Axiomatic Foundations of Multiplier Preferences

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Multiplier preferences

Expected Utility inconsistent with observed behavior

We (economists) may not want to fully trust any probabilistic model.

Hansen and Sargent: "robustness against model misspecification"

Multiplier preferences

Unlike many other departures from EU, this is very tractable:

Monetary policy – Woodford (2006)
Ramsey taxation – Karantounias, Hansen, and Sargent (2007)
Asset pricing: – Barillas, Hansen, and Sargent (2009)
– Kleshchelski and Vincent (2007)

Multiplier preferences

But open questions:

$\rightarrow$ Where is this coming from?
What are we assuming about behavior (axioms)?

Multiplier preferences

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$\rightarrow$ Relation to ambiguity aversion (Ellsberg's paradox)?

Multiplier preferences

But open questions:

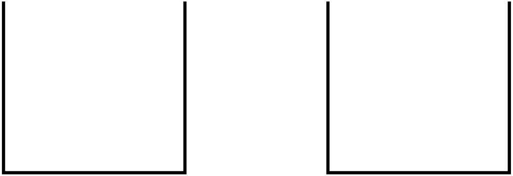
$\rightarrow$ Where is this coming from?
What are we assuming about behavior (axioms)?

$\rightarrow$ Relation to ambiguity aversion (Ellsberg's paradox)?

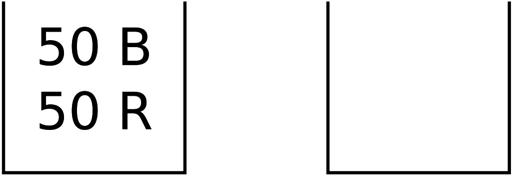
$\rightarrow$ What do the parameters mean (how to measure them)?

Sources of Uncertainty

Ellsberg Paradox



Ellsberg Paradox



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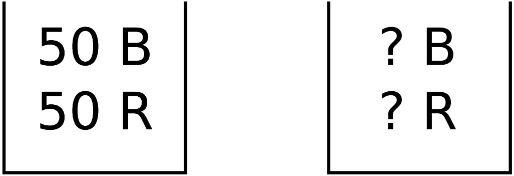


Ellsberg Paradox

50 B
50 R

? B
? R

B

R

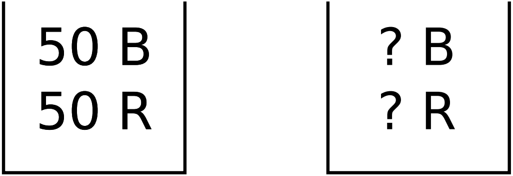
Ellsberg Paradox

50 B
50 R

? B
? R

B
||
R

Ellsberg Paradox



B	B
||
R	R

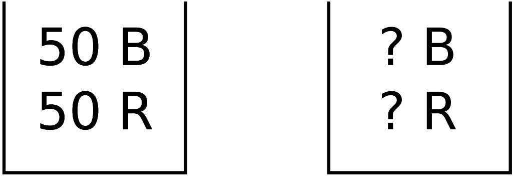
Ellsberg Paradox

50 B 50 R	? B ? R
B ‖ R	B ‖ R

Ellsberg Paradox

50 B 50 R		? B ? R
B	>	B
=		=
R	>	R

Sources of Uncertainty



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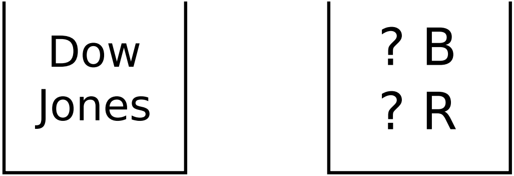

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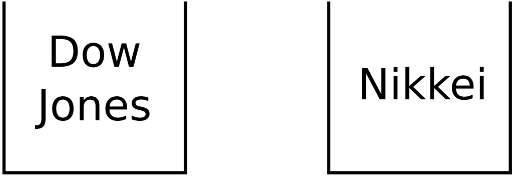

Sources of Uncertainty

Small Worlds (Savage, 1970; Chew and Sagi, 2008)

Issue Preferences (Ergin and Gul, 2004; Nau 2001)

Source-Dependent Risk Aversion (Skiadas)

Main Result

Within each source (urn) multiplier preferences are EU

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Within each source (urn) multiplier preferences are EU

But they are a good model of what happens between the sources

Criterion

Savage Setting

S – states of the world

Z – consequences

$f : S \to Z$ – act

Expected Utility

$u : Z \to \mathbb{R}$ – utility function

$q \in \Delta(S)$ – subjective probability measure

Expected Utility

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$V(f) =$ – Subjective Expected Utility

Expected Utility

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$V(f) = \qquad f_s \qquad$ – Subjective Expected Utility

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$V(f) = \quad u(f_s)$ – Subjective Expected Utility

Expected Utility

$u : Z \rightarrow \mathbb{R}$ – utility function

$q \in \Delta(S)$ – subjective probability measure

$V(f) = \int_S u(f_s)\, \mathrm{d}q(s)$ – Subjective Expected Utility

Multiplier preferences

q – reference measure (best guess)

Multiplier preferences

$$V(f) = \int u(f_s)\,\mathrm{d}p(s)$$

q – reference measure (best guess)

Multiplier preferences

$$V(f) = \min_{p \in \Delta(S)} \int u(f_s)\,\mathrm{d}p(s)$$

q – reference measure (best guess)

Multiplier preferences

$$V(f) = \min_{p \in \Delta(S)} \int u(f_s)\, \mathrm{d}p(s) + \theta\, R\,(p \| q)$$

q – reference measure (best guess)

Multiplier preferences

$$V(f) = \min_{p \in \Delta(S)} \int u(f_s)\, \mathrm{d}p(s) + \theta\, R\,(p \| q)$$

Kullback-Leibler divergence
relative entropy:
$R(p\|q) = \int \log\left(\frac{\mathrm{d}p}{\mathrm{d}q}\right) \mathrm{d}p$

q – reference measure (best guess)

Multiplier preferences

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Multiplier preferences

$$V(f) = \min_{p \in \Delta(S)} \int u(f_s)\, \mathrm{d}p(s) + \underset{\uparrow}{\theta}\; R\,(p \| q)$$

$\theta \in (0, \infty]$

$\theta \uparrow \Rightarrow$ model uncertainty $\downarrow$

$\theta = \infty \Rightarrow$ no model uncertainty

q – reference measure (best guess)

Observational Equivalence

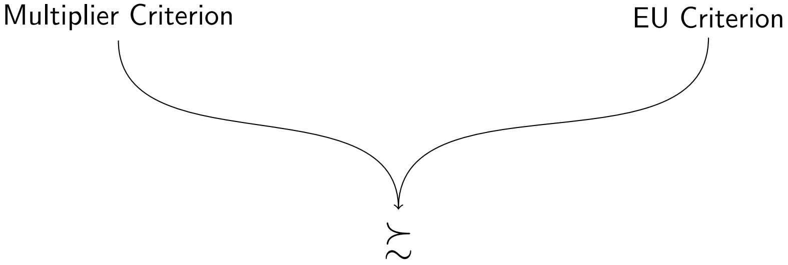
When only one source of uncertainty

Link between model uncertainty and risk sensitivity:

Jacobson (1973); Whittle (1981); Skiadas (2003)

dynamic multiplier preferences = (subjective) Kreps-Porteus-Epstein-Zin

Observational Equivalence

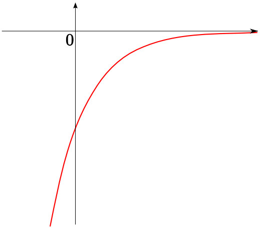


$\rightarrow$ u and θ not identified

$\rightarrow$ Ellsberg's paradox cannot be explained

Observational Equivalence

$$\phi_\theta(u) = \begin{cases} -\exp\left(-\frac{u}{\theta}\right) & \text{for } \theta < \infty, \\ u & \text{for } \theta = \infty. \end{cases}$$



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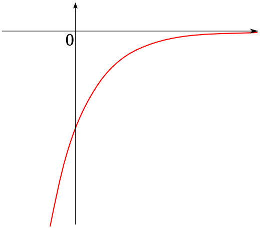


$\phi_\theta \circ u$ is *more concave* than u

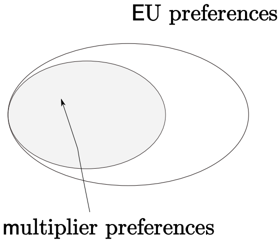
more risk averse

Observational Equivalence

Dupuis and Ellis (1997)

$$\min_{p \in \Delta S} \int_S u(f_s)\, \mathrm{d}p(s) + \theta R(p \| q) = \phi_\theta^{-1}\left(\int_S \phi_\theta \circ u(f_s)\, \mathrm{d}q(s) \right)$$

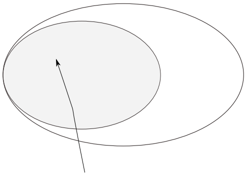
Observation (a) If $\succsim$ has a multiplier representation with (θ, u, q), then it has a EU representation with $(\phi_\theta \circ u, q)$.

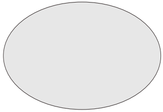


Observational Equivalence

Observation (a) If $\succsim$ has a multiplier representation with (θ, u, q), then it has a EU representation with $(\phi_\theta \circ u, q)$.

Observation (b) If $\succsim$ has a EU representation with (u, q), where u is bounded from above, then it has a multiplier representation with $(\theta, \phi_\theta^{-1} \circ u, q)$ for any $\theta \in (0, \infty]$.





Boundedness Axiom

Axiom There exist $z \prec z'$ in Z and a non-null event E, such that $wEz \prec z'$ for all $w \in Z$

Enriching the Domain: Two Sources

Enriching Domain

$f : S \rightarrow Z$ – Savage act (subjective uncertainty)

$\Delta(Z)$ – lottery (objective uncertainty)

$f : S \rightarrow \Delta(Z)$ – Anscombe-Aumann act

Anscombe-Aumann Expected Utility

$$f_s \in \Delta(Z)$$

$$\bar{u}(f_s) = \sum_z u(z) f_s(z)$$

Anscombe-Aumann Expected Utility

$$f_s \in \Delta(Z)$$

$$\bar{u}(f_s) = \sum_z u(z) f_s(z)$$

$$V(f) = \int_S \bar{u}(f_s)\, \mathrm{d}q(s)$$

Axiomatization

Variational Preferences

Multiplier preferences are a special case of *variational preferences*

$$V(f) = \min_{p \in \Delta(S)} \int \bar{u}(f_s)\, \mathrm{d}p(s) + c(p)$$

axiomatized by Maccheroni, Marinacci, and Rustichini (2006)

Multiplier preferences:

$$V(f) = \min_{p \in \Delta(S)} \int \bar{u}(f_s)\, \mathrm{d}p(s) + \theta R(p \| q)$$

MMR Axioms

A1 (Weak Order) The relation $\succsim$ is transitive and complete

MMR Axioms

A2 (Weak Certainty Independence) For all acts f, g and lotteries π, π' and for any $\alpha \in (0, 1)$

$$\alpha f + (1-\alpha)\pi \succsim \alpha g + (1-\alpha)\pi$$

$$\Updownarrow$$

$$\alpha f + (1-\alpha)\pi' \succsim \alpha g + (1-\alpha)\pi'$$

MMR Axioms

A3 (Continuity) For any f, g, h the sets
$\{\alpha \in [0,1] \mid \alpha f + (1-\alpha) g \succsim h\}$ and
$\{\alpha \in [0,1] \mid h \succsim \alpha f + (1-\alpha) g\}$ are closed

MMR Axioms

A4 (Monotonicity) If $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$

MMR Axioms

A5 (Uncertainty Aversion) For any $\alpha \in (0,1)$

$$f \sim g \Rightarrow \alpha f + (1-\alpha) g \succsim f$$

MMR Axioms

A6 (Nondegeneracy) $f \succ g$ for some f and g

MMR Axioms

Axioms A1-A6

$$\Updownarrow$$

Variational Preferences

MMR Axioms

A7 (Unboundedness) There exist lotteries $\pi' \succ \pi$ such that, for all $\alpha \in (0,1)$, there exists a lottery ρ that satisfies either $\pi \succ \alpha\rho + (1-\alpha)\pi'$ or $\alpha\rho + (1-\alpha)\pi \succ \pi'$.

A8 (Weak Monotone Continuity) Given acts f, g, lottery π, sequence of events $\{E_n\}_{n\geq 1}$ with $E_n \downarrow \emptyset$

$$f \succ g \Rightarrow \pi E_n f \succ g \text{ for large } n$$

MMR Axioms

Axioms A1-A6

$\Updownarrow$

Variational Preferences

Axiom A7 $\Rightarrow$ uniqueness of the cost function $c(p)$

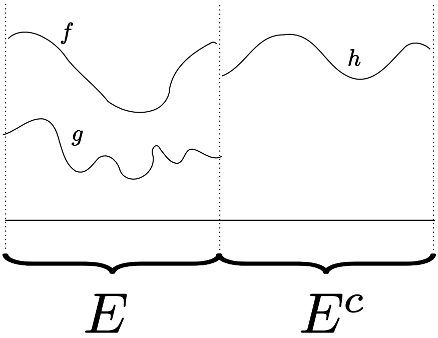
Axiom A8 $\Rightarrow$ countable additivity of p's.

Axioms for Multiplier Preferences

P2 *(Savage's Sure-Thing Principle)*

For all events E and acts $f, g, h, h' : S \to Z$

$$fEh \succsim gEh \Longrightarrow fEh' \succsim gEh'$$



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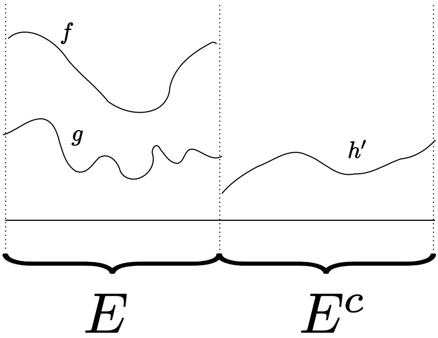
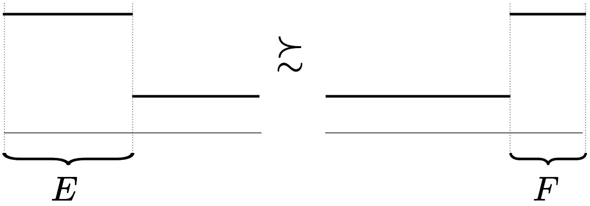


P4 (Savage's Weak Comparative Probability)

For all events E and F and lotteries $\pi \succ \rho$ and $\pi' \succ \rho'$

$$\pi E \rho \succsim \pi F \rho \Longrightarrow \pi' E \rho' \succsim \pi' F \rho'$$



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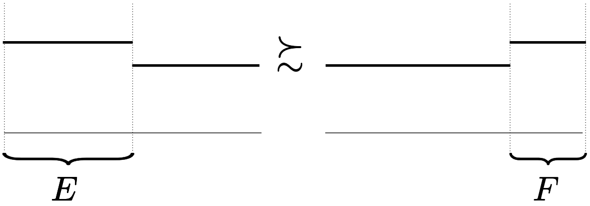

P6 *(Savage's Small Event Continuity)*

For all Savage acts $f \succ g$ and $\pi \in \Delta(Z)$, there exists a finite partition $\{E_1, \ldots, E_n\}$ of S such that for all $i \in \{1, \ldots, n\}$

$$f \succ \pi E_i g \quad \text{and} \quad \pi E_i f \succ g.$$

Main Theorem

Axioms A1-A8, together with P2, P4, and P6, are necessary and sufficient for $\succsim$ to have a multiplier representation (θ, u, q).

Moreover, two triples (θ', u', q') and (θ'', u'', q'') represent the same multiplier preference $\succsim$ if and only if $q' = q''$ and there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u' = \alpha u'' + \beta$ and $\theta' = \alpha \theta''$.

Proof Idea

Proof: Step 1

$\succsim$ on lotteries $\rightarrow$ identify u (uniquely)

MMR axioms $\rightarrow$ $V(f) = I\big(\bar{u}(f)\big)$

I defines a preference on utility acts $x, y : S \rightarrow \mathbb{R}$

$x \succsim^* y$ iff $I(x) \geq I(y)$

Where $I(x + k) = I(x) + k$ for $x : S \rightarrow \mathbb{R}$ and $k \in \mathbb{R}$

(Like CARA, but utility effects, rather than wealth effects)

Proof: Step 2

P2, P4, and P6, together with MMR axioms imply other Savage axioms, so

$f \succsim g$ iff $\int \psi(f_s)\,\mathrm{d}q(s) \geq \int \psi(g_s)\,\mathrm{d}q(s)$

$\pi' \succsim \pi$ iff $\psi(\pi') \geq \psi(\pi)$ iff $\bar{u}(\pi') \geq \bar{u}(\pi)$.

ψ and $\bar{u}$ are ordinally equivalent, so there exists a strictly increasing function ϕ, such that $\psi = \phi \circ \bar{u}$.

$f \succsim g$ iff $\int \phi\big(\bar{u}(f_s)\big)\,\mathrm{d}q(s) \geq \int \phi\big(\bar{u}(g_s)\big)\,\mathrm{d}q(s)$

Because of Schmeidler's axiom, ϕ has to be concave.

Proof: Step 3

$x \succsim^* y$ iff $\int \phi(x)\, \mathrm{d}q \geq \int \phi(y)\, \mathrm{d}q$

iff (Step 1) $x + k \succsim^* y + k$ iff $\int \phi(x + k)\, \mathrm{d}q \geq \int \phi(y + k)\, \mathrm{d}q$

So $\succsim^*$ represented by $\phi^k(x) := \phi(x + k)$ for all k

Proof: Step 4

So functions ϕ^k are affine transformations of each other

Thus, $\phi(x + k) = \alpha(k)\phi(x) + \beta(k)$ for all x, k.

This is Pexider equation. Only solutions are ϕ_θ for $\theta \in (0, \infty]$

Proof: Step 5

$$f \succsim g \Longleftrightarrow \int \phi_\theta\big(\bar{u}(f_s)\big)\,\mathrm{d}q(s) \geq \int \phi_\theta\big(\bar{u}(g_s)\big)\,\mathrm{d}q(s)$$

Proof: Step 5

$$f \succsim g \iff \int \phi_\theta\big(\bar{u}(f_s)\big)\,\mathrm{d}q(s) \geq \int \phi_\theta\big(\bar{u}(g_s)\big)\,\mathrm{d}q(s)$$

From Dupuis and Ellis (1997)

$$\phi_\theta^{-1}\left(\int_S \phi_\theta \circ \bar{u}(f_s)\,\mathrm{d}q(s)\right) = \min_{p\in\Delta S}\int_S \bar{u}(f_s)\,\mathrm{d}p(s) + \theta R(p\|\,q)$$

Proof: Step 5

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So

$$\min_{p\in\Delta S}\int \bar{u}(f_s)\,\mathrm{d}p(s) + \theta R(p\|q) \geq \min_{p\in\Delta S}\int \bar{u}(g_s)\,\mathrm{d}p(s) + \theta R(p\|q)$$

Interpretation

$V(f) = \int \phi_\theta\big(u(f_s)\big)\,\mathrm{d}q(s)$

(1)	$u(z) = z$	$\theta = 1$
(2)	$u(z) = -\exp(-z)$	$\theta = \infty$

Anscombe-Aumann

u is identified

$(1) \neq (2)$

Savage

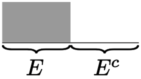
Only $\phi_\theta \circ u$ is identified

$(1) = (2)$

Ellsberg Paradox

$$V(f) = \int \phi_\theta\Big(\sum_z u(z) f_s(z) \Big) \, \mathrm{d}q(s)$$

Objective gamble: $\frac{1}{2} \cdot 10 + \frac{1}{2} \cdot 0 \rightarrow \phi_\theta\big(\frac{1}{2} \cdot u(10) + \frac{1}{2} \cdot u(0)\big)$

Subjective gamble: 

$\rightarrow \frac{1}{2}\phi_\theta\big(u(10)\big) + \frac{1}{2}\phi_\theta\big(u(0)\big)$

For $\theta = \infty$ objective $\sim$ subjective

For $\theta < \infty$ objective $\succ$ subjective

Measurement of Parameters

Ellsberg Paradox – Measuring Parameters

$$V(f) = \int \phi_\theta\Big(\sum_z u(z) f_s(z)\Big)\, \mathrm{d}q(s)$$

Certainty equivalent for the objective gamble:
$\phi_\theta\big(u(x)\big) = \phi_\theta\big(\frac{1}{2} \cdot u(10) + \frac{1}{2} \cdot u(0)\big)$

Certainty equivalent for the subjective gamble:
$\phi_\theta\big(u(y)\big) = \frac{1}{2}\phi_\theta\big(u(10)\big) + \frac{1}{2}\phi_\theta\big(u(0)\big)$

$x \to$ curvature of u

$(x - y) \to$ value of θ

Sources of Uncertainty

Multiplier Preferences

$$V(f) = \int \phi_\theta\Big(\sum_z u(z) f_s(z)\Big)\, \mathrm{d}q(s)$$

Anscombe-Aumann Expected Utility

$$V(f) = \int \Big(\sum_z u(z) f_s(z)\Big)\, \mathrm{d}q(s)$$

$$V(f) = \int \Big(\sum_z \phi_\theta\big(u(z)\big) f_s(z)\Big)\, \mathrm{d}q(s)$$

Second Order Expected Utility

Neilson (1993)

$$V(f) = \int \phi\Big(\sum_{z} u(z) f_s(z) \Big) \, \mathrm{d}q(s)$$

Ergin and Gul (2009)

$$V(f) = \int_{S_b} \phi\Big(\int_{S_a} u(f(s_a, s_b)) \, \mathrm{d}q_a(s_a) \Big) \, \mathrm{d}q_b(s_b)$$

Conclusion

Axiomatization of multiplier preferences

Multiplier preferences measure the difference of attitudes toward different sources of uncertainty

Measurement of parameters of multiplier preferences

Thank you

BARILLAS, F., L. P. HANSEN, AND T. J. SARGENT (2009):