate data. This model is at the heart of discrete choice econometrics. The heterogeneity of tastes (not just knowing average demand) is important for the firms (e.g. to choose the product mix) and to policymakers (who care about distributional effects).

What about stochastic choices of a single agent? Here there are more possible mechanisms:

1. Random Utility. Instead of a distribution of utilities in the population, we now have a distribution of utility realizations for a fixed agent.\(^{11}\) In choice tasks, the tastes of the agent fluctuate. In perception tasks, perceptions are random. For example Thurstone (1927) assumed that the perceived stimulus

\(^{11}\)This is similar to Harsanyi’s purification in game theory (Harsanyi, 1973a).
equals true stimulus plus a normally distributed error; which leads to what is now known as the **probit model**.

2. **Learning.** Here the agent’s tastes are fixed but their information evolves as they learn new things. The information arrival can be exogenous (passive learning) or chosen by the agent (active learning). In models of *experimentation* (as in the multi-armed bandit model) and *experience goods*, the agent learns something about the utility of a good each time they consume it. In models of *sequential sampling* and Drift-Diffusion Models (DDM), the agent chooses the sample size. In models of *rational inattention* or *costly information acquisition*, the information structure is chosen optimally at a cost.

3. **Random Consideration.** The agent’s tastes and information might be fixed, but they may not always pay attention to the same objects in the menu. If the attention process is random, it will lead the agent to consider different subsets of the menu (called consideration sets) from trial to trial, thereby generating random choices.

We therefore have two models of attention (endogenously choosing the information and being exogenously restricted to a subset of the menu). We will treat them in separate chapters.

In all of the above stories so far, choices are actually deterministic from the point of view of the agent. They know what their craving is today, or what they learned so far, or what they are paying attention to. Observed choices appear stochastic to the analyst as a result of the informational asymmetry between the two characters. In the following two stories, choices are random even in the agent’s eyes, so both our characters are on the same footing.

4. **Trembling Hands:** The agent cannot perfectly control their choice: there is a random implementation error or decision error. In some models this error is exogenous, while in others the agent may control mistakes at a cost. Observed randomness is then the result of a balance between the importance of choosing correctly and the cost of doing so.

5. **Deliberate Randomization.** The agent likes to randomize. They view each menu $A$ as the set of probability distributions $\Delta(A)$ over $A$ and pick a favorite one according to some non-expected utility preference over lotteries that may capture a wish to hedge their bets or aversion to regret.

This book starts with random utility. This is by far the most popular model to study population-level data: almost all of discrete choice econometrics and demand system theory stems from this model. Moreover, much of the classical decision theory work on stochastic choice is about random utility. A good understanding of this model is also a prerequisite for the other models.
1.5. Random Utility

There are three equivalent ways to formulate the model mathematically: (1) a probability distribution over preferences, (2) a probability distribution over utility functions, (3) a random utility function. It may seem like excessive formalism to define all three here but going forward it will be convenient to seamlessly switch between them, depending on the application or context, so I want you to get comfortable with all three.

Let $\mathcal{P}$ be the set of all strict preferences over a finite set $X$. Let $\mu \in \Delta(\mathcal{P})$ be a probability distribution over strict preferences. Here $\mu$ is either the distribution of preferences in the population, or the probability that governs the evolution of the preferences of the individual, depending on our interpretation of $\rho$. For any $A \in \mathcal{A}$ and $x \in A$ let

$$N(x, A) := \{\succ \in \mathcal{P} : x \succ y \text{ for all } y \in A\}$$

be the set of preferences that rationalize the choice of $x$ from $A$.

**Definition 1.6.** $\rho : A \rightarrow \Delta(X)$ is represented by a distribution over preferences if there exist $\mu \in \Delta(\mathcal{P})$ such that $\rho(x, A) = \mu(N(x, A))$ for all $A \in \mathcal{A}$ and $x \in A$.

In Chapter 2 we will discuss the axioms and uniqueness results. For now, notice that if we observe choices from only one menu $A$, then any $\rho$ has such a representation. We can just define the probability that $x$ is on top of the ranking to equal $\rho(x, A)$ for all $x \in A$ (the relative ranking of non-top items does not matter). There are no consistency conditions to check, so this construction works. It is the nontrivial menu variation that gives content to the representation.

1.5.1. Invariance of $\mu$. The key assumption is that the distribution $\mu$ does not depend on the menu $A$—it is a structural invariant of the model. If $\mu$ is allowed to depend on the choice set in an arbitrary way, then any s.c.f. $\rho$ can be trivially explained (why?).

Of course, there are some realistic situations in which $\mu$ depends on $A$. This is the case for some models of learning (both passive and active). Under “random consideration,” the menu offered to the agent can affect the “consideration set” and de facto make the utility of some items equal to $-\infty$.

Another possible complication occurs if the invariance assumption is actually satisfied by the data generating process, but violated in the observed sample because of the way the sample is collected. For example, menu $A$ could be the set of products available in country 1 and menu $B$ in country 2. The distribution of preferences is different in those countries and so the invariance assumption holds within each country (say if you give menu $B$ to people from country 1), but if you naively try to fit the same $\rho$ to both menus invariance will be violated. Similar examples involve self selection: the distribution of preferences between two brands of orange juice is different depending whether
the menu of choices is Whole Foods or Walmart because different people choose
to go to these stores.

For now, we will assume that the data generating process and our sample
is free of such effects. This assumption will let the analyst estimate \( \mu \) based on
choices from some incomplete set of menus \( \mathcal{A}^* \) and predict choice from a new
menu \( A \not\in \mathcal{A}^* \), for example when a new product is introduced.

1.5.2. Equivalent Definitions. A slightly different object than a distribu-
tion over preferences is a distribution over utilities. Our set \( N \) becomes
\[
N(x, A) := \{ U \in \mathbb{R}^X : U(x) \geq U(y) \text{ for all } y \in A \} \\
= \{ U \in \mathbb{R}^X : U(x) = \max_{y \in A} U(y) \}.
\]
Now \( N \) stands for the set of utility functions that rationalize the choice of \( x \)
from \( A \).

When \( X \) is finite it is without loss of generality to consider discrete mea-
sures over \( \mathbb{R}^X \), but sometimes it is convenient to use continuous distributions
that admit a density. In general, let \( \Delta(\mathbb{R}^X) \) be the set of Borel probability
measures over \( \mathbb{R}^X \). (For the purpose of understanding this book, you can just
think of this as containing all discrete and continuous distributions.)

**Definition 1.7.** \( \rho : \mathcal{A} \rightarrow \Delta(X) \) is represented by a distribution over
utilities if there exist \( \mu \in \Delta(\mathbb{R}^X) \) such that \( \rho(x, A) = \mu(N(x, A)) \) for all \( A \in \mathcal{A} \) and
\( x \in A \).

Yet another way to model this is to let utility be a random variable. Let
\((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, i.e., \( \mathcal{F} \) is a \( \sigma \)-algebra and \( \mathbb{P} \) is a probability
measure.\(^{12}\) Utility is a random function, i.e., \( \tilde{U} : \Omega \rightarrow \mathbb{R}^X \) is \( \mathcal{F} \)-measurable
random utility function. I will try to put a tilde on every random variable
(function, element, etc.). We can think of \( \Omega \) as things that are observable to
the agent but unobservable to the analyst. The event \( N \) is now written as
\[
N(x, A) := \{ \omega \in \Omega : \tilde{U}_{\omega}(x) \geq \tilde{U}_{\omega}(y) \text{ for all } y \in A \} \\
= \{ \omega \in \Omega : \tilde{U}_{\omega}(x) = \max_{y \in A} \tilde{U}_{\omega}(y) \}.
\]

**Definition 1.8.** \( \rho : \mathcal{A} \rightarrow \Delta(X) \) has a random utility representation if there
exists a random variable \( \tilde{U} : \Omega \rightarrow \mathbb{R}^X \) such that \( \rho(x, A) = \mathbb{P}(N(x, A)) \) for all
\( A \in \mathcal{A} \) and \( x \in A \).

The following is an easy adaptation of Theorem 3.1 in Block and Marschak
(1960), see also Regenwetter and Marley (2001).

**Proposition 1.9.** The following are equivalent for a finite \( X \):

1. \( \rho \) is represented by a distribution over preferences,

\(^{12}\)See, e.g., Chapter 1 of Williams (1991). If you are not familiar with measure-theoretic probabil-
ity, you can rely on your intuitive understanding of random variables.
1.6. Tie Breaking*

(ii) $\rho$ is represented by a distribution over utilities,
(iii) $\rho$ has a random utility representation.

**Proof.** Appendix A.1.2. □

Given this result, we will write $\rho \sim RU$ whenever any of the conditions above holds.

**Remark** 1.10. Material with an asterisk may be omitted at first reading. Proposition 1.9 holds for countable $X$ under appropriate definitions, see Cohen (1980). The equivalence between (ii) and (iii) holds for uncountable $X$ under appropriate technical conditions. For uncountable $X$ condition (i) is typically modified because preferences are typically assumed to be continuous, which implies that they are not strict (have nontrivial indifference curves). We will talk more about the infinite case later. ▲

I have not made a distinction between the three different definitions of the set $N(x, A)$ and I will not do so in the future. I do make a notational distinction between $P$, which is the probability measure on the probability space $\Omega$ that carries the random utility $\tilde{U}$, and $\mu$—which is the law of $\tilde{U}$. I will put a tilde on any random element.

1.6. Tie Breaking*

If $\rho$ is represented by a distribution over preferences, then ties are ruled out by construction. On the other hand, RU in principle allows for ties, but as mentioned above ties must occur with zero probability. More precisely, let $T_{xy} := \{ \omega \in \Omega : \tilde{U}_\omega(x) = \tilde{U}_\omega(y) \}$. If $\rho$ has a RU representation, then it must be that $P(T_{xy}) = 0$ for all $x \neq y$; otherwise, $\rho(x, y) + \rho(y, x) > 1$ (why?). This means that RU with ties does not lead to legitimate stochastic choice functions. I will refer to those without ties as as proper RU. Formally, $\tilde{U}$ is proper if for any menu $A$, with probability one, $\tilde{U}$ has a unique maximizer on $A$.\(^{13}\)

For various reasons it is sometimes convenient to allow for ties. Let’s take a RU representation that is not proper. One possible way to define $\rho$ is to use a tiebreaker. For instance, we could assume that the agent uniformly randomizes over the maximal elements (uniform tiebreaking). A more general notion of tiebreaking was introduced by Gul and Pesendorfer (2006a).

A GP-tiebreaker is a random utility function $\tilde{W} : \Omega_W \to \mathbb{R}^X$ that itself is proper. In a random utility representation with a tiebreaker, the agent first maximizes $\tilde{U}$ and then uses $\tilde{W}$ to break the ties. The state space is now $\Omega \times \Omega_W$ because the tie breaker needs its own state space, as the original one may not be rich enough to allow for a proper $W$.

**Proposition 1.11.** The following are equivalent when $X$ is finite:

\(^{13}\)This property is also called noncoincidence (Falmagne, 1983) or regularity (Gul and Pesendorfer, 2006a).
1.7. Additive Random Utility

(i) \( \rho \) has a proper RU representation
(ii) \( \rho \) has an improper RU representation with uniform tiebreaking
(iii) \( \rho \) has an improper RU representation with a GP-tiebreaker.

Proof. Appendix A.1.3. □

How do we know when to classify randomness in choice as true preference variation vs tiebreaking? This question has an intuitive answer in a dynamic model (Ahn and Sarver, 2013) because the two sources of randomness enter differently in the agent’s option value calculation (taste variation provides flexibility whereas tiebreaking is a matter of indifference). We will discuss this in Chapter 8.

Instead of using tiebreakers, other papers change the primitive and study stochastic choice correspondences or capacities: Barberá and Pattanaik (1986); Gul and Pesendorfer (2013); Lu (2016); Lin (2018); Piermont and Teper (2018). To a large extent this approach is “morally equivalent” to assuming tie breakers and I view the choice between them as a matter of convenience.

Remark 1.12. There is another way to define ties: Let \( T := \{ \omega \in \Omega : \tilde{U}_\omega(x) = \tilde{U}_\omega(y) \text{ for some } x \neq y \} \). Note that \( T = \bigcup_{x \neq y} T^{xy} \) so for finite \( X \) if \( \mathbb{P}(T^{xy}) = 0 \) for all \( x \neq y \), then \( \mathbb{P}(T) = 0 \) (and vice versa). But with uncountable \( X \), this new definition is stronger: when \( X \) is multidimensional we are forced to have \( \mathbb{P}(T) > 0 \) because all continuous preferences have well-behaved indifference curves, so for any fixed utility function there will be many points that are indifferent to each other. However, for any two fixed points, the probability that they will be indifferent could well be zero. ▲

1.7. Additive Random Utility

Other fields use an equivalent way of writing random utility, called additive random utility (ARU). This involves writing \( \tilde{U}(x) = v(x) + \tilde{\epsilon}(x) \), where \( v : X \to \mathbb{R} \) is a deterministic utility function, called the “representative utility” or “systematic utility” and \( \tilde{\epsilon} : \Omega \to \mathbb{R}^X \) is a “random utility shock,” which is private information of the agent.

In game theory, ARU is used as a model of smooth best responses. (Fudenberg and Levine, 1998; Hofbauer and Sandholm, 2002). In discrete choice econometrics, this way of writing things means that \( \tilde{\epsilon} \) contains all the information about the preference heterogeneity in the population. The interpretation is that the agent knows \( v \) and the realization of \( \tilde{\epsilon} \), while the analyst does not know the realization (and may potentially be able to estimate \( v \)).

If \( X \) is finite, then I will say that the distribution of \( \tilde{\epsilon} \) is smooth if it has a density. For infinite \( X \), it is smooth if for any menu \( A = \{ x_1, \ldots, x_n \} \) the joint distribution of \( (\tilde{\epsilon}(x_1), \ldots, \tilde{\epsilon}(x_n)) \) has a density. The following definition is based on McFadden (1973).
1.7. Additive Random Utility

Definition 1.13. \( \rho : \mathcal{A} \rightarrow \Delta(X) \) has an additive random utility (ARU) representation if it has a RU representation with \( \tilde{U}(x) = v(x) + \tilde{\epsilon}(x) \), where \( v : X \rightarrow \mathbb{R} \) and the distribution of \( \epsilon \) is smooth.

Note well that the distribution of \( \epsilon \) is independent of the menu: for each \( A \) we just select the corresponding coordinates.

The smoothness assumption guarantees that we have a proper RU representation, as it implies that ties occur with probability zero. It is worthwhile to notice though that there are proper RU representations which are of the form \( \tilde{U}(x) = v(x) + \tilde{\epsilon}(x) \) where \( \epsilon \) has a discrete distribution (take for example the one constructed in the proof of \((i) \Rightarrow (ii)\) in Proposition 1.9). McFadden’s (1973) general definition does not require the existence of a density, but as the following result shows, this assumption is without loss of generality and ARU and RU are equivalent. That is, even if we have a discrete distribution over utilities, we can “smoothify” it without loss of generality.

**Proposition 1.14.** \( \rho \sim RU \) if and only if \( \rho \sim ARU \).

**Proof.** Appendix A.1.4. \( \square \)

The construction used in this proof shows that it is also without loss of generality to assume that \( \tilde{\epsilon} \) has finite first moments.

ARU representations derive their strength from several powerful parametric special cases where the distribution of \( \epsilon \) is i.i.d. The most predominant is the extreme value distribution, which leads to logit.

**Definition 1.15.** \( \rho : \mathcal{A} \rightarrow \Delta(X) \) has a logit representation if it has a ARU representation where \( \tilde{\epsilon}(x) \) are i.i.d. across \( x \) with the Type 1 Extreme Value (TIEV) cdf \( G(\epsilon) = \exp(-\exp(-\epsilon)) \).

Another well-known model is probit, where the distribution of \( \tilde{\epsilon} \) is Normal. We will look more at such parameterizations in Chapter 3.

Often times it is assumed that the density in an ARU representation not only exists, but is everywhere positive. This ensures that all items are chosen with a positive probability (because arbitrarily large shocks can elevate even dominated alternatives).

**Axiom 1.16** (Positivity). \( \rho(x, A) > 0 \) for any \( x \in A \).

This property is important since keeping all probabilities positive leads to a non-degenerate likelihood function, which facilitates estimation; moreover, as argued by McFadden (1973), positivity cannot be refuted based on any finite data set.

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14TIEV, which is also known as the Gumbel distribution, is a whole class of distributions with mean and variance parameters. However, in economics TIEV typically means this particular member of the family.
1.7. Additive Random Utility

Positivity does not imply that $\epsilon$ has positive density (see Example A.1.5 in the Appendix). Li (2021) shows how to strengthen Positivity to ensure that there exists a ARU representation with positive density.

There are two interpretations of $\epsilon$: (1) preference heterogeneity that is unobserved by the analyst (after conditioning on observable characteristics of the agent), or (2) mistakes/errors on the part of the agent. The difference between the interpretations is that in the first case the preference shocks are welcomed by the agent (her tastes do actually change from time to time), while in the second case these shocks lead to choices that the agent disagrees with. While in predictive applications of the static model the two interpretations are largely equivalent, they differ when it comes to normative evaluations and have different predictions for dynamic behavior.

A case that is somewhat in between the two is one of imperfect perception. In the following example the agent sometimes makes mistakes, but they are doing the best they can given their imperfect information. As we will see in Chapter 5 this behavior is Bayes-optimal, so the shocks are welcomed by the agent (ex ante) despite sometimes leading to errors.

**Example 1.17 (Law of Comparative Judgment)**. Recall Example 1.5 with weight perception. Thurstone (1927) introduced the probit model to capture such behavior. Suppose for each weight $x$ the agent forms a subjective, imperfect, and random perception $\gamma(x) + \tilde{\epsilon}(x)$, where $\gamma$ is a strictly increasing function (typically assumed to be logarithmic) and $\tilde{\epsilon}(x) \sim N(0, \sigma_\epsilon^2)$ are independent across $x$. Faced with items $x$ and $y$, the agent chooses item $x$ if $\gamma(x) + \epsilon_x \geq \gamma(y) + \epsilon_y$ and chooses $y$ otherwise. A simple calculation reveals that

$$\rho(x, y) = \Phi \left( \frac{\gamma(x) - \gamma(y)}{\sigma_\epsilon \sqrt{2}} \right).$$

Thus, Thurstone’s model leads to S-shaped psychometric functions. It is easy to see that by setting $\gamma(x) = \log x$ the model explains diminishing sensitivity. However, it does not explain frequency-dependence because the distribution of $\epsilon$ is independent of the distribution of menus $\{x, y\}$.

Finally, the model cannot explain payoff-monotonicity either. This is because $\gamma(x)$ is not the payoff from choosing $x$, but instead a subjective perception of $x$. The magnitude of the payoff for guessing correctly does not enter Thurstone’s formula: his is “probit in perceptions”\(^\text{15}\)

\(^{\text{15}}\)One could imagine a “probit in payoffs”, where it’s the payoffs that get distorted. Let $v > 0$ be the payoff of guessing correctly. Then for $x > y$ we have $\tilde{U}(x) = v + \epsilon_x$ and $\tilde{U}(x) = 0 + \epsilon_y$. This leads to payoff-monotonicity, but induces a psychometric function that is a step function (as opposed to $S$-shaped), so we cannot capture the first two stylized facts. We will need a more fancy model to capture them simultaneously.

Remark 1.18. Suppose that our analyst has a theory that the utility function is deterministic and belongs to some class $\mathcal{U}$. RU and ARU suggest different approaches to taking this theory to data, i.e., building a stochastic model. We can either randomize over utilities $u \in \mathcal{U}$ or add stochastic $\epsilon$ to a fixed utility
1.8. Social Surplus

Our analyst often wants to evaluate the welfare. Under RU the natural way to do this is to set

$$V(A) := \mathbb{E} \left[ \max_{x \in A} \tilde{U}(x) \right].$$

This function captures the expected utility from the best item in the menu. McFadden (1973) called it the social surplus.\(^{16}\)

This function is key for dynamic optimization (Chapters 8 and 12). It also enters into nested logit (Section 3.7).

Under ARU, we have

$$V(A) := \mathbb{E} \left[ \max_{x \in A} v(x) + \tilde{\epsilon}(x) \right].$$

This formula makes sense only if we interpret \(\epsilon\) as unobservable preference shocks. If we think of them as decision errors of the agent, then there is no reason for them to enter welfare. In this case, it may be more appropriate to treat them as just driving behavior, but evaluate welfare using the undistorted preferences. For example a formula a la Strotz (1955) would look like:

$$V(A) = \mathbb{E}[v(\tilde{x})] = \sum_{x \in A} v(x) \rho(x, A),$$

where \(\tilde{x} = \arg \max_{x \in X} v(x) + \tilde{\epsilon}(x)\). A theory along these lines has been developed by Ke (2018).

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\(^{16}\)This should be called “consumer surplus” because the social surplus also includes the firms. But people call it the social surplus so I will do so here too.
Chapter 2

Basic Properties

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2.1. Statistical Models

As Proposition 1.14 says, the RU and ARU representations are equivalent in terms of behavior. But these are two different statistical models because they are parametrized differently. In RU the parameter is a distribution over utility functions \( \tilde{U} \) while in ARU the parameter is a pair: the deterministic utility function \( v \) and a distribution over shocks \( \tilde{\epsilon} \). As we introduce more models, each will be differently parametrized. In general, let \( \Theta \) be the space of parameters and let \( \{\rho_\theta \} \) be the family of s.c.f’s indexed by parameter \( \theta \). We hope that the agent’s true s.c.f. is a member of this family for some value of \( \theta \).

There are three basic questions about \( \{\rho_\theta \} \):

1. **Characterization.** Here we are interested in the image of the mapping \( \theta \mapsto \rho_\theta \), i.e., the set \( \{\rho \in \text{s.c.f} : \rho = \rho_\theta \text{ for some } \theta \in \Theta\} \). What kinds of distributions over data does the model allow? What things are ruled out? This is where axiomatic decision theory shines: We already saw how Sen’s \( \alpha \) axiom characterizes deterministic choice and we will see other axiomatic characterizations throughout this book.

2. **Identification.** Are the parameters pinned down uniquely? If uniqueness fails, then there are some \( \theta, \theta' \in \Theta \) such that \( \rho_\theta = \rho_{\theta'} \); in that case we can’t back out the parameter even if we had infinitely many observations. In econometrics the one-to-one property of \( \rho \) is called point-identification. If this fails, we have partial identification and the exercise is to understand the sets of \( \theta \) that lead to the same \( \rho \) (Manski, 2003).

3. **Comparative statics.** How does \( \rho_\theta \) change as \( \theta \) changes? Answering this question helps us understand what the parameter intuitively means. For example, under expected utility, the curvature of the Bernoulli utility function controls the degree of risk aversion.

There are a number of things the analyst can do with her model given a data set.

4. **Testing the Axioms.** Given an axiomatic characterization and a finite amount of data, how much confidence do we have in the fact that our axioms are satisfied? There is some classic literature on this topic and recently, there has been a renewed interest in this question.

5. **Statistical Inference.** Suppose that we have a finite data set. How do we estimate and quantify uncertainty about the parameter \( \theta \)?

6. **Counterfactual Prediction.** Suppose that you observe choices from menus \( A^1, \ldots, A^k \) and you want to predict choices from a new menu \( B \). It is important to realize that without a model that ties your hands

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17These are infinitely dimensional “parameters.” In most applications finite-dimensional parameterizations are used.
in some way, any probability distribution over $B$ is a legitimate prediction. It is only the model that will help us make a connection between behavior across menus. You can trust your prediction only as much as you trust your model (and the data). A frequentist econometrician would estimate the parameter $\hat{\theta}$ on $A^1, \ldots, A^l$ and plug it in to compute $\rho_\theta(x, B)$. A Bayesian econometrician would look at $\int \rho_\theta(x, B) \hat{\mu}(d\theta)$, where $\hat{\mu}$ is her posterior belief given the data. If the model is not identified, there will be many parameters that match the data, so our predictions will be set-valued (Manski, 2007). On the other hand, if our model is misspecified (i.e., the true $\rho$ does not correspond to $\rho_\theta$ for any value of $\theta$), then our prediction may be systematically wrong. We will see this in the blue-bus red bus example (Example 3.9).

We won’t talk much about (4), (5), and (6) in this book. These are not trivial topics, but I don’t know much about them.

To make (1) and (2) concrete, under random utility, $\Theta_{RU}$ is the set of probability measures $\mu$ on $\mathcal{P}$. In the additive random utility model $\Theta_{ARU}$ is the Cartesian product of the collection of all deterministic utility functions $v$ and the collection of all distributions over $\epsilon$.

### 2.2. Main Axiom: Regularity

In decision theory we like to characterize a given class of $\rho$ by a list of axioms, which are conditions expressed directly in terms of $\rho$, i.e., in the language of observables (agent’s choices) and avoid referring to any mathematical representation (e.g., there exists a utility function). This formulation makes them directly testable. Because axioms boil down the content of any model to the same language of observables, they can also help us see connections between different models. For example, sometimes they may share an axiom even though their functional forms look very differently.

In Section 1.2 we saw that a deterministic choice function is rationalized by a strict preference if and only if it satisfies Sen’s $\alpha$ axiom: for any $x \in A \subseteq B$ if $x = \chi(B)$, then $x = \chi(A)$. The following is a stochastic analogue of Sen’s $\alpha$.

**Axiom 2.1** (Regularity). If $x \in A \subseteq B$, then $\rho(x, A) \geq \rho(x, B)$.

This condition is sometimes also called “monotonicity”. In the case where $\rho$ is deterministic, it boils down exactly to Sen’s $\alpha$ (why?). Otherwise, it means that when we add new items to menu $A$, the choice probability of existing items has to go down to “make room” for new items.\(^{18}\)

\(^{18}\)Christensen and Connault (2019) restrict attention to local perturbations around a parametric family. Tebaldi, Torgovitsky, and Yang (2019) work with a discretized problem and compute the sharp bounds by solving finite-dimensional linear programs.

\(^{19}\)We need a pair of nested sets in our domain for regularity to have bite. For example, we cannot check it when we only observe binary menus. Axioms often lose bite when applied to smaller domains, see De Clippel and Rozen (2021).
Proposition 2.2. (Block and Marschak, 1960) If $\rho$ has a random utility representation, then it satisfies Regularity.

Proof. First note that if $x$ maximizes $U$ on $B$, then $x$ maximizes $U$ on $A$ (because $A$ is smaller). Thus, for any $y \in A \subseteq B$ we have $N(y, A) \supseteq N(y, B)$. Now, since all probability measures are set-monotone (they attach a bigger probability to bigger sets), we must have $\mu(N(y, A)) \geq \mu(N(y, B))$. □

By analogy to the deterministic case, one would hope that this axiom would be enough to rationalize any $\rho$ by random utility. This is true if $X$ has at most three elements, but not more generally.

Proposition 2.3 (Block and Marschak, 1960). Suppose that $|X| = 3$. A s.c.f. $\rho$ satisfies regularity if and only if $\rho \sim RU$.

Proof. Appendix A.2.1 □

To intuitively see why in general Regularity is not enough, recall that $\mathcal{P}$ is the set of strict preferences over $X$. Note that for each menu $A$ the sets $N(x, A)$ form a partition of $\mathcal{P}$ as $x$ ranges over $A$ (why?). The s.c.f. $\rho$ defines a probability distribution over the cells of this partition. We have as many partitions as there are menus in $\mathcal{A}$. Our axioms on $\rho$ need to ensure that all these probability distributions are consistent with each other and with a single $\mu$. This is a lot to ask.

There are several well-known violations of Regularity. Each of them is a “paradox” from the point of view of RU: a compelling behavior that cannot be accounted for. Many models have been written to rationalize such paradoxes and we will talk about some of them in future chapters. Regardless of how compelling you find those experiments, they are good illustrations of just exactly what the regularity axiom means. This also shows the power of the axiomatic method: it suffices to show that a very simple condition is violated instead of trying each possible distribution over utilities and establishing that none of them rationalizes the data.

Example 2.4 (Choice overload). Iyengar and Lepper (2000) set up tasting booths in two supermarkets. Customers could taste any number of jams and were able to eventually buy any variety of jam they wanted. In the first supermarket there were 6 varieties of jam in the tasting booth and 30% of the customers purchased some variety. In the second supermarket there were 24 varieties of jam (a superset of those 6) and only 3% of the customers made a purchase. Thus, the probability of choosing the outside option of not buying anything went up as the menu expanded. ▲

Example 2.5 (Classical Menu Effects). Huber, Payne, and Puto (1982) showed that adding a “decoy” option raises demand for the targeted option. See panel (a) of Figure 2.1, where adding to the menu $\{x, y\}$ any point that is dominated
by $y$ but not by $x$ will increase the choice probability of $y$. Intuitively, adding a decoy makes $y$ shine more by comparison so the agent is more likely to choose it over $x$. This is known as the \textit{decoy effect} or \textit{asymmetric dominance effect} and is extensively studied in the marketing literature, including situations in which it fails and regularity holds (Huber, Payne, and Puto, 2014). The \textit{compromise effect} is similar to the decoy effect, except like in panel (b) of Figure 2.1 we are now adding an option $z$ that makes $y$ a compromise between $x$ and $z$ (Simonson, 1989).

When there are four or more elements, Regularity is not enough to pin down RU and we need to add other axioms. We will talk about them in the next section, which is optional given that those axioms are more technical.

### 2.3. More Axioms*

The axioms that follow are admittedly complicated. Things get much simpler in the case where $X$ are lotteries because the shape of EU preferences restricts the $N(x, A)$ sets (Gul and Pesendorfer, 2006a). Another recent model with simple axioms is when $X$ is an ordered set and all preferences satisfy single-crossing (Apesteguia, Ballester, and Lu, 2017). Much of the earlier work was confined to the Luce/logit model, which is characterized by a simple yet restrictive axiom; we discuss this in Chapter 3.

Let’s get our hands dirty only a little bit at first. When $|X| = 4$, the additional axiom is still relatively simple.

**Axiom 2.6** (Supermodularity). If $x \in A \cap B$, then

$$\rho(x, A) + \rho(x, B) \leq \rho(x, A \cup B) + \rho(x, A \cap B).$$

Supermodularity means that the additional impact on the choice probability $x$ of adding items to the menu is decreasing (in terms of its absolute value) in the size of the menu. To see that, let $E := A \setminus B$, $F := A \cap B$, and $G := B \setminus A$.
and notice that the condition is equivalent to \( \rho(x, E \cup F) - \rho(x, E \cup F \cup G) \leq \rho(x, F) - \rho(x, F \cup G) \).

**Proposition 2.7** (Block and Marschak). Suppose that \( |X| = 4 \). A s.c.f. \( \rho \) satisfies regularity and supermodularity if and only if \( \rho \sim RU \).

The proof proceeds more or less along the same lines as the proof of Proposition 2.3, see Theorem 5.3 of Block and Marschak (1960): we want to ensure that the probabilities defined by \( \rho \) on the partitions generated by the \( N \)-sets extend to a well-defined \( \mu \). With more than 3 elements, the combinatorial structure of the \( N \)-sets is less nice. In particular, we never get to isolate the probability of a single element in \( P \). This is why we need the additional axiom.

The combinatorial structure of the \( N \)-sets gets complicated as the cardinality of \( X \) grows. Each time we add an element, we need to add another axiom. If we fix \( x \) and vary \( A \), then \( \rho(x, \cdot) \) defines a function on the collection of all events that contain \( x \). This collection is partially ordered. Regularity says that this function is “decreasing.” Supermodularity says that it is “convex.” Block and Marschak (1960) proposed an axiom that signs all the “derivatives” of \( \rho \).

**Axiom 2.8** (Block and Marschak). For all \( x \in A \)

\[
q(x, A) := \sum_{B \supseteq A} (-1)^{|B \setminus A|} \rho(x, B) \geq 0.
\]

Some intuition for the BM axiom can be gained from the proof of necessity. Under RU, \( q(x, A) \) turns out to be the probability of the event that \( x \) is best in \( A \) but everything outside of \( A \) is better than \( x \) (this follows from the inclusion-exclusion formula from combinatorics). As such, it’s probability is nonnegative.

**Theorem 2.9** (Block and Marschak). If \( \rho \sim RU \), then it satisfies the BM axiom

**Proof.** See Appendix A.2.2. \( \square \)

The proof tells us that we can write

\[
\rho(x, A) = \sum_{B \supseteq A} q(x, B).
\]

Regularity says that we always add probability to \( x \) when we drop another element from the menu. The BM axiom captures this idea while taking into account the partial order structure of set inclusion. In particular, we have \( q(x, X) = \rho(x, X) \): this is how much choice probability is being added to \( x \) in choice set \( X \). Now, when we move down to \( X \setminus \{a\}, q(x, X \setminus \{a\}) = p(x, X \setminus \{a\}) - q(x, X \setminus \{a\}) - q(x, X \setminus \{b\}). In other words, this is how much probability is added to the choice of \( x \) as we move to set \( X \setminus \{a\} \). Taking it one step further, \( q(x, X \setminus \{a, b\}) = p(x, X \setminus \{a, b\}) - q(x, X \setminus \{a\}) - q(x, X \setminus \{b\}) - q(x, X \setminus \{a, b\}) \). The choice sets \( X, X \setminus \{a\}, \text{ and } X \setminus \{b\} \) have each added \( q(x, X), q(x, X \setminus \{a\}), \text{ and } q(x, X \setminus \{b\}) \) to the choice probability of \( x \) respectively. So \( q(x, X \setminus \{a, b\}) \)
is exactly how much choice probability the choice set $X \setminus \{a, b\}$ has to add to the probability of $x$ to achieve $p(x, X \setminus \{a, b\})$. I thank Christopher Turansick for sharing this intuition with me.
There are two other axiomatizations of RU. I like them even less, as they explicitly refer to the representation. In both of them, as the first step you need to basically compute all the rationalizable deterministic choice functions. The advantage of those axioms is that the BM approach relies having \( \rho_{\text{observed}} \) for all menus. The advantage of the other two axioms is that they work on incomplete domains.

The first axiom was developed by McFadden and Richter (1971, 1990). Take the collection of item-menu pairs \( \{ (x, A) : x \in X \text{ and } A \in A \} \); let \( n \) be its size. For each preference \( \succ \in \mathcal{P} \) let’s form a vector of dimension \( n \) equal one if \( \succ \) chooses \( x \) from \( A \) and zero otherwise; let’s denote this vector by \( p_{\succ} \). Now it remains to check whether \( \rho \) is in the convex hull of those vectors (which is equivalent to RU).

Consider any set of points \( Q \) in \( \mathbb{R}^n \) and another point \( r \in \mathbb{R}^n \). It is easy to verify that \( r \) is in the convex hull of \( Q \) if and only if

\[
\langle r, \lambda \rangle \leq \max_{p \in Q} \langle p, \lambda \rangle
\]

for all \( \lambda \in \mathbb{R}^n \). Now we just need to apply this to \( Q = (p_{\succ})_{\succ \in \mathcal{P}} \) and \( r = \rho \). Actually, it turns out that it suffices to only check \( \lambda \in \mathbb{N}^n \), which directly translates to the following axiom.

**Axiom 2.10 (Axiom of Revealed Stochastic Preference (ARSP)).** For any \( k \) and for any sequence \( (x_1, A_1), \ldots, (x_k, A_k) \) such that \( x_i \in A_i \)

\[
\sum_{i=1}^{k} \rho(x_i, A_i) \leq \max_{\succ \in \mathcal{P}} \sum_{i=1}^{k} p_{\succ}(x_i, A_i).
\]

The second axiom was developed by Clark (1996). This is kind of the flip of the previous exercise. For each \( x \in A \) let’s now form a long vector with dimension equal to the set of preferences \( P \). Define \( p(x, A) \) to be the indicator function of the event \( N(x_i, A_i) \).

**Axiom 2.11 (Coherency).** For any \( k \) and any sequence \( (x_1, A_1), \ldots, (x_k, A_k) \) such that \( x_i \in A_i \), and for any sequence of real numbers \( \lambda_1, \ldots, \lambda_k \)

\[
\sum_{i=1}^{k} \lambda_i p(x_i, A_i) \geq 0 \implies \sum_{i=1}^{k} \lambda_i \rho(x_i, A_i) \geq 0.
\]

It’s easy to see that Coherency implies the BM axiom, by taking \( \lambda_i \in \{-1, 0, 1\} \). The intuition behind this axiom is a no-arbitrage argument. Imagine that \( \rho(x_i, A_i) \) is the cost of placing a bet on the event \( N(x_i, A_i) \). Suppose that we now have a complex bet on a combination of events that pays off a positive amount in every state of the world. This complex bet must cost a positive amount of money. This axiom is a restatement of de Finetti’s coherency condition, which ensures that a set function (here defined over all \( N \)-sets) can be extended to a probability measure (over all events), see, e.g. Pollard (2002).
Theorem 2.12 (Characterization of RU). The following conditions are equivalent for a s.c.f. $\rho$ on a finite set $X$:

(i) $\rho \sim RU$
(ii) $\rho$ satisfies the BM axiom
(iii) $\rho$ satisfies coherency
(iv) $\rho$ satisfies ARSP.

Moreover, (iii) and (iv) are equivalent to (i) even if the domain of $\rho$ is arbitrary.

Block and Marschak (1960) showed that their axiom is necessary. Sufficiency was proved by Falmagne (1978) and independently by Barberá and Pattanaik (1986). Other proofs of this theorem can be found in Fiorini (2004) using network flows, Monderer (1992) using cooperative game theory, and Chambers and Echenique (2016) using the Farkas lemma. A version of (i)–(ii) for infinite $X$ was discussed by (Cohen, 1980). The equivalence (i)–(iii) was proved Clark (1996), directly for infinite $X$. The equivalence (i)–(iv) was proved by McFadden and Richter (1990, 1971). A nice and simple proof of this equivalence is in Stoye (2019) and the infinite case was worked out by McFadden (2005) and Gonczarowski, Kominers, and Shorrer (2020).

2.4. Uniqueness/Identification

Even though RU and ARU coincide as classes of s.c.f.s, they differ as statistical models because they are different mappings with different domains. As a result of this, they have different identification properties.

2.4.1. Identification under RU. Since utility is unique only ordinally, we cannot hope to identify its distribution; at best, we can hope to pin down the distribution of ordinal preferences. This is possible when $X$ has only three alternatives.

Proposition 2.13. (Block and Marschak, 1960) If $|X| \leq 3$, then if $\mu$ is a distribution over preferences that represents $\rho$, then $\mu$ is unique.

Proof. See Appendix A.2.3.

Unfortunately, with more elements RU is not point-identified. The following example shows two distributions that induce the same s.c.f.

Example 2.14. (Fishburn, 1998) Suppose that $X = \{x, y, z, w\}$ and $\mu_1$ and $\mu_2$ are given as in Figure 2.2. Those two probability distributions lead to the same s.c.f. $\rho$ (why?). ▲

Not only are $\mu_1$ and $\mu_2$ different, but their supports are disjoint. The difficulty here is that although we can determine the probability that $x$ is
better than \( y \) and the probability that \( w \) is better than \( z \), we can’t determine the probability of those two events occurring at the same time.\(^{20}\)

Once we add lotteries, we will see that we can read those probabilities directly off of \( \rho \). Although the amount of correlation between different rankings is not pinned down; however, the marginal distributions over the rankings is. In other words, in RU the distribution of preferences is unique up to correlations.

**Theorem 2.15** (Falmagne 1978). If \( \mu_1 \) and \( \mu_2 \) are two distributions over preferences that represent the same \( \rho \), then for any \( x \in X \)

\[
\mu_1(x \text{ is } k\text{-th best in } X) = \mu_2(x \text{ is } k\text{-th best in } X)
\]

for all \( k = 1, \ldots, |X| \).

**Proof.** Appendix A.2.4. \( \square \)

2.4.2. Identification under ARU. The parametrization of discrete choice models involves a deterministic utility function \( v \) and a distribution over \( \tilde{\epsilon} \). It turns out that without additional assumptions, the utility is not identified at all (even ordinally). This is perhaps not very surprising because we can just absorb any \( v \) into \( \tilde{\epsilon} \) by defining

\[
\tilde{\epsilon}_2(x) := v_1(x) - v_2(x) + \tilde{\epsilon}_1(x).
\]

Remarkably, \( v \) is completely nonidentified even if we restrict the mean of \( \epsilon \) to be a zero vector. This was shown in a binary model by Manski (1988) and in general by Koning and Ridder (2003).\(^{21}\)

**Proposition 2.16.** Suppose that \( X \) is finite. If \( \rho \sim \text{ARU}(v_1, \tilde{\epsilon}_1) \), then for any \( v_2 \in \mathbb{R}^X \) there exists \( \tilde{\epsilon}_2 \) with \( \mathbb{E}\tilde{\epsilon}_2 = 0 \) such that \( \rho \sim \text{ARU}(v_2, \tilde{\epsilon}_2) \).

Moreover, for a fixed \( v \) the correlation structure of \( \tilde{\epsilon} \) is not pinned down. For example we can take i.i.d. \( \tilde{\epsilon} \) and shift it by a common constant \( \tilde{\eta} \) to make it as correlated as we want.

\(^{20}\)McClellon (2015) shows that non-uniqueness occurs in all RU that have a full support distribution. Turansick (2021) characterizes exactly when uniqueness occurs. Moreover, he shows that uniqueness of the distribution occurs if and only if its support is unique.

\(^{21}\)Their Theorem 1 does not state the restriction \( \mathbb{E}\tilde{\epsilon}_2 \) explicitly but their proof actually delivers it.
Proposition 2.17. If $\rho \sim ARU(v, \tilde{\epsilon})$, then for any random variable $\tilde{\eta} : \Omega \to \mathbb{R}$ we have $\rho \sim ARU(v, \tilde{\epsilon}')$ where $\tilde{\epsilon}'(x) = \tilde{\epsilon}(x) + \tilde{\eta}(x)$.

Proof. Appendix A.2.5.

The identification result is much improved within the i.i.d. class. I discuss this in the next chapter (Proposition 3.8). Another way to vastly improve identification is to add product characteristics (Section 10.3).
Chapter 3

More Models and Properties

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3.1. Luce and Logit

3.1.1. Logit. Logit is a special case of ARU where \( \eps(x) \) are i.i.d. extreme-value distributed (Definition 1.15). It leads to a very simple specification of choice probabilities.

\[
\rho(x, A) = \frac{e^{v(x)}}{\sum_{y \in A} e^{v(y)}}. \tag{3.1}
\]

3.1.2. Luce. A related model was introduced by Luce (1959). Here \( \mathbb{R}^{++} \) is the set of positive real numbers.

**Definition 3.1.** \( \rho \) has a Luce representation if there exists a function \( w : X \to \mathbb{R}^{++} \) such that for all \( A \in \mathcal{A} \)

\[
\rho(x, A) = \frac{w(x)}{\sum_{z \in A} w(z)}. \tag{3.2}
\]

Here, \( w(x) \) is the perceived strength of the stimulus \( x \) and choice probabilities are proportional to those perceptions.

Notice that the function \( w \) is unique up to multiplication by positive numbers. Likewise, the function \( v \) in the logit representation is unique up to adding a constant.

Mathematically, the probability distribution on \( A \) is the conditional of the probability distribution on the grand set \( X \). This makes it possible to characterize the model by very a simple axiom.

**Axiom 3.2** (Luce’s IIA). For all \( x, y \in A \cap B \) whenever the probabilities are positive

\[
\frac{\rho(x, A)}{\rho(y, A)} = \frac{\rho(x, B)}{\rho(y, B)}. \]

This axiom says that the ratio of the choice probabilities of \( x \) and \( y \) does not depend on what other elements are in the menu.\(^{22}\) Hausman and McFadden (1984) propose a statistical test of the axiom. (Estimate the model twice: on a full set of alternatives and on a subset. If IIA holds, the two estimates should not be statistically different.)

An equivalent version of this axiom is the following product rule.\(^{23}\) We will write \( \rho(B, A) := \sum_{x \in B} \rho(x, A) \) whenever \( B \subseteq A \). This is the probability that something from \( B \) gets chosen from the menu \( A \).

**Axiom 3.3** (Luce’s Choice Axiom). For all \( x \in A \subseteq B \)

\[
\rho(x, B) = \rho(x, A) \rho(A, B). \]

**Theorem 3.4.** The following are equivalent for any set \( X \):

\(^{22}\)This axiom is often written equivalently by imposing that \( B = \{x, y\} \).

\(^{23}\)These two axioms are equivalent even without positivity (Cerreia-Vioglio, Maccheroni, Marinacci, and Rustichini, 2017). Other models that obtain a Luce-like representation without positivity are Echenique and Saito (2019), Ahumada and Ülkü (2018), and Horan (2021).
3.1. Luce and Logit

(i) \( \rho \) satisfies Luce’s IIA and positivity

(ii) \( \rho \) satisfies Luce’s Choice Axiom and positivity

(iii) \( \rho \) has a Luce representation \( w \)

(iv) \( \rho \) has a logit representation \( v(x) = \log(w(x)) \).

Many decision theory models draw on the Luce-logit model.\(^{24}\) One cost of its tractability is that Luce-logit makes very strong assumptions about the substitution patterns between products. We will discuss the famous red bus blue bus problem in Section 3.2.3. Later we will discuss models that allow for more flexible patterns.

3.1.3. The Noise Parameter. Sometimes, we want to scale the variance of the extreme-value shocks \( \tilde{\epsilon} \). In logit with noise parameter \( \lambda \) we set \( \tilde{U}(x) := v(x) + \lambda \tilde{\epsilon}_x \), where \( \tilde{\epsilon}_x \) are i.i.d. TIEV. This leads to the choice probabilities:

\[
\rho(x, A) = \frac{e^{v(x)/\lambda}}{\sum_{y \in A} e^{v(y)/\lambda}}.
\]

If we let \( \lambda \to 0 \), then \( \rho \) converges to a deterministic choice function that is the argmax of \( v \) (ties are broken uniformly). On the other hand, as \( \lambda \to \infty \) choice becomes uniform. In the literature on neural networks equation (3.1) is known as the softmax.

3.1.4. Social Surplus. Logit has a nice closed-form expression for social surplus (Section 1.8). This is the so-called “log-sum” expression.\(^{25}\)

**Proposition 3.5.** In logit with noise parameter \( \lambda \) the social surplus equals

\[
V_\lambda(A) = \lambda \log \left( \sum_{x \in A} \exp(v(x)/\lambda) \right).
\]

Note here that as we increase the number of goods, the ex ante consumer welfare grows without bound.

3.1.5. Another form of IIA*. Luce’s IIA has a cardinal feel to it (we require products or ratios of probabilities to be equal to each other). Gul, Natenzon, and Pesendorfer (2014) consider an ordinal axiom that says that the ranking of probabilities does not change when adding/taking away other alternatives. This axiom relies only on ordinal information, the exact magnitudes of choice probabilities do not matter.

**Axiom 3.6** (GNP’s IIA). If \( A \cup B \) and \( C \cup D \) are disjoint, then

\[
\rho(A, A \cup C) \geq \rho(B, B \cup C) \implies \rho(A, A \cup D) \geq \rho(B, B \cup D).
\]

\(^{24}\)To mention a few, the random consideration model of Manzini and Mariotti (2014), the attribute rule model of Gul, Natenzon, and Pesendorfer (2014), and the additive perturbed utility model of Fudenberg, Iijima, and Strzalecki (2015).

\(^{25}\)For a history of this expression, see De Jong, Daly, Pieters, and Van der Hoorn (2007). A generalization of this formula to GEV is \( V(X) = \log H(e^{v_1}, \ldots, e^{v_n}) \).
The dagger symbol † means I am not stating all technical details. A serious reader will want to consult the original paper. The richness condition implies that \( X \) is infinite.

**Theorem†3.7.** *(Gul, Natzenzon, and Pesendorfer, 2014)* In the presence of a richness condition, \( \rho \) satisfies GNP’s IIA iff \( \rho \sim \text{Luce} \).

### 3.2. i.i.d. ARU

#### 3.2.1. The Model

Under logit, \( \tilde{\epsilon}(x) \) are i.i.d. across \( x \in X \), distributed TIEV. Thurstone’s model from Example 1.17 had i.i.d. Normal distributions. We will now consider the class of all distributions.

The question of axiomatizing the general class of i.i.d. ARU model remains, as far as I understand, open.

Recall that \( \rho(x,y) \) denotes the probability that \( x \) is chosen over \( y \), i.e., \( \rho(x,\{x,y\}) \). If \( \tilde{\epsilon}(x) \) are i.i.d. then we have

\[
\rho(x,y) = \mathbb{P}(v(x) + \tilde{\epsilon}(x) \geq v(y) + \tilde{\epsilon}_y) = \mathbb{P}(\tilde{\epsilon}_y - \tilde{\epsilon}(x) \leq v(x) - v(y)) = F(v(x) - v(y)), \tag{3.5}
\]

where \( F \) is the cdf of of the symmetric random variable \( \tilde{\eta} := \tilde{\epsilon}(y) - \tilde{\epsilon}(x) \). Models with such a representation for pairwise choices are called *Fechnerian*; we will discuss them in the next section.

#### 3.2.2. Identification

The i.i.d. ARU has much cleaner identification properties than ARU.

**Proposition 3.8.** Suppose that \( \rho \) has two i.i.d ARU representations with positive density: \((v_1, \tilde{\epsilon}_1)\) and \((v_2, \tilde{\epsilon}_2)\). Let \( F_i \) be the cdf of \( \tilde{\epsilon}_i(x) - \tilde{\epsilon}_i(y) \). If the range of \( v_1 \) is an interval, then there exist \( \alpha > 0, \beta \) such that \( v_2 = \alpha v_1 + \beta \) and \( F_2(t) = F_1(\alpha^{-1}t) \) for all \( t \in \{v_2(x) - v_2(y) : x, y \in X\} \).

**Proof.** Appendix A.3.2, which also has an extended discussion of other uniqueness properties.

The proposition says that if there is enough variation in \( v \), then \( v \) is unique up to a positive affine transformation and the distribution of \( \epsilon \) differences is unique up to a multiplicative factor. Notice that such sharp identification occurs only within the i.i.d. ARU class. As ARU, these models are not identified by Proposition 2.16. As RU, they are also not identified, see footnote 20.

Can knowing \( F \) pin down the distribution of \( \tilde{\epsilon} \) within the i.i.d. ARU class? Of course, the mean of \( \tilde{\epsilon} \) cancels out and does not affect the choice probabilities, so we won’t be able to pin it down. How about pinning down the distribution modulo the mean? This is true within some special classes of distributions,
such as normal and Poisson distributions, but not in general.\textsuperscript{26} I do not know what are the exact uniqueness properties of the distribution of $\tilde{c}$.

### 3.2.3. Blue Bus-Red Bus

The following paradox was originally conceived by Debreu (1960) as a critique of Luce’s IIA. It actually applies to all i.i.d. ARU models. McFadden adapted his example to transportation choices.

**Example 3.9.** People can commute by a train or a bus. There are two kinds of buses: a blue bus and a red bus. So $X = \{t, bb, rb\}$. Suppose that we observed that $\rho(t, bb) = \rho(t, rb) = \rho(bb, rb) = \frac{1}{2}$. If $\rho \sim \text{i.i.d. ARU}$, then $F$ is a cdf of a symmetric random variable, so we infer that $v(t) = v(bb) = v(rb)$ and predict that $\rho(t, X) = \frac{1}{3}$. But this doesn’t make much sense if you think that the main choice is between the modes of transportation (train or bus) and the bus color is just an icing on the cake. In that case we would like to have $\rho(t, X) = \frac{1}{2}$. (If you are still not convinced, imagine that there $n$ colors of buses. Would you insist on $\rho(t, X) \to 0$ as $n \to \infty$? If people behaved this way, a firm could capture the whole market by introducing a bunch of new products that are virtually identical to each other.) ▲

This is a paradox for i.i.d. ARU because in that model all three modes of transportation must be indifferent in terms of $v$. But really they are not indifferent, and the i.i.d. model just doesn’t have enough degrees of freedom to capture this. More generally, i.i.d. ARU with positive density satisfies the following axiom.

**Axiom 3.10** (Interchangeability). If $\rho(x, y) = 0.5$, then $\rho(x, z) = \rho(y, z)$ for all $z \neq x, y$.

This is precisely what goes wrong in Example 3.9. From the fact that $\rho(t, bb) = 0.5$ we simply cannot infer that those two are interchangeable. They could be very different objects that have different substitution patterns with a third object $rb$. i.i.d. ARU models (and Fechnerian models more generally) squash all of this onto a one-dimensional scale. This might make sense for perception of weight and other physical stimuli, but it is too simplistic to capture economic demand.

Note that this is a paradox only for i.i.d. ARU, not for RU in general. Here is a very simple RU that explains the paradox perfectly well.

\textsuperscript{26}This is the subject of decomposition theory, see Loève (1978). Feller (1957) gives an example of two symmetric distributions that lead to the same $F$, yet differ by more than just the scaling of $v$ (this is number iii of his Curiosities on page 506). I thank Jetlir Duraj for pointing me to this example. These are not full support distributions but perhaps such counterexamples also exist.
Example 3.11. Let $X = \{bb, rb, t\}$. Let $\mu$ assign weight $\frac{1}{4}$ to each of the following four orderings:

\[
\begin{align*}
t &\succ bb \succ rb \\
t &\succ rb \succ bb \\
bb &\succ rb \succ t \\
rb &\succ bb \succ t
\end{align*}
\]

Then each pairwise choice is fifty-fifty, but $\rho(bb, X) = \rho(rb, X) = \frac{1}{4}$ and $\rho(t, X) = \frac{1}{2}$.

Example 3.11 implicitly introduces correlation between $\epsilon$. To see that explicitly, replace each of the deterministic orderings in the above example with a logit. But in fact, as noticed by Christopher Turansick, these choice probabilities can be explained by RU with independent but not identical distribution of $\epsilon$. To see that, let $v = 0$ and $\epsilon_t$ equal $+10$ or $-10$ with equal probabilities, $\epsilon_{bb}$ equal $+1$ or $-1$ with equal probability and $\epsilon_{rb} = 0$. ▲

This illustrates a general phenomenon: we have one model (here RU) that has limited identifying power, but is large enough so that $\rho = \rho_\theta$ for some $\theta$. The other model (i.i.d. ARU) has much better identification but is misspecified $\rho \neq \rho_\theta$ for all $\theta$, so we should not be trusting our estimates. In general, we want our models specific enough to get good identification, but large enough to allow them to capture the true $\rho$.

3.3. Fechnerian Models

Fechnerian models come from the psychometric tradition, where choice randomness is often interpreted as decision error or discrimination error (see Examples 1.5 and 1.17). Whether $\rho$ is Fechnerian or not depends only on its restriction to binary menus.

**Definition 3.12.** We say that $\rho$ has a Fechnerian representation if there exist real functions $v : X \to \mathbb{R}$ and $F$ such that

$$\rho(x, y) = F(v(x) - v(y)).$$

Moreover, $F$ is strictly increasing on its domain, $D := \{v(x) - v(y) : x, y \in X\}$, and symmetric, i.e., $F(-k) = 1 - F(k)$ for all $k \in D$.\textsuperscript{27}

In a perception task $x$ is the true stimulus strength and $v(x)$ is the perceived stimulus strength. An appropriate choice of $F$ leads to S-shaped psychometric functions.

Equation (3.5) shows that any i.i.d. ARU model with positive density is Fechnerian because we can take $F$ to be the c.d.f. of the $\epsilon$ difference. The converse is true for finite $X$ but not true in general because not every $F$ is

---

\textsuperscript{27}In the early literature this model was sometimes called the strong utility model and its special case Luce is called the strict utility model.
decomposable as a difference of two i.i.d. random variables. In general, RU is not Fechnerian (Marschak, 1959). I do not know if the converse is true.

Davidson and Marschak (1959) showed that following axiom is necessary for Fechnerian representations.

**Axiom 3.13 (Quadruple Condition).**

\[
\rho(x, y) \geq \rho(w, z) \text{ if and only if } \rho(x, w) \geq \rho(y, z)
\]

Debreu (1958) proved that this axiom is also sufficient if a richness condition is assumed. This condition implies that \( X \) is infinite.

**Axiom 3.14 (Richness).** If \( \rho(x, y) \leq \alpha \leq \rho(z, y) \), then there exists \( w \in X \) such that \( \rho(w, y) = \alpha \).

This theorem uses our assumption that \( \rho(x, x) = \frac{1}{2} \) for all \( x \).

**Theorem 3.15 (Debreu 1958).** Suppose that \( \rho \) satisfies Richness. It has a continuous Fechnerian representation if and only if it satisfies the quadruple condition. Moreover, if \((v_1, F_1)\) and \((v_2, F_2)\) both represent \( \rho \), then there exists \( \alpha > 0 \) and \( \beta \in \mathbb{R} \) such that:

\[
v_2(x) = \alpha v_1(x) + \beta \text{ for all } x \in X,
\]

\[
F_2(\alpha t) = F_1(t) \text{ for all } t \in D_1,
\]

where \( D_1 = \{v_1(x) - v_1(y) : x, y, \in X\} \).

**Proof.** In Appendix A.3.3.

A related model is sometimes used in experimental economics (Harless and Camerer, 1994). Here, the agent makes a mistake with a fixed probability \( p \), regardless of how serious this mistake is.

**Example 3.16 (The Constant-Error Model).** Here \( \rho \) is given by the same formula as in Definition 3.12 with \( F \) given by

\[
F(t) = \begin{cases} 
  p & \text{if } t > 0 \\
  0.5 & \text{if } t = 0 \\
  1 - p & \text{if } t < 0,
\end{cases}
\]

where \( p \in (0, 0.5) \). This is not Fechnerian because the function \( F \) is not strictly increasing. The probability of making an error does not depend on how big the utility difference is.

---

28 For example, the uniform distribution is not decomposable, see Example A.3.1 in the Appendix.
29 This condition is sometimes called solvability or stochastic continuity.
3.4. Stochastic Transitivity*

We say that $x$ is stochastically preferred to $y$, denoted $x \succsim^* y$, if $x$ is more frequently chosen than $y$ in pairwise choice, i.e., $\rho(x, y) \geq 0.5$. Suppose that $x \succsim^* y$ and $y \succsim^* z$. What can we conclude about the frequency of choices between $x$ and $z$? Let $p := \rho(x, y) \geq 0.5$, $q := \rho(y, z) \geq 0.5$, and $r = \rho(x, z)$. We have:

- **weak stochastic transitivity** (WST) if $r \geq 0.5$
- **moderate stochastic transitivity** (MST) if $r \geq \min\{p, q\}$
- **strong stochastic transitivity** (SST) if $r \geq \max\{p, q\}$.

WST is simply transitivity of the relation $\succsim^*$. In general, WST does not imply RU and RU can violate WST, as the following example shows.

**Example 3.17** (Condorcet Paradox). Let $X = \{x, y, z\}$. Let $\mu$ assign weight $\frac{1}{3}$ to each of the following three orderings:

- $x \succ y \succ z$
- $y \succ z \succ x$
- $z \succ x \succ y$

Then we have $\rho(x, y) = \rho(y, z) = \rho(z, x) = \frac{2}{3}$, a violation of WST. If we interpret $\mu$ as a population of agents and $\rho$ as recording their vote fractions in a pairwise election then we have what is called the Condorcet paradox. ▲

This may make you relatively uninterested in WST and even stronger properties. But prominent models satisfy them and they are a point of reference in the literature. Notice for instance that WST is satisfied in Fechnerian models because $\rho(x, y) \geq 0.5$ iff $v(x) \geq v(y)$. Fechnerian models not only satisfy WST, but also SST (why?). In fact, SST is satisfied by a wider class.

**Definition 3.18.** $\rho$ has a **simple scalability** representation if

$$\rho(x, y) = H(v(x), v(y))$$

for some function $v : X \rightarrow \mathbb{R}$ and appropriately defined $H$.\(^{30}\)

Of course, Fechnerian representations are a special case. Simply scalable $\rho$s are characterized by a slight strengthening of SST, denoted $\text{SST}^\dagger$, where the inequality has to hold strictly if any of the inequalities in the premise hold strictly.

**Theorem\(^\dagger\) 3.19** (Tversky and Russo 1969). $\rho$ satisfies $\text{SST}^\dagger$ if and only if it has a simple scalability representation.

---

\(^{30}\) $H$ is strictly increasing in the first argument and strictly decreasing in the second argument and symmetric $H(s, t) = 1 - H(t, s)$.\]
A characterization of a slightly stronger version of MST, called MST†, was recently obtained by He and Natenzon (2018). These are the models that are represented by *moderate utility*, where

\[
\rho(x, y) = F \left( \frac{v(x) - v(y)}{d(x, y)} \right)
\]

for some \( v : X \to \mathbb{R} \), distance metric \( d : X \times X \to \mathbb{R}_+ \), and \( F : \mathbb{R} \to [0, 1] \) strictly increasing transformation, defined on an appropriate domain, that satisfies \( F(t) = 1 - F(-t) \).

**Theorem†3.20** (He and Natenzon 2018). \( \rho \) satisfies MST† if and only if it has a Moderate Utility representation.

**Remark 3.21.** (Rieskamp, Busemeyer, and Mellers, 2006) discuss experimental evidence on SST and WST. From a theoretical point of view, we should expect stochastic transitivity in some applications, like in one-dimensional perception tasks. But in settings with richer substitution patterns, like the blue-bus red-bus example (Example 3.9), we should not be surprised to see violations. Likewise, if we know that \( \rho \) represents aggregate behavior, like in the Condorcet example (Example 3.17), we should not expect WST to hold. ▲

The final property is known as the triangle axiom.

**Axiom 3.22** (Triangle). For any three distinct \( x, y, z \in X \)

\[
\rho(x, y) + \rho(y, z) \geq \rho(x, z).
\]

Marschak (1959) thought that Triangle is necessary and sufficient for RU on binary menus but in fact it is only necessary, see Cohen and Falmagne (1990), Gilboa (1990), and Marley (1990). Sprumont (2020) shows that Triangle is sufficient for an extension to all menus that satisfies Regularity.

### 3.5. Perturbed Utility∗

**Definition 3.23.** \( \rho \) has a *perturbed utility* (PU) representation if for each \( A \) the probability \( \rho(\cdot, A) \) solves

\[
\max_{p \in \Delta(A)} \sum_{x \in A} v(x) p(x) - c_A(p),
\]

where \( v \in \mathbb{R}^X \) is a deterministic utility function and \( c_A : \Delta(A) \to (-\infty, \infty] \) is the cost of implementing the probability mixture \( p \).

One interpretation is that the agent implements their choices with an error (trembling hands). The error can be reduced at a cost that depends on the tremble probabilities. Another interpretation is hedging against ambiguity (see Fudenberg, Iijima, and Strzalecki, 2015).
3.5. Perturbed Utility*

Of course, in its full generality every \( \rho \) has a PU representation because we can set \( c_A \) to be equal to zero on \( \rho(\cdot, A) \) and \( \infty \) everywhere else.\(^{31}\) Typically used specifications involve Additive Perturbed Utility, which has significant bite as it satisfies regularity.

**Definition 3.24.** \( \rho \) has an *additive perturbed utility* (APU) representation if it has a perturbed utility representation where

\[
c_A(p) = \sum_{x \in A} \phi(p(x))
\]

for some \( \phi : [0, 1] \rightarrow (-\infty, \infty] \) that is strictly convex and \( C^1 \) over \((0, 1)\). Moreover, it is an APU with *steep cost* if \( \lim_{q \rightarrow 0} \phi'(q) = -\infty \).

The steep cost condition ensures Positivity. It is satisfied for example in Harsanyi (1973b), who uses logarithmic costs \( \phi(t) = -\log(t) \) and Fudenberg and Levine (1995), who use negative entropy costs: \( \phi(t) = t \log t \). The quadratic costs \( \phi(t) = t^2 \) used by Ben-Akiva and Lerman (1985); Rosenthal (1989) are not steep and choices can violate Positivity. The entropy costs are special, as they lead to Luce/logit choice probabilities.

**Theorem 3.25** (Rockafellar 1970; Anderson, de Palma, and Thisse 1992). The following are equivalent:

(i) \( \rho \) has a logit representation with utility \( v \)

(ii) \( \rho \) has an entropy APU representation with utility \( v \)

**Remark 3.26.** The equivalence between entropy maximization and the “exponential tilting” formula \( (e^v(x)/\sum_y e^v(y)) \) shows up in related problems. In Chapter 6 we will see this for mutual information. This also occurs in models of ambiguity with relative entropy (Hansen and Sargent, 2008).

Outside of the entropy case APU and ARU can be different. In general the solution to APU is characterized by the Kuhn–Tucker theorem in the form of the first-order condition associated with the maximization problem in Definition 3.23: \( v(x) + \lambda(A) = \phi'(\rho(x, A)) \), where \( \lambda(A) \) are the Lagrange multipliers, one for each menu \( A \). Imagine a binary relation on menu-item pairs induced by \( \rho \). The first-order condition means that \( \rho \) satisfies APU with steep costs if and only if this induced relation has an additive representation.

An additive representation is guaranteed by the following axiom.

**Axiom 3.27** (Acyclicity). For any \( n \) and bijections \( f, g : \{1, ..., n\} \rightarrow \{1, ..., n\} \), such that \( x_k \in A_k \), and \( x_{f(k)} \in A_{g(k)} \) for all \( k = 1, \ldots, n \)

\[
\rho(x_1, A_1) > \rho(x_{f(1)}, A_{g(1)})
\]

\[
\rho(x_k, A_k) \geq \rho(x_{f(k)}, A_{g(k)}) \quad \text{for } 1 < k < n
\]

\(^{31}\)The situation is different in the model with attributes, which will be discussed in Section 10.8.
One intuition for the axiom is as follows: suppose you have $n$ xylophones and $n$ Accordions. You have a scale that records the weight of an $(x, A)$ pair. The sum of these weights should not depend on how you pair up the instruments.

Acyclicity implies that $\rho(x, A) \geq \rho(y, A)$ iff $\rho(x, B) \geq \rho(y, B)$, so the agent’s choice probabilities do not reverse due to “menu effects” (recall Example 2.5). The ranking that $\rho$ induces on $X$ is represented by the utility function $v$. Additionally, acyclicity implies that $\rho(x, A) \geq \rho(x, B)$ iff $\rho(y, A) \geq \rho(y, B)$. The ranking that $\rho$ induces over (nested) menus can be interpreted as “competitiveness” of menus and is represented by the $\lambda$.

Another condition that characterizes strict APU is a weakening of Luce’s IIA (Axiom 3.2)

**Condition 3.28** (Ordinal IIA). For some continuous and monotone function $f : (0, 1) \to \mathbb{R}_+$

$$\frac{f(\rho(x, A))}{f(\rho(y, A))} = \frac{f(\rho(x, B))}{f(\rho(y, B))}$$

for each menu $A, B \in A$ and $x, y \in A \cap B$.

**Theorem 3.29** (Fudenberg, Iijima, and Strzalecki 2015). The following are equivalent under Positivity:

(i) $\rho$ has an APU representation with steep cost

(ii) $\rho$ satisfies Acyclicity

(iii) $\rho$ satisfies Ordinal IIA

Positivity can be dropped by weakening Acyclicity. Moreover, on binary menus and under Positivity and technical conditions, Acyclicity holds if and only if there is a Fechnerian representation (see Proposition 1 of Fudenberg, Iijima, and Strzalecki, 2014). The uniqueness properties are inherited from the Fecherian model. Because of the Fechnerian property, APU is not a good explanation of the blue bus-red bus paradox.

### 3.6. Mixed Logit

Mixed logit is an average of logits with different $v$ functions.

**Definition 3.30.** $\rho$ has a *mixed-logit* representation if there exists a probability measure $\alpha$ over functions $v : X \to \mathbb{R}$ such that

$$\rho(x, A) = \int \frac{e^{v(x)}}{\sum_{y \in A} e^{v(y)}} \alpha(dv).$$
3.7. Nested Logit

Every $\rho$ with a mixed logit representation must have a RU representation (since every logit has one and a mixed logit is just another randomization over those). Conversely, if $\rho$ has a RU representation, then it can be approximated by a sequence of mixed logits (so mixed-logit is dense in RU).

**Proposition 3.31.** If $\rho \sim RU$, then there exists a sequence $\rho^n \sim$ mixed logit, such that $\rho^n(x, A) \to \rho(x, A)$ for all $A$ and $x \in A$.

**Proof.** A simple proof for finite $X$ is in Appendix A.3.4. For arbitrary $X$, see Theorem 3 of Gul, Natenzon, and Pesendorfer (2014) and its proof. □

This result shouldn’t be too surprising since every $\rho$ that has a RU representation is by definition a mixture of deterministic choice functions and each of those is a limit of logit choice functions with noise going to zero. In fact, there is nothing special about logit. The mixed i.i.d. probits is also dense in RU, and so is any other i.i.d. ARU model.

**Remark 3.32 (Mixture Models).** Mixed logit suggests a general class of mixed models, which we discussed already in Remark 1.18. Suppose that we have a class $C$ of $\rho$’s. A mixed-$C$ model contains all $\rho$ that can be written as $\rho(x, A) = \sum_{i=1}^{n} \alpha_i \rho_i(x, A)$ such that $\alpha_i \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$, and $\rho_i \in C$ (or an integral more generally). For example, if $C$ is the class of deterministic choice functions that satisfy Sen’s $\alpha$ condition, then the class of mixed $C$ equals the RU class. We get mixed logit if $C$ is the logit class. RU is closed under mixtures, whereas logit and i.i.d. ARU are not (Apesteguia and Ballester, 2017a). ▲

**Remark 3.33.** We could also think of mixed-logit as a random $v$ function $\tilde{v} : \Omega \times X \to \mathbb{R}$ so that $\rho(x, A) = \mathbb{E} \left[ \frac{e^{\tilde{v}(x)}}{\sum_{y \in A} e^{\tilde{v}(y)}} \right]$. Even though you could call $\tilde{v}$ a random utility, note that $\tilde{v}$ is not a RU representation of $\rho$. The RU representation of $\rho$ is $\tilde{U}(x) = \tilde{v}(x) + \tilde{\epsilon}(x)$. In other words, the distribution of $\tilde{U}$ is $\mu$ whereas the distribution of $\tilde{v}$ is $\alpha$. ▲

3.7. Nested Logit

This is an older solution to the blue bus-red bus paradox. Imagine that instead of a one shot choice from $\{t, bb, rb\}$ the agent is choosing first from $\{t, b\}$ and then conditional on choosing $b$ the choice is between $\{bb, rb\}$, like in Figure 3.1.

![Figure 3.1. A nested decision problem.](image)
3.8. Other models of correlated $\epsilon^*$

It’s clear how to model the second step within the logit framework, but to assign probabilities in the first stage we need to define the value of the menu $\{bb, rb\}$ so that we can compare it with the value of $\{t\}$. One idea is to use expected utility:

$$V_\lambda(A) := \mathbb{E} \left[ \max_{x \in A} v(x) + \lambda \epsilon(x) \right].$$

Intuitively, when evaluating the menu $A$, the agent does not know the realizations of $\epsilon(x)$ for $x \in A$ yet. They think that for each possible realization they will choose the optimal item in the menu and $V(A)$ is the expected value of this optimal strategy. In the context of nested logit, this expression is called the “inclusive value” (McFadden, 1981). For logit with noise parameter $\lambda$ we have $V_\lambda(A) = \lambda \log \left( \sum_{x \in A} \exp(v(x)/\lambda) \right)$.

Whereas in the basic logit model all $\epsilon$ are independent, “nested logit” can be thought of as allowing for correlation of $\epsilon$ within the nest (but not across the nest).

In nested logit, the set of alternatives is partitioned into nests $B_1, \ldots, B_k$. The conditional probability of choosing $x$ from a nest $B_i$ is

$$\frac{\exp(v(x)/\lambda_2)}{\sum_{y \in B_i} \exp(v(y)/\lambda_2)}$$

and the probability of choosing nest $i$ in the initial stage is

$$\frac{\exp(V_{\lambda_2}(B_i)/\lambda_1)}{\sum_{i=1}^{k} \exp(V_{\lambda_2}(B_i)/\lambda_1)},$$

where $\lambda_1, \lambda_2$ are the noise parameters for stage 1 and stage 2 respectively.\textsuperscript{32}

While nesting seems like a nice approach, the analyst has to decide about the nest structure. The advantage of mixed logit is that the correlation between utilities of different goods is being estimated instead of being imposed by the analyst upfront. Other approaches include Gul, Natenzon, and Pesendorfer’s (2014) model in which (generalized) nests are revealed from data. Axioms for nested logit are given by Fudenberg, Iijima, and Strzalecki (2014) and Kovach and Tserenjigmid (forthcominga).

3.8. Other models of correlated $\epsilon^*$

Mixed logit and nested logit implicitly allow $\epsilon$ to be correlated. There are a few other models that do that. Figure 3.2 illustrates their relationship.

\textsuperscript{32}My exposition follows Sec. 2.7.1 of Anderson, de Palma, and Thisse (1992). Chap. 4.2 of Train (2009) presents a slightly more general model that goes outside of the RU class.
3.8. Other models of correlated $\epsilon^*$

3.8.1. Multivariate Probit. Here $\epsilon \sim \mathcal{N}(0, \Sigma)$ where $\Sigma$ is the variance-covariance matrix, so we are confronting the correlation issue head-on. Closed forms for choice probabilities are missing, so this literature relies on simulation (see Chapter 5 of Train, 2009).

3.8.2. GEV. Nested logit can be represented as ARU where the $\tilde{\epsilon}$ are correlated. This distribution belongs to the class of generalized extreme value distributions proposed by McFadden (1981), which allows for even more flexible substitution patterns. Like mixed logit, GEV is dense in RU (Dagsvik, 1995). For more, see Appendix A.3.1.

Figure 3.2. The nesting of models that allow for correlation of $\epsilon$. 
Part 2

Risk and Learning
Chapter 4

Risk

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4.1. Expected Utility

Our agent will now be choosing between alternatives that involve risk, such as insurance plans, career paths, etc. We will call them lotteries. Formally, let $Z$ be the set of prizes. A lottery is a mapping $p : Z \to [0,1]$ such that $p(z) > 0$ for finitely many $z \in Z$ and $\sum_{z \in Z} p(z) = 1$. The assumption here is that the agent knows these probabilities perfectly (this will be relaxed in Chapter 5, which allows for subjective probabilities). Lotteries will be denoted $p, q, r \in \Delta(Z)$ and elements of $Z$ will be denoted by $z, z'$, etc. As a prerequisite, this section reviews deterministic choice between lotteries.

Our preference relation $\succsim$ is now defined on $X = \Delta(Z)$. As usual, we say that $U$ represents $\succsim$ if $U(p) \geq U(q) \iff p \succsim q$.

**Definition 4.1.** $U : \Delta(Z) \to \mathbb{R}$ is an expected utility (EU) function if

$$U(p) = \mathbb{E}_p u := \sum_{z \in Z} u(z)p(z)$$

for some function $u : Z \to \mathbb{R}$.

The function $u : Z \to \mathbb{R}$ is called the Bernoulli utility. The function $U : \Delta(Z) \to \mathbb{R}$ is called the von Neumann–Morgenstern utility, or the vNM utility for short.

### 4.1.1. Mixing Lotteries.

A key new thing to play with, now that we have more structure on $X$, is the mixing operator. Given any two lotteries, $p, q \in \Delta(Z)$ and a number $\alpha \in [0,1]$ we can define a new lottery

$$\alpha p + (1-\alpha) q \in \Delta(Z).$$

This lottery attaches to each prize $z$ the probability equal to $\alpha p(z) + (1-\alpha)q(z)$, so it’s a weighted average of the two lotteries.

One way to achieve this mixed lottery is to first toss a coin that with probability $\alpha$ lands on heads and $1-\alpha$ on tails. Then give the agent lottery $p$ if heads come up and $q$ if tails come up. Such a two-stage lottery, or compound lottery, is formally an object that lives in a different space $\Delta(\Delta(Z))$; if our preferences are defined just on one-shot lotteries $\Delta(Z)$, we need to think of $\alpha p(z) + (1-\alpha)q(z)$ as the reduced, or flattened lottery, where the probabilities are multiplied out.

### 4.1.2. The Axioms.

A key property of EU is “linearity in probabilities:”

$$U(\alpha p + (1-\alpha) q) = \alpha U(p) + (1-\alpha) U(q).$$

This property leads to the following axiom.

**Axiom 4.2 (vNM Independence).** For all $\alpha \in (0,1)$ and $r \in \Delta(Z)$

$$p \succsim q \text{ if and only if } \alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r.$$
4.1. Expected Utility

The agent likes \( p \) more than \( q \) if and only if they like a mixture of \( p \) with some lottery \( r \) more than a mixture of \( q \) and \( r \) as long as they are of identical proportions.

Since \( X \) is now uncountably infinite, it is useful to have some continuity.

**Axiom 4.3** (Archimedean Continuity). If \( p \succ q \succ r \) there exist \( \alpha, \beta \in (0,1) \) such that
\[
\alpha p + (1 - \alpha) r \succ q \succ \beta p + (1 - \beta) r.
\]

Another axiom that works is the following.\(^{33}\)

**Axiom 4.4** (Mixture Continuity). For any \( p, q, r \in \Delta(Z) \) the sets
\[
\{ \alpha \in [0,1] : \alpha p + (1 - \alpha) q \succ r \}\quad \text{and} \quad \{ \alpha \in [0,1] : \alpha p + (1 - \alpha) q \preceq r \}
\]
are closed in \([0,1]\).

**Theorem 4.5.** (Von Neumann and Morgenstern, 1944) A preference \( \succcurlyeq \) on \( \Delta(Z) \) has an EU representation if and only if it satisfies vNM Independence and Archimedean Continuity (or Mixture Continuity). Moreover, \( u \) is cardinally unique, i.e., whenever \( u_1, u_2 \) represent \( \succcurlyeq \), there exists \( a > 0, b \in \mathbb{R} \) such that 
\( u_2(z) = au_1(z) + b \) for all \( z \in Z \).

**Proof.** See Theorem 5.5 of Kreps (1988). \( \square \)

The idea behind cardinal uniqueness is similar to the measurement of temperature: Celsius and Fahrenheit are affine transforms of each other.\(^{34}\) Cardinal uniqueness makes it meaningful to talk about the curvature of \( u \). As we will soon see, its curvature will control risk aversion.

This theorem combines two of the exercises we talked about in Chapter 2: characterization and identification. The third exercise, comparative statics, is the topic of the next subsection.

4.1.3. Risk Aversion. We say that someone is risk averse if they demand insurance. Suppose that payoffs are monetary, i.e., \( Z \subseteq \mathbb{R} \), and for any lottery \( p \) define \( \bar{p} \) to be its expected monetary value. For any prize \( z \in Z \) let \( \delta_z \) denote the lottery that gives \( z \) for sure; I will sometimes call it a point mass on \( z \).

**Definition 4.6.** \( \succcurlyeq \) is risk averse if \( \delta_{\bar{p}} \succcurlyeq p \) for all \( p \in \Delta(Z) \).

Notice that the concept of risk aversion is defined for all risk preferences, not just expected utility. In the EU model there is a particularly nice characterization because it’s equivalent to concavity of \( u \).

\(^{33}\)There are other notions of continuity: the Calibration axiom and topological continuity on \( \Delta(Z) \) with the Euclidean topology for finite \( Z \) and weak* topology for \( Z \) infinite. In the latter case \( \Delta(Z) \) is the set of Borel probability measures and \( U(p) = \int u(z) dp(z) \) (see, e.g., Grandmont, 1972). Total nerds will keep in mind that \( \Delta(Z) \) is a subset of the dual space of \( \mathbb{R}^Z \), hence weak star.

\(^{34}\)This implies that \( U \) is also cardinally unique. We still have ordinal uniqueness of \( U \) from Chapter 1: for any strictly increasing function \( \phi \) the function \( \phi \circ U \) represents the preference. However this function will not be of the vNM form unless the function \( \phi \) is affine.
4.1. Expected Utility

Definition 4.7. \( u : Z \to \mathbb{R} \) is concave if
\[
u(\alpha x + (1 - \alpha)y) \geq \alpha u(x) + (1 - \alpha)u(y)
\]
for all \( x, y \in Z \), \( \alpha \in (0, 1) \) such that \( \alpha x + (1 - \alpha)y \in Z \).

Theorem 4.8. Suppose that \( \succsim \) has an EU representation with Bernoulli utility \( u \). \( \succsim \) is risk averse if \( u \) is a concave function.

Proof. The proof is a restatement of Jensen’s inequality. \( \square \)

Another definition helps us compare risk aversion of two individuals.

Definition 4.9. \( \succsim_1 \) is more risk averse than \( \succsim_2 \) if for all \( z \in Z \) and \( p \in \Delta(Z) \)
\[
\delta_z \succsim_2 p \implies \delta_z \succsim_1 p
\]
and
\[
\delta_z \succsim_2 p \implies \delta_z \succsim_1 p.
\]

If agent 2 chooses a sure thing over a lottery, then agent 1 who is more risk averse should also go for the sure thing.

Proposition 4.10 (Pratt 1978). Suppose that \( \succsim_1, \succsim_2 \) have an EU representation with Bernoulli utilities \( u_1, u_2 \). \( \succsim_1 \) is more risk averse than \( \succsim_2 \) iff \( u_1 = \phi \circ u_2 \) for some strictly increasing and concave function \( \phi \) whose domain is the range of \( u_2 \).

The price we have to pay for such nice comparative statics is that we restrict attention to a certain subclass (in this case expected utility). This is generally the case: we pay for identification by making assumptions. Ideally, we can test these assumptions by checking whether some axioms hold (in the case of expected utility, its the the vNM axiom).

4.1.4. Stochastic Dominance*. There are three main dominance relations over lotteries with monetary payoffs. They are transitive, but incomplete.

We say that lottery \( p \) first-order stochastically dominates (FOSD) lottery \( q \) if all EU preferences with an increasing \( u \) like \( p \) more than \( q \), denoted by \( p \succeq_{FOSD} q \). The relation \( \succeq_{FOSD} \) can be characterized by comparing the c.d.fs of the two distributions. Let \( F_p \) be the cdf of \( p \) and \( F_q \) the cdf of \( q \). We have \( p \succeq_{FOSD} q \) iff \( F_p(z) \leq F_q(z) \) for all \( z \in Z \). In other words, \( p \) is a definite improvement over \( q \): for any prize \( z \), the probability of getting at least \( z \) under \( p \) is higher than under \( q \).

We say that lottery \( p \) dominates \( q \) in the convex order if all EU preferences with a concave \( u \) like \( p \) more than \( q \). This is denoted \( p \succeq_{cx} q \) and implies that the expectations of those two lotteries are equal (why?). An alternative representation of this order involves comparing integrals of the \( F_p \) and \( F_q \). Another comparison is through a sequence of mean-preserving spreads.
4.1. Expected Utility

Finally, lottery $p$ second-order stochastically dominates (SOSD) lottery $q$ if all EU preferences with an increasing and concave $u$ like $p$ more than $q$.

For more on these see, Chapters 1, 3, and 4 of Shaked and Shanthikumar (2007) and Section 6.D of Mas-Colell, Whinston, Green, et al. (1995).

4.1.5. Popular Parameterizations. The two most used families are Constant Absolute Risk Aversion (CARA) and Constant Relative Risk Aversion (CRRA). The first one says that risk aversion over incremental wealth stays constant as we make the agent richer. The second one is a multiplicative version. In other words, CARA is shift-invariant while CRRA is scale-invariant.

Both families have single-dimensional parameterizations. We say that $u$ is in the CARA family if

$$u(z) = \begin{cases} \frac{-\exp(-\theta z)}{\theta} & \text{if } \theta \neq 0 \\ z & \text{if } \theta = 0 \end{cases}$$

for some parameter $\theta \in \mathbb{R}$. We say that $u$ is in the CRRA family if

$$u(z) = \begin{cases} z^{1-\theta} - 1 & \text{if } \theta \neq 1 \\ \ln(z) & \text{if } \theta = 1 \end{cases}.$$

In both cases, the parameter measures risk aversion: the more we bump it up, the more risk averse the agent becomes.

4.1.6. Non-expected Utility. There is a large literature on non-EU preferences, motivated by the Allais (1953) paradox and the related Common Ratio Paradox.

Example 4.11 (Common Ratio Paradox). Suppose we have the following four lotteries.

The Independence axiom implies that $q \succsim p$ if and only if $q' \succsim p'$. This is because $p' = .25p + .75\delta_0$ and $q' = .25q + .75\delta_0$. However in their experiment Kahneman and Tversky (1979) find that among 95 subjects 80 have the
preference $q > p$ while only 35 have the preference $q' > p'$, so preferences of a substantial fraction of subjects are inconsistent with EU.35

To accommodate such behavior, non-EU preferences relax linearity in probabilities. A number of classes of nonlinear functions $V : \Delta(Z) \to \mathbb{R}$ have been developed and axiomatized.36

4.2. Stochastic Models for Expected Utility

It is widely documented that individual choices between lotteries are actually stochastic (Mosteller and Nogee, 1951). Given a class of deterministic utility functions $U$ there are two basic ways to construct a stochastic model $\rho$: one is to randomize over utilities $u \in U$ and the other is to fix a particular $v \in U$ and add an error term $\epsilon$ (Remark 1.18). We will now apply these two ideas to $U =$ “EU preferences.”37

Our $\rho$ is defined on the collection of all finite subsets of $\Delta(Z)$.

Definition 4.12 (Random Expected Utility). $\rho$ has a Random Expected Utility (REU) representation if it has a proper RU representation where with probability one the realized utility satisfies Definition 4.1.

We can think of REU as a probability distribution over EU preferences or alternatively as a distribution over Bernoulli utilities $u$ or a distribution over vNM utilities $U$. When dealing with a parametric class such as CARA or CRRA, we can think of it as a distribution over the parameter space $\Theta$.

On the other hand, we have a model where shocks are added to the expected utility values.

Definition 4.13 (Additive Random Expected Utility). $\rho$ has an Additive Random Expected Utility (AREU) representation if it has a RU representation where $\tilde{U}(p) = \mathbb{E}_p v + \tilde{\epsilon}(p)$ for some deterministic $v : Z \to \mathbb{R}$ and smooth i.i.d. $\tilde{\epsilon}$.

This is the i.i.d. ARU model adapted to the world of lotteries: the deterministic utility function is expected utility and the error term (known to the agent at the time of choice) is i.i.d. across lotteries.

35This experiment has been replicated a number of times, using both within- and between-subjects designs. Notice that in a between-subject design we cannot test Axiom 4.2 because we don’t observe a preference relation, but instead a s.c.f. $\rho(q, p) = \frac{80}{95}$ and $\rho(q', p') = \frac{35}{95}$.


37Sometimes in the literature the first model is called “random parameter” whereas the second “random utility;” we will not use this terminology here. Both kinds of stochastic models were introduced Block and Marschak (1960) and Becker, DeGroot, and Marschak (1963). By the way, the latter is not the famous paper of the trio. The experimental technique of random incentive systems was introduced in (Becker, DeGroot, and Marschak, 1964).
REU and AREU are both special cases of RU but they are very different 
animals. By definition, under REU the realized utility $\tilde{U}$ is with 
probability one linear. On the other hand, as we will see in Section 4.5, in AREU the 
realized utility $\tilde{U}$ with probability one represents a preference that violates the 
vNM Independence axiom.

4.3. Random Expected Utility

In the world without lotteries, axioms for random utility are complicated and 
hard to interpret (Section 2.3). Gul and Pesendorfer (2006a) showed that the 
axioms get much simpler in the EU case: we just need Regularity, Continuity, 
and two versions of Independence. As we will see, lotteries also give us much 
better uniqueness properties.

The first version of Independence compares choices from a menu $A$ to 
choices from a menu $A$ that is mixed with some lottery $r$. Formally, define a 
new menu

$$
\alpha A + (1 - \alpha) r := \{ \alpha p + (1 - \alpha) r : p \in A \}.
$$

**Axiom 4.14 (Linearity).** For any $\alpha \in (0, 1)$ any $p \in A$ and $r \in \Delta(Z)$ $\rho(p, A) = \rho(\alpha p + (1 - \alpha) r, \alpha A + (1 - \alpha) r)$.

This axiom is necessary for REU because every vNM utility function is 
linear in probabilities, so $U(p) \geq U(q)$ iff $U(\alpha p + (1 - \alpha) r) \geq U(\alpha q + (1 - \alpha) r)$. This implies that

$$
N(p, A) = N(\alpha p + (1 - \alpha) r, \alpha A + (1 - \alpha) r),
$$

which implies that the choice probabilities are equal. Geometrically, $N(p, A)$ is 
one of the shaded angles in panel (a) of Figure 4.1, defined as the normal cone 
of the convex hull of $A$ at point $p$. It equals the other shaded angle because of 
the “corresponding angles theorem.”

![Figure 4.1](image.png)
In the next axiom the set \( \text{ext}(A) \) denotes the set of extreme points of \( A \), i.e., lotteries \( p \in A \) that cannot be represented as convex combinations of other lotteries in \( A \).

**Axiom 4.15** (Extremeness). \( \rho(\text{ext}A, A) = 1. \)

Extremeness is a consequence of linearity of the vNM utility and properness of RU.\(^{38}\)

Finally, we need a flavor of continuity. This is a technical axiom, so you can ignore the details. For a fixed menu \( A \) we treat \( \rho(\cdot, A) \) as a Borel probability measure on \( \Delta(Z) \) and equip the set of all Borel measures, \( \Delta(\Delta(Z)) \), with the weak* topology. We endow the set \( M(\Delta(Z)) \) with the Hausdorff metric.

**Axiom 4.16** (Continuity). The function \( \rho : M(\Delta(Z)) \to \Delta(\Delta(Z)) \) is continuous.

I say that continuity is a technical axiom because to reject it we need to observe an infinite number of menus. On the other hand, to reject the other axioms we just need a small number of menus (especially if we cook them up in a smart way). Of course, to verify that any those axioms are satisfied, we still need infinitely many menus.

**Theorem 4.17** (Gul and Pesendorfer, 2006a). Suppose that \( Z \) is finite. A s.c.f. \( \rho \) satisfies Regularity, Linearity, Extremeness, and Continuity if and only if \( \rho \sim \text{REU} \). In this case the measure \( \mu \) is unique over the twice-normalized Bernoulli utilities.\(^{39}\)

Recall Example 2.14, which showed that in general we can’t determine the probability that \( p \) is better than \( q \) and at the same time \( p' \) is better than \( q' \). The reason we are getting stronger uniqueness now is that mixing will allow us to determine this probability. We simply need to look at the probability that \( \frac{1}{2}p + \frac{1}{2}p' \) gets chosen from the set \( \{ \frac{1}{2}p + \frac{1}{2}p', \frac{1}{2}p + \frac{1}{2}q', \frac{1}{2}q + \frac{1}{2}p', \frac{1}{2}q + \frac{1}{2}q' \} \).

**Remark 4.18** (Uniqueness under REU). The event \( N(\alpha p + (1 - \alpha)q, \alpha A + (1 - \alpha)B) \) equals the intersection of \( N(p, A) \) and \( N(q, B) \). This is true for any \( \alpha \in (0, 1) \) because for any linear \( U : \Delta(Z) \to \mathbb{R} \) we have

\[
U(\alpha p + (1 - \alpha)q) \geq U(\alpha p' + (1 - \alpha)q') \quad \text{for all } p', q' \in B
\]

iff

\[
\alpha U(p) + (1 - \alpha)U(q) \geq \alpha U(p') + (1 - \alpha)U(q') \quad \text{for all } p', q' \in B
\]

iff

\[
U(p) \geq U(p') \quad \text{for all } p' \in A \quad \text{and} \quad U(q) \geq U(q') \quad \text{for all } q' \in B.
\]

\(^{38}\)Any fixed \( u \) will be maximized on the boundary of \( A \), see panel (b) of Figure 4.1. This may be an extreme point of \( A \), but it may also a nonextreme point (called exposed point) if the indifference curve happens to be parallel to the side of \( A \). But since we required REU to be proper, for any fixed \( A \), the probability of finding such a non-generic utility is zero.

\(^{39}\)More precisely, the measure \( \mu \) is unique on the Borel \( \sigma \)-algebra. Gul and Pesendorfer (2006a) have a weaker uniqueness result. The stronger claim follows from the Caratheodory’s extension theorem and the construction in Section S3.2 of the supplement of Ahn and Sarver (2013).
As you may recall from Chapter 1, in a proper RU representation ties must occur with probability zero. This may be confusing because any fixed EU preference over lotteries must have nontrivial indifference curves (as long as $|Z| \geq 3$). To make this work, in any proper REU representation every single preference has probability zero. This can be done by taking a distribution over preferences that is smooth enough so every particular preference gets zero probability and for any pair $x, y$ the probability of a tie is zero (for example take a positive density over the set $\mathbb{R}^Z$ of Bernoulli utilities).

4.4. Technical Aspects of REU*

To work with examples or applications, we might want to have a REU that puts positive probability on finitely many utilities and therefore is improper. To do so, Gul and Pesendorfer (2006b) introduce tiebreakers.

A technical wrinkle is that for tiebreakers Gul and Pesendorfer (2006b) relax the countable additivity assumption and require that they be only finitely additive. This is obtained by relaxing the continuity axiom to mixture continuity.

**Axiom 4.19** (Mixture Continuity). For any menus $A$ and $B$, the function $\alpha \mapsto \rho(\cdot, \alpha A + (1 - \alpha)B)$ is continuous.

**Theorem 4.20** (Gul and Pesendorfer, 2006b).

(i) $\rho$ has a finitely additive proper REU representation if and only if it satisfies Regularity, Linearity, Extremeness, and Mixture Continuity. In this case the measure $\mu$ is unique on the algebra generated by the sets $N(p, A)$.

(ii) $\rho$ has a finitely additive improper REU representation with a GP tiebreaker if and only if it has a finitely additive proper REU representation.

Ahn and Sarver (2013) study REU representations with finitely many vNM utilities. In the supplement to their paper they present a Finiteness axiom that guarantees that the REU representation is discrete (with a GP tiebreaker). The main idea hinges on their clever construction of a separating menu: for any utility $u$ in the support of $\mu$ there is a lottery $p_u$ such that $N(p_u, A) = \{u\}$, where $A = \{p_u : u \in \text{supp} \mu\}$. This guarantees that $\rho(p_u, A) = \mu(u)$. By using such constructions it is possible to behaviorally define “menus without ties” and “menus with ties.”

**Remark 4.21.** Recall uniform tiebreaking from Section 1.6. In the world of lotteries it was used for example by Loomes and Sugden (1995). This leads to a $\rho$ which satisfies Linearity but violates Extremeness (why?). ▲

**Remark 4.22.** The discussion here assumes that $Z$ is finite. Extensions to infinite $Z$ are obtained by Ma (2018), Frick, Iijima, and Strzalecki (2019), and Lu and Saito (2019). ▲
4.5. Comparison of REU and AREU

Table 1 summarizes the comparisons between the two models. The punchline is that if we believe in EU, then REU is the more appealing of the two models. AREU leads to non-EU behaviors of a particular form. If we are motivated by departures from EU, we may want to build those directly into the baseline model \( U \) (as in Section 4.6), instead of generating them as an artifact of the \( \epsilon \).

<table>
<thead>
<tr>
<th>REU</th>
<th>AREU</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{P}(\succsim \in EU) = 1 )</td>
<td>( \mathbb{P}(\succsim \in EU) = 0 )</td>
</tr>
<tr>
<td>Linearity, Extremeness can violate WST</td>
<td>Stochastic Independence, Positivity satisfies SST</td>
</tr>
<tr>
<td>FOSD monotone with prob 1</td>
<td>FOSD monotone with prob ( \in (\frac{1}{2}, 1) )</td>
</tr>
<tr>
<td>cvx-monotone with prob 1</td>
<td>non cvx-monotone with prob 1</td>
</tr>
</tbody>
</table>

Under REU the realized preference is EU with probability one, while under AREU this happens with probability zero. To see that, consider lotteries \( p, q \) such \( \mathbb{E}_p v = \mathbb{E}_q v \). Fix a realization of \( \tilde{\epsilon} \) and suppose w.l.o.g. that \( \tilde{U}(p) \geq \tilde{U}(q) \). For any \( \alpha \in (0, 1) \) and \( r \in \Delta(Z) \) we have \( \mathbb{E}_{\alpha p + (1-\alpha) r} v = \mathbb{E}_{\alpha q + (1-\alpha) r} v \). Because \( \tilde{\epsilon}(\alpha p + (1-\alpha) r) \) and \( \tilde{\epsilon}(\alpha q + (1-\alpha) r) \) are i.i.d. across \( \alpha \in (0, 1) \), by the exact law of large numbers, with probability one there exists \( \alpha \) such that \( \tilde{\epsilon}(\alpha p + (1-\alpha) r) < \tilde{\epsilon}(\alpha q + (1-\alpha) r) \).40

Moving to the second row of Table 1, AREU violates Extremeness.41 Likewise, it also violate Linearity (Axiom 4.14).

**Example 4.23** (Spurious Common Ratio Paradox). Recall Example 4.11, where

\[ .63 = \rho(p', q') > \rho(p, q) = .16. \]

It follows from Linearity that under REU those two choice probabilities are always identical. On the other hand, if we take a AREU model with a deterministic risk-averse EU function \( v \) then we get the qualitative preference pattern \( \rho(p', q') > \rho(p, q) \). However, both choice probabilities will be on the same side of \( \frac{1}{2} \), so AREU cannot really explain the common ratio paradox (Loomes, 2005).

In general, AREU satisfies the following axiom. As in Section 3.4, define the stochastic preference \( p \succsim^* q \) by \( \rho(p, q) \geq \frac{1}{2} \).

**Axiom 4.24** (Stochastic Independence). \( \succsim^* \) satisfies Axiom 4.2.

---

40See, e.g., Sun (2006) and Podczeck (2010). If you are uncomfortable with the exact law of large numbers, choose a discrete grid on the interval \([0, 1] \ni \alpha \) and notice that the desired \( \alpha \) exists with a probability approaching one as the number of grid points approaches infinity.

Note that this axiom is a weakening of Linearity. AREU satisfies the following axiom.

**Axiom 4.25 (Stochastic Continuity).** $\succeq^*$ satisfies Axiom 4.3.

A Luce model for choice over lotteries is also sometimes considered.\footnote{There are also Fechnerian versions of EU, see, e.g., Becker, DeGroot, and Marschak (1963) and Loomes and Sugden (1995).}

**Definition 4.26 (Luce Expected Utility).** $\rho$ has a *Luce Expected Utility* (LEU) representation if it has a Luce representation with $w(p) = h(E_p v)$ for some deterministic functions $v: Z \to \mathbb{R}$ and $h: \mathbb{R} \to \mathbb{R}_{++}$, where $h$ is strictly increasing on the convex hull of the set $\{v(z) : z \in Z\}$.

LEU is a special case of AREU if $h$ is the exponential function. It is worthwhile to notice that LEU is characterized by those two axioms (plus Luce’s IIA and positivity).

**Proposition 4.27 (Dagsvik 2008).** The following are equivalent:

(i) $\rho \sim LEU$

(ii) $\rho$ satisfies Positivity and Luce’s IIA, Stochastic Independence, and Stochastic Continuity (Axioms 1.16, 3.2, 4.24, and 4.25).

**Proof.** Appendix A.4.1.\footnote{Similar simple axiomatizations of Fechnerian EU have been obtained, see, e.g., Dagsvik (2008) who also shows what additional axioms guarantee the linearity of the function $h$. Blavatskyy (2008) offers a different axiomatization of the special case of Fechnerian EU with linear $h$; for a correction see Ryan (2015). Dagsvik (2015) explores the relationship between these axioms.}

The third row of the table presents another distinguishing feature of AREU and REU: stochastic transitivity. AREU satisfies SST, whereas REU typically violates even WST. Thus, rejections of stochastic transitivity are rejections of AREU, not of Expected Utility in general, see also Mellers and Biagini (1994).

The last two rows of the table refer to the stochastic dominance relations. The idea is easiest to see by implementing both models parametrically. Take for example CARA or CRRA preferences, both parametrized by $\theta \in \Theta$. Fix the “average” level risk aversion $\theta \in \Theta$ and let $\tilde{\epsilon}$ be a real-valued random variable that perturbs the risk aversion coefficient. Under REU this leads to the following choice probabilities for $p, q \in \Delta(Z)$:

$$\rho_{\theta}(p, q) := \mathbb{P}(E_p[u_{\theta + \tilde{\epsilon}}] \geq E_q[u_{\theta + \tilde{\epsilon}}]).$$

In the AREU version of the model there are still $\theta \in \Theta$ and a random variable $\tilde{\epsilon}$. However, $\epsilon$ does not perturb the coefficient of risk aversion, but the value of each lottery; that is $\epsilon \in \mathbb{R}^{\Delta(Z)}$. This leads to the following choice probabilities.

$$\rho_{\theta}(p, q) = \mathbb{P}(E_p[u_{\theta}] + \epsilon(p) \geq E_q[u_{\theta}] + \epsilon(q)).$$
Suppose that \( q \) strictly FOSD-dominates \( p \). Then under AREU \( \rho_\theta(p, q) > 0 \). On the other hand, under REU we have \( \rho_\theta(p, q) = 0 \) because all \( u \) are increasing. The difference between the two models becomes starkest when \( q \) is just a little bit better than \( p \): under AREU their choice probabilities will be about equal.\(^{44}\)

Suppose now that \( p \) is a mean-preserving spread of \( q \) and that \( \tilde{\epsilon} \) has full support. Under REU for any \( q \geq_{\text{cx}} p \) the choice probability \( \rho_\theta(p, q) \) is decreasing in \( \theta \).\(^{45}\) On the other hand, under AREU there exists \( \bar{\theta} \) above which the function \( \theta \mapsto \rho_\theta(p, q) \) is increasing in \( \theta \).\(^{46}\)

**Remark**\(^*\) 4.28. Perhaps we need a little bit of \( \epsilon \) to estimate the model, otherwise our likelihood function will be degenerate. This suggests a BLP-style model for choice under lotteries (BLP is discussed in Section 10.6). A model like this is sketched in Section 5.1.2. of Barseghyan, Molinari, O’Donoghue, and Teitelbaum (2018).

Instead, the literature in industrial organization typically uses REU models, where the randomness is purely population heterogeneity (each agent’s choices are deterministic), see, e.g., Einav, Finkelstein, Ryan, Schrimpf, and Cullen (2013); Handel (2013); Ho and Lee (2020). The only BLP-style model I’m aware of is Ho and Lee (2017). For a discussion of these issues see also DellaVigna (2018).

**4.6. Non-Expected Utility**

With \( \mathcal{U} = \text{EU} \) we can randomize over \( \mathcal{U} \) (and get REU) or fix \( u \in \mathcal{U} \) and add \( \epsilon \) shocks to the evaluation of each lottery (to get AREU). We will talk about both of those approaches in the context of non-EU, but first we notice that a new opportunity opens up, one that was absent under EU.

Machina (1985) noticed that even if the preference is fixed and deterministic, the agent may want to deliberately randomize over choies. This is because with nonlinear preferences the optimal lottery may be created as a mixture of those that belong to the menu. To implement such a mixture, the agent will toss a “mental coin” and thus the choices observed by the analyst will be stochastic (assuming that the optimal mixture doesn’t actually belong to the menu).

Suppose that the agent’s preferences are represented by a quasi-concave function \( V : \Delta(Z) \to \mathbb{R} \), as in Figure 4.2. If the menu is \( A = \{p, q\} \), the agent will weakly prefer any mixture \( r_\alpha = \alpha p + (1 - \alpha)q \). Let \( \alpha^* \) be a value of \( \alpha \) that maximizes \( V(r_\alpha) \). The agent would like to implement this lottery by choosing \( p \) with probability \( \alpha^* \) and \( q \) with probability \( 1 - \alpha^* \), so we have \( \rho(p, q) = \alpha^* \).

\(^{44}\)Such violations were first noticed by Becker, DeGroot, and Marschak (1963) and Loomes and Sugden (1995) in the context of Fechnerian EU models.

\(^{45}\)This probability can be positive because with some probability \( u_{\theta + \epsilon} \) is risk loving. However, as \( \theta \) increases, this probability goes down.

\(^{46}\)See, e.g., Wilcox (2008, 2011) and Apesteguia and Ballester (2017b).
Machina’s model was recently axiomatized by Cerreia-Vioglio, Dillenberger, Ortoleva, and Riella (2019). They also show that this class of \( \rho \) typically violates Regularity: as long as \( \succeq \) has a point of strict convexity. To see that, consider the lotteries \( p \) and \( q \) as above and the lottery \( r' \) that is close to \( r^* = \alpha^* p + (1 - \alpha^*)q \), but is FOSD-dominated by it. The choice from menu \( \{p, q, r'\} \) is \( p \) with probability \( \alpha^* \) and \( q \) with probability \( 1 - \alpha^* \); this is because tossing the mental coin implements the lottery \( r^* \) which is better than \( r' \). On the other hand, from the menu \( \{p, r'\} \) the agent will choose \( r' \) with probability close to one because choosing \( p \) without being able to choose \( q \) with the complementary probability is not very appealing.

Another question is what would be the choice if the optimal lottery does belong to the menu. Agranov and Ortoleva (2017) show that experimental subjects have a preference for \( r^* \) being in the menu, i.e., not having to toss the mental coin. Agranov and Ortoleva (2020) provide further evidence of preference for randomization by devising a version of multiple price lists that allows subjects express such a preference. See also Dwenger, Kübler, and Weizsäcker’s (2018) field study on university admissions. As far as I understand Machina-style preferences are not a good model of that, as they are silent whether a mental coin is better or worse than an actual one.

Mathematically, Machina’s model is similar to Perturbed Utility, where the agent maximizes expected utility plus a non-linear term (Section 3.5, see also Marley (1997)).

Let us now return to the two other ways deal with non-EU preferences: randomize over preferences or add noise to the value of each lottery. The first one was studied by (Lin, 2019a,b), who shows that if the random preferences belong to the betweenness class, stochastic choices still satisfy Regularity and

\footnote{As far as I know, nobody has tested those axioms yet. Hey and Carbone (1995) estimate a parametric (quadratic) version of this model (and find that it explains only 10% of subject’s choices).}
Extremeness. Kashaev and Aguiar (2022b) complement these results by studying random Rank-dependent Expected Utility. See also Melkonyan and Safra (2016) who study weak stochastic transitivity of such models. In mathematical psychology, researchers have developed standardized software for testing random non-EU models (Regenwetter, Davis-Stober, Lim, Guo, Popova, Zwilling, Cha, and Messner, 2014; Zwilling, Cavagnaro, Regenwetter, Lim, Fields, and Zhang, 2019).

The second approach was taken by Hey and Orme (1994) who considered i.i.d. ARU implementations of non-EU (without allowing the agent to toss any mental coins). A similar, but different method was introduced by Harless and Camerer (1994) who considered a constant-error choice rule (such as in Example 3.16). Ballinger and Wilcox (1997) derive tests for both implementations. Their experimental data is consistent with the first, but rejects the second. More recently, i.i.d. ARU versions of non-EU models were used by De Palma, Ben-Akiva, Brownstone, Holt, Magnac, McFadden, Moffatt, Piccard, Train, and Wakker (2008) and Barseghyan, Molinari, O’Donoghue, and Teitelbaum (2013). See also DellaVigna (2018).

48Those authors allow for unobservable heterogeneity (i.e., consider models where the parameters of the non-EU risk function vary from person to person), i.e., a mixture over i.i.d. ARU models. But importantly, the parameter is fixed within a person, so the stochastic choices of each individual suffer from the same monotonicity violations as those described in Section 4.5.
Chapter 5

Passive Learning

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5.1. The Bayesian Model

In RU models the agent’s observed choices \( \rho \) are stochastic because their tastes are randomly fluctuating. Another possible reason for random choices could be that the agent’s beliefs are fluctuating because they are learning new information over time (while for simplicity tastes stay fixed). What kinds of observable behavior does this lead to? Is it different than behavior captured by RU? The answer to this will depend on what we mean by “tastes” and “beliefs.”

In the simplest model the agent gets the same information regardless of the menu. This model can capture some forms of frequency-dependence that escape RU (similar to Example 1.5 but not exactly those).

We then move to a model where the agent’s information can depend on the menu. This is similar to making the distribution of utilities menu-dependent; if this dependence is arbitrary, we can explain any \( \rho \).

In the simplest version of the Bayesian model, studied in this chapter, the signal arrives whether the agent wants it or not. This is called passive learning, as opposed to active learning studied in Chapter 6, where the agent can choose what signal to acquire (and whether to acquire anything at all). Passive learning is a strong assumption and leads to counterfactual predictions. We spend time on it here partially as a building block for active learning, which puts some discipline on the dependence of information on the menu. (And partially to find out which predictions are counterfactual and which are not.)

5.1.1. Bayes Representations. Let \( S \) be the set of states: this is what the agent is learning about.\(^{49}\) The state is unknown to the agent but the agent gets information in the form of a message \( m \in M \). In each state \( s \) there is a probability distribution over possible messages. The function \( \beta : S \to \Delta(M) \) is called an experiment (or a signal structure, or a signal).\(^{50}\)

A Bayesian agent has a prior \( p \in \Delta(S) \) which represents their belief before getting the message. For each message \( m \in M \) there is a posterior belief \( q(\cdot|m) \in \Delta(S) \).\(^{51}\) When \( S \) and \( M \) are finite, this is given by the Bayes rule:

\[
q(s|m) = \frac{\beta(m|s)p(s)}{\sum_{s'} \beta(m|s')p(s')}
\]

as long as the denominator is positive.\(^{52}\) We will write \( q_m \in \Delta(S) \) for the vector \( q(\cdot|m) \) and \( \beta_s \in \Delta(M) \) for the vector \( \beta(\cdot|s) \).

\(^{49}\)In statistics it is customary to use \( \Theta \). In Chapter 2 we used \( \theta \) for the parameters of the model that our analyst is learning about. Since the analyst and the agent are two different people, we need another letter, to denote the states of the agent. Note that \( S \) is different from \( \Omega \) in a RU representation. The relationship between \( S \) and \( \Omega \) will become clear as you keep reading along.

\(^{50}\)For simplicity, assume that \( S \) and \( M \) is finite. In probability theory this object is known as a probability kernel, or Markov kernel. In statistics this is known as the likelihood function.

\(^{51}\)From Nature’s point of view, the state \( s \) is not random: it is fixed and simply unknown to the agent. Their beliefs \( p \) and \( q_m \) reflect that subjective uncertainty.

\(^{52}\)If \( \beta \) and \( p \) are densities, the above equation becomes: \( q(s|\cdot|m) = \frac{\beta(m|s)p(s)}{\int \beta(m|s')p(s') \, ds'} \). In general, we need to worry about regular conditional probabilities, see, e.g., Chapter 5 of Pollard (2002) or Chapter 10 of Dudley (2002).
Let $v : X \times S \to \mathbb{R}$ be a deterministic state-dependent utility function. For any belief $q \in \Delta(S)$ the expected utility of $x$ is denoted by $E_q v(x) := \sum_{s \in S} q(s) v(x, s)$. Our agent is faced with some menu $A \subseteq X$ and solves $\max_{x \in A} E_q v(x)$. Modulo ties, the agent’s choice conditional on $s$ is deterministic for each posterior. The observed choice probability is

$$\rho^s(x, A) = \beta_s \left\{ m \in M : E_q m v(x) = \max_{y \in A} E_q m v(y) \right\}.$$ 

In the eyes of the analyst the choice is stochastic because she does not observe the message. To deal with ties, we will use GP tiebreakers (Section 1.6).

**Definition 5.1.** $(\rho^s) \sim \text{Bayes}$ if there exists an experiment $\beta : S \to \Delta(M)$, a prior $p \in \Delta(S)$, and a utility function $v : X \times S \to \mathbb{R}$ such that $\rho^s$ is RU with state space $\Omega = M$, utility $\bar{U}(x, m) = E_q m v(x)$, probability $\mathbb{P} = \beta^s$, and a GP tiebreaker.

Even though this is a very natural representation, and one that is widely used, I do not know what the corresponding axioms are. These $\rho^s$ are connected to each other across $s$ because they all come from the same prior and same experiment, which suggests we need some axiom that ties them. There are answers with Anscombe–Aumann acts; we will discuss them in Section 5.3.2.

If the analyst does not observe $s$, then all she can see is the marginal: the average of $\rho^s$ over $s$ according to the true probability distribution (which is just like our old primitive). In principle, this true distribution may or may not be equal to the agent’s prior $p$. Under the rational expectations assumption this average is taken using the agent’s prior.

**Definition 5.2.** $\rho \sim \text{average Bayes}$ if there exists $(\rho^s) \sim \text{Bayes}$ and $\pi \in \Delta(S)$ such that

$$\rho(x, A) = \sum_{s \in S} \rho^s(x, A) \pi(s).$$

Moreover, we say that $\rho \sim \text{average Bayes with rational expectations}$ if there exists $(\rho^s) \in \text{Bayes}(p, \beta, v)$ such that

$$\rho(x, A) = \sum_{s \in S} \rho^s(x, A) p(s).$$

As we will see in the next section, these two definitions are equivalent to each other and to RU. This means that looking at average choices is not enough to test the rational expectations assumption. On the other hand, if the analyst has access to the conditional choice probabilities $\rho^s$, then she presumably also has access to the distribution $\pi \in \Delta(S)$. In this case, she can simply compare $\pi$ to the prior distribution revealed from $(\rho^s)$.

**5.1.2. Distribution over Posteriors.** An alternative model of learning bypasses the message space $M$ and looks directly at the agent’s ex ante distribution of posteriors. To see how this works, fix a Bayesian model. In any given state the experiment leads to a distribution over messages, and for each
message we have a posterior belief. Thus, in each state, if we integrate out the messages, there is a distribution over posteriors. Ex ante, the agent does not know the state yet, so the distribution over posteriors that they expect to have is the average of those distributions. This is denoted by $\mu \in \Delta(\Delta(S))$.

Given a prior $p$ and an experiment $\beta$ we get a distribution over posteriors $\mu$. The most important property of $\mu$ is that its average equals $p$; that is, beliefs don’t change on average. Formally, $\int q\mu(dq) = p$. This follows from the law of iterated expectations and is sometimes called the martingale property of beliefs. This holds only unconditionally. Conditional on $s$ the distribution over posteriors averages to a belief that is “closer” to $s$ than $p$.

We can go in both directions: for any $\mu$ there exists a prior (uniquely given by $p_\mu := \int q\mu(dq)$) and an experiment $\beta_\mu$ such that $\mu$ is induced by $p$ and $\beta$.

Thus, if we want to describe agent’s ex ante reasoning we can interchangeably use experiments or distributions over posteriors. Modulo ties, we have

$$\rho(x,A) = \mu\left(\left\{q \in \Delta(S) : \mathbb{E}_q v(x) = \max_{y \in A} \mathbb{E}_q v(y)\right\}\right).$$

**Definition 5.3.** $\rho \sim$ distribution over posteriors if there exists $\mu \in \Delta(\Delta(S))$ such that $\rho$ has a RU representation with $\Omega = \Delta(S)$, $\mathbb{P} = \mu$, and $\tilde{U}(x,q) = \mathbb{E}_q v(x)$ with a GP tiebreaker.

Notice, that this $\rho$ does not condition on the state, but rather is the unconditional distribution of choices.

**Proposition 5.4.** If $X$ is finite, then the following are equivalent:

(i) $\rho \sim$ RU.
(ii) $\rho \sim$ average Bayes,
(iii) $\rho \sim$ average Bayes with rational expectations,
(iv) $\rho \sim$ distribution over posteriors,

**Proof.** Appendix A.5.1. $\square$

The equivalence with RU breaks if we allow for a separation of tastes and beliefs (Section 5.3.1). Another way to break it is to give the agent many batches of trials, each batch with a different $p$. In a RU representation $\rho$ will be independent of $p$, whereas an average Bayes allows for frequency-dependence. A distribution over posteriors is a clumsy object when we have variation in $p$ because for each $p$ we get a different distribution over posteriors (each one has to average to its respective $p$). On the other hand, in a Bayes distribution the $\beta$ stays fixed. (Likewise distribution over posteriors is clumsy when we have

53Notice that we already used the symbol $\mu$ for the distribution of utilities. Now we are using it for the distribution of posteriors, which is a bit of an abuse of notation, but the two distributions play conceptually the same role.

54Given $\mu$, the experiment is not necessarily unique. Denti, Marinacci, and Rustichini (forthcoming) summarize a number of properties of the mapping between $\beta$ and $\mu$. 
state-dependent choice, as then we have a vector \((\mu^*)\) in our representation, instead of just one fixed \(\beta\).)

5.1.3. Action-Recommendations*. Consider a special kind of experiment, called action-recommendation, where the message space \(M\) equals the set of available actions \(A\). It is without loss to consider such experiments. Intuitively, for any experiment we can “glue together” all the messages that lead to the same action choice. This gives us a new experiment that just suggests an action to be taken. Taking this action is the optimal thing to do for the agent (why?), so this experiment leads to the same expected utility and same observed state-dependent choice frequencies as the original experiment.

Thus, for a fixed menu \(A\) the experiment is the stochastic choice function itself, \(\beta_s(x) := \rho^s(x, A)\). It is a useful simplification because we can read off the experiment directly from the observable. Note however that we now have a different experiment for each menu, whereas \(\beta\) stays fixed.

Suppose that the utility \(v\) and the prior \(p\) are given. The following condition says that the agent wants to follow the action-recommendation. It says that the expected utility of choosing \(x\) upon hearing the action recommendation \(x\) is higher than choosing some non-recommended action \(y\).

**Condition 5.5.** (Obedience) For all \(x\) such that \(\rho(x, A) > 0\) we have

\[
\sum_{s \in S} v(x, s)\rho^s(x, A)p(s) = \max_{y \in A} \sum_{s \in S} v(y, s)\rho^s(x, A)p(s).
\]

Obedience is used in game theory for defining correlated equilibrium (Bergemann and Morris, 2016). Another name for obedience is no improving action switches (Caplin and Martin, 2015). For the following proposition to hold we need to allow the tie breakers to be message-dependent.

**Proposition 5.6.** Suppose that \(S\) is finite, that \(p\) and \(v\) are given. For any fixed menu \(A\) the choice probabilities \(\rho^s(\cdot, A)\) have a Bayes representation if and only if obedience holds.

**Proof.** Section A.5.2
5.2. Examples

Prior. Even in controlled experiments different ways to induce a prior may not be equivalent (the two most popular techniques are to announce it and assume that the agent takes our announcement at face value, or run enough batches of trials to ensure the agent learns it). The observability requirement is less realistic if there are any unobservable utility or belief shocks.

Notice that with menu variation obedience is not enough. We need extra axioms that tie those choice probabilities across menus to ensure they are coming from the same experiment.

5.2. Examples

Example 5.7 (HR recruiter). The agent is a HR recruiter who is hiring an applicant based on an interview. The state of the world is the qualification of the applicant (high or low); let \( S := \{s_h, s_l\} \). This is what the recruiter ultimately cares about. The message is how the interview goes. It can go well or be a flop; let \( M = \{m_w, m_f\} \). Suppose that \( \beta(m_w|s_h) = \beta(m_f|s_l) =: b \in [0.5, 1] \) and let \( p := p(s_h) \). Then the recruiter’s posterior beliefs are

\[
q_{m_w} = \left( \frac{bp}{bp + (1-b)(1-p)}, \frac{(1-b)(1-p)}{bp + (1-b)(1-p)} \right) \\
q_{m_f} = \left( \frac{(1-b)p}{(1-b)p + b(1-p)}, \frac{b(1-p)}{(1-b)p + b(1-p)} \right).
\]

For example, if \( b = 0.5 \), then the experiment is completely uninformative so \( q_m = p \) for all \( m \). On the other hand, if \( b = 1 \), then the recruiter learns the true state perfectly: their belief is a point mass. In general, \( b \) measures the “strength” of the evidence.55

Notice that \( M \) are just “labels.” In particular, if we set \( b = 0 \), we also get a perfectly informative experiment, despite the fact that \( \beta \) tells the “opposite” of the true state. Our Bayesian agent is smart enough to invert the message. Likewise, an uninformative experiment is any constant function \( \beta \), not just the one that corresponds to \( b = 0.5 \).

Our HR recruiter’s choices are to hire \( x_h \) or not hire \( x_n \); let \( A := \{x_h, x_n\} \). The recruiter’s utility of hiring a qualified applicant equals 1 and an unqualified applicant, \( -1 \). The utility of not hiring is zero. Thus, applying the Bayes rule, the recruiter will hire after \( m_w \) if \( b + p > 1 \) and hire after \( m_f \) if \( p > b \).

Let’s pick values of \( p \) and \( b \) such that the recruiter wants to hire after \( m_w \) but not after \( m_f \). The analyst who observes the qualification of the applicant sees high-skilled applicants hired \( b \) percent of the time and low-skilled applicants being hired \( 1 - b \) percent of the time. If the analyst does not observe \( s \), they only observe the average proportion \( \pi b + (1 - \pi)(1 - b) \) of applicants who are

55Starting with Blackwell (1951), there is a tradition of measuring the informational content of \( \beta \) by a partial ordering; we will write \( \beta \geq \beta' \) if \( \beta \) is Blackwell-more informative than \( \beta' \). Section A.6.2 in the Appendix provides an overview. In this example, the Blackwell ranking coincides with the ranking of real numbers \( b \).
5.2. Examples

hired, where $\pi$ is the true proportion of highly qualified workers. On the other hand, if the analyst observes the result of the interview, then she sees the agent’s choices as deterministic: they will make the hire if the interview goes well and not make it otherwise.

Example 5.8 (Character Recognition). The Bayes representation is known in the perception literature as Signal Detection Theory (Tanner and Swets, 1954; Green and Swets, 1966).

In each trial the subject is briefly shown a character $m$ or $n$ and asked to identify it. Formally, $S = X = \{m, n\}$. Let $p$ be the subject’s prior. Let $M = \mathbb{R}$, so the subject gets a random perception of the sort “the character looks very much like $m$” or “the character only kind of looks like $m$”, etc. If $\beta(m|s)$ is the density of the experiment, Bayes rule says that the posterior is

$$
\frac{q(m|m)}{q(n|m)} = \frac{\beta(m|m) p(m)}{\beta(m|n) p(n)}.
$$

Let $p := p(m)$ and $\ell$ be the likelihood ratio: $\ell(m) := \frac{\beta(m|m)}{\beta(m|n)}$. Assuming that the payoff of correct guesses is one and incorrect zero, it follows that the agent chooses $m$ conditional on message $m$ if $\ell(m) > \frac{1-p}{p}$ and chooses $n$ if the opposite inequality holds. Let $L(k) := \{m \in M : \ell(m) > k\}$ and notice that $k > k'$ implies $L(k) \subseteq L(k')$. Thus, if $\rho$ has such a Bayes representation we have $\rho^{s,p}(m) = \beta(L(\frac{1-p}{p})|s)$, which is an increasing function of $p$.

![Figure 5.1. A ROC curve. Each point on the curve corresponds to a different prior $p$ for a fixed difficulty of the task. Making the task harder shifts the curve toward the diagonal.](image)

Old theories a la Fechner ruled out frequency-dependence because they predicted that behavior should be independent of the prior. But experiments show that typically behavior is monotone in the prior (Swets, 1973; Gescheider, 1997). Figure 5.1 shows how data looks like in a typical perceptual experiment.

If we reduce this to action-recommendations and think of $\rho$ as the experiment, then it may seem that the agent is choosing a different experiment for each prior, as in active learning. It is true that the agent is producing a different
experiment each time, but the experiment they are endowed with is fixed so according to our definitions learning is passive. A statistician would say that the agent is choosing a test, but the likelihood function is fixed.

A key issue is matching the predictive Bayes model with data from the experiment. Suppose that the true state (character) varies from trial to trial. In the experiment trials are batched, so that the frequency of characters \( \pi \) is constant within each batch but varies across batches. We often assume that in each batch \( p = \pi \). This is known as rational expectations. We will talk more about it in Chapter 8.\(^{56}\)

The simplest justification behind this assumption is that the subject habituates to each batch of trials. It’s conceivable that by trial 100 out of 250 they adapt to the current batch and believe that the prevalent distribution of \( s \) is governed by \( \pi \). A more detailed model would assume that the agent knows that \( \pi \) changes over time and rationally updates their beliefs about \( \pi \). In cognitive science this is sometimes called an “online prior” (Petzschner and Glasauer, 2011; Verstynen and Sabes, 2011; Cicchini, Anobile, and Burr, 2014).

**Example 5.9** (Law of Comparative Judgment from Bayes Rule). Recall Example 1.17, where we assumed that the agent is facing two items and incentivized to pick the one with higher weight. Thurstone’s model says that the agent picks the item with a higher perceived weight. In that model the agent is taking their perception at face value. We will now unpack this.\(^{57}\)

Suppose that the two items are distinguishable; for concreteness, one is on the left, the other is on the right, so \( X = \{l, r\} \). The state contains the true weights of the two items \( s = (s(l), s(r)) \). The message space is \( M = \mathbb{R}^2 \), so each message contains the weight perception of each of the items. In state \( s \) the agent gets a message

\[
\tilde{m}(x) = \gamma(s(x)) + \tilde{\varepsilon}(x),
\]

where \( \gamma \) is Thurstone’s transformation function and \( \tilde{\varepsilon}(x) \sim \mathcal{N}(0, \sigma^2_{\varepsilon}) \) are independent across \( x \).

The agent’s prior belief is that \( \gamma(s(x)) \sim \mathcal{N}(\mu_0, \sigma^2_0) \); for example if \( v \) is log this means that \( x \) is distributed log-normally.

The Bayes rule works here as follows. The posterior given message \( \tilde{m}(x) \) is that \( \gamma(s(x)) \sim \mathcal{N}(\mu_1, \sigma^2_1) \), where

\[
\mu_1(m) = \sigma^2_1 \left( \frac{\mu_0}{\sigma^2_0} + \frac{m}{\sigma^2_{\varepsilon}} \right) \quad \text{and} \quad \sigma^2_1 = \left( \frac{1}{\sigma^2_0} + \frac{1}{\sigma^2_{\varepsilon}} \right)^{-1},
\]

\(^{56}\)This name is used by economists to describe the idea that the agent’s beliefs are equal to the true data generating process. The data generating process in general includes behavior of “nature” as well as other agents, so the rational expectations assumption adds an equilibrium requirement. But in a one-person economy the two notions coincide.

\(^{57}\)Similar examples were given by Jazayeri and Shadlen (2010) and Polanía, Woodford, and Ruff (2019).
5.3. Random Tastes vs Random Beliefs

see, e.g., Lemma 15.7 of Williams (1991). Thus, the agent’s posterior is Normal, with a deterministic variance and mean that is a random quantity equal to a weighted average of the prior mean and the message realization. The weights depend on how informative the message is relative to the prior: for a fixed prior sending $\sigma_\epsilon \to \infty$ results in weight zero, and sending it to zero results in putting all the weight on the message.

In the language of distributions over posteriors, $\mu$ puts probability one on the collection of Normals with a fixed variance $\sigma_1^2$. Their mean is distributed $N(\mu_0, \sigma_0^2 - \sigma_1^2)$ (why?).

Given the payoff structure the agent chooses $l$ if their posterior probability that $s(l) > s(r)$ is bigger than a half. Thus, it is optimal for the agent to choose $l$ over $r$ iff the posterior mean of $\gamma(s(l))$ is greater than the posterior mean of $\gamma(s(r))$, i.e., $\gamma(s(l)) + \tilde{\epsilon}(l) > \gamma(s(r)) + \tilde{\epsilon}(r)$. This is exactly Thurstone’s probit, so it explains the same two of the stylized facts: S-shaped psychometric function and diminishing sensitivity (recall Example 1.17), but cannot give us payoff-monotonicity or frequency-dependence.\(^{58}\)

\[\text{Remark 5.10 (Noisy Coding and Efficient Coding). Fechner and Thurstone taught us to think of } m \text{ as an internal mental representation of the stimulus, a sort of noisy encoding of the truth (Khaw, Li, and Woodford, 2021). Noisy coding does not have to be efficient: in Examples 5.8 and 6.10 the signals are given exogenously, not optimized over. What is efficient is their usage (the action is Bayes-optimal given the message). In cognitive science efficient coding generally means optimally choosing the signal (e.g., the firing rate of neurons) subject to a metabolic cost (Barlow et al., 1961; Rustichini, Conen, Cai, and Padoa-Schioppa, 2017; Polanía, Woodford, and Ruff, 2019; Bucher and Brandenburger, 2021). This is the territory of active learning (Chapter 6).}\]

5.3. Random Tastes vs Random Beliefs

Proposition 5.4 shows that random tastes (RU) leads to the same behavior as random beliefs (average Bayes). This is because with state-dependent utility there isn’t really a separation between tastes and beliefs. Suppose that we have a preference represented by a belief $p$ and a state-dependent utility $v$. For any other $q \in \Delta(S)$ such that $q(s) > 0$ in all states where $p(s) > 0$, there exists $v'$ (given by $v'(\cdot, s)p(s)/q(s)$) such that $(v', q)$ is a state-dependent representation of the same preference. To overcome this problem it is often assumed that $v$ is state-independent.

5.3.1. Deterministic Choice. In this model the agent is choosing between acts, which are state-contingent payoffs. Let $Z$ be the set of primitive payoffs, like in Chapter 4. In the Savage (1972) model, an act is a mapping $f : S \to Z$

\[^{58}\text{We can get another form of frequency-dependence: if the subject is consistently presented with a heavier left item, then as opposed to Thurstone’s model, the subject will choose the left item with a higher frequency, much like in Example 5.8.}\]
that describes which prize the agent gets in every state. Following Anscombe and Aumann (1963), we will study acts \( f : S \to \Delta(Z) \). Here, in each state the agent gets a lottery over prizes in the set \( Z \). These acts are a bit contrived but they simplify the analysis considerably.\(^{59}\)

Typical acts are denoted \( f, g, h \) and the agent has a preference \( \succsim \) over acts. An Anscombe Aumann representation \( U(f) \) of \( \succsim \) consists of a prior \( p \in \Delta(S) \) and a vNM utility function \( v : \Delta(Z) \to \mathbb{R} \), such that

\[
U(f) = \mathbb{E}_p v(f) = \sum_{s \in S} v(f(s))p(s).
\]

The key assumption is that utility is state-independent, i.e., it does not depend on \( s \). State independence is heavily debated, see, e.g, Aumann and Savage (1987), Karni, Schmeidler, and Vind (1983), but it is an important identifying assumption because it pins down \( p \) uniquely and \( v \) cardinally.

The representation is characterized by two axioms. The first one relies on the mixing operation for acts. For any fixed \( \alpha \in [0, 1] \) an \( \alpha \)-mixture of two acts \( f \) and \( g \) is another act that in each state \( s \in S \) gives the lottery \( \alpha f(s) + (1 - \alpha)g(s) \). We will denote this act \( \alpha f + (1 - \alpha)g \). Formally, \( (\alpha f + (1 - \alpha)g)(s) := \alpha f(s) + (1 - \alpha)g(s) \).

Note that \( U \) is linear in those mixtures. As a consequence, the main axiom is Independence (Axiom 4.2) written using those mixtures.

**Axiom 5.11** (Independence). For all \( \alpha \in (0, 1) \) and \( r \in \Delta(Z) \)

\[
f \succsim g \text{ if and only if } \alpha f + (1 - \alpha)h \succsim \alpha g + (1 - \alpha)h.
\]

Independence only gives us a state-dependent representation. We need another axiom to get state-independence. For each \( s \in S \) with a slight abuse of notation we understand \( f(s) \) to be a constant act that pays off the same lottery \( f(s) \) in every state.

**Axiom 5.12** (Monotonicity). If \( f(s) \succsim g(s) \) for all \( s \in S \), then \( f \succsim g \).

Intuitively, if for each \( s \) you’d rather commit to getting \( f(s) \) in every state rather than \( g(s) \) in every state, then you should choose \( f \) over \( g \) before knowing which state is realized.

**Axiom 5.13** (Mixture Continuity). For any \( f, g, h \) the sets

\[
\{ \alpha \in [0, 1] : \alpha f + (1 - \alpha)g \succsim h \} \text{ and } \{ \alpha \in [0, 1] : \alpha f + (1 - \alpha)g \precsim h \}
\]

are closed in \([0, 1]\).

**Theorem 5.14.** A preference \( \succsim \) satisfies Axioms 5.11, 5.12, and 5.13 if and only if it has an Anscombe–Aumann representation. Moreover, if \( \succsim \) is nontrivial, then the probability is unique and utility is unique up to a positive affine transformation.

\(^{59}\)Actually, Anscombe and Aumann (1963) studied even more contrived acts. The exposition in this section is due to Fishburn (1970).
5.3. Random Tastes vs Random Beliefs

5.3.2. Stochastic Choice and Learning. Let’s go back to our agent, who is receiving information. In Section 5.1 we said this can be modeled by a distribution over posteriors $\mu \in \Delta(\Delta(S))$. To make things simple, assume that the utility is state-independent and deterministic. This way, all randomness in choice is driven by learning (variation in information) and not by randomness in tastes.\(^60\)

For any posterior $q$ the conditional expected utility of act $f$ is

$$\tilde{U}(f) = \mathbb{E}_q v(f) = \sum_{s \in S} v(f(s))q(s).$$

This theory was developed for the average s.c.f. (unconditional on state).

**Definition 5.15.** $\rho$ has an Anscombe–Aumann representation if there exist a random belief $\mu \in \Delta(\Delta(S))$, and a linear utility function $v : \Delta(Z) \to \mathbb{R}$ such that

$$\rho(f, A) = \mu\left(\left\{ q \in \Delta(S) : \mathbb{E}_q v(f) = \max_{g \in A} \mathbb{E}_q v(g) \right\}\right).$$

This is not well defined if there are ties. The original paper develops a novel approach to this problem, which I will not discuss but put a dagger † on all the results. A serious reader will want to consult the paper.

If $\rho$ has an Anscombe–Aumann representation, then it satisfies the Gul–Pesendorfer axioms (Axioms 4.14, 4.15, and 4.16) written using this new mixture operation. These axioms alone are not enough: they lead to a random linear $\tilde{U}$ defined over acts. We can write it as $\tilde{U}(f) = \sum_{s \in S} \tilde{v}(f(s), s)\tilde{q}$, but this would not add much because as discussed before, we can absorb the beliefs into tastes.

Lu (2016) showed precisely what other axioms $\rho$ has to satisfy.

The first axiom guarantees state-independence of $v$. Similarly to the deterministic version, it says that that if there is an act that is best in each state, then this act should be chosen ex ante before the state is known. As you recall, $f(s)$ is understood as a constant lottery that pays off $f(s)$ in every state. Likewise, $A(s)$ is a menu of such lotteries as $f$ varies over $A$. Let $A(s) := \{f(s) : f \in A\}$.

**Axiom 5.16.** If $\rho(f(s), A(s)) = 1$ for all $s \in S$ then $\rho(f, A) = 1$.

The next axiom guarantees that $v$ is deterministic and all the randomness in choice comes from random posteriors, not random tastes.

**Axiom 5.17.** If $A$ is a menu of constant acts, then $\rho(f, A) = 1$ for some $f \in A$.

**Theorem† 5.18** (Lu 2016). $\rho$ has an Anscombe–Aumann representation iff it satisfies Regularity and Axioms 4.14, 4.15, and 4.16 plus 5.16 and 5.17. Moreover, the information structure $\mu$ is unique and the utility function $v$ is cardinally-unique.

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\(^{60}\)This distinction is absent in the model without acts. If we assumed that $v(x, s)$ does not depend on $s$ in Section 5.1, this would have given us a model of deterministic choice.
5.3. Random Tastes vs Random Beliefs

As discussed before, when utility is state-dependent, the prior is completely unidentified. However, as Lu (2019) shows, the methodology developed in this section can be used to point-identify the prior under state-dependent utility. The key is to observe the agent’s behavior across two information structures.

Theorem 5.18 can used to characterize state-dependent Bayes representations (a version of Definition 5.1). Suppose that in each state $\rho^s$ satisfies all the axioms, so by Theorem 5.18 there is a distribution over posteriors $\mu^s$. Suppose for simplicity that each $\mu^s$ has finite support. Suppose that $\pi \in \Delta(S)$ is the objective distribution of states (identified from the data the analyst observes).

The key condition says that the Bayesian update of $\pi$ given message $q$ gives us precisely $q$, so the agent takes messages at face value.

**Condition 5.19 (Bayes Consistency).** For all $s \in S$ and for all $q$ that is in the support of at least one $\mu^s$

$$\frac{\pi(s)\mu(q|s)}{\sum_{s' \in S} \pi(s')\mu(q|s')} = q(s).$$

Another condition makes sure the utility is the same in each state.

**Condition 5.20 (State-Independence of Tastes).** For all $s \in S$, when restricted to constant acts, the s.c.f. is the same.

**Proposition 5.21.** Suppose that $\rho^s$ satisfies the axioms from Theorem 5.18 in each state and Conditions 5.19 and 5.20 hold. Then there exists an experiment $\beta$ and $v : \mathbb{Z} \to \mathbb{R}$ such that in each state modulo ties we have

$$\rho^s(f, A) = \beta^s(\{m \in M : \mathbb{E}_{q_m}v(f) = \max_{g \in A} \mathbb{E}_{q_m}v(g)\}).$$

This follows because we can take $M := \Delta(S)$ and define $\beta(q|s) := \mu(q|s)$. By Constant Utility, the utility functions are affine transformations of each other; take $v$ to be one of them. Then Bayes Consistency is just (5.1). A similar characterization is stated in Sorin (2002).

I hesitate to call Bayes Consistency an axiom because it uses derived objects, such as distributions over beliefs. Note that this proposition assumes rational expectations. To relax, we could put an existential quantifier into Condition 5.19. Duraj (2018) is an attempt at making this condition more pallatable.

5.3.3. Comparative Statics. What happens to choices when the experiment becomes more informative? We say that $\mu$ is more informative than $\mu'$ if $\mu$ Blackwell-dominates $\mu'$ (Section A.6.2). Intuitively, this means that under $\mu$ the posterior is more random than under $\mu'$. Since it’s the randomness of posterior that drives randomness in choice, one intuition is that more information means that choices are more random. Before we formalize this intuition, it’s important to realize that it does not hold conditional on the state. In fact, conditional on
5.3. Random Tastes vs Random Beliefs

the state the agent will learn more about it and will be more likely to pick the optimal action, which makes choices less random.

**Example 5.22.** Example 5.9 developed the Normal-Normal model of perception. Suppose that the true weights are $x > y$, and let $\gamma$ be the Thurstone curving function. The parameter $\sigma$ controls the noisiness of the experiment. In this model the agent chooses $x$ with probability $\Phi(\frac{\gamma(x) - \gamma(y)}{\sqrt{2} \sigma})$, so as $\sigma \to 0$ the probability of choosing $x$ converges monotonically to 1.\(^{61}\)

The following example shows that more information can also lead to less randomness in choice unconditional on the state.

**Example 5.23.** Let $S = \{s_1, s_2, s_3, s_4\}$. There are two acts $f$ and $g$ with payoffs given in Table 1 (suppose utility is linear).

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$g$</td>
<td>0</td>
<td>1</td>
<td>50</td>
<td>0</td>
</tr>
</tbody>
</table>

*Table 1. Payoffs in Example 5.23*

Suppose that the prior is $(.49, .01, .01, .49)$. Consider two information structures: in the first one, the agent learns perfectly whether the state is in the cell $\{s_1, s_2\}$ or $\{s_3, s_4\}$; the second information structure perfectly reveals the state.

Under the first partition with probability .5 the first cell is realized and leads the agent to deterministically choose $f$; with probability .5 the second cell is realized and the agent chooses $g$. Thus, for an analyst who does not observe the state (only has access to the average $\rho$) the observed choice probabilities are $(.5, .5)$. Under the second partition those choice probabilities are are $(.98, .02)$—much less random.\(^{\spadesuit}\)

Thus, for our intuition to work, we will have to make some assumptions. First, assume that we are in the Anscombe–Aumann setting. Let $\tilde{f}, \tilde{f}$ denote the best and worst acts (with state-independent utility we can choose them to be constant acts). Normalize their utilities at 1 and 0. The utility of act $f^\alpha := \alpha \tilde{f} + (1 - \alpha) \tilde{f}$ is deterministically equal $1 - \alpha$.

By adding $f^\alpha$ to any menu $A$ we can see what is the probability that the expected utility of some act from the original menu is above $1 - \alpha$. This helps us get a grasp on the distribution $\mu$. The test function of $A$ is the mapping $\alpha \mapsto \rho(A, A \cup \{f^\alpha\})$. For each menu this is an increasing function from $[0, 1]$ to itself; in fact it is a c.d.f.

\(^{61}\)Of course, conditional on a state where $y > x$ that probability converges to 0. (The speed of convergence depends on how close the weights are.)
5.4. Menu-Dependent Signals

We say that $\rho$ is *more random* than $\rho'$ if for any menu $A$ the test function of $\rho$ second-order stochastically dominates the test function of $\rho'$ (Section 4.1.4).

**Theorem 5.24** (Lu 2016). Suppose that $\rho$ has an AA representation $(v, \mu)$ and $\rho'$ has a AA representation $(v, \mu')$ such that $p_\mu = p_{\mu'}$. In this case $\rho$ is "more random" than $\rho'$ if and only if $\mu$ is more informative than $\mu'$.

This can be illustrated using the following example.

**Example 5.25.** The agent is a risk-neutral lender, who is faced with a pool of loan applications and has to decide whether to approve each applicant. Each applicant has a fixed probability of default. Let $S := \{0, 1\}$ denote whether there is default or not. Approving an applicant results in a payoff of zero for the lender if the applicant defaults and one if they don’t default, so it’s an act $a(1) = \delta_0, a(0) = \delta_1$. The payoff of declining an application is a constant act $d(s) = 1 - \alpha \in [0, 1]$. The lender has utility linear in money and uses a fixed information structure $\mu \in \Delta(\Delta(S))$ to learn about the likelihood of default $\bar{q}$ before making a decision. This implies that the agent will choose $a$ over $d$ iff $\bar{q}(0) \geq 1 - \alpha$.

The analyst is a regulator who wants to check whether the lender is following proper anti-discrimination policies, e.g., not taking into account demographic information when evaluating loan applications. The regulator cannot condition on the information available to the lender, so the observed approval probability is

$$\rho(a, d) = \mu(\{q \in \Delta(S) : q(0) \geq 1 - \alpha\}).$$

If the regulator is able to vary $\alpha$, she can uncover the cdf of $\mu$. Suppose there is another lender who is known not to condition on demographics. Our regulator can compare the variability of the two revealed $\mu$ and potentially raise a flag.

### 5.4. Menu-Dependent Signals

So far information was independent of the menu. But sometimes new items can provide new information that sheds light on existing items. Consider the following, admittedly silly, example.

**Example 5.26.** Suppose that our agent is choosing between entrées in a restaurant and their utility is given by

<table>
<thead>
<tr>
<th></th>
<th>$\bar{U}_s($steak tartare$)$</th>
<th>$\bar{U}_s($chicken$)$</th>
<th>$\bar{U}_s($fish$)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s =$ good chef</td>
<td>10</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>$s =$ bad chef</td>
<td>0</td>
<td>5</td>
<td>0</td>
</tr>
</tbody>
</table>

Moreover, if a restaurant serves fish, then the moment our agent enters the restaurant this provides an informative experiment about the quality of
the chef. If the agent enters the restaurant and the whole restaurant smells like fish, this means that the chef is bad (so our agent chooses ch). If the restaurant serves fish but there is no fishy smell in the air, the chef is good (and our agent chooses s). Thus an analyst who does not observe $\rho^*$ but just $\rho$ will record the choice frequencies as $\rho(st, \{st, ch, fi\}) = \rho(ch, \{st, ch, fi\}) = \frac{1}{2}$ and $\rho(fi, \{st, ch, fi\}) = 0$. On the other hand, in absence of $fi$ on the menu the agent gets no message and has to go by their prior and maximizes ex ante expected utility; this leads to $\rho(st, \{st, ch\}) = 0$ and $\rho(ch, \{st, ch\}) = 1$ (if the prior is uniform). Thus, menu-dependent information behaves very much like menu-dependent utility and can lead to a violation of the Regularity axiom.

**Definition 5.27.** $\rho$ has a *menu-dependent learning* representation if for each menu $A$ there exists $\mu^A \in \Delta(\Delta(S))$ and a utility function $v : S \to \mathbb{R}^X$ such that

$$\rho(x, A) = \mu^A \left\{ q \in \Delta(S) : \mathbb{E}_q v(x) = \max_{y \in A} \mathbb{E}_q v(y) \right\}.$$

**Theorem 5.28 (Safonov 2017).** Any $\rho$ that satisfies Positivity (Axiom 1.16) has a menu-dependent learning representation.\(^{62}\)

One way to add bite to the general model would be to consider state-dependent s.c.f’s, instead of just looking at the average. Another route is parametric: Natenzon (2019) developed a Bayesian Probit model, where the agent observes a Normal signal of the utility of each item in the menu. The prior belief on $v \in \mathbb{R}^X$ is that each $v(x)$ is Normal with mean $\bar{v}(x)$ and independent across $x \in X$. The message is $m(x) = v(x) + \epsilon(x)$, where $\epsilon$ is jointly Normal. The correlation of messages means that adding new items to the menu can shed light on the utilities of existing items and therefore different menus lead to different experiments.

Natenzon (2019) developed a behavioral notion of similarity that captures the ranking of correlation coefficients. In addition he developed identification results, and necessary axioms for the model (versions of moderate stochastic transitivity and the BM axiom.). He used the model to explain decoy effect, compromise effect, and similarity effects.\(^{63}\)

Note that in Bayesian Probit adding an item gives more information about the state. It might be interesting to work out what is the bite of such a monotonicity assumption in a model without the Normal assumptions.\(^{64}\)

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\(^{62}\)When Positivity is lifted, the model is equivalent to acyclicity of a certain revealed relation, where $x$ is ranked above $y$ if it is chosen with probability one from some menu containing $y$.

\(^{63}\)Natenzon (2019) uses the concept of phantom alternatives—ones that are not available for choice, but are seen by the decision maker and therefore convey the message attached to them. This is similar to Safonov’s (2017) assumption that information can be varied independently of the menu.

\(^{64}\)Alternatively, could we assume that adding items results in less information because of information overload? Does this have any bite?
5.5. Other Work on Learning*

5.5.1. Menu correlated with the state. So far, the state was uncorrelated with the menu. We considered the case where information depended on the menu but the true state was uncorrelated with the menu (the distribution of $s$ was the same for each $A$). What if there is such a correlation? This happens in the frog legs example (Example 1.3). Here the menu is directly correlated with the state. Of course, given Theorem 5.28, it should not be surprising that this model does not have any bite.

Kamenica (2008) considers a version of this model where consumers make inferences from menus and firms strategically exploit this. He shows that the model explains choice overload and compromise effect.

5.5.2. Other Related Work. Gabaix and Laibson (2017) use a multiperiod learning model to microfound “as-if” discounting and present bias. Even though the agent’s utility is not discounted, their choices appear to reveal impatience because signals about future objects are more noisy, so the agent relies more on her prior.

Woodford (2020) overviews the perception literature in much more detail than here. In addition to the two tasks discussed here (Examples 1.5 and 5.8), he discusses a reproduction task, directly aimed at measuring the agent’s subjective perception of a given stimulus. A Bayesian model can explain a number of stylized facts about this task: the estimate is biased toward the prior (the mean of the batch $\pi$), and the variance of the estimate is higher for higher batches (for details see Jazayeri and Shadlen, 2010).

The idea that agents perceive quantities imperfectly can be applied to stochastic choice between lotteries (such as in Chapter 4). Here the agent has a fixed Bernoulli utility function $u$ but perceives the probabilities of each lottery with an error. This model is analyzed in Khaw, Li, and Woodford (2021) and further experiments done by Frydman and Jin (2022); a related idea is Enke and Graeber (2019).

Finally, there is the classic line of work directly demonstrating biases in updating and probabilistic reasoning more broadly, such as: gambler’s fallacy, hot hand fallacy, base rate neglect, confirmation bias, law of small numbers, and non-belief in the law of large numbers, etc. This line of work is very rich in interesting phenomena.

There is a seeming contradiction between using the Bayesian model to explain a number of perceptual phenomena, and at the same time noting that it is violated in a number of ways.

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65Gabaix and Laibson (2017) and Khaw, Li, and Woodford (2021) are in a sense an application of a similar idea, to the domains of time and risk, respectively.
Chapter 6

Active Learning

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6.1. The Model and Motivating Examples

In Chapter 5, randomness of choices was generated by randomness in agent’s messages. The experiment was exogenous: the information structure was fixed once and for all and could not be affected by the actions of the agent. This is called passive learning.

Active learning happens when information is a margin of choice: it can be chosen by the agent (at least to some degree). The agent may decide what to learn about and how much.

While passive learning satisfies Regularity, active learning can violate it. This can happen because changing the menu changes incentives to learn about the state and therefore changes the distribution over preferences. This cannot happen when learning is passive.

Example 6.1. Suppose that the state of the world is determined by the number of red balls on the screen. There are two equally likely states $S = \{s_1, s_2\}$. Let $X = \{x, y, z\}$ and the state-dependent utility is given in Table 1. If the menu is $\{x, y\}$ then the payoff from distinguishing the states is low and the agent chooses $y$ with probability zero in each state, provided that the cost of learning the state is higher than 1. On the other hand, if the menu is $\{x, y, z\}$ then it might make sense to pay attention to the states and choose $y$ in state $s_2$ and $z$ in state $s_1$. Thus, there is a violation of Regularity in state $s_2$ (and also from the point of view of the analyst who does not observe the state: the agent chooses $y$ with probability a half (and $z$ with probability a half), thus violating regularity. Dean and Neligh (2017) document experimentally such violations.

\begin{table}[h]
\begin{center}
\begin{tabular}{c|cc}
      & $s_1$ & $s_2$ \\
\hline
$x$   & 50    & 50  \\
$y$   & 40    & 52  \\
$z$   & 100   & 0   \\
\end{tabular}
\end{center}
\caption{Payoffs in Example 6.1}
\end{table}

In this chapter we will study a general model of active learning, where the agent can buy any kind of information they please and there is a cost function defined over information structures. Intuitively, the agent’s optimization problem is:

$$\max_i [V^A(i) - \text{cost}(i)],$$

where $i$ stands for “information” and $V^A(i)$ is the value of information $i$ given by maximizing posterior expected utility over the menu $A$.$^66$ Representing the choice problem as maximizing utility minus cost is superficially similar to the PU model of Section 3.5. However, in that model the agent was maximizing over a different variable and also the interpretation of the model was somewhat different.
6.2. Value of Information

There are two prevailing interpretations of this model:

1. **Costly information acquisition**: The agent can run actual physical experiments, at a cost. For example, hire a geologist to drill in the ground. This approach has roots in the statistics literature (Wald, 1947; Bohnenblust, Shapley, and Sherman, 1949; Blackwell, 1951; Raiffa and Schlaifer, 1961) and is the basic model in microeconomics (Persico, 2000; Bergemann and Välimäki, 2002).

2. **Costly information processing aka Rational inattention**: Information is already out there in front of the agent. The cost represents the mental energy of processing this information. This approach is due to Sims (2003, 2006, 2010).

The literature on rational inattention traditionally uses a specific cost function from information theory, called mutual information. It leads to a tractable model, but is at odds with some stylized facts. For instance, in the weight perception task (Example 1.5) it leads to the psychometric function that is a step function (the error rate is a constant function of the weight difference: it depends only on its sign). This is because for the mutual information agent it is equally difficult to distinguish between two “nearby” states (e.g., weight difference of 1g) and between two “far away” states (e.g., a weight difference of 500g). Given that their payoff function treats “easier” and “harder” guesses equally, there is no reason for the agent to make an extra effort. We will delve more formally into this later in this chapter.

This can be fixed by choosing a different cost function. Section 6.4 maps out different classes of cost functions and discusses how they relate to each other. Section 6.5 discusses axioms.

Active learning is also present in dynamic models, such as sequential sampling that will be discussed in Chapter 9. Another example of active learning are *experience goods*, where the agent can learn about their utility through consuming the good (we will discuss them briefly in the next chapter).

6.2. Value of Information

6.2.1. Expected Payoff. So far we had an abstract function $V(i)$. We will now be more specific. Recall that we can define $i$ either as an experiment $\beta : S \rightarrow \Delta(M)$ or as a distribution over posteriors $\mu \in \Delta(\Delta(S))$. Moreover, given a prior $p$ and an experiment $\beta$ we get a distribution over posteriors, which we will denote by $\mu = p \oplus \beta$. Of course, for a fixed maximization problem the agent’s prior $p$ is fixed, so they are not choosing over all $\mu$ but only those that average out to $p$.

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67In decision theory this is referred to as *costly contemplation*, see, e.g., Ergin (2003), Ergin and Sarver (2010), De Oliveira, Denti, Mihm, and Ozbek (2016), de Oliveira (2019)
6.2. Value of Information

Define the interim utility of menu \( A \) under belief \( q \) to be: \( v^A(q) := \max_{x \in A} \mathbb{E}_q v(x) \).

The value of information can be written in two ways:

\[
V^A_p(\beta) = \sum_{s \in S} \sum_{m \in M} v^A(q_m) \beta(m|s) p(s),
\]

(6.1)

where \( p \in \Delta(S) \) is the agent’s prior, or:

\[
V^A(\mu) = \sum_{q \in \Delta(S)} v^A(q) \mu(q).
\]

(6.2)

We have \( V^A_p(\beta) = V^A(p \oplus \beta) \) for any \( p \) and \( \beta \). It depends on the prior because the value of a piece of information depends on what the agent already knows.

**Example 6.2.** Suppose that \( A = \{x, y\} \) and \( S = \{s_x, s_y\} \). The agent wants to match the state: \( v(x, s_x) = 1 = v(y, s_y) \) and otherwise zero. Consider Figure 6.1. Suppose that \( \beta \) is a binary experiment that is almost uninformative. Then if \( p \) is a prior, the resulting \( p \oplus \mu \) will put weight on two posteriors which are close to \( p \). If \( p = \frac{1}{2} \), then we are in the strictly convex part of \( v^A(q) \) and therefore the benefit of experiment \( \beta \) is positive: \( V^A_p(\beta) - v^A(p) > 0 \). On the other hand, if \( p \) is low enough, then both those posteriors will lie on the linear segment of the function \( v^A(q) \) and therefore the net benefit is zero.  

6.2.2. Cost. We will denote by \( h \) a cost function defined on experiments and by \( c \) a cost function defined on distributions over posteriors. Going forward, we will assume that all \( h \) and \( c \) are Blackwell-monotone, i.e., more information

---

If \( M \) is infinite we need to change the second sum in the first expression to an integral. If the support of \( \mu \) is infinite, we need to do the same with the second expression.

In general, \( v^A(q) \) is a convex and continuous function, so the benefit of any experiment is nonnegative. This is because information adds risk in the space of beliefs. This is related to one of the characterizations of the Blackwell theorem A.6.2.

In general, with finitely many actions there will be linear regions where the agent’s demand for information is zero, see Radner and Stiglitz (1984) and Chade and Schlee (2002).
is more costly. This is without loss of generality because if a less informative experiment costs more, then nobody would use it.\textsuperscript{71} It is also without loss to assume cost is normalized so that uninformative experiments are costless, i.e., \( c(\delta_p) = 0 \) for all \( p \in \Delta(S) \).

Recall that for any \( \mu \) there exists a prior (equal to \( p_\mu = \int q\mu(dq) \)) and an experiment \( \beta_\mu \) such that \( \mu = p_\mu \oplus \beta_\mu \).

Given a cost function \( h \) we can construct a cost function \( c_h(\mu) := h(\beta_\mu) \).\textsuperscript{72} Such a cost function is independent of the prior: \( c(p \oplus \beta) = c(p' \oplus \beta) \) for all full support \( p, p' \).\textsuperscript{73} But not all cost functions \( c \) are of this form.

**Definition 6.3.** A cost function \( c : \Delta(\Delta(S)) \rightarrow [0, \infty] \) is **prior-independent** if \( c(p \oplus \beta) = c(p' \oplus \beta) \) for all \( \beta \) and all full support \( p, p' \). A cost function \( c \) is **prior-dependent** if it is not prior-independent.

For an example of a prior-independent cost function, imagine that you are hiring an expert to perform a geological survey. The expert charges only based on the number of drillings, not on your prior belief. This makes it sound like in information acquisition problems the cost should be prior-independent. More on this later.

Should the cost of information processing be prior independent? Mutual information depends not only on the experiment, but also on the prior \( c(p \oplus \beta) \neq c(p' \oplus \beta) \), which means the mental cost of processing this information is different for people holding different beliefs. To me, prior-dependence is a useful working distinction between “costly information acquisition” and “rational inattention.” Other cost functions used here are also prior-dependent.

This distinction doesn’t matter if we only observe behavior for a fixed prior. But it matters if there is experimental variation in prior, or we observe behavior across time (yesterday’s posterior is today’s prior), or in game theoretic models where the prior is an equilibrium quantity.

### 6.2.3. Optimization

A Bayesian agent maximizes the value of information minus the cost. We can write the maximization problem as choosing an experiment, or equivalently, as choosing a distribution over posteriors.

\[
\max_{\beta \in \mathcal{E}} V_p^A(\beta) - c(p \oplus \beta) = \max_{\mu \in \Delta(\Delta(S))} \text{s.t. } \bar{\mu} = p V^A(\mu) - c(\mu) \tag{6.3}
\]

These maxima exist if we assume that \( c \) is lower-semicontinuous, \( S \) is finite, and \( M \) is compact.

\textsuperscript{71}The Blackwell order is introduced in Section A.6.2 in the Appendix. Formally, we can define another cost function where this experiment now costs as much as the more informative one. This will not change behavior because the value function (6.1) and (6.2) is Blackwell-monotone.

\textsuperscript{72}It doesn’t matter which \( \beta_\mu \) we choose, as by part (3) of Theorem A.6.4 they are all Blackwell-equivalent, so they have the same cost.

\textsuperscript{73}We say that \( p \) has full support if \( p(s) > 0 \) for all \( S \); we say that \( \mu \) has full support if \( p_\mu \) has full support.
Definition 6.4 (Active Learning). \((\rho^s)_s\) has an active learning representation if there exists a prior \(p \in \Delta(S)\), utility \(v : S \times X \to \mathbb{R}\), and cost \(c : \Delta(\Delta(S)) \to [0, \infty]\) such that
\[
\rho^s(x, A) = \beta^*_A \left( \left\{ m \in M : \mathbb{E}_{q_m} v(x) = \max_{y \in A} \mathbb{E}_{q_m} v(y) \right\} \mid s \right),
\]
where \(q_m\) is the Bayesian posterior given \(m\), \(\beta\) and \(p\) and \(\beta^*_A\) solves (6.3) for each \(A\).

Remark 6.5. In Definition 6.4, attention is perfectly tailored to the details of each choice problem \((A, v, p)\). But we know little about how fast it adjusts, except in some controlled experiments on perception. Maybe it is tailored to situations we are facing on average? Maybe agents make systematic mistakes in allocating it?

This optimality assumption should not be taken literally: this is a hard optimization problem. It is a convenient as-if assumption that adds structure to the model and is sensible at least in those cases where we’d expect attention to respond positively to incentives. If we want to replace the maximization problem with something else, we will need a theory of how attention is chosen (in light of Proposition 5.28, we’ll need some structure on the assignment of attention to decision problems).

Remark 6.6. In another formulation there is no cost function but a constraint: the agent solves \(\max_{i \in \Gamma} V^A(i)\), where \(\Gamma\) is some set of constraints on \(i\). For example \(\Gamma\) includes only normal experiments conditional on the state. Another example is passive learning where \(\Gamma\) is the set of all experiments weakly less informative than some fixed experiment.

For any fixed decision problem the two formulations are equivalent, but across decision problems the cost formulation is more general.

One implication of the constraint formulation is that scaling up the payoffs does not change behavior. With cost, the agent makes (weakly) better choices when stakes are raised. Dean and Neligh (2017) show in a perception-style experiment that error rates diminish when payoffs scale up, but the stylized fact in psychology is that stakes don’t matter.\(^{74}\)

Remark 6.7. For a fixed menu \(A\) this maximization problem can be simplified by focusing action-recommendations (where the message space \(M\) equals the set of available actions \(A\)). The class of such experiments will be denoted \(\mathcal{R}\). If \(c\) is Blackwell monotone then without loss the agent can restrict the maximization problem to \(\mathcal{R}\).

\[
\max_{\beta \in \mathcal{E}} V^A_p(\beta) - c(p \oplus \beta) = \max_{\beta \in \mathcal{R}} V^A_p(\beta) - c(p \oplus \beta).
\]

\(^{74}\)To see that cost is more general, set \(c(i) = 0\) if \(i \in \Gamma\) and infinity otherwise. This is similar to the literature on ambiguity aversion, where maximin preferences have a constraint and variational preferences have a cost (Gilboa and Schmeidler, 1989; Maccheroni, Marinacci, and Rustichini, 2006). Positive homogeneity is exactly what distinguishes these two classes. Likewise, in Hansen and Sargent (2008) there are “constraint preferences” and “multiplier preferences.”
6.3. Mutual Information

Intuitively, the action recommendation “glues” together all the messages that lead to the same action choice. This new experiment is less costly and leads to the same expected utility. Because of this, an optimal experiment is precisely equal to the state-dependent stochastic choice function \( \beta_s(x) = \rho_s(x) \). ▲

### Definition 6.8.

The mutual information of \( \mu \) is defined to be

\[
c(\mu) = \lambda \int_{\Delta(S)} [H(p_\mu) - H(q)] \mu(dq),
\]

where \( \lambda > 0 \) and \( H(q) = -\sum_{s \in S} q(s) \log_2 q(s) \) is the entropy of \( q \).

Intuitively, \( H(q) \) measures how much uncertainty is contained in a belief \( q \), so \( c(\mu) \) is the average reduction in uncertainty in beliefs.\(^{75}\)

The following lemma gives us a recipe to determine the choice probabilities: solve problem (6.4) and then then plug it into (6.5). The observed choice probability is given by a Luce-like formula, where the Luce choice probabilities are reweighed towards the average choice probabilities.

### Lemma 6.9.

Suppose that the cost function is mutual information with parameter \( \lambda \) and the prior is \( p \). Let \( \rho^*_s \) be the optimal choice probabilities and \( \rho^*_s(x, A) := \sum_{s \in S} \rho^*_s(x, A)p(s) \) be the average choice probability. Then \( \rho^*_s(\cdot, A) \) is a unique solution to

\[
\max_{\rho \in \Delta(A)} \sum_{s \in S} \log \left( \sum_{y \in A} \rho(y)e^{v(y,s)/\lambda} \right) p(s). \tag{6.4}
\]

Moreover, the conditional choice probabilities (and hence the optimal action-recommendation) are

\[
\rho^*_s(x, A) = \frac{\rho^*_s(x, A)e^{v(x,s)/\lambda}}{\sum_{y \in A} \rho^*_s(y, A)e^{v(y,s)/\lambda}}. \tag{6.5}
\]

Finally, if \( \rho^*_s(x, A) > 0 \), then the posterior belief in state \( s \) given action recommendation \( x \) equals

\[
\frac{p(s)e^{v(x,s)/\lambda}}{\sum_{y \in A} \rho^*_s(y, A)e^{v(y,s)/\lambda}}.
\]

### Proof.

See Proposition 2 of Steiner, Stewart, and Matějka (2017) which combines Theorem 1 and Lemma 1 of Matejka and McKay (2015) and Theorem 1 of Caplin and Dean (2013). Matejka and McKay’s (2015) result holds for an infinite state space \( S \).\(^{76}\) □

---

\(^{75}\)In information theory, this is the mutual information between the state and the message (which is implicit here), see Cover and Thomas (1991).

\(^{76}\)In fact, instead of a state-dependent utility function \( v \) they work directly with (not necessarily discrete) distributions over \( v \in \mathbb{R}^X \).
Lemma 6.9 makes it possible to axiomatically characterize behavior of an agent with mutual information cost. Matejka and McKay (2015) study choice between Savage acts and an analyst who observes the state-dependent s.c.f. $\rho^s$. Each of their two axioms is specific weakening of Luce’s IIA.

**Example 6.10.** Let’s apply Lemma 6.9 to the perception task (Example 1.5). Let $X = l, r$ and $S = \mathbb{R}^X$. Suppose that the prior is symmetric, i.e., $p(s(l) > s(r)) = p(s(l) < s(r))$. The payoff function is $v(l, s) = w$ if $s(l) > s(r)$ and zero otherwise, $v(r, s) = w$ if $s(r) > s(l)$ and zero otherwise.

Note first that the objective function (6.4) is symmetric in $\rho(l)$ and $\rho(r)$, so $\rho(l, r) = \frac{1}{2}$.

By (6.5), we have

$$
\rho^s(l, r) = \begin{cases} 
\frac{e^{w/\lambda}}{e^{w/\lambda}+1} & \text{if } s(l) > s(r), \\
0.5 & \text{if } s(l) = s(r), \\
\frac{1}{e^{w/\lambda}+1} & \text{if } s(l) < s(r).
\end{cases}
$$

Thus, for a fixed value of $s(r)$ the choice probability is a step function of $s(l)$.

### 6.4. Other Cost Functions

Technical note: For simplicity in this section we will work with a smaller domain. Following Bloedel and Zhong (2021), posteriors will be uniformly bounded away from mass points.

A generalization of mutual information replaces $H$ with an arbitrary convex and continuous function $L$. This class is referred to as uniformly posterior-separable (Caplin, Dean, and Leahy, 2022). Uniformly, because the function $L$ is the same for all priors. *Posterior separable* allows for such dependence.

**Definition 6.11.** A cost function $c$ is *uniformly posterior-separable* (UPS) if

$$
c(\mu) = \int_{\Delta(S)} [L(p_\mu) - L(q)]\mu(dq),
$$

where $L : \Delta(S) \to \mathbb{R}$ is convex and continuous. A cost function $c$ is *posterior-separable* (PS) if

$$
c(\mu) = \int_{\Delta(S)} [D(p_\mu, p_\mu) - D(q, p_\mu)]\mu(dq),
$$

where $D : \Delta(S) \times \Delta(S) \to \mathbb{R}$ is convex and continuous in the first variable.

UPS is characterized by the following condition. Suppose that we have an experiment $\beta'$ and then conditional on message $m$ we run another experiment

\(^{77}\)As discussed by Pomatto, Strack, and Tamuz (2018), this conclusion is not unique to the mutual information: it also holds for any prior-independent cost function $h$ that is invariant with respect to a permutation of the states and is convex as a function of the state-dependent action distributions ($\beta_\mu$).

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\(^{77}\)As discussed by Pomatto, Strack, and Tamuz (2018), this conclusion is not unique to the mutual information: it also holds for any prior-independent cost function $h$ that is invariant with respect to a permutation of the states and is convex as a function of the state-dependent action distributions ($\beta_\mu$).
This defines a new compound experiment $\beta_m'$. We could execute it step by step like a compound lottery or reduce it.

A cost function is \textit{indifferent to sequential learning} if the cost of each execution is the same. This condition has a dynamic flavor. We will talk more about dynamic models in Section 9.7.

**Proposition 6.12.** $c$ is UPS if and only if it is indifferent to sequential learning.

**Proof.** See Lemma 1 of Bloedel and Zhong (2021). □

PS is characterized by the following condition. Suppose that we have two experiments $\beta'$ and $\beta''$ and assume that the set of messages of the two experiments is disjoint so the agent always knows which one was used. Consider another compound experiment that with probability $\alpha$ executes $\beta'$ and otherwise $\beta''$. A cost function is indifferent to randomization if the direct cost of the compound experiment equals the expected cost of running the compound experiment.

**Proposition 6.13.** $c$ is PS if and only if it is indifferent to randomization.


Given that in the UPS class the function $L$ is prior-independent, it is tempting to think that such costs are prior-independent. But in fact the opposite is true.

**Proposition 6.14.** If $c \neq 0$ is bounded and UPS, then it is prior-dependent. In particular, mutual information is prior-dependent.

Consider a fixed experiment and a sequence of priors converging to a Dirac measure. The belief movement converges to zero along this sequence, so under UPS the cost converges to zero. In contrast, if the cost was prior-independent, then the sequence would be constant. For details, see Appendix A.6.1.\footnote{This result is Proposition 4 of Mensch (2018), Proposition 1 of Denti, Marinacci, and Rustichini (forthcoming), and footnote 38 of Bloedel and Zhong (2021). The fact that the mutual information cost is prior-dependent was discussed by Woodford (2012), Gentzkow and Kamenica (2014), and Che and Mierendorff (2019).}

The relationship between UPS, PS, and PI is illustrated in Figure 6.2. The figure also depicts some additional classes, which we discuss in the remainder of this section.

An important class of prior-independent cost functions was studied by Pomatto, Strack, and Tamuz (2018). They assume constant marginal cost (CMC). This consists of two axioms: the cost of running the experiment twice (i.i.d. conditional on $s$) is twice the cost of running it once and running an experiment with probability a half (and an null experiment with probability half) is equal
to half its cost. Notice that the first property is implied by indifference to sequential learning while the second is implied by indifference to randomization. Taken together with prior independence, these conditions plus continuity imply that $h$ has a weighted relative entropy (WRE) representation

$$h(\beta) = \sum_{s,s' \in S} \alpha_{s,s'} R(\beta_s || \beta_{s'}),$$

for some collection weights $(\alpha_{s,s'})_{s,s' \in S}$ in $\mathbb{R}_+$ where for any $\pi, \pi' \in \Delta(M)$ the relative entropy or Kullback-Leibler divergence is defined by $R(\pi || \pi') := \sum_{m \in M} \pi(m) \log \frac{\pi(m)}{\pi'(m)}$ if $\pi'(s) > 0 \Rightarrow \pi(s) > 0$ and $R(\pi || \pi') = 0$ otherwise. Pomatto, Strack, and Tamuz (2018) show that this class is PS with

$$D(q || p) = \sum_{s,s'} \beta_{s,s'} \frac{q_s}{p_s} \log \frac{q_s}{q_{s'}}.$$  

This is not UPS because it is prior-independent. Unlike mutual information, WRE leads to reasonable psychometric functions in Example 1.17.

If we take CMC (the two axioms) together with UPS, we get

$$c(p \oplus \beta) = \sum_{s,s' \in S} p(s) \alpha_{s,s'} R(\beta_s || \beta_{s'}).$$
6.5. Behavioral Characterizations

which was shown independently by Pomatto, Strack, and Tamuz (2018) and Bloedel and Zhong (2021). The latter paper dubs them total costs and studies interesting dynamic stability properties. The UPS representation of total costs is given by \( L(q) = \sum_{s,s'} \beta_{s,s'} q_s \log \frac{q_s}{q_{s'}}. \)

A classic prior-independent cost function is channel capacity, defined as \( h(\beta) := \max_{p \in \Delta(S)} \text{c}_{\text{mi}}(p \oplus \beta), \) where \( \text{c}_{\text{mi}} \) is mutual information. That is, the cost of the experiment is the maximal mutual information obtained by choosing over all priors.\(^79\) Woodford (2012) analyzed a series of examples using this cost.

The intersection of prior-independent and posterior-separable costs was characterized by Denti, Marinacci, and Rustichini (forthcoming). For any prior \( p \) and posterior \( q \) we can form the vector of likelihood ratios \( q(s)/p(s) \); let’s denote it by \( q/p \). The function \( c \) belongs to the intersection of those two classes if and only if it is likelihood-separable: \( c(\mu) = \int_{\Delta(S)} \phi(q/p_{\mu}) \mu(dq), \) where \( \phi : \mathbb{R}^S_+ \rightarrow \mathbb{R} \) is a continuous and sublinear function.

Hébert and Woodford (2021) propose and characterize a family of “neighborhood-based” cost functions that allow for the cost of learning about states to be affected by their proximity. This family contains mutual information and is a subset of uniformly posterior-separable class, so by Proposition 6.14 is prior-dependent. Such costs are flexible enough to generate S-shaped psychometric functions.

The intersection of total costs and neighborhood-based costs includes the Wald costs (Morris and Strack, 2019) and the Fisher Information.

Morris and Yang (2021) study coordination games where \( S = \mathbb{R} \) and each player is endowed with a prior-independent cost function. The key general properties are submodularity and a form of uniform continuity and the key running example is WRE.

Fosgerau, Melo, De Palma, and Shum (2020) study a general class of Bregman divergences. In a model with attributes (Section 10.2.2) they show how the resulting behavior is equivalent to a form of RU.

6.5. Behavioral Characterizations

A group of papers assumes that the analyst knows the agent’s utility function \( v \) and the prior \( p \). Caplin and Dean (2015) characterize the class of \( (\rho_s) \) induced by general cost functions. This involves obedience (Condition 5.5), which characterizes passive learning, and a new acyclicity condition.

To state this condition we need the concept of revealed distribution over posteriors. The condition will say that the agent’s utility net of cost cannot be improved by any counterfactual reallocation of distributions of revealed posteriors across menus.

\(^79\)According to Shannon’s coding theorem, if we think of \( \beta \) as a communication channel, then \( h(\beta) \) is the maximal transmission rate achievable with arbitrarily low error probability (Cover and Thomas, 1991).
Suppose the menu is $A$ and the agent faces an action-recommendation that in state $s \in S$ suggests action $x \in A$ with probability $\rho^s(x, A)$. Then by Bayes rule (5.1) the agent’s posterior belief is now

$$\hat{q}_x(s) = \frac{p(s)\rho^s(x, A)}{\sum_{s' \in S} p(s')\rho^{s'}(x, A)}.$$ 

In every state $s$ there is a distribution over those revealed posteriors. Notice that by construction it satisfies Bayes Consistency (Condition 5.19). Unconditionally on the state, the distribution over posteriors is an average of those according to the prior. Formally, $\hat{\mu}_A \in \Delta(\Delta(S))$ is

$$\hat{\mu}_A(q) := \sum_{x \in A} \sum_{s \in S} p(s)\rho^s(x, A),$$

where the sum over the empty set is zero.

Given any distribution over posteriors $\mu$ the agent’s net utility is $\int v^A d\mu - c(\mu)$. Consider now two menus $A$ and $B$. It may be that the agent’s gross utility of using $\hat{\mu}_B$ when choosing from menu $A$ is higher than using $\hat{\mu}_A$, that is $\int v^A d\hat{\mu}_B > \int v^A d\hat{\mu}_A$, because $\hat{\mu}_B$ is more informative but not worth the extra cost. Nevertheless, it must hold that

$$\int v^A d\hat{\mu}_A + \int v^B d\hat{\mu}_B \geq \int v^A d\hat{\mu}_B + \int v^B d\hat{\mu}_A.$$ 

This is because if we summed up the net payoffs instead of gross payoffs, then on each side of the inequality the costs sum up to the same thing, so they would cancel, and the assignment of $\hat{\mu}_A$ to $A$ is optimal given the cost. The NIAC condition generalizes this to longer cycles.

**Condition 6.15 (NIAC).** $\rho$ satisfies no improving action cycles (NIAC) if for any sequence of menus $A_1, \ldots, A_n$ such that $A_1 = A_n$ we have

$$\sum_{i=1}^{n-1} \int v^{A_i} d\hat{\mu}^{A_i} \geq \sum_{i=1}^{n-1} \int v^{A_i} d\hat{\mu}^{A_{i+1}}.$$ 

NIAC and obedience together characterize a general class of cost functions. NIAC is related to cyclic monotoncity (Section 10.5.3). As Theorem A.10.1 shows, cyclic monotoncity characterizes convex functions and this is why the cost function in Theorem 6.16 can be chosen to be convex.

**Theorem 6.16 (Caplin and Dean 2015).** Suppose that $S$ is finite, that $p$ and $v$ are given. It satisfies obedience and NIAC if and only if it has an active learning representation (Definition 6.4). In addition, the cost function can be chosen to be convex in $\mu$.

On this domain Denti (forthcoming) characterizes the PS class. He also characterizes the UPS class, along with Caplin, Dean, and Leahy (2022). Both papers offer characterizations of the mutual information case that are different from each other and from Matejka and McKay (2015). Dean and Neligh (2017)
design an experiment that uses this axioms and show that UPS is typically satisfied.

What about the prior-independent class? If we impose NIAC for each prior separately (each cycle has its fixed prior), then we’ll get a general cost function. But if we require NIAC to rule out cycles where both the menu and the prior can vary, then, as Denti (XXXX) shows, we get a prior-independent cost function.

Chambers, Liu, and Rehbeck (2020) characterize a general model without the additive separability between the value and cost of information.

As in the case of passive learning (Section 5.1.3), this approach assumes that the analyst knows the prior \( p \) and the utility function \( v \). As shown by Denti (forthcoming), if we state the problem in the Anscombe–Aumann setting (Section 5.3.1), the analyst just needs to know the prior, which she will under the rational expectations assumption.

The original theorem is formulated for Savage acts, but with a known state-dependent utility it is equivalent to Theorem 6.16. The original theorem assumes that \( \rho \) is given on a finite collection of menus and provides partial uniqueness results in the form of bounds on the cost function. The theorem extends to full domain (Caplin, Dean, and Leahy, 2017) and under a richness condition the cost function is pinned down uniquely.

6.5.1. Other Work. Another approach was taken by Lin (2017), who worked with Anscombe–Aumann acts. Here the analyst does not know the utility function nor the prior: they are recovered from the data. The analyst only observes the unconditional s.c.f. \( \rho \). His work builds on Lu’s (2016) characterization of passive learning on this domain (Section 5.3.2) and De Oliveira, Denti, Mihm, and Ozbek’s (2016) characterization of active learning on the domain of preferences over menus. He characterizes convex and Blackwell-monotone costs, and obtains essential uniqueness of cost. This means that in principle we don’t have to observe \((\rho^s)\) but only the average \( \rho \).

Yet another approach was taken by Ellis (2018) who restricts attention to partitional experiments (i.e., in each state of the world the message is deterministic). Conditional on the state the analyst observes the (deterministic) choice function. See also Van Zandt (1996).

For further reviews of this literature see Caplin (2016) and Mackowiak, Matejka, and Wiederholt (2018). Behavioral models introduce other elements, see Gabaix (2019). Gagnon-Bartsch, Rabin, and Schwartzstein (2020) ask whether behavioral agents will learn from their mistakes and which biases are “attentionally stable.”
Part 3

Dynamic Choice
Chapter 7

Dynamic Choice

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7.1. Stochastic Utility

Dynamic models have important applications to the consumer side of the economy (durable goods purchases, labor supply, education, fertility, and retirement), as well as to the firm side (investment decisions, patent renewals).

Suppose that we are tracking agent’s choices over time. In each period \( t \) they are choosing \( x_t \) from menu \( A_t \). We will now explore the ways in which their choices are connected across periods.

For now, let’s set aside the fact that the agent might be forward-looking: incorporating their own future choices into their current decision. Such dynamic optimality adds an additional restriction in the form of the Bellman equation; we will return to this in Chapter 8. For now, we focus on phenomena that exist whether agents are sophisticated or myopic (or anything in between). These phenomena are backward-looking in nature and have to do with observed choices appearing history-dependent, or correlated over time. Consider the following example.

![Figure 7.1. Conditioning on Past Choices.](image)

**Example 7.1 (History-Dependence).** Suppose that \( r \) is habit-forming drug and \( d \) is not. We would expect to see history-dependent choice behavior:

\[
\rho_{t+1}(r_{t+1}|r_t) > \rho_{t+1}(r_{t+1}|d_t)
\]

(7.1)

because taking \( r_t \) makes the agent crave \( r \) more in the future. In terms of the representation, we would capture this by letting \( U_t \) depend not only on \( x_t \) but also on lagged consumption. This is sometimes called state-dependence.

But history dependence can also occur in a more subtle way. Suppose that you had to predict someone’s vote in the presidential election. In forming your prediction it would make sense to condition on how this person’s voted previously (if you had access to such data). We would expect (7.1) to hold because political preferences are persistent over time, not because past votes
are habit-forming. In terms of the representation, we have
\[ \tilde{U}_t(x_t) = \tilde{v}(x_t) + \tilde{\epsilon}_t, \]
where \( \tilde{\epsilon}_t \) is a transitory i.i.d. shock and \( \tilde{v} \) is a persistent preference, unobservable to the analyst. If someone votes Republican in year \( t \), they reveal that probably \( \tilde{v}(r) > \tilde{v}(d) \), so they are more likely to vote republican next period. It is important to realize that in this case choices appear correlated to the analyst because of asymmetric information (the agent knows their \( \tilde{v} \), the analyst does not), and not because \( x_{t-1} \) enters \( \tilde{U}_t \).

One way to see the difference between state-dependence and asymmetric information is to imagine that choices in period 1 are exogenously randomized by the analyst. In this case, we would expect history-dependence to go away in the voting example, but not in the drug example.

For now, we will assume away state-dependence and focus on asymmetric information. We discuss habit in Section 7.4.

For now, we will also assume that the menus \( A_t \) and \( A_{t+1} \) are determined exogenously, i.e., drawn independently of past choices. I will call this the simple domain. This means that the agent’s choice in period \( t \) does not influence the menu of options available in period \( t + 1 \). For example, the set of candidates today is unaffected by previous votes of our agent. This assumption is also made by the literature on brand choice dynamics in marketing and economics, where in each period \( t \) the agent chooses a brand \( x_t \) from some exogenously determined menu \( A_t \).

Assumption 7.2 (Simple Domain). First the menu \( A_1 \) is drawn, then the agent learns \( \tilde{U}_1 \) and chooses \( x_1 \in A_1 \) to maximize it. The analyst observes that choice. Then the new period begins, the menu \( A_2 \) is drawn independently of \( x_1 \), the agent learns \( \tilde{U}_2 \) and chooses \( x_2 \in A_2 \) to maximize it, the analyst observes that.

If we take the sample size to infinity, we will get the joint choice probability
\[ \rho(x_1, x_2; A_1, A_2) \]
for all menus \( A_1, A_2 \). Equivalently, we could record the marginal probability \( \rho_1(x_1, A_1) \) and a conditional choice probability
\[ \rho_2(\cdot | x_1, A_1). \]

This formulation is useful in the recursive formulation for forward-looking agents. This gives us a s.c.f. in period \( t = 1 \) and a s.c.f. in period \( t = 2 \) conditional on each history. Here, when looking at period 2 choices the analyst conditions on the fact that in period 1 the choice \( x_1 \) was made from menu \( A_1 \).

\[80\] Similar limiting assumptions are made in the literature on panel data, where joint choice probabilities are assumed to be identified from the data. If \( \rho \) represents choices of a single individual, we need many trials for each history and we need to observe many histories. This is asking for a lot, but can be done in some perceptual experiments.
7.1. Stochastic Utility

The representation needs to keep track of the distribution of preferences over time. The utility in each period is random \( \tilde{U}_t : \Omega \times X_t \rightarrow \mathbb{R} \) and by definition independent of past choices. Define the event

\[
N(x_t, A_t) := \{ \tilde{U}_t(x_t) = \max_{y_t \in A_t} \tilde{U}_t(y_t) \}.
\]

**Definition 7.3.** \( \rho \) has a Stochastic Utility (SU) representation if there exists a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and utility functions \( \tilde{U}_t : \Omega \rightarrow \mathbb{R} \times X_t \) such that

\[
\rho(x_1, x_2; A_1, A_2) = \mathbb{P}(N(x_1, A_1) \cap N(x_2, A_2)).
\]

Alternatively, \( \rho_1(x_1, A_1) = \mathbb{P}(N(x_1, A_1)) \) and

\[
\rho_2(x_2, A_2|x_1, A_1) = \mathbb{P}(N(x_2, A_2)|N(x_1, A_1))
\]

for each \((x_1, A_1)\) such that \( \rho_1(x_1, A_1) > 0 \). As in the static model, we assume that \( \mathbb{P} \) does not depend on the menu pair.

Examples of SU abound. In panel data econometrics we have \( \tilde{U}_t(x_t) = \tilde{v}(x_t) + \tilde{\epsilon}_t(x) \), where \( \epsilon_t \) are i.i.d. over time and alternatives; the agent’s “type” \( v \) is drawn at \( t = 1 \) and is perfectly persistent. This is very similar to the random effects model in econometrics, except that here we do not have any covariates. We will discuss the model with covariates in Chapter 12 and here rely on menu-variation. Another interpretation is that we are observing choices of a pair of individuals whose utilities are subject to correlated shocks.\(^{81}\)

In models of brand choice, \( \tilde{U}_t \) follows a Markov process and the transition matrix can be estimated based on how persistent the choices \( \rho_t \) are.\(^{82}\)

A knife-edge case is when \( \tilde{U}_t \) is independent over time. In this case we don’t need to condition on histories. Equivalently, this means that the joint is the product of the marginals. This puts a lower bound on history dependence. In the next section, we investigate the upper bound.

The following example shows that we don’t always have a simple domain. We will discuss the more general case in Section 7.5.

**Example 7.4** (Spurious Violation of Regularity). Consider choices of supermarket customers depicted in Figure 7.2. If we naively define \( A := \{ \text{medium, cheap} \} \) and \( B := \{ \text{premium, medium, cheap} \} \), then we code this as a violation of regularity because \( \rho(\text{medium}, A) < \rho(\text{medium}, B) \). This happens because the distribution of preferences is different between menus \( A \) and \( B \) as a result of self-selection: agents of different types select differently into the two supermarkets.

---

\(^{81}\)See e.g., Chambers, Masatlioglu, and Turansick (2021). A recent decision theoretic paper studying these kinds of models with menu-variation is Dardanoni, Manzini, Mariotti, Petri, and Tyson (2022).

\(^{82}\)For example, Jeuland (1979); Keane (1997); Dubé, Hitsch, and Rossi (2010); Seetharaman (2004); Dew, Ansari, and Li (2020), and references therein.
7.2. Axioms for SU

Because we ruled out state-dependence, the realization of \( U_2 \) does not depend on \( x_1 \) as a function. This implies that the marginal choice distribution in period \( t = 2 \) is independent of the menu in period \( t = 1 \).

**Axiom 7.5** (Marginal Consistency). For any \( A_2 \) the marginal choice distribution

\[
\sum_{x_1 \in A_1} \rho_2(x_2, A_2 | x_1, A_1) \rho_1(x_1, A_1)
\]

does not depend on \( A_1 \).

We already implicitly assumed a flavor of Marginal Consistency by writing \( \rho_1(x_1, A_1) \) in a way that does not depend on the future menu \( A_2 \). Under this notational assumption, Marginal Consistency is equivalent to a Marginality axiom on the joint distribution (Chambers, Masatlioglu, and Turansick, 2021).

While Marginal Consistency is a necessary consequence of state-independence, it is by itself not strong enough to rule out state-dependence. The example in Section C.2 of Chambers, Masatlioglu, and Turansick (2021) shows a \( \rho \) represented by a state-dependent SU, such that \( \rho \) satisfies Marginal Consistency yet cannot be written as SU.

What about other axioms? It is clear that the first period choice \( \rho_1 \) is RU and that the second period choice \( \rho_2(\cdot | x_1, A_1) \) is RU conditional on each history. But this is not enough: conditional choice probabilities after different histories are tied to each other.
Axiom 7.6 (Joint Regularity). For any $x_t \in A_t \subseteq B_t$.

$$\rho(x_1, x_2; A_1, A_2) \geq \rho(x_1, x_2; B_1, B_2).$$

Joint regularity implies Regularity conditional on each history (including period $t = 1$). This puts restrictions on how much the conditional choice probability can depend on history. It implies that for any $A_1 \subseteq B_1$

$$\rho(x_2, A_2|x_1, B_1) - \rho(x_2, A_2|x_1, A_1) \leq 1 - \frac{\rho(x_1, A_1)}{\rho(x_1, B_1)}, \quad (7.2)$$

so the closer the histories, the closer the conditional choice probabilities.

Instead of Regularity, we could have a joint version of Supermodularity.

Axiom 7.7 (Joint Supermodularity). For any $x_t \in A_t \subseteq B_t$.

$$\rho(x_1, x_2; A_1, A_2) + \rho(x_1, x_2; B_1, B_2) \geq \rho(x_1, x_2; A_1, B_2) + \rho(x_1, x_2; B_1, A_2).$$

Theorem 7.8 (Li 2021). When $|X_1| \leq 3$ and $|X_2| \leq 3$, SU is characterized by Axiom 7.5 and Axiom 7.7.

Chambers, Masatlioglu, and Turansick (2021) showed that when $|X_1| \leq 3$ or $|X_2| \leq 3$, SU is characterized by Axiom 7.5 and strengthening of Axioms 7.6 and 7.7 based on the BM Axiom 2.8. A BM-style axiomatization for higher cardinalities is an open question. Chambers, Masatlioglu, and Turansick (2021) also show that regardless of the cardinality of $X_t$, and regardless of the collection of observable menus, SU can be characterized by a joint version of ARSP (Axiom 2.10). Li (2021) shows that the same is true using Coherence (Axiom 2.11).

To understand SU better, we will introduce another axiom which will be useful going forward. Intuitively, we get history-dependent choices because past choices reveal something to the analyst about the current utility of the agent, which is a useful predictor of the future one. If two histories reveal similar information, they should lead to similar choices going forward because we are getting a similar selection of agents in each case, like equation (7.2). To gain the intuition behind this, consider the following example where two histories reveal the exact same amount of information.

Example 7.9. We have $A_1 = \{x, y\}$, $B_1 = \{x, y, z\}$ and $A_2 = \{x, y\}$. The choice probabilities are given in Figure 7.3. Think of this as $z$ being introduced to the market in period 1. Given the choice probabilities in Figure 7.3 we can conclude that, under SU, $z$ does not steal any customers from $x$, it only steals customers from $y$ (why?). Given this pattern of substitution, period 2 choices of consumers who chose $x$ in $t = 1$ should be the same in each market (suppose for simplicity that $z$ is unavailable in $t = 2$). This is because it’s exactly the same selection of people who are making this choice. On the other hand, the choices of people who previously chose $y$ are different in the two situations, as these represent different selections of customers. In particular, given that $z$
steals only from $y$ the types that chose $y$ from $A_1$ are a mixture of the types who chose $y$ from $B_1$ and those who chose $z$ from $B_1$. ▲

**Axiom 7.10** (Contraction History Independence). For all $x_1 \in A_1$ if

(i) $A_1 \subseteq B_1$ and 
(ii) $\rho_1(x_1, A_1) = \rho_1(x_1, B_1) > 0$,

then $\rho_2(x_2, A_2|x_1, A_1) = \rho_2(x_2, A_2|x_1, B_1)$ for all $x_2 \in A_2$.

Both (i) and (ii) are crucial for the conclusion to hold. In particular, it is not enough to just require (i) because in general the analyst learns weakly more after history $(x_1, B_1)$ as $x_1$ had to beat more alternatives to be chosen.

In general, when (ii) fails, we can only say that $\rho_2$ will be close after the two histories that are close. Joint Regularity gives us a one-sided flavor of this in (7.2). Axiom 7.1.1 has genuine distances.

### 7.3. Axioms with Lotteries

Like in the static model, axioms look nicer if we add lotteries and restrict to EU. We will call this model *Stochastic Expected Utility*.

In this model the sets $N(p_t, A_t)$ are linear so the REU axioms are satisfied conditional on all observable histories (including $t = 1$). A mirror implication is linearity of $\rho$ in the conditioning event. Choosing $p_1$ from $A_1$ reveals the same information as choosing option $\lambda p_1 + (1 - \lambda) q_1$ from menu $\lambda A_1 + (1 - \lambda) \{q_1\}$, so conditioning on either of these observations leads to the same prediction (see also the discussion after Axiom 4.14 and Remark 4.18). This is captured by the following axiom.

**Axiom 7.11** (Linear History Independence).

$$\rho_2(\cdot, \cdot|p_1, A_1) = \rho_2(\cdot, \cdot|\lambda p_1 + (1 - \lambda) q_1, \lambda A_1 + (1 - \lambda) \{q_1\})$$
for all $p_1 \in A_1$ such that $\rho_1(p_1, A_1) > 0$, all $q_1$ and $\lambda \in (0, 1)$.

A stronger version of this is mixing with a menu $B_1$ instead of a singleton.

**Axiom† 7.12 (Strong Linear History Independence).**

$$
\rho_2(\cdot, \cdot | p_1, A_1) = \rho_2(\cdot, \cdot | \lambda p_1 + (1 - \lambda) B_1, \lambda A_1 + (1 - \lambda) B_1)
$$

for all $p_1 \in A_1$ such that $\rho_1(p_1, A_1) > 0$, all $B_1$ and $\lambda \in (0, 1)$.

The dagger symbol (†) is here because I am not telling you what it means to mix with a menu. Axiom 7.12 implies Axiom 7.11 and Axiom 7.5.

There are two axiomatizations. In the first axiomatization there are more “conditional” axioms. To avoid conditioning on probability zero events, the distribution of each $\tilde{U}_t$ is discrete and there are tiebreakers $(\tilde{W}_t)_{t=1}^T$ as in Section 4.4. Technically speaking the first theorem is stated on the domain of decision trees (Section 7.5), but it holds on the simple domain as well.

**Theorem† 7.13 (Frick, Iijima, and Strzalecki 2019).** $\rho$ has a Stochastic Expected Utility representation with finite support if and only it satisfies

(i) Contraction History Independence

(ii) Strong Linear History Independence†

(iii) The GP axioms: Regularity, Linearity, Extremeness, Mixture Continuity, and Finiteness† conditional on each history

(iv) History-Continuity.†

The second axiomatization assumes joint Regularity instead of conditional Regularity. This allows them to weaken the assumptions that discipline history dependence: Strong Linear History Independence and Contraction History Independence. All axioms can be expressed in terms of the joint distribution, which shows that on the simple domain and without state-dependence the actual timing of choice does not matter.

**Theorem† 7.14 (Chambers, Masatlioglu, and Turansick 2021).** $\rho$ has a Stochastic Expected Utility representation if and only it satisfies

(i) Joint Regularity

(ii) Linear History Independence†

(iii) The GP axioms: Linearity and Extremeness conditional on each history

(iv) Joint Mixture Continuity.†

### 7.4. State-Dependence

So far, we allowed history-dependence only to the extent implied by self-selection and ruled out state-dependence. We assumed that from the point of view of the agent, past choices are irrelevant because $\tilde{U}_2$ depends on $x_2$ only and is independent of $x_1$. This was partially captured by Marginal Consistency.
7.4. State-Dependence

(Axiom 7.5). This precludes things like habit formation and other psychological effects.\(^{83}\) This also precludes things like switching costs (Pakes, Porter, Shepard, and Calder-Wang, 2020).

**Example 7.15** (Experience Goods). Other examples of consumption-dependence include models of experimentation or experience goods (see, e.g., Erdem and Keane, 1996; Crawford and Shum, 2005).

Suppose that there are three products \(X = \{x, y, z\}\) and two periods. Each of the goods can be a match (give utility one) or a mismatch (give zero utility) for the agent. The agent does not know whether a product is a match or not before trying it out. For each product the probability of a match is \(\alpha > 0.5\) and the three goods are independent.

The optimal strategy of the agent is to pick the product at random in the first period and stick with it in the second period if it turns out to be a match and switch to one of the other product otherwise. This strategy yields the following choice probabilities:

\[
\rho_2(x_2 = x|x_1 = x) = \alpha > \frac{1 - \alpha}{2} = \rho_2(x_2 = x|x_1 = y). \quad \Box
\]

Notice that history-dependence in this example occurs due to a different force than the informational asymmetry discussed so far. The example is cooked up so that the period-1 choice of \(x\) does not reveal anything about the “type” of the agent: it is purely random. Nevertheless, the observed choices look history-dependent. This is because of what is called state-dependence.

What these examples have in common is what Heckman (1981) calls structural state-dependence: period-2 utility \(\tilde{U}_2\) depends on the period-1 choices \(x_1\) (either directly through tastes, or through beliefs). Heckman’s (1981) term for the kinds of history-dependence that come from pure self-selection is spurious state-dependence. There is an extensive literature in econometrics that studies state-dependence, including Chamberlain (1993) and Honoré and Kyriazidou (2000). In such models we have

\[
\rho_2(x_2, A_2|x_1, A_1) = \mathbb{P}\left(\tilde{U}_2(x_2; x_1) = \max_{y_2 \in A_2} \tilde{U}_2(y_2; x_1) \mid N(x_1, A_1)\right).
\]

Distinguishing between structural and spurious state-dependence is a key problem. If mere self-selection leads to history-dependent choices, then how much should the analyst attribute to state-dependence? How can we let the data speak on this issue?

A controversial example comes from psychology. According to the famous cognitive dissonance theory, people change their preferences to rationalize past choices: rejected alternatives are devalued and the chosen ones are bumped up.

7.4. State-Dependence

(Brehm, 1956; Harmon-Jones and Mills, 1999). However, as pointed out by Chen (2008) and Chen and Risen (2010), the prevalent method used to test this theory suffers from a spurious state-dependence problem.

Example 7.16 (Cognitive Dissonance). There are three periods. In period 1 the subjects rate \( n \) items on a discrete numerical scale. In period 2 the experimental group chooses between two equally rated items; let \( x \) be the chosen item and \( y \) an unchosen one. The control group makes a choice between items that are ranked or rated far from each other or does not make a choice at all. In period 3 the subject is asked to choose between \( y \) and \( z \)—another item that was initially rated as equal to \( x \) and \( y \). The main empirical finding is that in the treatment group subjects are more likely to choose \( z \) over \( y \) while there is no systematic tendency in the control group.

This finding is typically attributed to cognitive dissonance because it looks as if people rationalize their rejection of \( y \) by devaluing it. However, Chen (2008) shows that it can be explained purely by spurious state-dependence. To see that, note that even though \( x, y, z \) receive the same numerical rating, the rating system is discrete, so the agent may have preferences over them. In period 2 the analyst observes the event \( N(x, \{x, y\}) = \{xyz, xzy, zxy\} \). Assuming a uniform distribution over rankings, conditional on \( N(x, \{x, y\}) \) the probability that \( z \) is above \( y \) is \( \frac{2}{3} \) (in the control group there is no conditioning so the probability remains at \( \frac{1}{2} \)).

We could use Marginal Consistency to test for cognitive dissonance. Suppose that menus \( A, B \) partition the grand set \( X \). Suppose that in period 2 we randomly assign \( A \) or \( B \) and in period 3 the subject faces \( X \). If cognitive dissonance is true, we should observe that subjects that faced \( A \) in period 2, will choose from \( A \) more frequently than subjects who faced \( B \). At the same time, a pure cognitive dissonance theory would predict that if we partition \( X \) into singleton menus in period 2, then choices in period 3 will be history-independent.

In general, Heckman (1981) and the literature that follows develop stochastic choice models and econometric techniques that tease apart structural from spurious state-dependence. We will instead take a different route and assume that there exist lotteries which serve as perfect randomized controlled trials. Consider the following example.

Example 7.17 (Habit Formation). Consider a pharmaceutical researcher who wants to determine whether drug \( x \) is habit forming. The other drug, \( y \) is known not to be. If the researcher has only access to the decision tree in the left panel of Figure 7.4, she won’t be able to determine how much of the observed serial correlation in choices to attribute to selection and how much to habit-formation.

\(^{84}\)Another, more popular, version of the experiment elicits strict rankings instead of numerical ratings, but a similar argument shows that the observed cognitive dissonance is spurious, provided that subjects make small mistakes when reporting their rankings.
This can be solved if the researcher can randomly assign $x$ and $y$ in the first period. If the lottery in the first period is independent of everything else, then the population assigned $x$ is the same as those assigned $y$ and therefore if $\rho_2(x|x) > \rho_2(x|y)$, we can conclude that the drug is habit forming. This is illustrated in the middle panel of the figure. The idea of random assignment is routinely used in econometrics. The convention introduced by (Raiffa, 1968) is to denote decision nodes by squares and chance nodes, i.e., lotteries, by circles.

![Figure 7.4. Teasing out structural and spurious state-dependence.](image)

In general, lotteries are an idealized randomizing device, but they do sometimes occur in reality. The next section formulates an axiom based on lotteries that rules out structural state-dependence.

### 7.5. Decision Trees

We have so far assumed away a key feature of dynamics, which is that choices made today shape the menu of choices available tomorrow. This is true in the classic consumption-savings problem where the amount consumed in period $t$ influences the income available in period $t + 1$. Classic discrete choice examples include studies of fertility and schooling choices (Todd and Wolpin, 2006), engine replacement (Rust, 1987), patent renewal (Pakes, 1986), or occupational choices (Miller, 1984). To model this in generality, we will enhance the description of each alternative and define $x_t = (z_t, A_{t+1})$, where $z_t$ is an immediate consumption (or payoff) and $A_{t+1}$ is a menu of choices available in the next period.

In order to also incorporate lotteries, we will define what are known as decision trees. This is a canonical domain in dynamic decision theory (Kreps and Porteus, 1978). Now $x$ is a lottery over pairs like $(z_t, A_{t+1})$. Formally,

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85 For example, schools ration their seats via lotteries, a fact that is widely exploited in the empirical literature on school choice to generate quasi-experimental variation, e.g., Abdulkadiroglu, Angrist, Narita, and Pathak (forthcoming); Angrist, Hull, Pathak, and Walters (forthcoming); Deming, Hastings, Kane, and Staiger (2014); Deming (2011).

86 A small technical difference is that Kreps and Porteus (1978) look at Borel instead of simple lotteries, and compact instead of finite menus. Their paper is often remembered for temporal lotteries: the important special case of decision trees where all the decision nodes are singletons. However, general decision trees are also defined and analyzed in Sections 1–3 of that classic paper.
there are finitely many time periods $t = 1, \ldots, T$ and let $Z$ be a finite set of instantaneous consumptions. Each period-$t$ menu is a finite set of lotteries over the period-$t$ outcome space $X_t$. The spaces $X_t$ are defined recursively: in all periods $t < T$ the outcome space $X_t := Z \times A_{t+1}$ consists of pairs of current-period consumption and next-period continuation menus; the set of period-$t$ menus is $A_t := \mathcal{A}(\Delta(X_t))$, where $\mathcal{A}(Y)$ is the collection of finite subsets of $Y$.\footnote{In this notation, our old friend $\mathcal{A}$ is now $\mathcal{A}(X)$.} To close this construction, the outcome space in the final period $T$ is just $X_T = Z$, as there is no continuation menu in the terminal period.\footnote{Thus, in a sense, the sets $X_t$ are getting “smaller” as time goes by. This construction can be extended to an infinite time horizon, see, e.g., \cite{Gul2004}; here $X_t$ is constant over time.}

As opposed to the simple domain (Assumption 7.2) here only $A_1$ is exogenous. All subsequent menus are chosen by the agent (and possibly randomized). More precisely, the chronology works as follows: each trial is defined by a menu $A_1$ of lotteries $p_1$. In period 1 the agent chooses a lottery $p_1$, which subsequently resolves giving the agent immediate consumption $z_1$ and menu $A_2$. The new period begins with history $(A_1, p_1, z_1)$ and the agent chooses $p_2$ from $A_2$, then $(z_2, A_3)$ is realized according to $p_2$, and the cycle continues. Like on the simple domain, we could be sampling from a population of individuals or sampling the same agent over and over again in a stationary environment.

Our simple domain is a subset of decision trees where the agent cannot choose future menus. Formally, there exists some lottery $\pi_{t+1}$ over menus $A_{t+1}$ and each $p_t \in A_t$ is a product measure of a lottery over $Z$ and $\pi_{t+1}$.

The choices that occur with positive probability under $\rho_1$ define the set of all period-1 choice histories: pairs $(p_1, A_1)$ such that $\rho_1(p_1, A_1) > 0$. Conditioning on choice histories will allow our analyst to take care of self-selection. As discussed in Example 7.17, the analyst might also want to keep track of consumption histories: triples $(z_1, p_1, A_1)$ such that $\rho_1(p_1, A_1) > 0$ and $p_1(z_1) > 0$.

Conditioning on consumption histories allows for a simple axiom that rules out state-dependence.

**Axiom 7.18 (State Independence).** For all $p_1 \in A_1$ with $p_1(z_1), p_1(z_1') > 0$ we have

$$\rho_2(\cdot, A_2 | z_1, p_1, A_1) = \rho_2(\cdot, A_2 | z_1', p_1, A_1).$$

\cite{Frick2019} showed that under Axiom 7.18 Theorem 7.13 offers a characterization of SU on decision trees.

If we go outside of the simple domain, we will have a limited observability problem. In Example 7.4, if the agent chooses to go to supermarket $A$, we only observe their choices from menu $A$. We do not have access to the choices they would make from the menu $B$. We cannot extrapolate from choices of those who go to $B$ because they are a different population with different preferences. In the extreme case, if we do not know anything about the selection mechanism, then we do not learn anything from those choices.
When lotteries are absent, there is only one observable menu after each history, so limited observability is very severe. The following example shows that by adding lotteries we can overcome the limited observability problem and extrapolate across histories.

**Example 7.19 (School Choice).** In period 1, parents decide to enroll their child in one of two schools, which differ along many decision-relevant dimensions. Upon enrolling, in period 2, parents must choose between a number of after-school care options: $H$ (home); $Q$ (high quality after-school); or $B$ (basic after-school program offered *only* by school 1). Thus, choosing school 1 leads to period-2 menu $\{H, Q, B\}$, whereas school 2 leads to menu $\{H, Q\}$.

![Decision Tree](image)

**Figure 7.5.** Limited Observability.

This situation is illustrated by the decision tree in the left panel of Figure 7.5. There is limited observability, similar to Example 7.4: we don’t know how parents who select to school 1 would choose from the menu $\{H, Q\}$. We need to overcome this problem if we want to make policy recommendations about eliminating option $B$ in school 1.

The inclusion of lotteries allows us to do so. Consider the decision tree in the right panel of figure 7.5. Here seats to school 1 are allocated by a lottery and the student gets admitted with probability $\lambda$, while with probability $1 - \lambda$ they must go to school 2. If preferences in period 1 are EU, then the event in which the agent chooses school 1 in the decision tree on the left is precisely the same as the event in which the agent chooses the lottery in the decision tree on the right. In the axiomatization of Frick, Iijima, and Strzalecki (2019), Linear History Independence ensures that choice between $\{H, Q\}$ are independent of the value of $\lambda$. ▲
Chapter 8

Dynamic Optimality

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8.1. The Bellman Equation

In Chapter 7 we defined stochastic utility (SU), where the agent in each period $t$ maximizes a random utility function $\tilde{U}_t$. This is a very general model that does not take a stance on whether the agent thinks about the future: it allows for both myopic and forward-looking agents. Typically, economists assume that $\tilde{U}_t$ additionally satisfies the Bellman equation, meaning that the continuation value is the (correct) expectation of the future utility.

Assume that $\Omega = \Omega_1 \times \Omega_2 \times \cdots$, where in period $t$ the agent observes $\omega_t \in \Omega_t$. The vector $\omega^t$ describes information known by the agent at time $t$. Each alternative is $x_t = (z_t, A_{t+1})$, i.e., it involves a payoff today and a continuation menu for tomorrow. In other words, we have deterministic decision trees (Section 7.5).

**Definition 8.1.** $(\tilde{U}_t)$ satisfies the Bellman equation if

$$U_t(z_t, A_{t+1}, \omega^t) = u_t(z_t, \omega^t) + \delta \mathbb{E} \left[ \max_{x_{t+1} \in A_{t+1}} \tilde{U}_{t+1}(x_{t+1}, \omega^{t+1}) \mid \omega^t \right]$$

(8.1)

where $u_t : \Omega^t \to \mathbb{R}$ is a random flow utility (also called felicity) and $\delta \in [0, 1]$ is the discount factor.

This means that preferences are additively separable over flow utility, and continuation value

$$\tilde{V}_t(A_{t+1}) := \mathbb{E} \left[ \max_{x_{t+1} \in A_{t+1}} \tilde{U}_{t+1}(x_{t+1}) \mid \omega^t \right],$$

which captures the fact that the agent is forward looking (unless $\delta = 0$). The formula for $\tilde{V}$ is a generalization of the social surplus formula (Section 1.8). A Bellman agent is using that formula to evaluate their own future welfare (conditional on their current information). By applying the equation recursively it follows that the agent is looking into all the future periods. Notice that $(\tilde{U}_t)$ satisfies the Bellman equation vacuously on our simple domain (Assumption 7.2) because the agent was forbidden from affecting the future.

This chapter studies dynamic optimality without state-dependence. We will study dynamic optimality for Markov Decision Problems in Chapter 12.

This specific form of $V_t$ has two important implications: preference for flexibility and rational expectations. First, the agent foresees that they will learn something between periods $t$ and $t+1$ and adapt their action optimally to this new information. Therefore, adding more options to the menu is always weakly better because they may be useful in some situations and they cannot hurt in any situation. The following example illustrates this.

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89The analyst does not have access to this information. Her information is much coarser: she knows only as much as she can infer from agent’s past choices via the events $N(h_t)$. A similar situation occurs in dynamic mechanism design, see, e.g., Pavan, Segal, and Toikka (2014), where the mechanism designer gradually learns information about the agent’s type by looking at their choices.
Example 8.2 (Sun glasses and rain coat). Suppose that on Saturday you are going on a trip and on Friday you are packing your bag. You can bring: your sun glasses \{g\}, your rain coat \{c\} or both items \{g, c\}. By packing \{g, c\} you de facto delay the choice between \(g\) and \(c\) until Saturday. If you don’t choose that menu, then you are de facto choosing between \(g\) and \(c\) on Friday. Of course, it’s better to make an informed choice, so you bring both items.\(^{90}\)

Suppose that \(t = 1\) is Friday and \(t = 2\) is Saturday. In period \(t = 2\) you will learn the weather and pick the most appropriate item from your bag. What should you put in your bag?

You have some weather-dependent utility function \(\tilde{u}_2\). There are no more periods after \(t = 2\), so we just set \(\tilde{U}_2 := \tilde{u}_2\). Since we are not interested in what you are consuming while you are packing your bag, we set \(\tilde{u}_1 := 0\). For simplicity set \(\delta := 1\).

The Bellman equation says that:

\[
\tilde{U}_1(\{g\}) = \mathbb{E}[\tilde{u}_2(g)|\omega_1],
\]

\[
\tilde{U}_1(\{c\}) = \mathbb{E}[\tilde{u}_2(c)|\omega_1],
\]

\[
\tilde{U}_1(\{g, c\}) = \mathbb{E}[\max\{\tilde{u}_2(g), \tilde{u}_2(c)\}|\omega_1].
\]

There are two ways to see that the last expression dominates. The component-wise maximum always has a better expectations, i.e., \(\max\{\tilde{u}_2(g), \tilde{u}_2(c)\} \geq \tilde{u}_2(g)\) in each state of the world \(\omega\) and taking (conditional) expectations preserves this inequality (and likewise for \(c\)). Alternatively, notice that \(\max: \mathbb{R}^n \rightarrow \mathbb{R}\) is convex and then the conclusion follows from conditional Jensen’s inequality.\(^{\blacktriangle}\)

The second implication of the Bellman equation are rational expectations: the agent’s belief over future states is correct, i.e., it corresponds to the true data generating process \(\mathbb{P}\). In general, there could be one distribution that governs the true variability of preferences (say in our example, the true probability of rain) and another that represents the agent’s subjective belief (say the weather forecast). (8.1) says that those two are the same. This assumption seems innocuous in the weather forecast example.\(^{91}\) It seems strong in other settings: consumers predicting their future income streams, firms predicting profitability of new products, etc.

We will discuss rational expectations further in Section 8.3. Notice that this assumption imposes a consistency condition between \(\rho_t\) and \(\rho_{t+1}\). On the other hand, preference for flexibility is purely a condition on the shape of \(\tilde{U}_t\), so it will manifest itself by additional axioms imposed on \(\rho_t\) alone. Given that, we discuss option value first (in the next section).

\(^{90}\)If delay is costly, then there is a tradeoff between the option value and the delay cost. This typically generates a probability distribution over the choice to delay. We will discuss such models in Chapter 9.

\(^{91}\)Assuming that the weather forecast is asymptotically calibrated, see, Dawid (1982), and Ol- szewski (2015).
Models used in the dynamic discrete choice literature in econometrics and industrial organization specify the Bellman equation somewhat differently. While similar in spirit, those models are not equivalent: many specifications, such as dynamic logit, violate preference for flexibility and associated notions. We discuss this in Chapter 12.

Remark 8.3. In general, the discount factor $\delta$ is not identified. This is because $u_t$ can depend on $t$ in an arbitrary way, so it could absorb $\delta$. However, $\delta$ is identified for some important special cases, for example under stationarity, where $u_t(z_t, \omega^t) = u(z_t, \omega_t)$ for some time-invariant function $u$.

8.2. Preference for Flexibility

There is always option value: it’s good to keep our options open (as long as it is costless). If the agent is directly choosing between a menu and its superset, they will always take the superset, except if there is a tie—if the new items are so bad that they are dominated in each state of the world by something already in the menu.

Like in Example 8.2 we will make the following simplifications: there are two periods and consumption is suppressed in period $t = 1$ so that we observe choices over menus. Moreover, suppose that there is no private information in period $t = 1$ so that choices between menus are deterministic in the eyes of the analyst.\footnote{Given that we have eliminated $u_1(z_1, \omega_1)$ from the Bellman equation, this would be true under a weaker assumption that $\omega_2 \perp \omega_1$.} This allows us to capture observed choices by a preference relation $\succsim_1$ instead of a s.c.f. $\rho_1$.

As a further simplification, let’s forget about $\rho_2$ for now and just focus on choices made in period $t = 1$. This allows us to drop the subscripts and write things like $A \succsim B$ instead of $A_2 \succsim_1 B_2$.

8.2.1. Preferences over Menus. Our primitive is $\succsim$ defined over $A(X)$. On this domain, the Bellman equation boils down to the following.

Definition 8.4. $\succsim$ has a Koopmans representation if there exists a random utility $\tilde{U} : \Omega \rightarrow \mathbb{R}^X$ such that

$$ V(A) = \mathbb{E} \left[ \max_{x \in A} \tilde{U}(x) \right] $$

represents $\succsim$.

Koopmans (1964) asks what axioms on $\succsim$ pin down this representation. The key axiom says that bigger menus are better.

Axiom 8.5 (Preference for Flexibility). If $A \supseteq B$, then $A \succsim B$.

This is almost all that we need: we just need to discipline ties now.

Axiom 8.6 (Modularity). If $A \sim A \cup B$, then $A \cup C \sim A \cup B \cup C$.\footnote{This is almost all that we need: we just need to discipline ties now.}
Intuitively, if adding $B$ to $A$ doesn’t create any value, this must be because items in $B$ are statewise dominated by items in $A$. Adding $C$ to both sides does not change that.

**Theorem 8.7 (Kreps 1979).** Suppose that $Z$ is a finite set. A preference $\succsim$ satisfies Preference for Flexibility and Modularity if and only if it has a Koopmans representation.

**Remark 8.8.** Preference for flexibility makes sense in an idealized model. But in real life, it can be violated in many ways. For example, if there is *choice overload*, then going through the options in the menu and making a decision is costly to the agent. If they anticipate this cost, they may prefer to rule out options right away (presumably at some cost too, but suppose that it’s sunk).

Other situations involve temptation, where preferences at $t$ and $t+1$ (about $x_{t+1}$) disagree with each other. There are several ways to resolve this conflict. The agent could be *sophisticated* and perfectly foresee their future preferences (Strotz, 1955). Or they could try to resist temptation by exerting *costly self-control* (Gul and Pesendorfer, 2001). Or they could be *naive* and think there is no conflict whatsoever. Agents who are sophisticated or have costly self-control will violate the preference for flexibility axiom because they may want to commit to exclude options from the menu that are tempting and harmful. Naive agents will satisfy it. They think they think they satisfy the Bellman equation, but they do not: they violate rational expectations. The large axiomatic literature on temptation is summarized in Lipman and Pesendorfer (2013).

8.2.2. Preferences over Menus of Lotteries. Kreps’s theorem is very elegant, but has very weak uniqueness properties. Perhaps this should not come as a surprise, given the weak uniqueness properties of RU we discussed in Section 2.4, but the distribution of $\tilde{U}$ is not identified, nor is the set of preferences they represent.\footnote{As Kreps (1979) shows, the situation is even worse than in the RU case.}

Adding lotteries helped with RU, and it does here as well. Following Dekel, Lipman, and Rustichini (2001), henceforth DLR, let $X = \Delta(Z)$ and assume that preferences $\succsim$ are defined on all nonempty and compact subsets of $X$.

**Definition 8.9.** $\succsim$ has a DLR representation if there exists a random expected utility $\tilde{U} : \Omega \to \mathbb{R}^{\Delta(Z)}$, such that

$$V(A) = \mathbb{E} \left[ \max_{p \in A} \tilde{U}(p) \right]$$

represents $\succsim$.

Instead of Modularity, we will now have axioms that rely on the lottery structure which allows us to mix menus in the following way:

$$\alpha A + (1 - \alpha)B := \{ \alpha p + (1 - \alpha)q : p \in A \text{ and } q \in B \}.$$
8.3. Rational Expectations

Axiom 8.10 (Menu Independence). If $A \succ B$, then for all $C$ and all $\alpha \in (0, 1]$ we have $\alpha A + (1 - \alpha)C \succ \alpha B + (1 - \alpha)C$.

This is just the vNM axiom (mixing menus instead of lotteries).

Axiom 8.11 (Archimedean continuity). If $A \succ B \succ C$, then there exist $\alpha, \lambda \in (0, 1)$ such that $\alpha A + (1 - \alpha)C \succ B \succ \lambda A + (1 - \lambda)C$.

Theorem 8.12 (DLR). A preference $\succsim$ has a DLR representation if and only if it satisfies Preference for Flexibility, Menu Independence, and Archimedean continuity.


Adding lotteries to the domain helps with uniqueness in the following sense. In each state $\omega \in \Omega$ the agent has a preference over lotteries $\succsim_\omega$ represented by $\tilde{U}(. \, \omega)$. Let $P(\tilde{U}) := \{\succsim_\omega : \omega \in \Omega\}$. In other words, $P(\tilde{U})$ is the support of the distribution over preferences induced by $\tilde{U}$.

Theorem† 8.13 (DLR). If there is a representation with a finite state space, then the sets $P(\tilde{U})$ coincide for all representations.

Proof. See Theorem 1, parts B and C of Dekel, Lipman, and Rustichini (2001). This result holds under the additional assumption that all states are relevant. In the infinite case, the closures of those spaces coincide.

Remark 8.14. This sort of uniqueness is weaker than the one we have for REU representations, where the probability distribution over $P(\tilde{U})$ was pinned down uniquely (Theorem 4.17). Here we can only pin down its suport. To identify the subjective probability uniquely, Dillenberger, Lleras, Sadowski, and Takeoka (2014), add an objective state space to this model and study choices between menus of Anscombe–Aumann acts. In the next section we will discuss another route: coupling preferences over menus in period $t = 1$ with choices out of menus in period $t = 2$. As Ahn and Sarver (2013) show, this gives even more uniqueness.

8.3. Rational Expectations

Another assumption behind the Bellman Equation is that the agent has rational expectations. In the weather example (Example 8.2) this means that the weather forecast is correct (on average). Whenever the states are objective,
8.3. Rational Expectations

Rational expectations imposes consistency between the objective frequency of states and the agent’s beliefs revealed by \( \rho \) (as in Chapter 5).

In general, states may be subjective to the agent and unobservable to the analyst. In this case, rational expectations imposes a connection between \( \rho_t \) and \( \rho_{t+1} \).

The simplest case is when the analyst observes a deterministic preference \( \succsim_1 \) on menus and a s.c.f. \( \rho_2 \). Formally, \( \rho_2 \) is defined on \( X_2 = \Delta(Z_2) \) and \( \succsim_1 \) on \( X_1 = A(X_2) \).

Suppose that we have a DLR preference \( \succsim_1 \) and a REU s.c.f. \( \rho_2 \).

Definition 8.15. We say that \( (\succsim_1, \rho_2) \) is a Ahn–Sarver pair if \( \succsim_1 \) has a DLR representation \( (\Omega, \mathcal{F}, P_1, \tilde{U}_2) \) and that \( \rho_2 \) has a REU representation with \( (\Omega, \mathcal{F}, P_2, \tilde{U}_2) \) and \( \tilde{U}_2 \) has finitely many possible realizations.

Here \( P_1 \) is the belief that the agent holds in period \( t = 1 \) and \( P_2 \) is the true data generating process that drives random variation in agent’s period \( t = 2 \) choices. These two probability measures can be very different for an arbitrary pair \( (\succsim_1, \rho_2) \). For example, the agent may be over-optimistic about option value, or over-pessimistic. Rational expectations is when the agent’s expectations are exactly right.

Definition 8.16. We say that a Ahn–Sarver pair \( (\succsim_1, \rho_2) \) has Rational Expectations if the distribution of \( \tilde{U}_2 \) under \( P_1 \) is the same as under \( P_2 \).

Ahn and Sarver (2013) found a sharp axiom that captures this property.

Axiom† 8.17 (Sophistication). For any menu without ties \( A_2 \cup \{p_2\} \)

\[
A_2 \cup \{p_2\} \succsim_1 A_2 \iff \rho_2(p_2, A_2 \cup \{p_2\}) > 0
\]

This axiom says that the agent wants to include additional options into the menu if and only if they actually then choose them at least some of the time.

Theorem† 8.18. (Ahn and Sarver, 2013): \( (\succsim_1, \rho_2) \) has Rational Expectations iff it satisfies Sophistication.

To understand Sophistication better, assume that it holds only in one direction. First imagine that

\[
A_2 \cup \{p_2\} \succsim_1 A_2 \implies \rho_2(p_2, A_2 \cup \{p_2\}) > 0.
\]

Whenever the agent wants to include a new option in period 1, they then choose it at least some of the time in period 2. But there are options the agent is sometimes choosing in period 2 that they do not value in period 1. This is because of unforeseen contingencies. There are scenarios the agent does not imagine happening, so they do not value flexibility on those dimensions.

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\(^95\)It is without loss of generality to assume that they share the measurable space \( (\Omega, \mathcal{F}) \) and period-2 utility function \( \tilde{U}_2 \).

\(^96\)In other words, \( P_1 \) and \( P_2 \) are equal except for payoff-irrelevant events.
Now imagine the opposite:
\[ A_2 \cup \{p_2\} \succ_1 A_2 \iff \rho_2(p_2, A_2 \cup \{p_2\}) > 0. \]
Whenever it’s sometimes worth choosing an option in period 2, the agent wants to include it in period 1. Yet the agent insists on including some options that never end up getting used. This may be because some scenarios that they are imagining are completely impossible.\(^{97}\)

The Rational Expectations assumption helps with identification. As discussed in Section 8.2, looking at \(\succ_1\) alone identifies just the support of the distribution over ordinal risk preferences. Theorem 4.17 says that just looking at \(\rho_2\) identifies that distribution. This still means that in each state we can multiply the utility by a positive constant \(\alpha\) and add a constant \(\beta\) to it, where both constants can depend on the state. Remarkably, Ahn and Sarver (2013) show that putting \(\succ_1\) and \(\rho_2\) together the constant \(\alpha\) has to be state-independent.

Finally, it is important to realize that the rational expectations assumption becomes much weaker if we don’t observe a preference \(\succ_1\) over menus, but instead just a preference over lotteries (singleton menus). Thus, now our primitive is an EU preference \(\succ_1\) over lotteries and a REU s.c.f. \(\rho_2\). Given that there are finitely many possible realizations of \(\tilde{U}_2\), we have finitely many possible EU preferences in period 2: \(\succ_1^1, \ldots, \succ_1^n\).

**Definition 8.19.** We say that there is a preference reversal if there exist \(p, q \in \Delta(Z)\) such that \(p \succ_1 q\) and \(q \succ_2 p\) with at least one of the preferences \(\succ_1^1, \succ_2^1, \ldots, \succ_2^n\) strict.

**Theorem 8.20.** (Strack and Taubinsky, 2021) \((\succ_1, \rho_2)\) has Rational Expectations iff there are no preference reversals.

### 8.4. Recent Axiomatic Work*

Frick, Iijima, and Strzalecki (2019) take as their primitive the collection \((\rho_t)\). In other words, they allow for intermediate payoffs and more importantly for private information in all periods. They find the stochastic versions of DLR axioms and Sophistication that are equivalent to Definition 8.1. Frick, Iijima, and Strzalecki (2019) also axiomatize a multiperiod learning model where tastes are not allowed to vary over time, but only beliefs.

Lu and Saito (2019) unpack the simple domain Assumption 7.2. As discussed above, the simple domain makes sense under the time-separability assumption that is built into the Bellman equation. Lu and Saito (2019) study a model where time-separability is violated, as in Kreps and Porteus (1978) and Epstein and Zin (1989). In such models the continuation menu will affect

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\(^{97}\)Another explanation is that the agent enjoys a pure freedom of choice, or perhaps they are aware that there are some unforeseen contingencies but some of them are imaginary and do not actually exist.
current risk attitudes. They show that the analyst’s estimates of the function $u$
are biased because they are contaminated by the nonlinear continuation utility.

Lu and Saito (2018) study static choices between consumption streams. In their model the randomness in choices is driven by preference shocks to
discounting attitudes: the felicity function is deterministic, but the discount
factor is stochastic. They provide an axiomatic characterization of this model.
They also look at an extension where the analyst observes the average of $\rho_t$ in
each period $t$, that is the unconditional choice probabilities.

Following Ahn and Sarver (2013), the two papers by Ahn, Iijima, Sarver,
and Yaouanq (2019); Ahn, Iijima, and Sarver (2020) study a pair $(\succ, \rho)$, where
$\succ$ is a preference over menus and $\rho$ is a stochastic choice function from menus.
These papers look at models where the rational expectations assumption is
violated, and in particular focus on naivete for time-inconsistent preferences.
Chapter 9

Response Times

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9.1. New Variable and New Questions

So far, we looked at what the agent chose. In this chapter we will turn our attention to a new variable: how much time it takes them to make a choice.

In psychology these are known as response times. They are routinely collected for perceptual tasks such as those discussed in Section 1.3.98 A historic example is Jung’s (1910) word association test used for revealing subject’s emotional states. An example from contemporary psychology is the implicit association test.99 The increased availability of data about online and in-app behavior makes it a potentially fruitful object to study in economics.

One question you can ask using this primitive is whether difficult problems take more time than easy ones. The models studied in this chapter predict that in the weight perception task (Example 1.5), the reaction times are longer if the two items are closer in weight.

A second question is whether fast decisions are better (or more accurate). Presumably, taking more time with the decision has some benefits, for example the agent receives more information about the items; this is known as the speed-accuracy tradeoff. Based on this, we would expect the opposite: the more time the agent spends on the decision, the better it is. In other words, if we forced the agent to stop at time $t$, accuracy would be increasing in $t$.

However, our agent’s decision when to stop is endogenous. Waiting may have some opportunity costs, so the decision to stop will depend on how much the agent expects to learn (the option value of waiting). If they get an informative signal, they may want to stop, but they will continue if the signal is noisy. If the “standard of proof” used for stopping lowers over time, the agent makes worse and worse choices over time. This is indeed the pattern observed in perceptual tasks.100 A similar patterns appears for economic choices.101

To study these effects formally, we will set up a model of sequential sampling, where the agent is optimally choosing when to stop (and what to choose in the event of stopping). This is a model of active learning because by choosing the stopping boundary, the agent de facto decides how much to learn. In Wald’s model (binary prior) the optimal boundary is constant in time, so fast choices are equally good as slow ones. But this is an exception: with other priors the optimal boundary is time-dependent.

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98See, e.g., Luce (1986), Gold and Shadlen (2007), and Ratcliff and McKoon (2008).
100See, e.g., Swensson (1972); Luce (1986); Ratcliff and McKoon (2008); Shadlen, Hanks, Churchland, Kiani, and Yang (2006).
101For example, Reutskaja, Nagel, Camerer, and Rangel (2011), Fehr and Rangel (2011), Krajibich, Armel, and Rangel (2010) study choice tasks, i.e., experiments where subjects choose between consumption items. While the definition of “better choices” is clear in perceptual tasks, this is not the case for choice tasks, see Remark 9.6.
9.2. Optimal Stopping

Papers in psychology and cognitive science often times use a “reduced-form” model where the boundary is not optimally chosen by the agent, but instead exogenously specified by the analyst (and estimated). The Drift-Diffusion Model (DDM) uses a constant boundary.

Remark 9.1 (The dual system hypothesis). In other situations the opposite pattern is true: fast decisions are impulsive, instinctive, and often wrong, while slow decisions are deliberate, cognitive, and often right (Kahneman, 2011). The evidence behind this focuses mostly on choices where to arrive at the correct answer, the agent needs to solve a puzzle of some sort, or “think about the problem the correct way” (Rubinstein, 2007; Krajbich, Bartling, Hare, and Fehr, 2015; Caplin and Martin, 2016). In contrast, in simple perception experiments, the gut feeling is often correct and if we start doubting ourselves, there is a good chance we are off base.

Of course, economic choices involve a combination of both kinds of processes. The initial wave of experiments focused on choices between snack items, which seem to be closer to perceptual choices and therefore it is perhaps not surprising that decreasing accuracy was observed.

Remark 9.2 (Bandits). In another popular model of dynamic learning the agent continually experiments with each alternative and potentially flips back and forth each period. This agent is now solving a different dynamic problem (Gittins, 1979; Gittins, Glazebrook, and Weber, 2011; Weitzman, 1979; Keller, Rady, and Cripps, 2005; Doval, 2018). Bandit models are applied for example to experience goods (Example 7.15). The analyst has a different primitive than in reaction times experiments: a choice probability is observed conditional history of past consumption. Little is known about stochastic choices induced by optimal choice rules in this environment. The deterministic decision theory literature includes Piermont and Teper (2019), Hyogo (2007), and Cooke (2017).

9.2. Optimal Stopping

9.2.1. Primitive. The set of time indices is $\mathcal{T} = [0, \infty)$ or $\mathcal{T} = \{0, 1, 2, \ldots\}$. In an idealized situation with a lot of data the analyst has access to the joint distribution $\rho \in \Delta(A \times \mathcal{T})$, for some collection of (typically binary) menus $A \in \mathcal{A}$.

For any menu $A$ we can decompose the joint distribution $\rho$ into the distribution of response times and the conditional choice probability $\rho_t \in \Delta(A)$ for each $t \in \mathcal{T}$. For simplicity, assume that the response times distribution has full support, so the conditional choice probabilities $\rho_t$ can be computed.

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102 This is either sampling each individual a limited number of times (maybe even once) and using a mixture model that accounts for heterogeneity in tastes and/or stopping dynamics. Or sampling one individual many times over.
Typically $\rho_t$ will depend on $t$, i.e., there will be some correlation between time and choice. Complete history-independence is a knife-edge case.

It is useful to think of our agent as solving the stopping problem depicted in Figure 9.1, with the binary menu $A = \{\ell, r\}$. At each $t$, the agent can pick one of the items immediately, or they can ponder the decision a bit more.

![Figure 9.1. A stopping tree.](image)

### 9.2.2. Dynamic Learning.

Let $S$ be the state space, as in Chapters 5 and 6. At each time $t$ the agent receives a message $M_t \in \mathcal{M}_t$ and $M^t := (M_1, \ldots, M_t)$ denotes the history of messages up to time $t$.\(^{103}\) The agent has a prior on $S$ and the distribution of $M^t$ depends on $s \in S$ (typically i.i.d. conditional on $s$). It is useful to consider the probability space $\Omega = S \times (\times_{t \in T} \mathcal{M}_t)$, so that the message and the stopping times are random variables carried by $\Omega$.

The message process can be interpreted in various ways: recognition of objects on the screen in a lab experiment or some form of introspection, e.g., retrieving pleasant or unpleasant memories or coming up with reasons pro and con. In decision tasks it is assumed that the signal strength depends on the true underlying utility, for perceptual tasks it depends on the difficulty of the task. Consistently with Chapters 5 and 6, we will treat messages as unobserved by the analyst.\(^{104}\)

The agent has a state-dependent utility $v : S \to \mathbb{R}^X$. In period $t$ their conditional expected utility of choosing $x$ is $U_t(x) := \mathbb{E}[v(x)|M^t]$. The agent’s optimal choice from menu $A$ is given by the choice function $\chi_t = x$ iff $U_t(x) > U_t(y)$ for all $y \in A$.\(^{105}\) Given that $M^t$ is private, observed choices are stochastic.

---

\(^{103}\)In this chapter we will use upper case $M$ for messages and $\mathcal{M}$ for the set in which they live.

\(^{104}\)This was also done in early applications (Ratcliff, 1978). In animal experiments, some neuroscientists record neural firing, both single neurons (Hanes and Schall, 1996) and populations of them (e.g., Ratcliff, Cherian, and Segraves, 2003).

\(^{105}\)For expositional purposes we assume away ties; in parametric models discussed below they will happen with probability zero.
9.2.3. Optimal Stopping. Let \( \tau \) be the time at which the agent stops and makes a choice. Stopping is *exogenous* if the agent is forced to make a decision precisely at a time that is controlled by the analyst. Formally, \( \tau \) is a random variable that is independent of \( M_t \). But as we discussed before the agent can decide themself when to stop, so stopping is *endogenous*. Thus, \( \rho_t \) is the distribution of \( \chi_t \) *conditional* on the event that the agent stopped, i.e., \( \{ \tau = t \} \). Thus, \( \rho_t \) is based on a selected sample.

To model this we will use the notion of a stopping time. The event that the agent stops before time \( t \) is fully determined by the message history up to time \( t \) and does not depend on the future values of the message. Let \( \Sigma^t \) denote the collection of message histories after which the agent stops at time \( t \).

**Definition 9.3.** A *stopping time* \( \tau \) is a mapping \( \tau : \Omega \rightarrow \mathcal{T} \) such that for each \( t \) we have \( \tau(\omega) = t \) iff \( M_t \in \Sigma^t \).

Given a stopping time \( \tau \), the agent’s choices are \( \chi_{\tau} \) and their ex ante expected utility is \( \mathbb{E}[U_{\tau}(\chi_{\tau})] \). The analyst observes \( \rho^s \), the distribution of \( (\chi_{\tau}, \tau) \) in state \( s \).

The stopping time is chosen optimally. There are two main flavors of the model: additive time-cost and discounting. In the first one, there is a deterministic cost of waiting, a non-decreasing function \( c : \mathcal{T} \rightarrow \mathbb{R} \). The optimal stopping time \( \tau^* \) solves:

\[
\max_{\tau} \mathbb{E}[U_{\tau}(\chi_{\tau}) - c(\tau)].
\]

In statistics, this is known as sequential sampling: the analyst can buy additional data (experiments) at a cost. The special case of *additive-linear* time cost is often used where \( c(t) = \gamma t \) for some \( \gamma > 0 \). On the other hand, under *discounting*, there is \( \delta \in (0, 1) \) such that \( \tau^* \) solves

\[
\max_{\tau} \mathbb{E}[\delta^\tau U_{\tau}(\chi_{\tau})].
\]

**Remark 9.4.** In both of these versions the experiment \( (M_t) \) is fixed and cannot be chosen by the agent. Despite this, there is an element of active learning, in the sense of Chapter 6, because waiting longer gives more information (at a cost). The scope for attention allocation is limited here: our agent cannot pay “more attention” to one item than the others. Section 9.7 discusses papers where the agent optimizes over \( (M_t) \).

9.2.4. Accuracy. **Accuracy** is the probability of making the correct choice conditional on stopping at time \( t \).

**Definition 9.5.** Let \( x^s := \arg \max_{x \in A} v(x, s) \) be the correct choice in state \( s \). **Accuracy** is defined as \( \alpha^s(t) := \rho^s_t(x^s, A) \).

---

106Formally, for \( \tau \) to be well defined, we need the following condition. Let \( \Gamma^t \) be the continuation set, i.e., complement of \( \Sigma^t \). The condition is \( \Sigma^{t+1} \subseteq \Gamma^t \times \mathcal{M}_{t+1} \).

107In pairwise choice, accuracy is a decreasing function of time if and only if the distribution of stopping times of error choices dominates the distribution of correct choices in the sense that the ratio of their densities is an increasing function.
Here “correct” means “ex post correct,” i.e., conditional on the state. Of course, the agent does not know the state, but they are doing the best they can: their decisions are dynamically optimal and therefore correct conditional on information at time $t$.

**Remark 9.6.** This definition conditions on the true state of the world. This makes sense in the domain of perceptual tasks, where the analyst knows which choice is objectively correct in each trial (e.g., which weight is heavier). In choice tasks, preferences are subjective, so even if the analyst is conditioning choices on $s$, she may not know whether $v(x, s) > v(y, s)$ or the opposite. Typically an additional elicitation of preferences is made as a proxy for the true $v$.\(^{108}\) Alternatively, with a lot of data, accuracy could be defined by modal choice (accuracy only depends on the sign of the utility difference which is revealed by which item is chosen with the higher probability).

### 9.3. Wald’s Model

In the *Wald model* the agent is choosing from a binary menu $A = \{\ell, r\}$. There are two states $S = \{s_\ell, s_r\}$. We have $v(x, s) = 1_{\{s = s_x\}}$ and $c(t) = \gamma t$ for $\gamma > 0$. Conditional on the state $s$, messages are i.i.d. $M_t \sim \mathcal{N}(\delta(s), \sigma^2)$, where $\delta(s_\ell) = \alpha$ and $\delta(s_r) = -\alpha$. Because of this assumption, it is sufficient for the agent to keep track of the running sum $\bar{M}_t := M_1 + \cdots + M_t$, instead of the whole vector $M^t$.

The process $\bar{M}^t$ is a random walk with unknown drift ($\alpha$ or $-\alpha$) that the agent is learning about. By Bayes rule, the posterior log-likelihood ratio is

$$L_t := \log \frac{\mathbb{P}(s_\ell | \bar{M}^t)}{\mathbb{P}(s_r | \bar{M}^t)} = \log \frac{\mathbb{P}(s_\ell)}{\mathbb{P}(s_r)} + \bar{M}^t \frac{2\alpha}{\sigma^2}. \quad (9.1)$$

Given the payoffs, the agent chooses $\ell$ whenever $\mathbb{P}(s_\ell | \bar{M}^t) > \mathbb{P}(s_r | \bar{M}^t)$, which by formula (9.1) holds whenever $\bar{M}^t > \frac{\sigma^2}{2\alpha} \log \frac{\mathbb{P}(s_\ell)}{\mathbb{P}(s_r)} =: w$. Thus, in state $s_\ell$, if we force the agent to choose at time $t$ they will choose $\ell$ with probability $\mathbb{P}(\bar{M}^t > w) = 1 - \Phi \left( \frac{w - \alpha}{\sqrt{2t} \sigma} \right)$, where $\Phi$ is the cdf of the standard Normal distribution. If the prior is symmetric, this function is increasing in $t$, which formalizes the intuitive reasoning behind the speed-accuracy tradeoff.\(^{109}\)

It turns out that in the Wald model the speed-accuracy tradeoff is exactly offset by optimal stopping, so that on balance the error rate is independent of stopping time.

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\(^{108}\)For example, Krajbich, Armel, and Rangel (2010) for each subject elicit a rating of each $x \in X$ on the scale -10, ..., 10 and equate “accurate” with “higher ranked.” Likewise, Oud, Krajbich, Miller, Cheong, Botvinick, and Fehr (2016) elicit willingness to pay for each item. Of course, the subject’s rating may itself be stochastic, so this approach should be treated only as the first step. In addition, this approach is probably not well suited in situations where preference reversals are present (Tversky and Thaler, 1990). See also a discussion in Khaw, Li, and Woodford (2021).

\(^{109}\)Notice a subtlety: a nonmonotonicity occurs the agent believes that $\ell$ is more likely. For small values of $t$ they mostly go by their prior and correctly choose $\ell$. For large values of $t$, their posterior will correctly put a high weight on $\ell$, so they will choose correctly as well. But for intermediate values of $t$, there will be signal realizations that make the agent change their mind since the agent puts some weight on the signal they could incorrectly choose $r$. 

---
Theorem 9.7. In the Wald model there exists $k > 0$ such that
\[
\tau^* = \min\{t \geq 0 : |L_t| \geq k\},
\] (9.2)
where $(L_t)$ is given by (9.1). Moreover, if the prior is symmetric, $\tau^*$ can also be written as
\[
\tau^* = \min\{t \geq 0 : |\bar{M}_t| \geq b\}
\] (9.3)
for some $b > 0$.

**Proof.** See Arrow, Blackwell, and Girshick (1949). \(\square\)

Thus, under $\tau^*$, the agent stops the first time the posterior hits a time-invariant boundary (9.2) and for symmetric priors this is equivalent to the the first time the signal hits a time-invariant boundary (9.3).

The key here is that the boundary is constant in time. One reason behind this that there are only two states. Suppose that $\bar{M}_t$ is small after a long $t$. The agent is certain that the drift is $\alpha$ or $-\alpha$, so this signal is interpreted as pure noise (and the agent continues). On the other hand, if there are more states, e.g., the agent is learning about the difficulty of the task, or about the stakes, then there are many possible values of the drift, so $\bar{M}_t \approx 0$ is now interpreted as carrying some information (for example that the task is difficult, or that the stakes are low) and the agent may want to stop. We will see this later on in more detail.

The constancy of the boundary in the space of beliefs (9.2) does not depend on the assumption that signals are Normally distributed. It holds for any distribution of $M_t$ as long as they are i.i.d. What is key is that there are only two states.\(^{110}\)

Equation (9.3) has no reference to the belief process. Simply, the agent stops the first time the signal process $\bar{M}_t$ hits $b > 0$ and chooses $\ell$ if it hits the upper boundary and $r$ if it hits the lower boundary. This makes it tempting to forget about optimization altogether and think of $\tau = \min\{t \geq 0 : |\bar{M}_t| \geq b\}$ as a “reduced-form” model of reaction times. The continuous time version is known as the Drift-Diffusion Model (DDM).

DDM (and the Wald model) are a good description of some perceptual tasks, where the drift of $\bar{M}_t$ can take one of two possible values. However, they do not really make sense in situations where there are more than two possible states. For example in a weight perception task (Example 1.5), the drift could correspond to the weight difference between the two objects. This makes the state space larger than two points and renders a constant boundary suboptimal. Thus, an application of DDM to such tasks is inconsistent with

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110 Another important assumption that makes $b$ constant is that the marginal cost is constant. If the current belief equals $p \in [0, 1]$, then it does not make any difference how much time has elapsed so far: it’s as if the agent is starting with a prior equal to $p$. On the other hand, if the agent gets tired over time, the cutoff will be changing over time.
the rational expectations assumption (that the agent’s prior equals the data generating process).\textsuperscript{111}

9.4. The Drift-Diffusion Model

DDM was brought to psychology by Stone (1960) and Edwards (1965) to study perception and by Ratcliff (1978) to study memory retrieval. It is now a well established benchmark in psychology and neuroscience. In this model the cumulative signal is a diffusion: time is continuous \( T = [0, \infty) \) and

\[
\tilde{M}^t = t\delta + \sigma B_t, \tag{9.4}
\]

where \( B_t \) is a standard Brownian motion\textsuperscript{112} This ensures that \( \tilde{M}^t \sim \mathcal{N}(t\delta, t\sigma^2) \), just like in the discrete-time version.

**Definition 9.8.** Fix \( A = \{\ell, r\} \). The s.c.f. \( \rho \in \Delta(A \times T) \) has a DDM representation if there exists \( \delta \in \mathbb{R} \) and \( \sigma, b > 0 \) such that (9.4) holds and \( \rho \) is induced by \( \chi \) and \( \tau \) where

\[
\chi_t = \ell \text{ iff } \tilde{M}^t \geq b, \\
\tau = \inf \{t \geq 0 : |\tilde{M}^t| \geq b\},
\]

In this case we write \( \rho \sim DDM(\delta, \sigma, b) \).

To see how this fits with the previous section, note that in the (continuous time version of the) Wald model we have \( \rho^{s_1} \sim DDM(\alpha, \sigma, b) \) and \( \rho^{s_r} \sim DDM(-\alpha, \sigma, b) \).

The assumption that \( M_t \) is a diffusion is key. For a general process \( M_t \), Definition 9.8 is vacuous, see, e.g., Jones and Dzhafarov (2014) and Fudenberg, Strack, and Strzalecki (2015).

**Theorem 9.9 (Gambler’s Ruin Problem).** If \( \rho \sim DDM(\delta, \sigma, b) \), then the parameters are unique up to a common positive scalar multiple. Moreover, \( \rho \) is a product measure, i.e., \( \rho_t \) is constant in \( t \). Furthermore, for any \( t \in T \)

\[
\rho_t(\ell) = \frac{e^{\delta b/\sigma^2}}{e^{\delta b/\sigma^2} + e^{-\delta b/\sigma^2}}
\]

and

\[
\mathbb{E} [\tau] = \frac{b}{\delta} \tanh \left( \frac{b\delta}{\sigma^2} \right),
\]

where \( \tanh \) is the hyperbolic tangent function; \( \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \).

\textsuperscript{111}There are similar experiments, where in each trial the subject faces a screen where a fraction \( \kappa \) of dots is moving in a coherent direction (left or right), while others are moving randomly, see, e.g., Bogacz, Brown, Moehlis, Holmes, and Cohen (2006) and Drugowitsch, Moreno-Bote, Churchland, Shadlen, and Pouget (2012). A similar design was used by Dean and Neligh (2017). The fraction \( \kappa \) (the difficulty of the task) vary from trial to trial.

\textsuperscript{112}A standard Brownian motion starts at zero, has continuous sample paths, and has independent normally distributed increments, meaning that for \( t' > t \) the value of \( B_{t'} - B_t \) is distributed \( \mathcal{N}(0, t' - t) \).
Since $\rho_t$ is independent of $t$, to describe $\rho$ we can just look at the marginals on $A$ and $T$. We are already familiar with the marginal on $A$: it’s the psychometric function. The theorem above says it is of a Luce variety. The new object is the \textit{chronometric function}: the mapping $\delta \mapsto E[\tau]$.\footnote{Just to clarify, the psychometric function and the chronometric function are functions of $\delta$. In contrast, accuracy is a function of $t$. Theorem 9.9 does not fully pin down the distribution of $\tau$. There are Fourier series expressions for the cdf of $\tau$, see Chapter 10, Section 4 of Feller (1957) and Smith (1990).} This is a decreasing function of $|\delta|$, which means that difficult trials ($|\delta|$ small) take longer than easy trials ($|\delta|$ large).

A comparison of these two functions (Figure 9.2) illustrates the usefulness of including response times in estimation. The psychometric function is flat where the chronometric function is steep and vice versa (see e.g., Clithero and Rangel, 2013).

![Figure 9.2. The psychometric (left) and chronometric functions (right). Here $\delta$ varies over the interval $[-2, 2]$ and $b = \sigma = 1.$](image)

The fact that DDM produces constant accuracy was recognized from the beginning and we will discuss many extensions.

DDM and its various extensions have recently been used to study choice tasks.\footnote{Roe, Busemeyer, and Townsend (2001), Clithero and Rangel (2013), Krajbich, Armel, and Rangel (2010), Krajbich, Lu, Camerer, and Rangel (2012), Milosavljevic, Malmaud, Huth, Koch, and Rangel (2010), Reutskaja, Nagel, Camerer, and Rangel (2011).} Here, DDM is often applied across menus. Let $A$ be the collection of all binary menus. The analyst observes the family $\rho^A \in \Delta(A \times T)$ such that $\rho^A \sim \text{DDM}(v(x) - v(y), \sigma, b)$ for $A \in A$, where $v : X \rightarrow \mathbb{R}$ is the utility function of the agent.

The formulas given in Theorem 9.9 put a lot of discipline on the DDM. In particular, the choice probabilities are of the Luce variety:

$$\rho(\ell) = \frac{e^{\lambda v(\ell)}}{e^{\lambda v(\ell)} + e^{\lambda v(r)}},$$

where $\lambda := \frac{2b}{\sigma^2}$.

Thus, the marginal on choices satisfies Luce’s IIA (Axiom 3.2). Moreover, Baldassi, Cerreia-Vioglio, Maccheroni, Marinacci, and Pirazzini (2020) show that expected reaction times can be captured axiomatically. (Given that their
9.5. Time-Dependent DDM

Axioms are just on the expectations, there exist $\rho$ inconsistent with DDM that nevertheless satisfy the axioms.)

Fudenberg, Newey, Strack, and Strzalecki (2020) show how to pin down the behavioral content of any DDM (including those with time-varying boundaries which we will discuss in the next section).

Like in perception tasks, the application of DDM to consumption tasks across menus is inconsistent with the rational expectations assumption. The agent who faces a randomly chosen pair of items from $X$ should have a prior belief given by the distribution of utilities of all items in $X$, which is not a binary prior (as long as $X$ has more than two elements). This will make the optimal boundary time-dependent.

9.5. Time-Dependent DDM

The time-dependent DDM relaxes the constant boundary assumption of DDM while keeping all the other assumptions intact. This section takes $b$ as given; the next section microfound it.

**Definition 9.10.** Fix $A = \{\ell, r\}$. The s.c.f. $\rho \in \Delta(A \times T)$ has a **time-dependent DDM representation** if there exists $\delta \in \mathbb{R}$ and a boundary $b : T \to \mathbb{R}_+$ such that $\rho$ is induced by $\chi$ and $\tau$

$$\chi_t = \ell \text{ iff } \bar{M}^t = b(t)$$

$$\tau = \inf \{t \geq 0 : |\bar{M}^t| \geq b(t)\},$$

$$\bar{M}^t = \delta t + \sigma B_t,$$

where $B_t$ is a standard Brownian motion. In this case we write $\rho \sim DDM^+(\delta, \sigma, b)$.

Similarly to DDM, here the chronometric function is decreasing in $|\delta|$.

DDM has a constant boundary and constant accuracy. In DDM$^+$, accuracy $\alpha$ exactly mimics the monotonicity properties of the boundary $b$.

**Theorem 9.11** (Fudenberg, Strack, and Strzalecki 2018). Suppose that $\rho \sim DDM^+(\delta, \sigma, b)$. Then $\alpha$ is increasing iff $b$ is increasing, $\alpha$ is decreasing iff $b$ is decreasing, and $\alpha$ is constant iff $b$ is constant.

To see the intuition behind this result, suppose that $b$ is a decreasing function. This means that there is a higher bar to clear for smaller $t$ than for larger $t$. So, conditional on $\delta$, if the agent stopped early, $|\bar{M}^t|$ must have been high and therefore the chance of making the correct choice is higher than if the agent stopped late, where $|\bar{M}^t|$ was low.$^{115}$ More formally, it can be shown that if $\rho$

---

$^{115}$ There is a complication, as we need to condition on the event that the boundary has not been crossed before. However, because of the symmetry of the boundary, the same “number” of paths cross the upper and the lower boundary, so the conditioning event does not matter. Formally, a Brownian bridge argument is used.
has a time-dependent DDM representation, then
\[ \frac{\rho_t(\ell)}{\rho_t(r)} = \exp\left( \frac{\delta b(t)}{\sigma^2} \right). \]

9.6. Chernoff’s model

The previous section analyzed time-dependent DDM without worrying about the optimality of the boundary. In this section we will study a model similar to Wald’s, except that the prior of the agent is Normal.

The state space is \( S = \mathbb{R}^A \) and we assume that the prior is such that \( s(x) \sim \mathcal{N}(m_0^x, \sigma_0^2) \) independently across \( x \). It’s easiest to start with a model where the agent gets a signal \( M^t_x \) about each dimension \( x \) of \( s \)
\[ M^t_x = ts(x) + \sigma B^x_t, \]
where the Brownian motions \( B^x_t \) are independent across \( x \).

For example in a choice task, \( A \) are different goods and \( v(x, s) = s(x) \) is the true utility of good \( x \) (unknown to the agent). The posterior is in the same family as the prior: \( s(x) \sim \mathcal{N}(m^t_i, \sigma^2_t) \), where
\[ m^i_t := \frac{m^0_i \sigma^2_0}{\sigma^2_0} + \frac{M^t_i \sigma^2_t}{\sigma^2_t} \quad \text{and} \quad \sigma^{-2}_t := \sigma^{-2}_0 + t\sigma^{-2}_t \quad (9.5) \]
This is a continuous-time version of the Normal updating formula in Example 5.9. The posterior mean is combination of the prior mean and the signal. The posterior precision evolves deterministically and linearly in time.

To sum up, the Chernoff model is an optimal stopping model with a Normal prior, diffusion signal, payoff \( v(x, s) = s(x) \), and additive-linear time cost.

**Proposition 9.12** (Fudenberg, Strack, and Strzalecki 2018). In the Chernoff model there exists a decreasing function \( k : \mathcal{T} \rightarrow \mathbb{R} \) such that
\[ \tau^* = \inf \{ t \geq 0 : |m_t| \geq k(t) \}, \]
where \( m_t := M^t_\ell - M^t_r \) is the posterior mean difference process given by (9.5). Moreover, if \( m_t = 0 \), then there exists \( b : \mathcal{T} \rightarrow \mathbb{R} \) such that
\[ \tau^* = \inf \{ t \geq 0 : |\bar{M}^t| \geq b(t) \}, \]
where \( \bar{M}^t = \bar{M}^t_\ell - \bar{M}^t_r \).

The net signal is a sufficient statistic for our agent’s decision because of the diffusion assumption and the additive cost assumption; it is not sufficient in models with discounting, where the absolute level of the signals matters as well as their difference.

The key intuition for this result is non-stationarity: with a Normal prior the option value is decreasing. Suppose that the agent observes \( \bar{M}^t_\ell \approx \bar{M}^t_r \) after a long \( t \). With a Normal prior, the agent will conclude that they are close to

\[ \text{Chernoff’s (1961) original formulation involved regret-minimization. It can be shown that this formulation is equivalent (Fudenberg, Strack, and Strzalecki, 2018).} \]
indifferent between the two items (because the posterior will converge to the point mass on the indifference point) and stop, given that there is not much left to learn. This is different from the Wald model, where there is no room for such an interpretation of the signal. The prior does not allow for indifference, so the posterior will be uniform over the two states: the point that actually maximizes the option value of learning. Hence the agent behaves as if $M_t$ is pure noise and starts from scratch.

The theorem is silent on the monotonicity properties of $b$. For some parameter values it is decreasing, but for some it is hump-shaped. However, it can be shown that accuracy is always decreasing in expectation (according to agent’s prior).

An application of this model to choice tasks yields DDM$^+$ by setting $\delta$ to be a monotonic function of the true utility difference.

A similar model can be used for perception tasks. The only difference is that $v(x,s) = 1\{s > s_x\}$, i.e., the agent is rewarded for guessing correctly but the payoff is independent of the (unknown) difficulty of the task.\textsuperscript{117} Drugowitsch, Moreno-Bote, Churchland, Shadlen, and Pouget (2012) numerically estimate such a model. Tajima, Drugowitsch, Patel, and Pouget (2019) extend their computational framework beyond binary menus. These papers also allow for the marginal cost to be non-constant. Relaxing this assumption fully, results in a vacuous model.

**Theorem\textsuperscript{†} 9.13 (Fudenberg, Strack, and Strzalecki 2018).** For any $b$ there exists a (potentially nonlinear) cost function $c$ such that $b$ is the optimal solution to the stopping problem.

**Remark 9.14 (Is the boundary optimal?)**. A common intuition in behavioral economics is that people “overthink” decision problems that don’t matter and “underthink” those that are important. Whether this intuition is consistent with our models depends how we define “problems that don’t matter” and those that are “important.”

Let’s first focus on choice tasks where the drift equals the difference in utilities. Here problems that don’t matter are those where $|v(ℓ) - v(r)|$ is small and problems that are important are those where the utility difference is large. As mentioned before, in DDM and DDM$^+$ the reaction time is longer for low $|\delta|$ and shorter for higher $|\delta|$. Thus, it is optimal to spend more time on problems that don’t matter than on those that are important (because important problems are easier). To show that people spend too much time on problems that don’t matter we need something more fancy than just that comparative static.

\textsuperscript{117}Some perception experiments incentivize difficult tasks less than easy tasks, which qualitatively leads to a similar payoff structure as in choice tasks, see e.g., Oud, Krajbich, Miller, Cheong, Botvinick, and Fehr (2016).
Oud, Krajbich, Miller, Cheong, Botvinick, and Fehr (2016) designed experiments where subjects are sometimes forced to make decisions after a set amount of time elapsed. This de facto introduces a new boundary that collapses at zero at some point in time. On trials with such an intervention subjects perform better, on average, than on trials when the response time is freely chosen by them. This shows that the original boundary couldn’t have been optimal. This pattern is true for value-based choices as well as perceptual choices (where subjects are incentivized more on easy trials and less on hard trials).

The assumption that important problems are easier is a consequence of the assumption that the drift is equal to (or a monotonic function of) the utility difference. But there are situations where important decisions are actually difficult (e.g., choosing a retirement plan). Here the optimal strategy would be to spend even more time on problems that matter and even less time on easy, inconsequential decisions, exactly the opposite of the “behavioral” intuition.

For another example, suppose you got admitted to PhD programs at Stanford and MIT. How long will you think about your choice? Now imagine that your choice set is MIT and a university you never heard of. Presumably you would make that decision much faster. Presumably, this is because you expect to learn much less in the second case than in the first, so this is perhaps better modeled as a comparative static in $\sigma$, not in $\delta$.

9.7. Dynamic attention

9.7.1. Optimal dynamic attention. This is a very active area of research and it should be its own chapter, but the literature is moving faster than I can catch up, so this is just a brief introduction.

We can distinguish three types of problems.

- Pure stopping problem (choose $\tau$, Chapter 9):
- Pure attention (choose $(M^t)$, Chapter 6)
- Joint optimization (choose both $\tau$ and $(M^t)$)

In the pure stopping problem, the agent cannot direct their attention. But they can decide how much overall information to get. Notice that by choosing $\tau$ we are facto choosing a distribution over posteriors $\mu$. Morris and Strack (2019) show that with a diffusion signal and binary state space any $\mu$ can be obtained by an appropriate choice of $\tau$.

A natural question is what if we set the cost of $\mu$ to the expectation of the cost under $\tau$: $E \int_0^\tau c(q_t)dt$, where $q_t$ is agent’s posterior. This induced cost function is prior-dependent because it depends on the expectation of $\tau$ under the prior. Morris and Strack (2019) show that the induced cost function
9.7. Dynamic attention

is posterior separable and moreover all PS cost functions can be written this way.\footnote{With more than two states not all $\mu$ are achievable, but the cost for any $\tau$ is still PS in the induced $\mu$ and total when the flow cost is constant.}

An interesting special case is constant $c$, which gives a special case of total cost, where the weights are symmetric. In the general class of total costs Bloedel and Zhong (2021) allow the agent to choose any sequential experiment, not just ones based on diffusions.

An interesting open question is what is the class of static costs that arises from sequential learning with WRE.

Mutual information corresponds to the case where the flow cost is posterior-dependent and equals the variance of the posterior.

Hébert and Woodford (2017) show a similar reduction to a static separable problem in the joint optimization problem.

Che and Mierendorff (2019) study the joint optimization problem with two states by restricting the class of signals to be Poisson. They find that coexistence of two strategies is optimal: a contradictory strategy that seeks to challenge the prior and a confirmatory strategy that seeks to confirm the prior. Zhong (2018) shows that Poisson signals are optimal under discounting. Rustichini (2020) studies a model where the signals are Lévy processes.

For Normal signals, Fudenberg, Strack, and Strzalecki (2018) show that it is always optimal to pay equal attention to alternatives (or switch between them infinitely often), under a parametric restriction on the tradeoff between the informativeness of each signal. Liang, Mu, and Syrgkanis (2022) generalize this by allowing the prior to have an arbitrary covariance matrix. Liang and Mu (2020) find conditions under which the dynamically optimal strategy is close to the myopic strategy. Ke and Villas-Boas (2016) study joint optimization with two states per alternative.

Steiner, Stewart, and Matějka (2017) study optimal attention with mutual information and evolving state. They apply their general solution to the study of time-varying accuracy. Miao and Xing (2020) generalize these results to uniformly posterior-separable cost functions.

Woodford (2014) solves a optimal attention problem (with a constant boundary) and shows that optimal behavior leads to decreasing accuracy.

Duraj and Lin (2019) study an agent who has the option to buy a fixed signal (either at a cost, or experimentation takes time, so the problem is discounted). They show an axiomatic characterization and uniqueness results.

9.7.2. Attentional DDM. In lab experiments we can record additional data, such as eye movements (Krajbich, Armel, and Rangel, 2010; Krajbich and Rangel, 2011; Krajbich, Lu, Camerer, and Rangel, 2012; Gaia Lombardi, 2020; Callaway, Rangel, and Griffiths, 2020). That literature uses the Attentional
DDM model, which is an extension of DDM that incorporates attention. In those models “attention” is an exogenous process that does not condition on the signal $Z_t$. Evidence is accumulated only for the item that is currently paid attention to. This makes the model easy to estimate and take to data. I don’t know of microfoundations for such models and how they relate to models of dynamic attention allocation.

9.8. Other Models from Psychology

9.8.1. Full DDM. *Full DDM* or *extended DDM* is a mixture of constant-boundary DDMs where mixing occurs over three parameters: the drift $\delta$, the starting point $M_0$, and the initial latency $T_0$ (inaction period, so that the analyst observes the realizations of $T_0 + \tau$).\(^{119}\) We have encountered mixture models in static domain (the mixed logit model, see Section 3.6). We interpreted this as pooling over subjects. Here, the randomization is often done for a fixed experimental subject.

**Example 9.15** (Ads). To interpret that, imagine that there is a non-skippable ad in the phone app that you are using. This add adds an initial latency (of a deterministic length) and endows the agent with a signal, thus randomly moving the agent’s belief at the beginning of the decision phase.

Let $T_0$ be the length of the advertisement and the drift be $\kappa(v(\ell) - v(r))$. Then

$$M_0 \sim \mathcal{N}(T_0\kappa(v(\ell) - v(r)), T_0).$$

*Chiong, Shum, Webb, and Chen* (2018) estimate this model and simulate what would happen in a counterfactual, where the marketing company allows for skippable ads.\(^\Diamond\)

The above example is slightly different from the way the full DDM is typically used. The example suggests drawing $T_0$ first and then $M_0$ conditional on $T_0$. In practice, the distribution of $M_0$ is independent of $T_0$.

The full DDM model is extremely popular in applications, as it introduces more parameters that can be estimated and in particular allowed for time-varying accuracy. Axiomatic underpinnings of extended DDM are unknown, but like with simple DDM the difficulty lies in capturing the distribution of stopping times by a nice axiom.

9.8.2. Race Models or Accumulator Models. *Race models*, otherwise known as *accumulator models* assume a different signal $\tilde{M}_x^t$ for each $x \in X$ instead of looking at the net signal. Here evidence can be accumulated at different speeds and there can be correlation between the signals, see, e.g., *Vickers* (1970). *Pike* (1966) studies finite Markov models. The simple DDM belongs to

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the Markov class (the stopping region in $\mathbb{R}^2$ is $\{(x_\ell, x_r) : |x_\ell - x_r| \geq b\}$). The following example assumes a Poisson process.

**Example 9.16** (Eureka Moment). Here each $\bar{M}_x^t$ is an independent Poisson process with intensity $e^{v(x)}$. The agent stops the first time one of these processes hits the value 1 and picks the corresponding $x$. It is easy to show (see Appendix A.9.1) that such defined $\rho \in \Delta(A \times T)$ is a product measure, where the marginal on choices is of the Luce form

$$\rho_t(x) = \frac{e^{v(x)}}{\sum_{y \in A} e^{v(y)}}$$

and the marginal distribution of $\tau$ is exponential with parameter $\sum_{x \in A} e^{v(x)}$.

Given the closed forms, this model is equally easy to fit as the DDM and it extends beyond binary menus. A model like this was proposed by Audley (1960); see also Lensman (2020) who shows that the constant accuracy prediction depends on the assumption that the boundary equals one. For any higher value accuracy is decreasing.$^{120}$

Note that constant boundary is not optimal for a model where $M_t^x$ is a Poisson process with intensity proportional to the utility of $x$.

Smith and Vickers (1988) derive the stopping probabilities for a general class of accumulator models. An even more general model was proposed by Marley (1989). Here, each item $x$ is associated with a random time $T_x$ and the agent chooses the item whose time comes first (and chooses at that time).

A special kind of accumulator models are Linear Ballistic Accumulator (LBA) models, see Brown and Heathcote (2008) and Terry, Marley, Barnwal, Wagenmakers, Heathcote, and Brown (2015). In those models the paths of $M_t^x$ are linear and randomization is over the starting point and the angle of the path (uniform and Normal, respectively).

As shown by Jones and Dzhafarov (2014) without those parametric assumptions LBA has no predictive content, see also Marley and Colonius (1992). Webb (2019) shows that for accumulator models, this marginal is governed by a static ARU model with attributes like in Definition 10.6 but where the distribution of $\epsilon$ is allowed to depend on $v$.

### 9.8.3. Mean Reverting Stimulus or Leaky Models.

Busemeyer and Townsend (1993) study an accumulator model where instead of $\bar{M}_t$ being a diffusion with a constant drift, they let it be a mean-reverting (Ornstein-Uhlenbeck) process:

$$d\bar{M}_t^t = (\delta - \gamma \bar{M}_t^t)dt + \sigma dB_t.$$

A mean-reverting accumulator model, sometimes called as “leaky accumulator” and is the subject of “Decision-Field Theory.” A nice exposition of these and

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$^{120}$ A peculiar feature of this model is that adding options reduces the reaction time. This may seem counterintuitive, given what we know about choice overload. In perceptual tasks there is the “Hick–Hyman Law” which says that the average decision time increases logarithmically in the menu size (Luce, 1986).

9.8.4. Recent papers in economics. Alós-Ferrer, Fehr, and Netzer (2021) take as a primitive the marginal over choices $\rho \in \Delta(A)$ together with a distribution over stopping times. They look at a generalization of ARU where the distribution of $\epsilon$ is menu-dependent. The response time is a deterministic function of the realization of $\epsilon$.

Epstein and Ji (2020) study learning in a diffusion setting where there is prior ambiguity. Auster, Che, and Mierendorff (2022) look at a similar setting with Poisson signals.


Duraj and Lin (2019) and Duraj and Lin (2021) work on the domain of stopping trees and provide a decision-theoretic analysis of the general model presented in Section 9.2, both the additive cost and discounted versions. As mentioned before, Baldassi, Cerreia-Vioglio, Maccheroni, Marinacci, and Pirazzini (2020) give a partial axiomatization of DDM with a constant boundary. Fudenberg, Newey, Strack, and Strzalecki (2020) give necessary and sufficient conditions for an arbitrary boundary.
Part 4

Discrete Choice
Chapter 10

Discrete Choice

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10.1. Attributes

So far, our alternatives were some abstract items \( x \in X \). In *discrete choice* theory the alternatives are described by their *attributes*. We already saw something like that. In the weight preception task (Example 1.5) each item was identified with its weight (one-dimensional attribute). In Chapter 4, a lottery was a vector of probabilities of each prize (a vector of attributes). More generally, for each alternative \( x \) let \( \xi_x \) be a vector of its attributes, which lives in some space \( \Xi \subseteq \mathbb{R}^n \). These are also called *characteristics* or *hedonics*, see, e.g., Hotelling (1929); Griliches (1961); Lancaster (1966). In econometrics they are called *regressors* or *predictors*. Sometimes \( \xi \) can be controlled by the researcher (in a lab experiment), but sometimes it cannot (market prices are endogenous).

Let \( \xi := (\xi_x) \) describe the characteristics of all the items. I will assume that the menu is fixed and equal to \( X \). This is typically true in the literature, although there are some exceptions (Buchholz, Doval, Kastl, Matějka, and Salz, 2020). The primitive is a function that maps the profile of attributes of all goods to the choice probabilities. Recall that \( \Xi^X \) is the set of all functions from \( X \) to \( \Xi \).

**Definition 10.1.** A *s.c.f. with attributes* is a function \( \rho : \Xi^X \to \Delta(X) \). We will write \( \rho(x, \xi) \) to mean the probability of choosing \( x \) when the vector of attributes is \( \xi \).

For example, in applications to consumer demand the analyst observes a number of markets. In each market \( k \) she records the attributes of all goods \( \xi^k \in \Xi^X \) and the market shares \( s^k \in \Delta(X) \). We assume here that each market is large enough so that \( s^k = \rho(\xi^k) \) for all \( k \). By observing enough markets, we can trace out the function \( \rho \) pretty well. If price is one of the attributes, then we can study price elasticities and substitution patterns between various goods. In demand applications it is quite important to explicitly include an outside good. This makes it possible to model market size and makes price elasticities more realistic (Berry, 1994)

**Remark 10.2** (Comparison to menu-variation). Though there is no explicit menu variation in this model, if an attribute of \( x \) changes, this in fact does change the menu because now the new “version” of \( x \) is included and the old one is absent. For this reason, any s.c.f. with attributes can be represented by our usual s.c.f. with menu variation defined on a restricted set of menus with a fixed size.

Because of this close association, I denote the two objects by the same letter \( \rho \). However, I will keep distinguishing them as functions, \( \rho : A \to \Delta(X) \) and \( \rho : \Xi^X \to \Delta(X) \), because the two have a different mathematical structure and different results have been proved about them.
10.2. Independent Additive Random Utility

Remark 10.3 (Label-invariance). Sometimes an additional assumption is made that all relevant information about alternatives is encoded in their characteristics, so $x$ are just labels that do not affect choice (Manski, 1977; McFadden, 1981). Formally, for any permutation $\pi$ of $X$ and any $\xi$ define $\xi^\pi$ by $\xi^\pi_x := \xi_{\pi(x)}$. McFadden’s (1981) assumption PC5.2 says that $\rho(x, \xi) = \rho(\pi(x), \xi^\pi)$. Taken literally, label-invariance is very restrictive. For example, $X$ are brands and consumers have a preference for some brand $x$ even if there is another brand $y$ with all physical attributes exactly identical $\xi(x) = \xi(y)$. This problem goes away if we include brands as characteristics. ▲

Remark 10.4 (Continuous choice). There is a related model that also takes prices as regressors, but allows agents to continuously choose the quantities of each good, like in the classical setting of GARP (Afriat, 1967). The analyst observes the population frequency of each chosen consumption bundle. McFadden (2005) shows how to transform this continuous problem to a discrete RU problem. Kitamura and Stoye (2018) take this insight further and construct a test of this model. Smeulders, Cherchye, and Rock (2021) show that implementing this test is NP-hard. There is an older literature that does not keep track of frequencies of each chosen bundle and instead only keeps track of the average (aggregate) demand, and therefore has to deal with the representative consumer, see, e.g., Blundell, Browning, and Crawford (2003). ▲

10.2. Independent Additive Random Utility

10.2.1. IARU. The utility of item $x$ depends on the vector $\xi_x \in \Xi$ of observable attributes of $x$, but not on attributes of other items. (Utility can potentially also depend on the attributes of the consumer, but we will abstract from that here).

Definition 10.5. Let $X$ be a finite set. $\rho : \Xi^X \rightarrow \Delta(X)$ is represented by IARU if

$$\rho(x, \xi) = \mathbb{P}\left(\left\{\omega \in \Omega : \tilde{U}(x, \xi_x)(\omega) = \max_{y \in X} U(y, \xi_y)(\omega)\right\}\right),$$

where the agent’s random utility equals

$$\tilde{U}(x, \xi_x) = v_x(\xi_x) + \tilde{\epsilon}(x),$$

where $v_x : \Xi \rightarrow \mathbb{R}$ is a deterministic utility function and $\tilde{\epsilon}$ is a random vector independent of $\xi$ with a smooth distribution.

The independence of $\epsilon$ and $\xi$ is the analog of the independence assumption from the model with menu variation, where $\epsilon$ was independent of the menu (Sections 1.5.1 and 1.7).\footnote{Notice that while with menu-variation the independence assumption can be expressed both for RU and ARU, with attribute-variation this notion only makes sense for ARU. In RU, the distribution of utilities must depend on $\xi$, otherwise $\rho$ will be constant in $\xi$.}
In applications independence often fails. For example, prices are endogenous because firms endogenously adjust them in response to demand. Demand is driven by utility shocks of consumers, so prices are correlated with $\epsilon$. We will discuss endogeneity in Section 10.7; for now we will assume that the econometrician has solved this problem for us and handed us the $\rho$ that satisfies independence.

**10.2.2. Simple IARU.** In applications, the distribution of $\epsilon$ is often fixed and not estimated. The econometrician focuses on estimating the functions $v$. For each candidate $v$ we can compute the choice probabilities $\rho$. We then check how far those are from the observed data and choose another $v$ that gets us closer (iterate till we converge). A key object here is the mapping between $v$ and $\rho$, we will write this as $\rho(x, v)$. It is a special case of IARU, where the utilities are “magically” observed by the analyst.

**Definition 10.6.** $\rho : \mathbb{R}^X \rightarrow \Delta(X)$ has a *simple IARU* representation if there exists a random variable $\tilde{\epsilon}$ with values in $\mathbb{R}^X$ distributed smoothly and independently of $v$, such that

$$\rho(x, v) = \mathbb{P}(v(x) + \tilde{\epsilon}(x) = \max_{y \in X} v(y) + \tilde{\epsilon}(y)).$$

Another situation where simple IARU occurs is when utility is quasilinear in prices, except now with a minus sign.\(^{122}\)

**10.2.3. Logit.** Logit means that $\tilde{\epsilon}_x$ are i.i.d. TIEV. In a simple version of the model $v$ is an affine function,

$$v_x(\xi) = \beta_0^x + \sum_{i=1}^n \beta^i \xi_x^i = \langle \beta, \xi \rangle,$$

$n$ is the dimension of $\Xi$ and where in the last expression $\langle \cdot, \cdot \rangle$ is the inner product and we added an extra entry to $\xi$ equal to one (to capture the fixed effects $\beta^0$). This is known as the *logistic regression*. Polynomial functions are also used, as are other functional forms.

The early literature (McFadden, 1975; Train, 1986) discussed something called the *mother logit*. Here, we have $\tilde{U}(x, \xi) = v_x(\xi) + \tilde{\epsilon}_x$, that is the function $v_x$ can depend on the characteristics of not only good $x$ but also other goods. It is easy to see any s.c.f. with attributes that satisfies Positivity has a mother-logit representation.

**10.3. Identification**

If $\rho \sim IARU$, then the utility function $v$ and the distribution of $\epsilon$ are identified (up to a normalization), see Theorems 2, 3, and 4 of Matzkin (1993) (see also

\(^{122}\)Such models are a special case of IARU where $\Xi = \mathbb{R}$ and $v(x, \xi) = w(x) - \xi_x$ for some deterministic function $w$, or potentially $\Xi$ could also include observable attributes other than price. Jaffe and Weyl (2010) shows that this model cannot generate linear demand.
The WDZ Lemma says that in the simple IARU model the choice probability $\rho$ is the gradient of the social surplus function (Williams, 1977; Daly and Zachary, 1979; McFadden, 1981). The WDZ lemma allows us to compute welfare by integrating the choice probabilities (if we can observe a sufficiently rich variation in $v$). Formally, we have

$$V(v) = \mathbb{E}[^\text{max}_x v(x) + \tilde{\epsilon}(x)].$$

**Lemma 10.7 (WDZ).** Suppose that $X$ is finite. If $\rho \sim$ simple IARU and $\tilde{\epsilon}$ has finite first moments, then:

(i) For any $v \in \mathbb{R}^X$ the associated social surplus $V(v)$ is finite

(ii) The function $V : \mathbb{R}^X \rightarrow \mathbb{R}$ is differentiable and convex

(iii) $\rho = \nabla V$.

Equation (iii) can be directly verified in case of logit. By the log-sum expression (3.4) we have $V(v) = \log \sum_{x \in X} e^{v(x)}$, so taking the partial derivative of $V$ with respect to the utility of good $y$ we get

$$\frac{\partial V(v)}{\partial v(y)} = \frac{e^{v(y)}}{\sum_{x \in X} e^{v(x)}} = \rho(x; v).$$

For a formal proof, see Shi, Shum, and Song (2018). Intuitively, part (iii) is the envelope theorem. Its analog in production theory is Hotelling’s (1932) lemma, which says that the quantity produced equals the derivative of the profit function. The analog in classical demand theory is Roy’s (1947) identity.

The WDZ lemma is a step towards axiomatizing simple IARU because being a gradient of some function imposes some constraints on $\rho$. This is the content of the WDZ theorem.
10.5. The WDZ Theorem

10.5.1. The Theorem. For the purpose of this section let \( \frac{\partial \rho_x}{\partial v_y}(v) \) denote the derivative of \( \rho(x, v) \) with respect to the \( y \)-th coordinate of \( v \) (taken at \( v \)). For any \( v \) and \( k \in \mathbb{R} \) the function \( v + k \) assigns utility \( v(x) + k \) to item \( x \).

**Theorem 10.8 (WDZ).** Suppose that \( \rho : \mathbb{R}^X \rightarrow \Delta(X) \) is \(|X|\)-times continuously differentiable. Then \( \rho \sim \) simple IARU iff it satisfies:

(i) **translation invariance:** \( \rho(v) = \rho(v + k) \) for all \( v \in \mathbb{R}^X \), \( k \in \mathbb{R} \)

(ii) **zero limit demand:** \( \lim_{v_x \rightarrow -\infty} \rho(x, v) = 0 \) for all \( v_x \in \mathbb{R}^X \setminus \{x\} \)

(iii) **symmetric partials:**
\[
\frac{\partial \rho_x}{\partial v_y}(v) = \frac{\partial \rho_y}{\partial v_x}(v) \text{ for all } v \in \mathbb{R}^n \text{ and } x \neq y
\]

(iv) **gross substitutes:**
\[
\frac{\partial \rho_x}{\partial v_y}(v) < 0 \text{ for } x \neq y \text{ and all } v \in \mathbb{R}^X
\]

(v) **alternating signs of partials:**
\[
(-1)^k \frac{\partial^k \rho_{x_0}}{\partial v_{x_1} \cdots \partial v_{x_k}}(v) > 0
\]
for all \( v \in \mathbb{R}^X \) and for each \( k = 2, \ldots, |X| - 1 \) and each set of \( k + 1 \) distinct elements \( \{x_0, \ldots, x_k\} \subset X \).

**Proof.** See Appendix A.10.1. \( \square \)

10.5.2. Intuition Behind These Conditions.

**Translation invariance** means that shifting all utilities by a constant does not change the choice. This is because what matters are utility differences, not absolute levels.

**Zero limit demand** says that by sufficiently lowering the utility of \( x \) we can reduce demand for \( x \) as much as we want (holding the utilities of \( y \neq x \) constant). This is because the distribution of \( \tilde{\epsilon} \) is fixed so for \( x \) to be chosen \( \tilde{\epsilon}_x \) must clear a higher and higher bar, and the probability of such a tail event goes to zero.

**Symmetric partials** is similar to the symmetry of the Slutsky matrix in the classical demand theory.\(^{123}\) It is equivalent to \( \rho = \nabla V \) for some differentiable function \( V \).

**Gross substitutes** means that demand for good \( x \) decreases if the utility of good \( y \neq x \) increases.

\(^{123}\)See a discussion in McFadden (1981) in whose model the agent is also consuming some perfectly divisible commodities and has a Gorman-style utility, which guarantees that the indirect utility is quasilinear in prices.
10.5. The WDZ Theorem

*Alternating signs* is a stronger version of gross substitutes, similar to the exclusion-inclusion formula in Axiom 2.8.

10.5.3. Relationship Between These Conditions. Symmetric partials together with gross substitutes imply that the Jacobian of $\rho$ is symmetric and positive semi-definite, see Hofbauer and Sandholm (2002).

We have symmetric partials and positive semi-definiteness if and only if $\rho$ is the gradient of a convex function.\(^{124}\)

As we know from the WDZ Lemma, that function, will turn out to be the social surplus function.\(^{125}\)

Another way to capture both symmetric partials and positive semi-definiteness is *cyclic monotonicity*.\(^{126}\)

**Definition 10.9.** $\rho$ satisfies *cyclic monotonicity* (cm) if for any $k$ and any sequence of values $v_1, \ldots, v_k \in \mathbb{R}^X$ where $v_{k+1} = v_1$

$$\sum_{i=1}^{k} \langle \rho(v_i), v_i - v_{i+1} \rangle \geq 0.$$  

Note that (cm) is discrete in nature and thus it may be easier to test on a finite data set, as it does not rely on small variations in $v$.

Related conditions were used to axiomatize the GEV model (Smith, 1984). In Section 10.8 we’ll see that *perturbed utility* satisfies (i), (ii), (iii), and positive definiteness, but not the stronger conditions (iv) and (v).

10.5.4. Tests of These Conditions. Of course we can’t test these conditions directly, as we do not observe $v$. However, if utility is quasilinear in prices, then prices play the same role as utilities and the conditions below can be tested.\(^{127}\)


In economic applications prices are often correlated with unobserved taste shocks, so care needs to be taken when testing conditions (i)-(v) and (cm), as they rely on independence.

\[^{124}\text{This follows from Theorem 10.9 of Apostol (1969) and Theorem 35 of Fenchel (1953).}\]
\[^{125}\text{Actually, the WDZ Lemma assumes finite moments of } \tilde{\epsilon} \text{ but this assumption does not automatically follow from Theorem 10.8. While this assumption was w.l.o.g. with menu-variation, it is not with attribute-variation. In this case, if moments are infinite, } \rho \text{ is a gradient of another function, which is defined even in that case. For more on the issue of finite moments, see Fosgerau, McFadden, and Bierlaire (2013).}\]
\[^{126}\text{See Theorem A.10.1 in the Appendix.}\]
\[^{127}\text{Since in the quasilinear model prices enter with a negative sign, the signs in conditions (ii), (iv), and (v) need to be adapted; in particular the term } (-1)^k \text{ drops from condition (v).}\]
10.6. Patterns of Substitution

Suppose that we are in a market setting where each good $x$ has a price $p_x$. Let $p$ be the vector $(p_x)_{x \in X}$. There may be other attributes but for simplicity we will abstract from them. Suppose that utility is quasilinear in prices so that $v(x, p_x) = w(x) - p_x$, where $w \in \mathbb{R}^X$ is a fixed and deterministic utility function and $p_x$ is the price of good $x$. We know from the WDZ theorem (Theorem 10.8) that goods are gross substitutes: if the price of good $x$ increases, the demand for all other $y \neq x$ goes up (and the demand for $x$ decreases).

A particularly stark example is the logit model, which has *proportional substitution*: for any $x \neq z$

$$\frac{\partial \rho_x}{\partial p_z} = \rho(x, p) \rho(z, p)$$

This is a very strong prediction, which is obviously counterfactual. For example, a full-size car $x$ and a mid-size car $y$, must have the same elasticity with respect to the price of a compact car $z$. This inflexibility of the model is another manifestation of the blue bus-red bus problem (Example 3.9). A less restrictive implication holds for IARU with i.i.d. $\tilde{\epsilon}$.

**Proposition 10.10.** If $\rho \sim IARU$ with quasilinear prices like above and i.i.d. $\tilde{\epsilon}$, then $\rho(x, p) = \rho(y, p)$ implies that $x$ and $y$ have the same elasticity with respect to the price of a third good $z$.

**Proof.** See Appendix A.10.2

This is less restrictive, but still seems counterfactual. As Berry and Haile (2021) stress, “this is a bug, not a feature.” For example, if a full-size car $x$ and a mid-size car $y$ have the same market share, they must have the same elasticity with respect to the price of a compact car $z$.

The conclusion can be escaped by relaxing the i.i.d. assumption and introducing some correlation into $\epsilon$. This can be done directly, by estimating the covariance matrix of $\tilde{\epsilon}$, but can be hard when there are many alternatives. Nested models, as in Section 3.7, are another route. Yet another route are mixed models, which we will discuss now. They are more tractable and intuitive because they can be interpreted as heterogenous tastes for product characteristics and they generate more intuitive substitution patterns.

10.6.1. Random Coefficient Models. Under logit, we have

$$\tilde{U}(x, \xi) = \langle \beta, \xi_x \rangle + \tilde{\epsilon}_x,$$

where the coefficients $\beta$ are deterministic (Section 10.2.3).

In random coefficients models the coefficients $\beta$ are random to reflect the heterogeneity of unobserved individual characteristics.

$$\tilde{U}(x, \xi) = \langle \tilde{\beta}, \xi_x \rangle + \tilde{\epsilon}_x$$
You can think of this as a special case of mixed logit (Section 3.6), where the randomization over the coefficients implements a mixture over linear utility functions \( v \). Early papers include Daly and Zachary (1975); Boyd and Mellman (1980); Cardell and Dunbar (1980).

Fox, il Kim, Ryan, and Bajari (2012) show that the distribution over \( \tilde{\beta} \) is identified nonparametrically. Fox, Kim, Ryan, and Bajari (2011) and Fox, il Kim, and Yang (2016) develop nonparametric finite-mixture estimators. Nonparametric estimators often rely on tuning parameters for statistical guarantees that can be sometimes hard to select in practice.

Note that there are two ways we can write such a model: (1) as a mixed logit representation; here the preference heterogeneity is explicitly modeled by the mixing distribution and the shocks \( \tilde{\epsilon} \) are i.i.d. (2) as IARU, where the \( v \) function is deterministic (the average of the mixing distribution) and all the preference heterogeneity is hidden in the distribution of \( \tilde{\epsilon} \).

There is also mixed polynomial logit.

**Definition 10.11.** \( \rho \) has a mixed polynomial logit representation if it has a mixed logit representation where all functions \( v \) are polynomials of \( \xi \).

McFadden and Train (2000) show that if \( \Xi \) is a compact set, then any \( \rho \) with a RU representation is a limit of \( \rho \)'s with mixed polynomial logit representation. The additional difficulty here (over mixed logit in Proposition 3.31) is approximating the \( v \) function by polynomials.

Saito (2018) offers an axiomatization of mixed polynomial logits, where the polynomials are of degree at most \( d \). Under the assumption that \( \Xi \) is finite, Chang, Narita, and Saito (2022) show that the convergence result of McFadden and Train (2000) may not hold if we insist that there exists a uniform bound on the order of all the polynomials.

Random coefficient models were also studied for mixed probit (Hausman and Wise, 1978). Brownstone and Train (1998) only add random intercepts (the multiplicative coefficients are deterministic). When the mixture over the intercepts is normal, this mixed probit is simply a probit.

**10.6.2. Pure Characteristics and Address Models.** Berry and Pakes (2007) consider a pure characteristics model which is a mixture over deterministic utility functions. (You can also think of this as a limit of mixed polynomial logit representations with the noise parameter going to zero). For example, random expected utility from Chapter 4 is a pure characteristics model. Pure characteristics is explicitly a RU representation (a randomization over utility functions). In contrast, random coefficients is only implicitly RU (a randomization over logits, each being itself a randomization over utility functions). Broadly speaking, random coefficients and pure characteristics models are similar, the key difference being that in former there is positive demand for dominated products. A formal connection was drawn by Lu and Saito (2022).
Hotelling (1929) assumed that each agent has a deterministic utility with a blisspoint. The address models, which generalize the idea of a blisspoint, are reviewed in Chapter 4 of Anderson, de Palma, and Thisse (1992).

10.6.3. Complementarities? While random coefficients models can help us avoid some unrealistic patterns of substitution, all goods are still gross substitutes. This is basically because those models can be written as IARU that are quasilinear in prices, so the WDZ theorem applies and its condition (iv) says that the derivative of demand for good $x$ with respect to the price of good $y$ is negative. Are there models where goods are complements? Here is a simple example that shows that models of attention (like in Chapter 6) can have this property.

**Example 10.12.** Suppose there are three goods $X = \{x_1, x_2, x_3\}$, and two states $S = \{s_1, s_2\}$. The utility of good $x_i$ in state $s_j$ equals $v(x_i, s_j) - p_i$, where $p_i$ is the price of good $i$. Table 1 lists the values of $v$.

<table>
<thead>
<tr>
<th></th>
<th>$v(x, s_1)$</th>
<th>$v(x, s_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>$x_3$</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

The agent can pay a cost $1 - \alpha$ of observing the true state (or remain with their uniform prior at no cost). Suppose that initially the price of all three goods is zero. The agent chooses to learn the state and the observed choice probabilities are $(\frac{1}{2}, \frac{1}{2}, 0)$. Suppose that the price of good $x_2$ increases by one. The agent now chooses not to observe the state and the choice probabilities are $(0, 0, 1)$.$^{128}$

10.7. Dealing with Endogeneity*

For the purpose of this exercise, it is useful to think of the state as $(\omega, \xi)$, where $\xi$ is observable to the analyst (and agent), while $\omega$ is only observable to the agent. This way, $\rho(\cdot, \xi)$ is the choice probability conditional on observing $\xi$.

**Definition 10.13.** Let $X$ be a finite set. $\rho : \Xi^X \rightarrow \Delta(X)$ is represented by ARU with attributes if there exists a deterministic utility function $v : X \times \Xi \rightarrow \mathbb{R}$ and a probability space $(\Omega \times \Xi^X, \mathcal{F}, \mathbb{P})$ as well as a random variable $\tilde{\epsilon}$ with values in $\mathbb{R}^X$ such that for any $\xi$ the distribution of $\tilde{\epsilon}$ is smooth and

---

$^{128}$This example is due to Nicola Rossaia.
\[ \rho(x, \xi) = \mathbb{P}\left\{ \omega \in \Omega : v(x, \xi_x) + \tilde{\epsilon}(x, \xi_x, \omega) = \max_{y \in X} v(y, \xi_y) + \tilde{\epsilon}(y, \xi_y, \omega) \right\} | \xi \]  

So far, we have been maintaining the assumption that the distribution over preference shocks is independent of the attribute \( \xi \). However, this assumption is often violated. A common example is when there are demand shocks that in equilibrium influence the prices. If such shocks are unobservable for the analyst, independence is violated. In this section I will drop the tilde from random variables that are observable by the analyst.

Let \( p_x \) be the price of good \( x \) and let \( \tilde{\eta}_x \) be the (unobservable to the analyst) demand shifter. For simplicity, price is the only observable attribute. The random utility equals \( \tilde{U}(x, p_x) = \tilde{\delta}_x + \tilde{\gamma}_x \), where \( \tilde{\gamma}_x \) is i.i.d. TIEV and

\[ \tilde{\delta}_x = v_x - p_x + \tilde{\eta}_x \] (10.1)

Suppose that the analyst observes \( K \) independent markets and the realization of \( \eta_x \) is constant within each market but i.i.d. across markets.\(^{129}\) Each market is large enough so that the observed market share of good \( x \) equals the theoretical choice probability, i.e.,

\[ s^k_x(p^k) = \frac{e^{\tilde{\delta}^k_x}}{\sum_{y \in A} e^{\tilde{\delta}^k_y}}. \]

Given the invertibility of logit, the analyst can recover \( \delta^k \) from \( s^k \) up to a normalization. Let \( 0 \in X \) be the outside good and set \( \delta^k_0 := 0 \) and

\[ \delta^k_x := \log s^k_x - \log s^k_0. \] (10.2)

In principle, this could work for any invertible model, in particular any simple IARU (see Section 10.3).

Suppose, for the sake of the example, that whenever there is a positive demand shock, firms can capture all the benefits from it by increasing prices, so that \( p^k_x = \tilde{\pi}^k_x + \eta^k_x \), where \( \tilde{\pi}^k_x \) is unobservable to the analyst and independent of everything else. This plus equation (10.1) implies that \( \delta^k_x = v_x - \tilde{\pi}^k_x \). So an analyst who naively regresses market shares on prices will “estimate” a zero own-price elasticity. ▲

In such cases, we use instrumental variables. Suppose that we have an independent source of variation in prices, for example shocks to firm’s costs. We will say that \( z \) is an instrumental variable if:

- \( z \) is correlated with \( p \) (instrument relevance)
- \( z \) is independent of the \( \eta \) (instrument validity).\(^{130}\)

\(^{129}\)The independence assumption is just for simplicity and is often lifted, see, e.g., the discussion on pp. 617–618 of Berry, Linton, and Pakes (2004).

\(^{130}\)A weaker assumption is used that \( \mathbb{E}[\tilde{\eta}_x | z] = 0 \) for all \( x \in X \).
Following Berry (1994), suppose that in each market $k$ we observe a vector of prices $p^k$ and non-price attributes $\xi^k$. The random utility is $\tilde{U}(x, p_x, \xi_x) = \tilde{\delta}_x + \tilde{\gamma}_x$, where $\tilde{\gamma}_x$ is indep. of $(\tilde{\xi}, \tilde{p}, \tilde{\eta})$ and distributed TIEV i.i.d. across $x$ and

$$\tilde{\delta}_x = \langle \beta, \xi_x \rangle - \alpha p_x + \tilde{\eta}_x.$$  \hspace{1cm} (10.3)

Because of the same inversion argument, the analyst can deduce $\delta^k$ from market shares $s^k$, so for any choice of $\alpha, \beta$ she imputes a value of $\eta^k$ using equations (10.2) and (10.3). Now, crucially, she picks $\hat{\alpha}, \hat{\beta}$ to minimize the sample correlation between imputed $\eta^k$ and observed $z^k$ (as $k$ varies over the markets). Berry (1994) discusses the details of this approach in the above logit case, as well as nested logit (recall Section 3.7) and vertical differentiation models. Here, inversions different than (10.2) must be used (see a general theorem in his appendix).

The IV approach can also be combined with other models. Berry, Levinsohn, and Pakes (1995) combine it with random coefficients (Section 10.6.1). Here

$$\tilde{\delta}_x = \langle \tilde{\beta}, \xi_x \rangle - \tilde{\alpha} p_x + \tilde{\eta}_x,$$  \hspace{1cm} (10.4)

so that the coefficients $\tilde{\alpha}$ and $\tilde{\beta}$ are random (vary within each market) and independent of $(\tilde{\eta}, \tilde{\xi}, \tilde{p})$. There is a large literature on estimation of BLP, with many applications to various settings.\footnote{See, e.g., Nevo (2000), Ackerberg, Benkard, Berry, and Pakes (2007), and Shum (2016). For identification results, Berry and Haile (2009, 2014). See also Berry and Haile (2021) for a review.}

The IV approach can also be combined with the pure characteristics model (Berry and Pakes, 2007).

Remark 10.14. Other relaxations of independence have been studied in the econometrics literature. For example, Manski (1975, 1985) assumed that conditional on a value of $\xi$ the distribution of $\tilde{\epsilon}$ has zero median and that it is i.i.d. over $X$. Matzkin (1993) relaxed the median assumption. Note that the assumption that $\epsilon$ is independent of $\xi$ is different than the assumption that conditional on $\xi$ the distribution of $\epsilon$ is i.i.d. over $X$. In the earliest applications (McFadden, 1973) they are both made at the same time. Here only the i.i.d. assumption is made, whereas the dependence of $\epsilon$ on $\xi$ is relaxed. In general, without independence, the model is vacuous. \hfill \blacktriangle

10.8. Perturbed Utility*

Recall perturbed utility from Section 3.5. Now, instead of menu-variation, we have attribute-variation. This version of PU was used in game theory by: Hofbauer and Sandholm 2002, Mattsson and Weibull (2002), and van Damme and Weibull (2002). This class can be characterized by a weakening of the WDZ conditions: dropping alternating signs and weakening gross substitutes to positive definiteness. Thus, this is a weaker model than simple ARU, but
still has quite a bit of bite. (Recall that with menu-variation the general form of PU had no bite.)

**Definition 10.15.** We say that \( \rho \sim \text{simple PU} \) if \( \rho(\cdot, v) \) solves

\[
\max_{p \in \mathrm{int} \Delta(X)} \sum_{x \in X} v(x)p(x) - c(p)
\]

where \( c : \mathrm{int} (\Delta(X)) \rightarrow \mathbb{R} \) is such that at each point the Jacobian of \( c \) is positive definite on \( \{v \in \mathbb{R}^n : \sum_x v(x) = 0\} \) and the norm of its gradient approaches infinity near the boundary of \( \Delta(X) \).

**Theorem 10.16 (Hofbauer and Sandholm 2002).** Suppose that \( \rho : \mathbb{R}^X \rightarrow \Delta(X) \) is continuously differentiable. \( \rho \sim \text{simple PU} \) if and only if it satisfies translation invariance, zero limit demand, symmetric partials, and positive definiteness.

**Proof.** This follows from the steps in the proof of Theorem 2.1 of Hofbauer and Sandholm (2002). \( \Box \)

A recent study of identification of PU with attributes is Allen and Rehbeck (2019). In their model each individual solves

\[
\max_{p \in \Delta(X)} \sum_{x \in X} p(x)v_x(\xi_x) - c(p, \epsilon),
\]

where \( \xi_x \) are observed attributes of good \( x \) and \( \epsilon \) is unobservable heterogeneity. This is observationally equivalent to

\[
\max_{p \in \Delta(X)} \sum_{x \in X} p(x)v_x(\xi_x) - \bar{c}(p),
\]

which enables them to prove a generalization of the WDZ lemma from which they are able to identify utility indices, changes in average indirect utility, and obtain bounds for counterfactuals.
Chapter 11

Random Consideration

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11.1. Introduction

So far, the agent has considered all the items on the menu. In this chapter, the agent will pay attention only to a subset of the menu, called the consideration set. Here “attention” is exogenous to the agent—it is perhaps determined by advertising. Another interpretation is random product availability (unobservable to the analyst).

The consideration set is randomly drawn from some distribution. This distribution can depend on the menu offered and/or on attributes $\xi$, such as prices, branding, advertising, etc. We will start with the menu-variation literature and talk about attribute-variation later on. Technically, we could have discussed menu-variation already in Chapter 3, but I wanted to present it side by side with attribute-variation (even though the two literatures don’t talk to each other as much as they might want to).

11.2. Models with Menu Variation

Typically, in this literature one selected item is the status quo or the outside option. Choosing this item, denoted $o \in X$ means falling back on the status quo: not making a choice at all and sticking with the default. Here $A^o$ is the collection of all menus that contain the status quo.

**Definition 11.1 (Random Consideration Set).** A random consideration set is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random mapping $\tilde{\Gamma} : \Omega \times A^o \to A^o$ such that $\mathbb{P}(\tilde{\Gamma}(A) \subseteq A) = 1$ for all $A \in A^o$. We will define its distribution by $m_A(C) := \mathbb{P}(\tilde{\Gamma}(A) = C)$.

This definition assumes that $o$ is always considered. For each possible realization of the consideration set $\tilde{\Gamma}(A)$, the agent maximizes a random utility function $\tilde{U}$ on the set $\tilde{\Gamma}(A)$.

**Definition 11.2 (Random Consideration).** $\rho \sim RC$ if there exists a random consideration set $\tilde{\Gamma}$ and a random utility function $\tilde{U}$ such that

$$\rho(x, A) = \mathbb{P}\left(x \in \tilde{\Gamma}(A) \text{ and } \tilde{U}(x) = \max_{y \in \tilde{\Gamma}(A)} \tilde{U}(y)\right).$$

As recognized by Manski (1977), this model does not have any bite because all the randomness in choice can be attributed to the randomness of the consideration set. Unless we impose more assumptions, it will be impossible to separately identify the variation in utility and consideration.

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132 In Chapter 6, “attention” was a margin of choice: the agent paid attention only if it made sense to do it given the cost. A recent paper of Caplin, Dean, and Leahy (2019) builds a link between these two approaches.

133 A large literature exists where the consideration set is a deterministic function of the menu, see e.g., Hauser and Wernerfelt (1990), Masatlioglu, Nakajima, and Ozbay (2012), Bordalo, Gennaioli, and Shleifer (2013) and citations therein.
For simplicity many models assume that utility is deterministic and focus the analysis entirely on $\Gamma$. Various assumptions about $\Gamma$ are made. Manzini and Mariotti (2014) assumed that items belong to $\Gamma$ independently of each other: each $x \in A$ belongs to $\Gamma(A)$ with probability $\gamma_x$, independently over $x$.

**Definition 11.3 (Independent Random Consideration).** $\rho \sim IRC$ if $\rho \sim RC$ with

$$m_A(C) = \prod_{x \in C} \gamma_x \prod_{x \in A \setminus C} (1 - \gamma_x)$$

for each $C \in A^o$ such that $C \subseteq A$, where the numbers $(\gamma_x)_{x \in X}$ are all between zero and one and independent of the menu $A$ and $\gamma_o = 1$.

In particular, under IRC the probability that the consideration set consists just of the default option equals $\prod_{x \in A \setminus \{o\}}(1 - \gamma_x)$.

The independence assumption seems strong. In particular, it rules out the following simple example.

**Example 11.4 (Sleeping Agent).** The agent is in one of two states (asleep, or awake). When asleep, they only pay attention to $o$. When awake, they consider the whole menu $A$. The probability $\alpha$ that the agent wakes up is independent of the menu. ▲

More generally, we could imagine that a set $\tilde{B} \subseteq X$ of options gets generated at random and the consideration set equals the intersection of $\tilde{B}$ with the menu. In the above example $\tilde{B} = X$ with probability $\alpha$ and $\tilde{B} = \{o\}$ with probability $1 - \alpha$. This model was considered by Aguiar (2017) \(^{134}\).

**Definition 11.5.** (Constant Random Consideration) $\rho \sim CRC$ if $\rho \sim RC$ with $\Gamma(A) = \tilde{B} \cap A$ for some random menu $\tilde{B} : \Omega \rightarrow A^o$.

**Remark 11.6.** Any $\rho$ with a IRC representation has a CRC representation. Any $\rho$ with a CRC representation has a RU representation. But we can explain any $\rho$ if we allow the distribution of $\tilde{B}$ in CRC to depend on the menu. ▲

Another way to relax IRC and allow for correlation of objects within the consideration set was proposed by Brady and Rehbeck (2016). Here, the distribution of $\tilde{B}$ is defined by Luce weights for each menu.

**Definition 11.7 (Luce-Random Consideration).** $\rho$ has an LRC representation if it has a RC representation with

$$m_A(C) = \frac{\alpha_C}{\sum_{C' \subseteq A} \alpha_{C'}}$$

for each set $C \subseteq X$ and the numbers $(\alpha_C)_{C \in A^o}$ are positive and independent of the menu $A$.

\(^{134}\)The preference over menus version of this was considered by Barberà and Grodal (2011), see also the supplementary appendix of Brady and Rehbeck (2016).
In the world with menu-variation, a given \( \tilde{\Gamma} \) is of the IRC variety if and only if it is at the same time CRC and LRC.

**Proposition 11.8.** A random consideration set \( \tilde{\Gamma} \) satisfies Definition 11.3 if and only if it satisfies Definitions 11.5 and 11.7.

**Proof.** See Proposition 3 of Kovach and Suleymanov (2021). □

**Remark 11.9.** Proposition 11.8 is just at the level of representations. We cannot immediately conclude from it that \( \rho \sim IRC \) iff \( \rho \sim CRC \) and \( \rho \sim LRC \). Conceivably, \( \rho \) could have a CRC representation with \( \tilde{\Gamma}_1 \) and a LRC representation with \( \tilde{\Gamma}_2 \). Without enough uniqueness, they don’t have to be the same so we can’t immediately invoke Proposition 11.8. We will shortly discuss uniqueness results for these classes, but importantly this uniqueness holds only within each class. I do not know if the result holds at the level of \( \rho \). ▲

IRC, CRC, and LRC have been axiomatized respectively by Manzini and Mariotti (2014), Aguiar (2017), and Brady and Rehbeck (2016). Here is the uniqueness result.

**Proposition 11.10.** Suppose \( \rho \sim RC \) with a deterministic utility function that ranks \( o \) last.

(i) If \( \rho \) has a IRC representation, then the utility function is ordinally unique and the set of probabilities \( (\gamma_x)_{x \in X} \) is unique.

(ii) If \( \rho \) has a CRC representation, then the utility function is ordinally unique and the distribution of \( \tilde{C} \) is unique

(iii) If \( \rho \) has a LRC representation, then the utility function is ordinally unique and the distribution \( \alpha \) is unique.

**Proof Sources:** Part (i) is Theorem 1 of Manzini and Mariotti (2014). Part (ii) is Corollary 1 of Aguiar (2017). Part (iii) is Theorem 3.1 of Brady and Rehbeck (2016). The assumption that \( \rho \) is defined on all menus \( A_o \) can be somewhat relaxed. In part (i) the domain of \( \rho \) needs to contain sets of the form \( \{x, y, z, o\} \) and be closed under set-inclusion, In part (iii) the domain of \( \rho \) needs to contain sets of the form \( \{x, y, o\} \) and be closed under set-inclusion. □

**Remark 11.11.** Uniqueness obtains only within these classes. Since IRC and CRC also have RU representations, all the choice variation could be attributed to taste variation. Thus, a \( \rho \) can have a RC representation with some deterministic utility function \( v \) and at the same time a RU representation with a random utility \( \tilde{U} \). In this situation, the analyst is not in the position of deciding which representation is the “true” one, unless she makes some assumptions about unobservables, for example insists that there is absolutely no taste variation. We will see that this is different with attribute-variation. ▲
Cattaneo, Ma, Masatlioglu, and Suleymanov (2020) further relax the properties of $\tilde{\Gamma}$, while keeping the assumption that preferences are deterministic. The only restriction they impose on attention is a form of regularity.

**Definition 11.12** (Monotone-Random Consideration). $\rho \sim MRC$ if $\rho \sim RC$ with $\tilde{\Gamma}$ such that for any $C \subseteq A$ and $x \in A \setminus C$ we have

$$m_A(C) \leq m_{A \setminus \{x\}}(C).$$

Thus, removing an element from $A$ makes it more likely for the agent to pay attention to a given menu $C$. I will call such models monotone random consideration models (MRC). This assumption is satisfied by the IRC, CRC, and LRC models and many other examples discussed by Cattaneo, Ma, Masatlioglu, and Suleymanov (2020). This class of representations provides further insight into the issue of identification. The paper defines revealed preference by a violation of regularity: $x \succ^* y$ if adding $y$ to a menu causes $x$ to be chosen strictly more often from that menu. They show that in the MRC class $x \succ^* y$ if and only if for all monotone representations the utility of $x$ is above $y$. Moreover, they show that the MRC class is characterized by acyclicity of $\succ^*$.

For example, take $\rho$ that is RU. Under the definitions above, the relation $\succ^*$ is empty, so acyclic and $\rho$ has a MRC representation. However, it has a multitude of them, to the point that the utility function is not identified at all. This in particular applies to the special case of $\rho$ in the IRC class. On the other hand Manzini and Mariotti (2014) define their revealed preference in the opposite way $x \succ^{**} y$ if adding $x$ to a menu causes $y$ to be chosen strictly less often from that menu. Their uniqueness result (Proposition 11.10) says that the utility function is unique ordinally and represents $\succ^{**}$. This again shows that identification is always relative to the class of models we are considering.

Kashaev and Aguiar (2021) and Aguiar, Boccardi, Kashaev, and Kim (2021) relax the deterministic utility assumption. They study uniqueness properties of subclasses of MRC. They also construct statistical tests and design an experiment to tell various classes apart. Gibbard (2021) also studies uniqueness in the model where both utility and consideration are random.

Manzini and Mariotti (2018) axiomatize a model where the distribution over preferences depends directly on the menu without the intermediation of the consideration set. They assume that $\mu_A = \alpha_A \delta_u + (1 - \alpha_A) \delta_v$, where $u, v : X \to \mathbb{R}$ are independent of the menu and $\alpha_A$ is menu-dependent; see also Manzini, Mariotti, and Petri (2019).

### 11.3. Models with Attribute Variation

Applied papers use versions of RC with attribute variation and a fixed menu $X$. Let $\Xi$ be the set of possible attribute profiles.

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135 They drop the status quo from the domain. Horan (2019) also studies models without $o$. 
11.3. Models with Attribute Variation

Definition 11.13. \( \rho \sim \text{RCwithattributes} \) if there exists a random consideration set \( \tilde{\Gamma} : \Omega \times \Xi \to M^o \) and a random utility function \( \tilde{U} : \Omega \to \mathbb{R}^{X \times \Xi} \), quasilinear in prices, such that

\[
\rho(x, \xi) = \mathbb{P}\left( \tilde{U}(x, \xi) = \max_{y \in \tilde{\Gamma}(\xi)} \tilde{U}(y, \xi) \right)
\]

Like with menu-variation, the general model has no bite.

Historically, the first special case is nested logit (Section 3.7). Here the nest is randomly drawn according to Luce probabilities and then choice from each nest is also Luce. Intuitively, this involves two assumptions: 1) the consideration sets form a partition of \( X \), 2) the probabilities of drawing different consideration sets and the probabilities of choosing from those consideration sets are driven by the same underlying function \( v \). A number of models have been proposed in the transportation literature that relax the first assumption but keep the second one: Swait (2001), Wen and Koppelman (2001) and Cascetta and Papola (2001), Cantillo and de Dios Ortúzar (2005), and Calastri, Hess, Choudhury, Daly, and Gabrielli (2019).

The rest of the models relax both. IRC was studied by Swait and Ben-Akiva (1987); Ben-Akiva and Boccara (1995); Goeree (2008); Van Nierop, Bronnberg, Paap, Wedel, and Franses (2010). Here \( \gamma_x(\xi_x) \) is a function only of the characteristics of item \( x \) but not the other items. For example, in Goeree (2008) \( \gamma \) can depend on the level of advertising.

The “sleeping agent” was studied by Ho, Hogan, and Scott Morton (2017); Hortaçsu, Madanizadeh, and Puller (2017); Heiss, McFadden, Winter, Wuppermann, and Zhou (2016). Here the probability of waking up is a function only of the characteristics of the status quo (but not the other items).

Abaluck and Adams-Prassl (2021) study identification properties of a hybrid of those two models, under the assumption that utility is quasilinear in prices. While IARU satisfies the symmetry of the partials condition (Theorem 10.8), they show that nontrivial attention leads to asymmetric partials. In fact, they show that attention can be identified from those asymmetries.

The paper contains an interesting proof-of-concept experiment. Imagine that for each menu \( A \) we randomly draw \( \tilde{\Gamma}(A) \) and we make sure that the agent considers all of these items before making a choice according to \( \rho \). If we now lump those choices together, the recorded stochastic choices will be

\[
\rho^*(x, A) = \mathbb{E}[\rho(x, \tilde{\Gamma}(A))] = \sum_{C \subseteq A} \rho(x, C) m_A(C).
\]

Suppose we now feed this \( \rho^* \) to the model. Will we will we recover the correct set of weights \( m^* \)? Abaluck and Adams-Prassl (2021) run a laboratory experiment where they exogenously vary subjects’ attention and confirm that correct \( m \) is recovered.
If we don’t impose the restrictions on \( \Gamma \) like those above, then utility and counterfactual choices are only partially identified. For some, this may be enough of a reason to opt for one of the parametric classes. Instead, Barseghyan, Molinari, and Thirkettle (2021) and Barseghyan, Coughlin, Molinari, and Teitelbaum (forthcoming) allow for arbitrary correlation between consideration sets and preferences and only restrict the cardinality of \( \Gamma \) from below. In general, their approach is computationally challenging because of the presence of an infinitely dimensional nuisance parameter \( m_X(\cdot | \xi) \). For an overview of the literature on preference estimation with unobserved choice set heterogeneity see Crawford, Griffith, and Iaria (forthcoming).

11.4. Other “behavioral” models

In Simon (1956) the agent \emph{satisfices}: goes through a menu in some order and stops the first time they hit an item that is “good enough” (see also Rubinstein and Salant, 2006). Aguiar, Boccardi, and Dean (2016) study a model where the order is random and unobservable to the analyst. Here, with a deterministic preference and threshold, all the choice variability is attributed to this random order.

Tversky (1972a,b) studies a model of \emph{elimination by aspects} (EBA).\footnote{Becker, DeGroot, and Marschak (1963) sketched a version of this model where a subset of aspects can be considered at the same time.} In this model, each alternative is described by binary characteristics (aspects). The agent randomly picks an aspect and eliminates all items from the menu that do not possess this aspect. The process continues with a randomly picked aspect until there is only one item left or all items have the same aspects and such a tie is broken uniformly. EBA is a special case of RU, but there are no known axiomatizations of it. Gul, Natenzon, and Pesendorfer (2014) axiomatize a closely-related attribute rule.

\emph{Limited memory} is similar to limited consideration (Yegane, 2021). There is an active literature in behavioral economics on this (Bordalo, Gennaioli, and Shleifer, 2020) and of course in psychology (Kahana, 2012).

Other papers include: Echenique, Saito, and Tserenjigmid (2018), Kovach and Tserenjigmid (forthcomingb), and Koida (2018).
Chapter 12

Dynamic Discrete Choice

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12.1. Panel Data

In Chapter 7 we studied dynamic choices of a “myopic” agent, whose utility is given by some stochastic process. That exposition focused on the case with menu variation. In econometrics, the menu is fixed over time but there is variation in the attributes of each good $x$. I will use $\xi_{xt}$ to denote this vector and $\xi_t$ and $\xi_x$ will stand for stacked vectors. Following the static definition (Definition 10.5), we have

$$\tilde{U}_t(x) = v(x, \xi_{xt}) + \tilde{\epsilon}_t(x).$$

Here, the function $v$ is just like in the static model (deterministic, typically linear or polynomial, with coefficients identical for all agents). The stochastic process ($\tilde{\epsilon}_t$) can be either i.i.d. or involve permanent unobservable heterogeneity, i.e.,

$$\tilde{\epsilon}_t(x) = \tilde{\alpha}(x) + \tilde{\eta}_t(x), \quad (12.1)$$

where $\tilde{\alpha}$ is drawn once for the agent at the beginning of time, and typically $\tilde{\eta}_t$ is i.i.d. and independent of $\alpha$. This approach allows for unobserved heterogeneity, but only in the levels (the coefficients of $v$ are not random).

Estimation and identification are covered in Chamberlain (1984). If $\eta_t$ are TIEV i.i.d. over time and alternatives, then the linear coefficients of $v$ are identified without imposing any restrictions on the distribution of $\tilde{\alpha}$ conditional on $\xi$. Manski (1987) relaxes the i.i.d. assumption to full support and stationarity (conditional on $\xi$ and $\tilde{\alpha}$ the distribution of $\tilde{\epsilon}$ is the same in each time period, but they are allowed to be correlated serially, with $\alpha$, and across alternatives) and shows that inference is possible if there is enough variation in $\xi$. Chamberlain (2010) shows that having enough variation in $\xi$ is necessary for inference, and even under this assumption inference is slow unless we are in the TIEV family.\(^\text{137}\)

The econometric problem with panel data is non-trivial because past choices of the agent may appear as if they influence future choices even if they actually only carry information about their unobservable “type” $\tilde{\alpha}$ (we discussed this in Section 7.1). In section 7.4 we discussed how to distinguish “spurious” and “structural” state-dependence.

12.2. Markov Decision Problems

In Chapter 8 we introduced forward-looking agents, who anticipate their future choices. Their utility satisfies the Bellman equation that ties together $\tilde{U}_t$ and $\tilde{U}_{t+1}$. The theoretical approach is presented in Section 8.1. Section 8.2 shows that such agents like bigger menus because they give more option value, so they like to make decisions as late as possible. There, we used the domain of decision trees (Section 7.5), which is a dynamic extension of menu variation.

\(^{137}\)A separate literature deals with the case of continuous, as opposed to discrete, choices. The approach of Kitamura and Stoye (2018) can be extended to allow for arbitrary serial correlation of utilities (Kashaev and Aguiar, 2022a).
The econometric approach uses Markov Decision Problems, which is a dynamic extension of attribute variation. With respect to the panel data model, the agent is now able to control the future distribution of $\epsilon_t$ by choosing actions today and their utility takes this into account.

This model was introduced by Rust (1994), see also Aguirregabiria and Mira (2010) and Abbring (2010) for exposition. In a Markov Decision Problem (MDP) in each period $t$ the state $s_t \in S$ is revealed to the agent. The set of available actions in state $s_t$ is given by $A(s_t)$. The agent has a state-dependent utility $v_t(x_t, s_t)$. There is a transition probability over $s_{t+1}$ that depends on the current action $x_t$ and the current state $s_t$, thus choices made in period $t$ affect both current payoffs as well as the distribution over future states (and therefore future menus and future utilities). The transition probability is known by the agent and estimated by the analyst under a set of assumptions.

The MDP is partially observed by the analyst. The state has two components $s_t = (\xi_t, \epsilon_t)$ where $\xi_t$ is observed by the analyst while $\epsilon_t$ is private to the agent. The conditional independence (CI) assumption says that the transition probability satisfies

$$\mathbb{P}[\xi_{t+1}, \epsilon_{t+1}|\xi_t, \epsilon_t, x_t] = \mathbb{P}[\epsilon_{t+1}|\xi_{t+1}] \cdot \mathbb{P}[\xi_{t+1}|\xi_t, x_t].$$

This means two things: (1) conditional on $\xi_t$, $\epsilon_t$ is i.i.d. over time, (2) conditional on $\xi_t$ and $x_t$, $\xi_{t+1}$ is independent of $\epsilon_t$.

Because CI rules out persistence in unobservables, the choice probability can be written without conditioning on the history of past choices

$$\rho(x_t, \xi_t) = \mathbb{P}\left[\tilde{U}_t(x_t, \xi_t) = \max_{y_t \in A(\xi_t)} \tilde{U}_t(y_t, \xi_t) \bigg| \xi_t\right],$$

where the utility of action $x_t$ equals

$$\tilde{U}_t(x_t, \xi_t) = v(x_t, \xi_t) + \epsilon_t(x_t) + \delta \mathbb{E}\left[\max_{x_{t+1} \in A(\tilde{\xi}_{t+1})} \tilde{U}_{t+1}(x_{t+1}, \tilde{\xi}_{t+1}) \bigg| x_t, \xi_t\right].$$

The literature typically couples CI with the i.i.d. assumption $\epsilon_t(x)$ and $\epsilon_t(y)$ are i.i.d. and independent of $\xi_t$ (i.i.d.) A combination of CI and i.i.d. with the additional parametric TIEV assumption gives dynamic logit. Following Rust (1987), dynamic logit is often used in the literature due to its tractability.

**Example 12.1 (Bus Engine Replacement).** Rust (1987) studied the choices of Harold Zurcher, the superintendent of a bus company. Zurcher is managing a fleet of buses, each characterized by current milage $\xi_t \in [0, \infty]$. In each period for each bus, Zurcher can make a replacement decision. Replacing the engine, $x_t = 1$, means resetting the current mileage to zero, at a cost $RC$. Not replacing, $x_t = 0$, preserves the current mileage. There is a maintenance cost $c(\xi_t)$. Let $\theta$ be the vector of parameters of $c$ that also includes RC. The state variable next period is $\xi_{t+1} = \xi_t(1 - x_t) + \eta_{t+1}$, where the mileage increments, $\eta_{t+1}$, are random and independent of the current decision. Since $\xi$ is observable,
this process can be separately estimated by the econometrician. Zurcher’s value function solves:

\[ V(\xi_t; \theta) = \max \left\{ -c(0; \theta) - RC(\theta) + \epsilon_t(1) + \delta \mathbb{E}[V(\eta_{t+1}; \theta)], \right. \]
\[ \left. -c(\xi_t; \theta) + \epsilon_t(0) + \delta \mathbb{E}[V(\xi_t + \eta_{t+1}; \theta)] \right\}, \]

where \( \epsilon_t(1) \) and \( \epsilon_t(0) \) are i.i.d. TIEV.

Rust’s original approach was to use dynamic programming to compute \( V(\cdot; \theta) \) for each value of \( \theta \) to obtain \( \rho(\cdot; \theta) \) and then estimate \( \theta \). This was simplified by using an inversion by Hotz and Miller (1993) and Hotz, Miller, Sanders, and Smith (1994). Roughly speaking, by the WDZ Lemma (Lemma 10.7) the continuation value \( V \) can be directly computed from the choice probabilities. This method works for general i.i.d. models. See, e.g., Shum (2016) for exposition. 138

12.3. Serial Correlation of \( \tilde{\epsilon} \)

The CI assumption rules out persistent unobservables. One way to relax this assumption would be to say that utility in each period depends on the agent’s “type” (which they privately learn in period 1), but each type of agent is also subject to i.i.d. shocks to actions, exactly like in the panel data formulation (12.1). Such a formulation was used for example by Lee (2013).

The so called Eckstein–Keane–Wolpin models combine permanent unobserved heterogeneity with transitory shocks that are allowed to be correlated across actions, see, e.g., Example 2 of Aguirregabiria and Mira (2010).

Another way to relax CI is to endow \( \epsilon \) with a Markov structure, as in Pakes (1986).

Example 12.2. Figure 12.1 illustrates a simple example of patent renewal. The firm can renew the patent at cost \( c_t \). The instant reward for renewing is \( \tilde{\epsilon}_t \). Not renewing the patent makes it lapse forever. Thus, renewing gives an immediate payoff of \( \tilde{\epsilon}_t - c_t \) plus the option value of renewing in the future.

The econometrician knows \( c_t \) and wants to estimate the option value assuming that the distribution of rewards follows a first-order Markov process. We have

\[ V_t(\tilde{\epsilon}_t) = \max \{0, \tilde{\epsilon}_t - c_t + \delta \mathbb{E}[V_{t+1}(\tilde{\epsilon}_{t+1})|\tilde{\epsilon}_t]\} \]

There is actually no state variable in this model, except for \( t \). Therefore, CI boils down to the assumption that \( \tilde{\epsilon}_t \) is i.i.d. over time. But this assumption may not make sense in the context of patents because we expect there to be heterogeneity across patents.

138 A similar inversion approach was used to deal with endogeneity in Section 10.7. A related result in decision theory is Theorem 3 of Lu (2016).
Finally, general models of unobservable serially correlated state variables impose almost no structure on $\tilde{\epsilon}$ (Norets, 2009; Hu and Shum, 2012).

12.4. Identification

In general $\delta$ is not identified (Manski, 1993; Rust, 1994). This is because we can define a new utility function $\hat{v}(x, \xi) := v(x, \xi) + \delta E[V(\xi')|x, \xi]$ and set $\hat{\delta} = 0$.

To discuss identification of $v$, let’s assume that $\delta$ is known to the analyst. Let $k$ be the cardinality of $\Xi$ and $n$ be the cardinality of $X$. There are $kn$ utility parameters. We observe $kn$ conditional choice probabilities, but they have to sum up to one for each $\xi$. So to get point identification we need to make $k$ normalizing assumptions.

One typical approach is to fix an alternative $x_0$ and set $v(x_0, \xi) = 0$ for all $\xi$. Other normalizations include: exclusion restrictions (setting some elements of $v$ equal to each other) or parametric restrictions. A vast literature on identification includes Hotz and Miller (1993); Taber (2000); Magnac and Thesmar (2002); Norets and Tang (2013); Kasahara and Shimotsu (2009) and is summarized by Abbring (2010)

Abbring and Daljord (2020) show that local point identification holds. Under an exclusion restriction, a range of $\delta$ identified and for each $\delta$ there is a unique $v$.

Other partial identification approaches involve imposing shape restrictions on $v$, e.g., monotonicity, concavity, supermodularity, or obtaining bounds on parameter values (Honoré and Tamer, 2006). Even if parameters are partially identified, it is sometimes possible to point-identify the counterfactuals (Kalouptsidi, Scott, and Souza-Rodrigues, 2021; Kalouptsidi, Kitamura, Lima, and Souza-Rodrigues, 2021).
12.5. Dynamic Logit

It is relatively easy to axiomatically characterize dynamic logit if we dispense with the observable states $\xi_t$. This way each action consists of a payoff today and a continuation menu tomorrow. This is just like decision trees from Section 7.5. On this domain, the MDP (12.3) becomes what I will call Additive Stochastic Utility (ASU).

\[
\tilde{U}_t(z_t, A_{t+1}) = v_t(z_t) + \delta \mathbb{E} \left[ \max_{x_{t+1} \in A_{t+1}} \tilde{U}_{t+1}(x_{t+1}) \right] + \tilde{\epsilon}_t(z_t, A_{t+1}),
\]

with deterministic utility functions $v_t \in \mathbb{R}^{Z_t}$, discount factor $\delta \in [0, 1]$, and random payoff shock $\tilde{\epsilon}_t : \Omega^t \to \mathbb{R}^{X_t}$.

Note that in ASU $\delta$ is not identified because we can set

\[
\bar{\epsilon}_t'(z_t, A_{t+1}) := \delta \mathbb{E} \left[ \max_{x_{t+1} \in A_{t+1}} \tilde{U}_{t+1}(x_{t+1}) \right] + \epsilon_t(z_t, A_{t+1})
\]

and $\delta' := 0$.

Dynamic logit is ASU plus the i.i.d. and TIEV assumptions. Due to its tractability, this model is a workhorse for estimation.\(^{139}\) The tractability comes from the “log-sum” expression (3.4) for inclusive value/social surplus. (As we saw in Section 3.7, this formula is also the reason why nested logit is so tractable).

Fudenberg and Strzalecki (2015) showed that the main axiomatic consequences of these assumptions are Luce’s IIA (Axiom 3.2, period by period) and the analogues of Preference for Flexibility (Axiom 8.5) and Sophistication (Axiom 8.17).

**Axiom 12.3** (Weak Preference for Flexibility). For all $t$ if $B_{t+1} \supset A_{t+1}$ s.t. $B_{t+1} \neq A_{t+1}$ and $A_t := \{(z_t, B_{t+1}), (z_t, A_{t+1})\}$

\[
0 < \rho_t((z_t, A_{t+1}), (z_t, B_{t+1})) < \frac{1}{2}.
\]

Weak Preference for Flexibility holds for all i.i.d. representations with unbounded support. Compared with Preference for Flexibility (Axiom 8.5), which says that in pairwise choice $(z_t, B_{t+1})$ is chosen with probability one, here this probability is strictly less than one (because the support of $\tilde{\epsilon}_t$ is unbounded).

**Axiom 12.4** (Recursivity). For all $t$

\[
\rho_t((z_t, A_{t+1}), (z_t, B_{t+1})) \geq \frac{1}{2}
\]

\[\Downarrow\]

\[
\rho_{t+1}(A_{t+1}, A_{t+1} \cup B_{t+1}) \geq \rho_{t+1}(B_{t+1}, A_{t+1} \cup B_{t+1})
\]

Recursivity says that \((z_t, A_{t+1})\) is chosen more frequently at time \(t\) than \((z_t, B_{t+1})\) if and only if an alternative from \(A_{t+1}\) is chosen more frequently at time \(t + 1\) than an alternative from \(B_{t+1}\). This means that the stochastic preference (Section 3.4) has no reversals.

Recursivity leverages the “log-sum” expression (3.4) and is specific to the TIEV assumption and does not hold for all i.i.d. models.

Fudenberg and Strzalecki (2015) also show that all the parameters of the model, i.e., \(\delta\) and \(v\) are identified under a stationarity assumption on \(v\). I think this result extends to all of i.i.d. models. This sharp result comes from the rich variation in intertemporal problems that may be absent in the field but can be easily incorporated in experimental settings.

**Remark 12.5.** A number of recent decision theory papers use dynamic logit as a building block, such as the dynamic attribute rule of Gul, Natenzon, and Pesendorfer (2014). Other papers view \(\tilde{\epsilon}_t\) as errors, not utility shocks. In Fudenberg and Strzalecki (2015) errors lead to “choice aversion” (each menu is penalized by a function of its size). This makes the agent averse to bigger menus and leads to stochastic versions of the Set-Betweenness axiom which is studied in the literature on temptation and self-control (Gul and Pesendorfer, 2001; Dekel, Lipman, and Rustichini, 2009). Ke (2018) offers a dynamic model of mistakes (agent evaluates each menu by the expectation of the utility under her own s.c.f.).

### 12.6. Consequences of i.i.d.

The part of i.i.d. that we will be focusing on is that \(\epsilon\) is i.i.d. across alternatives. (Once we eliminate the covariates, i.i.d. over time is guaranteed by CI.)

The consequences described here hold not only for i.i.d. \(\tilde{\epsilon}_t\), but also under certain forms serial correlation, for example with permanent unobserved heterogeneity, where we have a mixture of i.i.d. models that inherits its properties.

What follows hinges on somewhat artificial situations, where the agent is offered a direct choice between two nested menus, or offered an option to defer choice at no cost. The following remarks may thus be only of theoretical interest. However, I think some caution may be warranted when applying the i.i.d. assumption to practical situations when alternatives have payoffs with very different time profiles. For example, Frick, Iijima, and Strzalecki (2019) show how i.i.d. models can lead to biased parameter estimates even in decision problems that do not involve pure option value.

The clear advantage of i.i.d. is that the likelihood function is non-degenerate, whereas the more strucured models from Chapter 8 put zero probability on some events that may very well happen.

**12.6.1. Preference for Flexibility.** If i.i.d. holds, then each alternative \(x_t = (z_t, A_{t+1})\) gets its own realization of the \(\epsilon\)-shock. In particular even if \(B\)
dominates $A$ the two random variables $\tilde{\epsilon}_t(z_t, A_{t+1})$ and $\tilde{\epsilon}_t(z_t, B_{t+1})$ are i.i.d.
This implies that dominated choices will be made with positive probability, as Weak Preference for Flexibility (Axiom 12.3) says. Mechanically this makes sense because small menus sometimes get a shock that outweighs their smaller value. However, from a theoretical point of view, it seems reasonable to assume instead bigger menu gets chosen with probability one, as asserted by Preference for Flexibility (Axiom 8.5). However, as discussed by Rust (1987), models that predict zero choice probabilities for some alternatives will be impossible to estimate.

The i.i.d. assumption is a solution to a practical problem, but is it the right solution? Consider what happens if we increase the variance of $\epsilon$. Intuitively, increasing the variance of shocks in period $t+1$ should increase the attractiveness of the bigger menu, as it now offers more option value. For example, suppose that $v_{t+1}(x) = 0$ for all $x$ and consider a singleton menu versus a menu of two items. If $\epsilon_t$ are i.i.d. normal with mean zero and standard deviation $\lambda$, then the expected value of the first one is zero, whereas the expected value of the second one is increasing in the variance.\(^{140}\) Given this logic, one would expect that the bigger menu gets chosen with a higher probability as $\lambda$ increases. However, there is another effect: increasing $\lambda$ automatically brings today’s choice probabilities closer to a half (even though there is no consumption today). It turn out that the second effect wins.

**Proposition 12.6** (Frick, Iijima, and Strzalecki 2019). Suppose that there are two periods, $A_2 = \{z_2\}$, $B_2 = \{z_2, z'_2\}$, and $A_1 = \{(z_1, A_2), (z_1, B_2)\}$ for some fixed $z_1, z_2, z'_2$ such that $v(z'_2) > v(z_2)$. Assume that $\epsilon$ are i.i.d., scaled multiplicatively by $\lambda > 1$, then the probability $\rho_1 ((z_1, A_2), A_1)$ strictly increases in $\lambda$.

As you recall, the i.i.d. assumption leads to unrealistic predictions about substitution patterns (Section 10.6) and about choices over lotteries (Section 4.5). This manifests itself in dynamic settings as well: this time, the i.i.d. assumption leads to unrealistic predictions about option value. In the static setting these problems can be fixed by appropriately disciplining the $\tilde{\epsilon}$, such as in the random characteristics model and random expected utility model. The model from Chapter 8 is a dynamic version of that.

To summarize Chapter 8: if we think of $\epsilon$ as representing shocks to utilities, then menus cannot directly impact today’s utility, only indirectly via the expectation of tomorrow’s utility. The shocks to continuation menus should be carried by the expectation operator. This is the maintained assumption in Pakes (1986) and Taber (2000). This assumption implies that bigger menus are chosen with probability one.

\(^{140}\) Intuitively, if we increase the noise, the chance that we get at least one outlier gets higher. The exact expression is: $\frac{\lambda}{\sqrt{\pi}}$. 
In principle the agent could have additional private information about continuation value that is separate from shocks to immediate utility. Formally speaking, the expectation in (12.4) should be conditional on the private information of the agent. For that reason, equation (8.1) has conditional expectations. This could be rewritten as part of the $\epsilon$, but such $\epsilon$ is not i.i.d. because (8.1) implies bigger menus are always chosen.

12.6.2. Postponing Decisions. Consider a slight variation on this problem.

Example 12.7 (Preference for making choices late). Suppose that you are packing your bag for a trip that starts on Saturday. It is now Friday morning ($t = 1$) and can decide to pack the bag right after you finish scrolling through the news ($t = 2$), or wait till Saturday morning ($t = 3$).

Suppose you can pack your sun glasses or your rain coat (but not both) and the payoffs are like in Example 8.2.

If you decide to pack today, then the menu you are choosing is $A_{2}^{\text{now}} = \{\{g\}, \{c\}\}$ and if you pack on Saturday you will face the menu $A_{2}^{\text{later}} = \{\{g, c\}\}$. Your period $t = 1$ choice, illustrated in Figure 12.2, is between those two menus.

The agent from Chapter 8 will have a deterministic preference $A_{2}^{\text{later}} \succ_{1} A_{2}^{\text{now}}$. To see that, note that we have

\[
\bar{U}_{1}(A_{2}^{\text{later}}) = \mathbb{E} \left[ \mathbb{E} \left[ \max \left\{ \tilde{u}_{3}(g), \tilde{u}_{3}(c) \right\} \big| \omega^{2} \right] \big| \omega^{1} \right] \\
\geq \mathbb{E} \left[ \max \left\{ \mathbb{E} \left[ \tilde{u}_{3}(g) \big| \omega^{2} \right], \mathbb{E} \left[ \tilde{u}_{3}(c) \big| \omega^{2} \right] \right\} \big| \omega^{1} \right] = A_{2}^{\text{now}}.
\]

This holds because of conditional Jensen’s inequality, which is exactly the same reason why the agent has preference for flexibility, cf. Example 8.2. (The preference for delaying the choice will be strict, unless the agent thinks that
Friday’s forecast will not improve until after Saturday morning, in terms of accuracy. ▲

Instead, the i.i.d. agent packs early with probability bigger than a half.

**Proposition 12.8** (Fudenberg and Strzalecki 2015; Frick, Iijima, and Strzalecki 2019). If \((\rho_t)\) has a i.i.d. representation with \(\delta < 1\), then
\[
\frac{1}{2} < \rho_1 (A^\text{now}_2, A_1) < 1.
\]

Moreover, if \(\epsilon\) is scaled by \(\lambda > 1\), then \(\rho_1 (A^\text{now}_2, A_1)\) strictly increases (modulo ties).

This is mechanically true under i.i.d. because the agent receives the \(\epsilon\) not at the time of consumption, but at the time of decision, even if the decision has only delayed consequences. Thus, in a sense, making decisions early allows them to get the \(\max \epsilon\) earlier. To illustrate that, consider the special case where \(v_3(g) = v_3(c) = v\). Here, we have
\[
V(A^\text{later}_2) = \delta^2 v + \delta^2 E[\max\{\tilde{\epsilon}_3(g), \tilde{\epsilon}_3(c)\}]
\leq \delta^2 v + \delta E[\max\{\tilde{\epsilon}_2(g), \tilde{\epsilon}_2(c)\}] = V(A^\text{now}_2),
\]
so by (3.5) the probability of packing now is more than a half. This is suggestive of a preference for commitment, which is usually associated with choice overload or self-control problems (see also Remark 8.8). One might argue that consumers do suffer from some of those behavioral issues, but what about profit-maximizing firms? If consumers are behavioral, do we want a structural model of that, instead of a mechanical side-effect?

\[\text{Fudenberg and Strzalecki (2015) show that a modification of dynamic logit leads to the opposite prediction: late choices are more frequent. However this coincides with the agent liking smaller menus more, so does not address the issues discussed in Section 12.6.1.}\]
Part 5

Appendices
Appendix A

Additional Material and Proof Sketches

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A.1. Chapter 1

A.1.1. Ordinal Uniqueness.

Proposition A.1.1. Functions \( U_1, U_2 \) represent the same preference \( \succsim \) on \( X \) (of any cardinality) if and only if there exits a strictly increasing function \( \phi : R_1 \to \mathbb{R} \) such that \( U_2(x) = \phi(U_1(x)) \) for all \( x \in X \), i.e., \( U_2 = \phi \circ U_1 \). Here \( R_1 \) is the range of \( U_1 \) defined by \( \{ U_1(x) : x \in X \} \).

A.1.2. Proof of Proposition 1.9. (i) \( \Rightarrow \) (ii): Suppose that \( \rho \) has a distribution over preferences representation \( \mu \in \Delta(\mathcal{P}) \). For each preference \( \succsim \) enumerate the elements of \( X \) so that \( x_n \succ x_{n-1} \succ \cdots \succ x_1 \) and define a corresponding utility function by \( U_{\succsim}(x_k) := k \). By construction, \( U_{\succsim} \) represents \( \succsim \), so to complete the proof it suffices to define the distribution over utilities \( \hat{\mu} \in \Delta(\mathcal{U}) \) by setting \( \hat{\mu}(U_{\succsim}) = \mu(\succsim) \) for all \( \succsim \in \mathcal{P} \) and \( \hat{\mu}(U) = 0 \) otherwise. We have
\[
\rho(x,A) = \mu(\{ \succsim \in \mathcal{P} : x \succsim y \text{ for all } y \in A \}) \\
= \hat{\mu}(\{ U_{\succsim} : \succsim \in \mathcal{P} \text{ and } U_{\succsim}(x) = \max_{y \in A} U_{\succsim}(y) \}) \\
= \hat{\mu}(\{ U \in \mathcal{U} : U(x) = \max_{y \in A} U(y) \}).
\]

(ii) \( \Rightarrow \) (iii): Suppose that \( \rho \) has a distribution over utilities representation \( \mu \in \Delta(\mathbb{R}^n) \). Define \( \Omega := \mathbb{R}^n, \mathcal{F} := \mathcal{B}, \mathbb{P} := \mu, \text{ and } \tilde{U}_\omega(x) := \omega(x) \) for all \( \omega \in \Omega \). Notice that the mapping \( \tilde{U} : \Omega \to \mathbb{R}^X \) is measurable because for any closed set \( G \subseteq \mathbb{R}^X \) it follows from the above definition that \( \tilde{U}^{-1}(G) = \{ \omega \in \Omega : \tilde{U}_\omega \in G \} = G \). Thus,
\[
\rho(x,A) = \mu(\{ U \in \mathbb{R}^n : U(x) = \max_{y \in A} U(y) \}) \\
= \mathbb{P}(\{ \omega \in \Omega : \tilde{U}_\omega(x) = \max_{y \in A} \tilde{U}_\omega(y) \}).
\]

(iii) \( \Rightarrow \) (i): Suppose that \( \rho \) has a random utility representation \( (\Omega, \mathcal{F}, \mathbb{P}, \tilde{U}) \). For each complete and transitive preference \( \succsim \) define the event \( E_{\succsim} := \{ \omega \in \Omega : U_\omega \text{ is represented by } \succsim \} \). Notice that \( E_{\succsim} \in \mathcal{F} \); this is because the set of utility functions \( \mathcal{U}_{\succsim} \) that represents \( \succsim \) is a closed set (an intersection of closed sets of the form \( \{ U \in \mathbb{R}^X : U(x) \geq U(y) \} \)) and \( E_{\succsim} \) is the inverse image of \( \mathcal{U}_{\succsim} \) under a measurable function.

For any \( \succsim \in \mathcal{P} \) define \( \mu(\succsim) := \mathbb{P}(E_{\succsim}) \). Because \( \tilde{U} \) is proper, we have that \( \mathbb{P}(E_{\succsim}) = 0 \) for any \( \succsim \notin \mathcal{P} \), so \( \mu \in \Delta(\mathcal{P}) \). Therefore, we have
\[
\rho(x,A) = \mathbb{P}(\{ \omega \in \Omega : \tilde{U}_\omega(x) = \max_{y \in A} \tilde{U}_\omega(y) \}) \\
= \mu(\{ \succsim \in \mathcal{P} : x \succsim y \text{ for all } y \in A \}).
\]

A.1.3. Proof of Proposition 1.11. (i) \( \Rightarrow \) (ii): If \( \rho \) has a RU representation, then ties occur with probability zero, so it doesn’t matter how we break them.
(ii) ⇒ (iii): Uniform tie breaking is equivalent to GP tiebreaking where \( \tilde{W} \) represents a uniform distribution over all strict orders over \( X \).

(iii) ⇒ (i): First, rescale \( \tilde{U} \) so that the utility gaps between each consecutive items are large enough. Then break any ties according to a rescaled version of \( \tilde{W} \) so that we don’t exceede these gaps. Formally, for any fixed \( \omega \), the minimum difference between two distinct values of \( \tilde{U}_\omega \) is 1. That is,

\[
\tilde{U}_\omega(x) \neq \tilde{U}_\omega(y) \Rightarrow |\tilde{U}_\omega(x) - \tilde{U}_\omega(y)| \geq 1.
\]

Now rescale \( \tilde{W} \) so that, for any fixed \( \omega \), the maximum difference between two distinct values of \( \tilde{U}_\omega \) is strictly less than 1. Finally, note that

\[
\rho(x, A) = \mathbb{P} \left( \left\{ \omega \in \Omega : \tilde{U}_\omega(x) + \tilde{W}_\omega(x) \geq \tilde{U}_\omega(y) + \tilde{W}_\omega(y) \ \forall y \in A \right\} \right).
\]

A.1.4. Proof of Proposition 1.14. Let \( \rho \) admit a discrete choice representation and let \( \mu \) be the distribution of \( \tilde{\epsilon} \). Since \( \mu \) has a density, there are no ties, i.e., for all distinct \( x, y \in X \) we have

\[
\mu(\epsilon : v(x) + \tilde{\epsilon}(x) = v(y) + \tilde{\epsilon}(y)) = 0.
\]

Conversely, assume now that \( \rho \) has a RU representation. By Proposition 1.9, there exists a probability distribution \( \mu \) over strict preferences \( P \) such that \( \rho(x, A) = \mu(N(x, A)) \). Let \( n \) be the cardinality of \( X \) and for any \( \succ \in P \) and \( i = 1, \ldots, n \) let \( x_\succ(i) \) denote the \( i \)-th ranked element of \( X \).

Define \( v(x) = 0 \) for all \( x \in X \). We need to find a probability measure \( \mathbb{P} \) over \( \mathbb{R}^X \) such that for each event of the type \( A_\succ = \{ \epsilon \in \mathbb{R}^X : \epsilon(x_\succ(1)) > \epsilon(x_\succ(2)) > \cdots > \epsilon(x_\succ(n)) \} \) we have

\[
\mathbb{P}(A_\succ) = \mu(\succ).
\]

To do so, for each \( \succ \) take a probability measure with first moments and density \( \gamma_\succ \) and support equal to the closure of \( A_\succ \), for example a truncated Normal probability distribution. Define the probability measure

\[
\mathbb{P}(\cdot) = \sum_{\succ \in P} \mu(\succ) \gamma_\succ(\cdot).
\]

This measure has first moments and a distribution.

A.1.5. Example. Let \( X = \{ x, y, z \} \) and let \( \rho \) be represented by distribution over preferences representation

\[
\mu(x > y > z) = \mu(y > x > z) = \mu(z > x > y) = \frac{1}{3}.
\]

It is easy to verify that \( \rho \) satisfies Positivity. Moreover, notice that \( \rho(z, \{ y, z \}) = \rho(z, \{ x, y, z \}) \).

To see that \( \rho \) does not have a ARU representation with a positive density, fix some \( v : X \to \mathbb{R} \) and let

\[
E := \{ \epsilon \in \mathbb{R}^X : v(x) + \epsilon(x) > v(z) + \epsilon(z) > v(y) + \epsilon(y) \}.
\]
Notice that this is a set of positive Lebesgue measure; however, we have
\[ 0 = \rho(z, \{ y, z \}) - \rho(z, \{ x, y, z \}) = \mathbb{P}(E). \]

\section*{A.2. Chapter 2}

\subsection*{A.2.1. Proof of Proposition 2.3}

Let \( X = \{ x, y, z \} \). I will write \( xyz \) to denote the order \( x \succ y \succ z \). With this notation, we have \( P = \{ xyz, xzy, yxz, yzx, zxy, zyx \} \).

The situation is simple enough that we can define \( \mu \) “by hand.” For example, to define \( \mu(xyz) \) note that \( N(y, \{ y, z \}) = \{ xyz, yxz, yzx \} \) and \( N(y, X) = \{ yxz, yzx \} \). Those two sets differ exactly by what we want, so we can define
\[
\mu(xyz) := \rho(y, \{ y, z \}) - \rho(y, X).
\]

This is in fact the only way to define \( \mu \) because \( \rho(y, \{ y, z \}) = \mu(\{ xyz, yxz, yzx \}) = \mu(\{ xyz \}) + \mu(\{ yxz, yzx \}) = \mu(\{ xyz \}) + \rho(y, X) \). We can define \( \mu \) on the remaining elements of \( P \) in the same way.

\[
\begin{align*}
\mu(xyz) &:= \rho(z, \{ y, z \}) - \rho(z, X) \\
\mu(yxz) &:= \rho(x, \{ x, z \}) - \rho(x, X) \\
\mu(yzx) &:= \rho(z, \{ x, z \}) - \rho(z, X) \\
\mu(xzy) &:= \rho(x, \{ x, y \}) - \rho(x, X) \\
\mu(zyx) &:= \rho(y, \{ x, y \}) - \rho(y, X).
\end{align*}
\]

Regularity ensures that all six numbers are non-negative. Moreover, it is easy to see that \( \mu \) adds up to one (the sum of the numbers on the right hand side is \( 3 - 2 \)).

Finally, to see that \( \mu \) is a random utility representation of \( \rho \) we need to show that \( \rho(a, A) = \mu(N(a, A)) \) for any \( a \in A \). This is also easy to see; for example
\[
\mu(\{ xyz, xzy \}) = \mu(xyz) + \mu(xzy) = [\rho(y, \{ y, z \}) - \rho(y, X)] + [\rho(z, \{ y, z \}) - \rho(z, X)] = 1 - \rho(y, X) - \rho(z, X) = \rho(x, X). \]
Likewise
\[
\mu(\{ xyz, xzy, zxy \}) = \mu(xyz) + \mu(xzy) + \mu(zxy) = [\rho(y, \{ y, z \}) - \rho(y, X)] + [\rho(z, \{ y, z \}) - \rho(z, X)] + [\rho(x, \{ x, y \}) - \rho(x, X)] = 1 + \rho(x, \{ x, y \}) - 1 = \rho(x, \{ x, y \}). \]

\subsection*{A.2.2. Proof of Theorem 2.9}

This proof is due to Fiorini (2004). We start with a bit of combinatorics. Let \((T, \leq)\) be a finite partially ordered set and suppose a real function \( f : T \to \mathbb{R} \) is given. Define the function
\[
F(t) := \sum_{s \geq t} f(s),
\]
so \( F \) is a discrete “integral” of \( f \). Then \( f \) is a “derivative” of \( F \) and we can recover it by using what is called the Möbius inversion: \( f(t) = \sum_{s \geq t} m(t, s) F(s) \), where \( m : T \times T \to \mathbb{R} \) is the Möbius function.

In the case where \((T, \leq)\) is the lattice of subsets with \( \subseteq \subseteq \), the Möbius function equals \( m(t, s) = (-1)^{|s| - |t|} \), see, e.g., Theorem 25.1 and equation (25.5) of Van Lint and Wilson (2001).

Now for the actual proof. First, define the sets
\[
N^*(x, A) := \{ z \in P : z \succ x \succ y \text{ for all } z \in A^c \text{ and } y \in A, y \neq x \}.
\]
Observe that $N^*(x, A) \subseteq N(x, A)$ and that $N(x, A) = \bigcup_{B \supseteq A} N^*(x, B)$. Moreover, this union is disjoint. Thus, $\rho(x, A) = \sum_{B \supseteq A} \mu(N^*(x, B))$. Thus, by the Möbius inversion we have

$$
\mu(N^*(x, A)) = \sum_{B \supseteq A} (-1)^{|B \setminus A|} \rho(x, B).
$$

Given that $\mu$ is a probability measure, these sums must be nonnegative. In fact, RU holds if and only if there exists a $\mu$ such that the displayed equation holds (this is stated as Proposition 7.3 in Chambers and Echenique (2016)). Finally, a note that some sources define the BM polynomials recursively: starting from the grand set $q(x, X) := \rho(x, X)$ and

$$
q(x, A) := \rho(x, A) - \sum_{B \supseteq A} q(x, B).
$$

\[ \square \]

A.2.3. Proof of Proposition 2.13. This follows from the proof of Theorem 2.3 because the value of $\mu$ on each point of $\Omega$ is pinned down uniquely by the choice probabilities, for example, $\mu(xyz) := \rho(y, \{y, z\}) - \rho(y, X)$. \[ \square \]

A.2.4. Proof of Proposition 2.15. This is Theorem 5 of Falmagne (1978). If $\mu_1$ and $\mu_2$ are RU representations of the same $\rho$, then they agree on the collection of sets $N(x, A)$. Moreover, as in A.2.2 we can use the Möbius inversion to show that they agree on the collection of sets $N^*(x, A)$. This yields our desired conclusion because

$$
\{ \succ \in P : x \text{ is k-th best in } X \} = \bigcup_{A \in A: x \in A, |A| = |X| - k + 1} N^*(x, A).
$$

This proves the theorem. Notice that in fact there is a stronger sense of uniqueness: the probabilities of all $N^*$ sets are identified. \[ \square \]

A.2.5. Proof of Proposition 2.17.

$$
v(x) + \tilde{\epsilon}_\omega(x) \geq v(y) + \tilde{\epsilon}_\omega(y) \text{ for all } y \in A, \omega \in \Omega
$$

iff

$$
v(x) + \tilde{\epsilon}_\omega(x) + \eta_\omega \geq v(y) + \tilde{\epsilon}_\omega(y) + \eta_\omega \text{ for all } y \in A, \omega \in \Omega.
$$

\[ \square \]

A.3. Chapter 3

A.3.1. GEV. Let $X = \{x_1, \ldots, x_n\}$. The joint cdf equals

$$
G(\epsilon) = \exp(-H(e^{-\epsilon_1}, \ldots, e^{-\epsilon_n})),
$$

where the function $H : \mathbb{R}_+^n \to \mathbb{R}_+$ is:

- homogenous of degree $\alpha$ for some $\alpha > 0$
- satisfies $t_x \to \infty$ with fixed $t_{-x}$ implies $H(t) \to \infty$
for any distinct \(x_1, \ldots, x_k\) the crosspartial \(\frac{\partial^k H}{\partial x_1 \cdots \partial x_k}\) is positive for odd \(k\) and negative for even \(k\).

The advantage of this class is that the choice probabilities are given in closed form by

\[
\rho(x, X) = \alpha^{-1} \frac{\partial}{\partial v_x} \ln H(e^{v_1}, \ldots, e^{v_n})
\]

We get logit by setting \(H(t_1, \ldots, t_n) := \sum_{i=1}^n t_i^\alpha\). We get nested logit by setting \(H(t_1, \ldots, t_n) := \sum_{i=1}^k (\sum_{x \in B_i} t_x^\alpha)^{\alpha_2}\) where \(\{B_1, \ldots, B_k\}\) the nest structure (a partition of \(X\)). When \(\alpha_1 < \alpha_2\), then this \(H\) satisfies the above conditions, which shows that nested logit with such parameters has a RU representation. For more on this class, see Section 2.7.2 of Anderson, de Palma, and Thisse (1992), and Section 4.6 of Train (2009).

### A.3.2. Proof of Proposition 3.8.

Note that we have two Fechnerian representations \((v_1, F_1)\) and \((v_2, F_2)\) that satisfy Richness (Axiom 3.14). By Theorem 3.15, there exists \(\alpha > 0\) and \(\beta \in \mathbb{R}\) such that \(v_2 = \alpha v_1 + \beta\) and \(F_2(\alpha t) = F_1(t)\) for all \(t \in D_1\).

Weaker results can be obtained in the finite \(X\) case:

(i) If \(F_1 = F_2\), then there exists \(\beta \in \mathbb{R}\) such that \(v_2(x) = v_1(x) + \beta\) for all \(x \in X\).

(ii) If \(v_1 = v_2\), then \(F_1\) and \(F_2\) coincide on the set \(\{v(x) - v(y) : x, y \in X\}\).

To prove (i), let \(F\) be the shared cdf of \(\tilde{\epsilon}_x - \tilde{\epsilon}_y\) for \(x \neq y\). We have

\[
\rho(x, y) = F(v_1(x) - v_1(y)) = F(v_2(x) - v_2(y)),
\]

for all \(x \neq y\). The function \(F\) is strictly increasing since \(\tilde{\epsilon}\) has a positive density (why?), so

\[
v_1(x) - v_1(y) = v_2(x) - v_2(y),
\]

for all \(x \neq y\). Thus, \(v_1(x) - v_2(x)\) is a constant function of \(x\).

To prove (ii), we have

\[
F_1(v(x) - v(y)) = \rho(x, y) = F_2(v(x) - v(y)).
\]

Part (i) says that if we know that the distribution of \(\epsilon\) is the same, then \(v\) is pinned down uniquely up to an additive constant. Similarly, part (ii) says that if we know that \(v\) is the same, then the distribution of \(\epsilon\) differences is the same (on the relevant domain).

**Example A.3.1.** Suppose that \(X = [-\frac{1}{2}, \frac{1}{2}]\) and \(\rho(x, y) = \frac{1}{2}(1 + x - y)\). This \(\rho\) has a Fechnerian representation where \(v(x) = x\) and \(F\) is the cdf of a uniform distribution on \([-1, 1]\). Moreover, it can be checked that \(\rho\) in our example satisfies the Richness condition in Theorem 3.15, so \(F\) is pinned down up to the scale factor. But \(F\) cannot be the c.d.f. of the difference of two i.i.d. random variables. The characteristic function of the difference of two i.i.d.
random variables is a real and non-negative function $|\varphi(t)|^2$, where $\varphi$ is the characteristic function of one of them. But the characteristic function of $F$ equals $\sin(t)/t$, which takes negative values.

In this example $X$ was infinite. In the finite case $F$ is pinned down only on finitely many points, so perhaps there is more leeway? ▲

A.3.3. Proof of Theorem 3.15. The necessity of the quadruple condition is trivial. Sufficiency follows from Debreu’s (1958) theorem, which says that there exists $v : X \to \mathbb{R}$ such that for any $x, y, z, w \in X$

$$\rho(x, y) \geq \rho(z, w) \iff v(x) - v(y) \geq v(z) - v(w). \quad (A.2)$$

It remains to conjure up the $F$ function. Expression (A.2) defines a preference $\succeq$ on $X \times X$ with two representations $(x, y) \mapsto \rho(x, y)$ and $(x, y) \mapsto v(x) - v(y)$. By ordinal uniqueness (Proposition A.1.1) there exists a strictly increasing function $F : D \to \mathbb{R}$ such that $\rho(x, y) = F(v(x) - v(y))$.

The existence of $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $v_2 = \alpha v_1 + \beta$ follows from Debreu (1958). This implies that $F_1(v_1(x) - v_1(y)) = \rho(x, y) = F_2(v_2(x) - v_2(y)) = F_2(\alpha(v_1(x) - v_1(y)))$.

To prove that $F$ is continuous, we need to prove a converse to the intermediate value theorem: that every increasing function with the “intermediate value property” is continuous. This is a known result but we will prove it from scratch because at this point we don’t know if $D$ is an interval. (Maybe it would be easier to prove that, but I don’t know how to.)

Toward contradiction, suppose that $F$ is discontinuous at some point $d \in D$. This means there is a sequence $d_n \to d$ such that $F(d_n)$ does not converge to $F(d)$. Without loss we can restrict attention to a subsequence such that $d_n < d_{n+1} < d$ for all $n$.

Since $F$ is increasing, the sequence $F(d_n)$ is increasing and bounded from above by $F(d)$, so it has a limit. Let $q := \lim_n F(d_n)$. Pick any number $q^* \in (q, F(d))$. We will invoke the Richness axiom to show that there must exist $d^*$ for which $F(d^*) = q^*$. This means that $d_n < d^* < d$ for all $n$ which is a contradiction because $d_n \to d$.

So it just remains to invoke Richness and find $d^*$. Let $d_1 = v(x_1) - v(y_1)$ and $d = v(x) - v(y)$. If either $v(x_1) = v(x)$ or $v(y_1) = v(y)$ then apply Richness directly. Otherwise, are four cases to check:

1. $v(x_1) < v(x)$ and $v(y_1) < v(y)$. Then $v(x_1) - v(y_1) < v(x) - v(y)$, so $\rho(x_1, y_1) < q^* < \rho(x, y_1)$ and by Richness there exists $x^* \in X$ such that $\rho(x^*, y_1) = q^*$. Define $d^* := v(x^*) - v(y_1)$.

2. $v(x_1) > v(x)$ and $v(y_1) < v(y)$. Then $v(x_1) - v(y_1) < v(x) - v(y)$, so $v(y_1) - v(x_1) < v(y) - v(x_1)$ and $F(v(y_1) - v(x_1)) < 1 - q^* < F(v(y) - v(x_1))$, so by Richness there exists $y^* \in X$ such that $\rho(y^*, x_1) = 1 - q^*$. Define $q^* := v(x_1, y^*)$.
(3) $v(x) > v(x_1)$ and $v(y_1) > v(y)$. Then there are two subcases:
(a) $v(x_1) - v(y) < v(x) - v(y_1)$. Then either $\rho(x_1, y) < q^* < \rho(x, y)$
or $\rho(x_1, y_1) < q^* < \rho(x, y_1)$ (or both). Each of those subcases can be dealt with analogously to case 1.
(b) $v(x) - v(y_1) < v(x_1) - v(y)$. Then again there are two subcases, which can be dealt with analogously to case 2.
(4) $v(x_1) < v(x)$ and $v(y_1) > v(y)$. This can be dealt with analogously to case 3.

A.3.4. Proof of Proposition 3.31. Suppose that $X$ is finite and let $\mu$ be a RU representation of $\rho$ that has a finite support (there must exist one). For any $v$ in the support of $\mu$ and any $n$ define the logit $\rho_{v,n}(x, A) := \frac{\exp(nv(x))}{\sum_{y \in A} \exp(nv(y))}$. Define the mixed logit $\rho_n(x, A) := \sum_v \rho_{v,n}(x, A)\mu(v)$ and let $\rho_{v,*}(x, A)$ be one if $x$ is the argmax of $v$ over $A$ and to zero otherwise. Note that $\lim_{n \to \infty} \rho_{v,n}(x, A) = \rho_{v,*}(x, A)$. Thus,

$$\rho(x, A) = \sum_v \rho_{v,*}(x, A)\mu(v) = \lim_{n \to \infty} \sum_v \rho_{v,n}(x, A)\mu(v) = \lim_{n \to \infty} \rho_n(x, A).$$

A.4. Chapter 4

A.4.1. Proof of Proposition 4.27. Let $X = \Delta(Z)$. By Theorem 3.4, $\rho$ has a Luce representation

$$\rho(p, A) = \frac{w(p)}{\sum_{q \in A} w(q)}$$

for some function $w : X \to \mathbb{R}_+$. In particular, the function $w$ represents $\succeq^*$. By Theorem 4.5, $\succeq^*$ has an EU representation $U(p) = E_p v$ for some $v : Z \to \mathbb{R}$. Let $U(X)$ denote the range of $U$. By ordinal uniqueness, there exists a strictly increasing function $h : U(X) \to \mathbb{R}$ such that $w(p) = h(E_p v)$. Question: Can we drop “stochastic continuity?” In other words, if we have $\succeq$ on $\Delta(Z)$ that satisfies Independence and has some utility representation, does this imply that $\succeq$ has an EU representation?

A.5. Chapter 5

A.5.1. Proof of Proposition 5.4. average Bayes with rational expectations $\Rightarrow$ average Bayes: The former is a special kind of the latter.

average Bayes $\Rightarrow$ distribution over posteriors: For any Borel set of beliefs $B \subseteq \Delta(S)$ define the probability that the posterior will land in that set

$$\mu(B) := \sum_{s \in S} \pi(s) \beta_s \left( \{ m \in M : q_m \in B \} \right),$$
where \( q_m \) is the posterior belief given message \( m \). By definition of average Bayes, modulo ties, we have

\[
\rho(x, A) = \sum_{s \in S} \pi(s) \beta_s \left( \left\{ m \in M : \mathbb{E}_{q_m}[v(x)] = \max_{y \in A} \mathbb{E}_{q_m}[v(y)] \right\} \right)
\]

so using the definition of \( \mu \)

\[
\rho(x, A) = \mu \left( \left\{ q \in \Delta(S) : \mathbb{E}_{q}[v(x)] = \max_{y \in A} \mathbb{E}_{q}[v(y)] \right\} \right).
\]

The new tiebreaker needs to be an average of the old tie breakers over states.

**Distribution over posteriors \( \Rightarrow \) RU:** By definition.

RU \( \Rightarrow \) **average Bayes with rational expectations:** Wlog, there are no ties and we can take \( \Omega \) in the RU representation to be finite. Define \( S := \Omega \) (notice that in average Bayes the analyst does not observe \( s \) so we can take it to be whatever we want, in contrast with state-dependent Bayes). Define \( v(x, s) := \tilde{U}_s(x), p = \mathbb{P}, M := \Omega, \beta(\cdot | \omega) := \delta_\omega \). This way the agent learns their utility perfectly and \( q_s = \delta_s \). Thus, for a fixed, \( x \in A \) we have

\[
\rho(x, A) = \mathbb{P}(N(x, A)) = p(s \in S : v(x, s) = \max_{y \in A} v(y, s)) = \sum_{s \in S} p(s) \beta_s (m \in M : \mathbb{E}_{q_m}[v(x)] = \max_{y \in A} \mathbb{E}_{q_m}[v(y)]).
\]

**A.5.2. Proof of Proposition 5.6.**

**Bayes \( \Rightarrow \) obedience:** Fix \( A \) and suppose that \((\rho^*) \sim \text{Bayes}(p, \beta, v)\). Then

\[
\rho^*(x, A) = \beta(M_x | s), \tag{A.3}
\]

where

\[
M_x = \{ m \in M : \sum_{s \in S} v(x, s)p(s|m) \geq \sum_{s \in S} v(y, s)p(s|m) \text{ for all } y \in A \}.
\]

Consider now the action recommendation \( x \in A \). Upon hearing it, the agent’s posterior is the average of all the posteriors in \( M_x \):

\[
p(s|M_x) = \frac{p(s, M_x)}{p(M_x)} = \frac{\sum_{m \in M_x} p(s|m)p(m)}{p(M_x)} = \sum_{m \in M_x} p(s|m)\lambda(m),
\]

where \( \lambda(m) = \frac{p(m)}{p(M_x)} \). Thus, taking the average of the inequalities in the definition of \( M_x \) with weights \( \lambda(m) \) gives us

\[
\sum_{s \in S} v(x, s)p(s|M_x) \geq \sum_{s \in S} v(y, s)p(s|M_x) \text{ for all } y \in Y. \tag{A.4}
\]

Notice that \( p(s|M_x) = \frac{\beta(E_x | s)p(s)}{D_x} \), where \( D_x = \sum_{s' \in S} \beta(E_x, s')p(s') \). Plugging in (A.3) gives us

\[
p(s|M_x) = \frac{\rho^*(x, A)p(s)}{D_x}
\]

Substituting to (A.4) and multiplying both sides by \( D_x \) gives us obedience.
Obedience ⇒ Bayes: Fix menu $A$ and define $M := A$ and $\beta(x|s) := \rho^*(x,A)$. Obedience implies that upon hearing $x$ the agent wants to choose $x$, so modulo ties in the Bayes representation the set $M_x$ equals $\{x\}$. Thus the Bayes representation implies that the probability of choosing $x \in A$ in state $s$ equals $\beta(x|s)$, which as we know is the actual choice probability $\rho^*(x,A)$.

As mentioned in the text, to deal with ties we need to allow a different tiebreaker after each message: if $x,y \in M_x$ then the tiebreaker needs to put probability zero on choosing $y$. \hfill \Box

A.6. Chapter 6

A.6.1. Proof Sketch of Proposition 6.14. Fix $s \in S$ and let $p_n$ be such that $p_n(s) \to 1$. Fix an experiment $\beta$ and let $\mu_n := p_n \oplus \beta$. Notice that we have $\mu_n \to \delta_s$ (in the weak$^*$ topology). If $c$ is prior independent, then $c(p \oplus \beta) = c(p_n \oplus \beta)$ for all $n$. We will now show that under UPS, the right hand side converges to zero, which means that $c(p \oplus \beta) = 0$ and since $\beta$ was arbitrary, this implies that $c$ is identically equal to zero.

Suppose that $c$ is UPS. We have $c(\mu_n) = \int [L(q) - L(p_n)]\mu_n(dq)$. Consider first the expression $\int L(q)\mu_n(dq)$. Since $\mu_n \to \delta_s$, we have that $\int L(q)\mu(dq) \to L(\delta_s)$, assuming that $L$ is bounded. Likewise, $L(p_n) \to L(\delta_s)$ since $L$ is continuous.

A.6.2. The Blackwell Theorem. There are many equivalences known as the Blackwell theorem. Most of them characterize an incomplete ranking of experiments $\beta \geq \beta'$. Others characterize a ranking over posteriors $\mu \geq \mu'$. Finally, the two orders are connected, which is why I am using the same symbol to denote them.\footnote{One can think of a Blackwell-monotone cost function as inducing a completion of that ranking; various cost functions inducing various completions. The completion depends on how the agent’s brain is wired or on the schedule of the expert the agent is hiring to obtain the information.}

This section collects results from many sources and is incomplete. We assume that $M$ is rich enough so that by varying $\beta$ we can trace out all $\mu$.\footnote{This happens when $M$ contains a copy of $\Delta(S)$, see Lemma 1 of Denti, Marinacci, and Rustichini (forthcoming).} The set $S$ is finite; all measures are Borel.

First, we introduce the notion of garbling that lets us compare two experiments. Intuitively, $\beta'$ is a garbling of $\beta$ if we can first generate the signal $m$ according to $\beta$ and then add “noise” to it.

**Definition A.6.1 (Garbling).** $\beta' : S \to \Delta(M')$ is a garbling of $\beta : S \to \Delta(M)$ if there exists a mapping $G : M \to \Delta(M')$ such that

$$\beta'(m'|s) = \sum_{m \in M} \beta(m|s)G(m'|m).$$

\footnote{This happens when $M$ contains a copy of $\Delta(S)$, see Lemma 1 of Denti, Marinacci, and Rustichini (forthcoming).}
In the infinite case the garbling $G$ must satisfy additional measurability properties (formally, it has to be a probability kernel such that $\beta' = \beta \otimes G$, see Chapter 5 of Pollard (2002)).

The key here is that the distribution $G(\cdot|m)$ is independent of the state, so it does not carry any additional information: it is pure noise.

Next, we need a notion of dilation that lets us compare two distributions over posteriors. Intuitively, $\mu'$ is a dilation of $\mu$ if it is a mean-preserving spread of it: we generate $q$ according to $\mu$ and then in its place plug in a distribution over posteriors that averages to $q$.

**Definition A.6.2** (Dilation). $\mu'$ is a dilation of $\mu$ if for some $D : \Delta(S) \to \Delta(\Delta(S))$ such that $q = \sum_{q' \in \Delta(S)} q' D(q'|q)$ for all $q$ we have

$$\mu'(q') = \sum_{q \in \Delta(S)} \mu(q) D(q'|q).$$

If the supports are infinite, we require $D$ to be a probability kernel such that $\mu' = \mu \otimes D$ and $q = \int_{\Delta(S)} q' D(dq'|q)$.

**Definition A.6.3** (Achievable Payoff Profiles). For any menu $A$ define a behavioral strategy to be a mapping from signals to mixed actions $\sigma : M \to \Delta(A)$. Let $\Sigma$ be the set of behavioral strategies; the advantage of considering mixtures is that this set is now convex. The expected payoff of $\sigma$ (given $\beta$) in state $s$ is

$$v(\sigma, s; \beta) := \left\{ \sum_{m \in M} \sum_{x \in A} v(x, s) \sigma(x|m) \beta(m|s) \right\}$$

and let $v(\sigma; \beta)$ be the profile of such payoffs as $s$ ranges over $S$. Let $\text{AEP}(\beta) := \{ v(\sigma; \beta) : \sigma \in \Sigma \}$ be the set of achievable expected payoff profiles given $\beta$.

**Theorem**† A.6.4 (Blackwell).

1. The following are all equivalent definitions of $\beta \geq \beta'$:
   (i) $\beta'$ is a garbling of $\beta$
   (ii) $V_p^A(\beta) \geq V_p^A(\beta')$ for any decision problem $A, v, p$
   (iii) For any decision problem $\text{AEP}(\beta) \supseteq \text{AEP}(\beta')$.
   (iv) $\sum_{s \in S} \phi(q_m) \beta(m|s)p(s) \geq \sum_{s \in S} \phi(q_m) \beta'(m|s)p(s)$ for any convex and continuous function $\phi : \Delta(S) \to \mathbb{R}$ and any prior $p$

2. The following are equivalent definitions of $\mu \geq \mu'$:
   (i) $\mu$ is a dilation of $\mu'$.
   (ii) $V^A(\mu) \geq V^A(\mu')$ for any decision problem $A, v$
   (iii) $\int_{\Delta(S)} \phi(q) \mu(dq) \geq \int_{\Delta(S)} \phi(q) \mu'(dq)$ for any convex and continuous function $\phi$

3. Moreover, the following are equivalent
   (i) $\beta \geq \beta'$
   (ii) $p \oplus \beta \geq p \oplus \beta'$ for some full support $p \in \Delta(S)$

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144 Notice the similarity to the Anscombe–Aumann acts.
(iii) \( p \oplus \beta \geq p \oplus \beta' \) for all full support \( p \in \Delta(S) \)

**Remark A.6.5.** Notice that the Blackwell ordering of informativeness is expressed in opposite ways for \( \mu \) and for \( \beta \). To make \( \beta \) less informative we need to apply a garbling, i.e., also add risk to it. Intuitively, this makes sense: adding uncorrelated noise makes the message less informative. To make \( \mu \) more informative we need to apply a dilation, i.e., also add risk to it.\(^{145}\) This may seem at first confusing, but recall that in the world of distributions over posteriors having no information means a point mass on some \( p \) and adding information means splitting that mass point into a random posterior. A perfectly informative signal splits \( p \) into the vertices of the simplex. ▲

**Remark A.6.6.** Notice that if \( \mu \geq \mu' \) and \( \mu' \geq \mu \), then it must be that \( \mu = \mu' \). On the other hand, for any \( \beta \) there exist many Blackwell-equivalent \( \beta' \) (for example ones that have a different signal space, so different “labels”). But they are equivalent in the sense that \( p \oplus \beta = p \oplus \beta' \) for all full support \( p \). ▲

There is a vast literature on this topic, starting with Bohnenblust, Shapley, and Sherman (1949) and Blackwell (1951, 1953). A nice summary of the Blackwell theorem is given by Le Cam (1996). Torgersen (1991) is a very very dense book on this topic. Section 2 of Denti, Marinacci, and Rustichini (forthcoming) contains valuable material on the properties of the \( \oplus \) operator (its invertibility, and mixture-linearity).

**Remark A.6.7.** Intuitively, the force described in this section is the “dual” of the force behind preference for flexibility (Section 8.2). There, the agent is comparing information structures, here they are comparing menus, but the objective function is the same in both cases. ▲

### A.7. Chapter 7

In the main text it is shown that Joint Regularity implies (7.2). In fact, a stronger property holds for SU.

**Axiom A.7.1** (Bounded History Dependence). If \( x_1 \in A_1 \subseteq B_1 \), then for all \( x_2 \in A_2 \)

\[
\left| \rho_2(x_2, A_2|x_1, A_1) - \rho_2(x_2, A_2|x_1, B_1) \right| \leq 1 - \frac{\rho_1(x_1, B_1)}{\rho_1(x_1, A_1)}.
\]

This axiom says that conditional choice probabilities are “Lipschitz continuous” in the probability of histories.

**Proposition A.7.2.** If \( (\rho_1, \rho_2) \) has a SU representation, then it satisfies Bounded History Dependence and therefore Contraction History Independence.

\(^{145}\)Recall the convex order over lotteries over money. Condition 2(iii) is a multi-dimensional extension of this idea
Proof of Proposition A.7.2. I thank Ricky Li for helping sharpen this proof. Let \( E := N(x_2, A_1) \), \( F := N(x_1, A_1) \), and \( G := N(x_1, B_1) \). The axiom implies that \( G \subseteq F \). Let \( H := F \setminus G \). We have

\[
|P(E|F) - P(E|G)| = |P(E|G)P(G|F) + P(E|H)P(H|F) - P(E|G)|
= |P(E|G)(P(G|F) - 1) + P(E|H)P(H|F)|
= | - P(E|G)P(H|F) + P(E|H)P(H|F)|
= P(H|F)|P(E|H) - P(E|G)|
\leq P(H|F) = 1 - P(G|F) = 1 - \frac{P(G)}{P(F)} \]

\[ \square \]

A.9. Chapter 9

A.9.1. A Calculation behind Example 9.16. Let \( w^x := e^{v(x)} \). Let \( Y^x_t \) be independent Poisson processes with intensities \( w^x \) respectively. Let \( Y^x_t := \sum_{x \in A} Y^x_t \). The stopping time is the first time the process \( Y_t \) hits value 1. By Theorem 18.2 of Gravner (2017), \( Y_t \) is a Poisson process with intensity \( w^A := \sum_{x \in A} w^x \). By Proposition 18.1 the distribution of the stopping time is exponential with parameter \( w^A \). By Example 18.5 of Gravner (2017), the conditional choice probabilities are of the Luce form. \[ \square \]

A.10. Chapter 10

A.10.1. Proof of Theorem 10.8. This theorem is usually stated with more generality by not assuming continuous differentiability: Theorem 5.1 of McFadden (1981), Theorem 3.1 of Anderson, de Palma, and Thisse (1992), Theorem 3 of Koning and Ridder (2003), Corollary 8 of Fosgerau, McFadden, and Bierlaire (2013), or Theorem 1 of Yang (2021).

Continuous differentiability of \( \rho \) simplifies exposition as it saves us yet another condition, which implies that the distribution of \( \tilde{\epsilon} \) is in some sense smooth.

Theorem 10.8 follows from Theorem 3.1 of Anderson, de Palma, and Thisse (1992) because if \( \rho \) is continuously differentiable, then conditions (iv) and (v) imply their condition P1, their condition P2 is our (iii), their condition P3 is our (i), and their P4 is our (ii). \[ \square \]

A.10.2. Proof of Proposition 10.10. Let \( G \) be the CDF of \( \tilde{\epsilon}_x \). By definition, we can write for any \( x \in X \) and any \( p \in \mathbb{R}^X \)

\[
\rho(x, p) = \mathbb{P}(w(x) - p_x + \tilde{\epsilon}_x \geq w(x') - p_{x'} + \tilde{\epsilon}_{x'} \quad \forall \ x' \neq x)
= \int_{e_x = +\infty}^{e_x = -\infty} \prod_{x' \neq x} G(w(x) - p_x + e_x - w(x') + p_{x'}) g(e_x) de_x.
\]
This formula implies that if \( \rho(x, \bar{p}) = \rho(y, \bar{p}) \) for some \( x, y \in X \) and \( \bar{p} \in \mathbb{R}^X \), then \( w(x) - \bar{p}x = w(y) - \bar{p}y \).

This in turn implies that, \( \rho(x, (\bar{p}_{-z}, p_z)) = \rho(y, (\bar{p}_{-z}, p_z)) \) as a function of \( p_z \) for a fixed value of \( \bar{p}_{-z} \); thus their derivatives in \( p_z \) must coincide as well. \( \square \)

### A.10.3. Cyclic Monotonicity.

Blume (2008) attributes the concept of cyclic monotonicity to Hotelling (1929). Rockafellar (1966) says that cyclic monotonicity “can be viewed heuristically as a discrete substitute for two classical conditions: that a smooth convex function has a positive semi-definite second differential, and that all circuit integrals of an integrable vector field must vanish.” The latter condition is equivalent to \( \rho = \nabla V \) for some differentiable function \( V : \mathbb{R}^X \to \mathbb{R} \), which by Theorem 10.9 of Apostol (1969) is equivalent to symmetric cross-partial derivatives.

We have the following result.

**Theorem A.10.1.** Suppose that \( \rho : \mathbb{R}^X \to \Delta(X) \) is continuously differentiable. The following conditions are equivalent:

(a) \( \rho \) satisfies cyclical monotonicity

(b) \( \rho \) satisfies symmetric partials and the Jacobian of \( \rho \) is positive semi-definite

(c) \( \rho = \nabla V \) for some convex and differentiable function \( V : \mathbb{R}^X \to \mathbb{R} \).

Moreover, the equivalence of (a) and (c) holds for any continuous function.

**Proof.** (c)\( \Rightarrow \) (b) Symmetric partials follows from Schwartz’s theorem (also known as Young’s theorem), Theorem 9.41 of Rudin (1976). Positive semi-definiteness follows from Theorem 35 of Fenchel (1953).

(b)\( \Rightarrow \) (c) By Theorem 10.9 of Apostol (1969), condition (iii) \( \rho = \nabla V \) for some potential function \( V : \mathbb{R}^X \to \mathbb{R} \). Thus the Jacobian of \( \rho \) is the Hessian of \( V \). By Theorem 35 of Fenchel (1953), if the Hessian is positive semidefinite, then \( V \) is a convex function.

(c)\( \Rightarrow \) (a) Follows from Theorem 24.8 of Rockafellar (1970).

(a)\( \Rightarrow \) (c) I thank Terry Rockafellar for helping me with this proof. By Theorem 24.8 of Rockafellar (1970), cyclic monotonicity implies that \( \rho \subseteq \nabla V \) for some closed, proper convex function \( V : \mathbb{R}^X \to \mathbb{R} \). By Theorem 12.17 of Rockafellar and Wets (2009), the mapping \( \nabla V : \mathbb{R}^n \to \mathbb{R}^n \) is monotone according to their Definition 12.1. By their Example 12.17 a continuous function is maximally monotone, which implies that \( \rho = \nabla V \). The conclusion follows from Theorem 25.1 of Rockafellar (1970). \( \square \)

An analogous result for deterministic demand systems with quasilinear utility was obtained by Nocke and Schutz (2017).
Appendix B

Bibliography


——— (2020): “Ranges of Preferences and Randomization,” . 62


Apesteguia, J., and M. Ballester (2017a): “Stochastic Representative Agent,” working paper. 46


Auster, S., Y.-K. Che, and K. Mierendorff (2022): “Prolonged Learning and Hasty Stopping: the Wald Problem with Ambiguity.” . 132


Barlow, H. B., et al. (1961): “Possible principles underlying the transformation of sensory messages,” Sensory communication, 1(01). 72


Bergemann, D., and J. Välimäki (2002): “Information acquisition and efficient mechanism design,” Econometrica, 70(3), 1007–1033. 82


BLACKWELL, D. (1951): “Comparison of experiments, proceedings of the second Berkeley symposium on mathematical statistics and probability,”. 69, 82, 176


B. Bibliography


——— (2022): “Rationally inattentive behavior: Characterizing and generalizing Shannon entropy,” Journal of Political Economy, 130(6), 000–000. 87, 91


B. Bibliography


B. Bibliography


Dagsvik, J. K. (1995): “How large is the class of generalized extreme value random utility models?,” Journal of Mathematical Psychology, 39(1), 90–98. 48


B. Bibliography


Dean, M., and N. L. Neligh (2017): “Experimental tests of rational inattention,” . 81, 85, 91, 123


B. Bibliography


——— (2017): “Myopia and Discounting.” 79


Gravner, J. (2017): Lecture notes for introductory probability. 177


B. Bibliography


KRAJIBICH, I., B. BARTLING, T. HARE, AND E. FEHR (2015): “Rethinking fast and slow based on a critique of reaction-time reverse inference.” Nature Communications, 6(7455), 700. 118


KRISHNA, V., AND P. SADOWSKI (2016): “Randomly Evolving Tastes and Delayed Commitment,” mimeo. 112


LOOMES, G. (2005): “Modelling the stochastic component of behaviour in experiments: Some issues for the interpretation of data,” Experimental Economics, 8(4), 301–323. 59


——— (1990): “A historical and contemporary perspective on random scale representations of choice probabilities and reaction times,” *Journal of Mathematical Psychology*, 34(1), 81–87. 43


B. Bibliography


MENSCH, J. (2018): “Cardinal representations of information,” Available at SSRN 3148954. 88


B. Bibliography


Safonov, E. (2017): “Random choice with framing effects: a Bayesian model,” . 78


SPRUMONT, Y. (2020): “The triangular inequalities are sufficient for regularity,” *mimeo*. 43


B. Bibliography


TANNER, W. P., AND J. A. SWETS (1954): “A decision-making theory of visual detection.,” *Psychological review*, 61(6), 401. 70


TRAIN, K. (1986): *Qualitative choice analysis: Theory, econometrics, and an application to automobile demand*, vol. 10. MIT press. 137


B. Bibliography


Verstynen, T., and P. N. Sabes (2011): “How each movement changes the next: an experimental and theoretical study of fast adaptive priors in reaching,” *Journal of Neuroscience*, 31(27), 10050–10059. 71


Appendix N

Notation

\(x, y, z \in X\) — alternatives (in the grand set)
\(A, B, C \in \mathcal{A}\) — menus
\(\rho(x, A)\) — the probability that \(x\) is chosen from \(A\)
\(\rho : \mathcal{A} \to \Delta(X)\) — stochastic choice function (s.c.f.)
\((\Omega, \mathcal{F}, \mathbb{P})\) — probability space
\(\tilde{U} : \Omega \to \mathbb{R}^X\) — random utility
\(\mu \in \Delta(\mathbb{R}^X)\) — distribution over utilities (the law of \(\tilde{U}\) under \(\mathbb{P}\))
\(s \in S\) — state (in the state space)
\(m \in M\) — message (in the message space)
\(\beta : S \to \Delta(M)\) — experiment, lives in the set \(\mathcal{E}\)
\(\mu \in \Delta(\Delta(S))\) — distribution over posteriors (intentional abuse of notation)
\(f : S \to \Delta(Z)\) — an Anscombe–Aumann act
\(h : \mathcal{E} \to [0, \infty]\) — cost function defined on experiments
\(c : \Delta(\Delta(S)) \to [0, \infty]\) — cost function defined on distribution over posteriors