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ON CHOOSING AN OPTIMAL TECHNOLOGY*

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This paper is concerned with the problem of choosing, on the enterprise level, a least cost technology for producing a bill of goods. Given certain simplifying assumptions appropriate to a long-run partial equilibrium environment, the problem is shown to be of a form which is solvable with any one of several techniques. Algorithms are presented and discussed. Proofs of their computational properties are provided, and implications for economic theory and decentralized decision making are noted. Finally, the basic model is expanded to consider the effects of modifying some of the economic assumptions.

1. The Basic Model

The setting for the problem to be presented is a type of activity analysis production model, which for convenience will hereafter be referred to as a "production system." A production system is basically an input-output model which has been modified in three directions.

First, there can exist alternative processes for producing some or all of the commodities. Second, the assumption of one primary input, "labor" in the Leontief case, is dropped. Instead, there can exist some or all commodities which can be purchased by the production system in any amount for a prescribed price.[1] Typically these "exogenous commodities" might include labor services of varying skills or levels (each level of labor service considered to be a separate commodity), raw materials (at a positive price if they must be purchased), possibly some fixed capital services, and perhaps even some intermediate or final goods.

The third modification generalizes what is usually a static model by including capital items in the production relations. Imputation is necessary to convert the the productive services rendered by capital stocks into ordinary production flows. The amount of capital amortized in one production period is imputed to be the sum of physical depreciation plus the "opportunity loss" incurred by using capital internally instead of renting it out at the going rate of interest. This imputation holds for circulating as well as for fixed capital.[2]

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[1] The production system is aimed at approximating an open industrial complex in a partial equilibrium setting where the prices of purchased materials can be treated as fixed. The question of which commodities are to be considered exogenous is entirely relative to the particular production system under consideration.

[2] In the case of circulating capital (intermediate materials), the opportunity loss is often considered to be zero. It was Lange who first emphasized that consistent notation required flow coefficients to be equal to the amortization on stock requirements, [10] p. 314.

Otherwise, the usual assumptions are obeyed by the production system. There is no joint production. Each homogeneous commodity is produced or purchased by linear processes or activities with fixed coefficients. For convenience, the period of production is taken to be one year; thus all quantities are on a *per annum* basis. This is obviously not a restrictive assumption.

In order to maintain a consistent conceptual scheme, we create an activity associated with the option of purchase (if such an option exists). This activity uses up no other commodities per unit of output and has associated with it a direct cost of unit operation equal to the purchasing cost per unit. We assume that the only activities which can have positive direct costs of operation are purchasing activities.

Some economic terms will be used in a broad sense throughout this paper. The word "commodity" will denote factors of production, raw materials, intermediate goods, or final goods. The words "capital item" will refer to commodities in their role as productive agents in contrast to their status as final goods. The words "to obtain" will mean "to produce or purchase".

The following mathematical conventions will be followed. With two vectors A and B, both of dimension n, we write $A \geqq B$ if each component of A is greater than or equal to the corresponding component of B. We write $A \geq B$ if $A \geqq B$ and $A \neq B$. Finally, we write $A > B$ if the full inequality holds component by component. The vector cross-product operator "$\times$" takes two n-vectors into one and is defined as follows: $A \times B$ is that n-vector whose i^{th} component is the product of the ith components of A and B.

Let there be a total of n commodities involved in the production system. By convention, we create n different departments, each one in charge of obtaining one of the commodities. Available to department i are m_i ($\geqq 1$) different activities which could be used in obtaining commodity i.

The k_i^{th} activity of department i ($1 \leqq i \leqq n$; $1 \leqq k_i \leqq m_i$) can be summarized by the triple

$$\{B_i^{k_i}, \mu_i^{k_i}, c_i^{k_i}\}$$

$B_i^{k_i}$ is an n component non-negative column vector of annual stock requirements, the jth component of which denotes the amount of capital item j which must be employed in order to produce a unit of commodity i in a year by activity k_i.

$\mu_i^{k_i}$ is a non-negative "amortization vector," the jth component of which denotes the annual amortization of capital item j when engaged in the production of one unit of commodity i by activity k_i. We can decompose each component of $\mu_i^{k_i}$ into the sum of a physical depreciation rate and an "opportunity" interest rate. Opportunity interest rates are internal discounting rates which presumably measure the net rental which could be obtained on a given capital item in its best alternative use outside the production system. Thus, in using a given capital item, the production system is considered to lose not only that amount of capital which physically vanishes in the process of production but, as well, that own net rate of return which could have been earned had the capital item been elsewhere

employed. Both depreciation and interest rates may differ among various capital items, or even among the same capital items when used in different modes of production, but they are all assumed to be known and constant. If a capital item is circulating capital (e.g., intermediate materials), typically depreciation will be one. If it is a fixed capital item (e.g., a lathe), typically depreciation is imputed to be less than one because there will presumably be some capital left after a year's operation.

$c_i^{k_i} \geqq 0$ is a scalar which specifies the direct or exogenous cost of operating the k_i^{th} activity of department i at the unit level. By convention, $c_i^{k_i} = 0$ for all activities other than the purchasing activity, whose capital stock and depreciation vectors are both zero.

The "composite requirements vector" $P_i^{k_i}$ is defined by the following equation

$$P_i^{k_i} = \mu_i^{k_i} \times B_i^{k_i}. \tag{1}$$

The components of the composite requirements vector denote the amounts of the various commodities which are amortized or "used up" in producing a unit of commodity i by technique k_i in one year.[3] Because $\mu_i^{k_i} \geqq 0$ and $B_i^{k_i} \geqq 0$, it follows that $P_i^{k_i} \geqq 0$.

2. The Primal Problem

The goal of the production system is to produce a given non-negative final demand in a year at minimum total exogenous cost (including the opportunity cost of tying up capital for a year). Disregarding the short run requirement that only an existing amount of equipment is initially available, the constraints of the system are that long-run production relations must be satisfied. The relevant decisions are whether to purchase a given commodity or to produce it; if the latter, which method of production to use, and in either case, at what levels to maintain the various activities.

In linear programming format, the problem is one of finding non-negative numbers $\{X_i^{k_i}\}$ $i = 1, \cdots, n$; $k_i = 1, \cdots, m_i$ which

$$\text{minimize} \quad \sum_{i=1}^{n} \sum_{k_i=1}^{m_i} c_i^{k_i} X_i^{k_i} \tag{2}$$

$$\text{subject to} \quad \sum_{i=1}^{n} \sum_{k_i=1}^{m_i} X_i^{k_i}(U_i - P_i^{k_i}) \geqq d \tag{3}$$

where $X_i^{k_i}$ is the level of operation of department i's k_i^{th} activity, $d \geqq 0$ is the n component column vector of final demands, and U_i is the unit vector with $+1$ as its ith component.

[3] While it is conceptually necessary to make all three modifications upon the classical linear economic model in order to make it operational enough to deal with the analysis of an integrated enterprise, only the first modification represents a substantive change. The second modification merely identifies "money" rather than "labor" as the primary factor, and, as we have just shown, the presence of capital stocks, introduced by the third modification, can be formulated in terms of the usual flow analysis.

The present framework could easily be expanded to handle *spatial* production analysis with linear transportation costs; so long as we were to follow the convention of differentiating commodities by their place of origin, as well as their physical characteristics, the mathematics would be identical.

The problem (2), (3) is called "strongly feasible" if there exists at least one $d > 0$ such that the problem is feasible for that d. We will always assume that the problem under consideration is strongly feasible.[4]

Lemma 1: Let $d > 0$ be a demand vector for which the problem (2), (3) is strongly feasible. Corresponding to d there exists an optimal solution having the property that each department is operating exactly one activity vector at a positive level.

Proof: The objective function is bounded from below because $c_i^{k_i} \geqq 0$. Since a feasible solution exists, an optimal feasible solution must also exist. Because the linear programming formulation involves vectors with n components, the theory of linear programming implies the existence of an optimal solution which contains no more than n activities operating at positive levels.[5] The condition $P_i^{k_i} \geqq 0$ implies that *any* feasible solution of (3) must contain at least one activity from each department, or else the positive final demands cannot be met (the problem (2), (3) cannot be degenerate). The conclusion follows.

A "technology" is any collection of n activities, one per department. Associated with any technology is an n by n non-negative "technology matrix"

$$P = (P_1^{k_1}, \cdots, P_n^{k_n})$$

and an n component non-negative direct cost row vector

$$c = (c_1^{k_1}, \cdots, c_n^{k_n}).$$

A technology is called "productive" if it can produce *any* demand vector $d \geqq 0$ at non-negative finite levels of its activities.[6]

The following theorem limits the search for the solution of (2), (3) to one of the $\Pi_{i=1}^n m_i$ possible technologies.

Theorem 1: If the problem is strongly feasible, there exists a productive technology which is optimal for all possible final demand vectors d satisfying[7] $d \geqq 0$.

Proof: Suppose initially that $d > 0$ is a vector for which the problem is strongly feasible. By Lemma 1 there is an optimal solution with the property that it involves exactly one activity from each department at a positive level. Let k_i^* be the number of this activity from department i. Let $P^* = (P_1^{k_1^*}, \cdots, P_n^{k_n^*})$. P^* is called an optimal technology matrix. Because the technology P^* is capable of producing $d > 0$, it can be shown (see Appendix 1 for a proof) that $(I - P^*)^{-1}$ exists and is non-negative in each entry. The optimal basis is still feasible if we

[4] If it were not, certain commodities could not be produced under our restrictions, and the system as a whole would not be viable. If every commodity can be purchased at some (possibly very high) price, the system is strongly feasible at any interest rate.

[5] See [4] p. 75 th. 2, or [3] p. 120 th. 1.

[6] An alternative approach to assuming strong feasibility would have been to postulate the existence of at least one productive technology. The assumption of strong feasibility was made because it is more intuitively appealing and because it appears to be a weaker assertion. Actually, in the context of this model, it can be shown that either assumption implies the other.

[7] In the static case with labor as the only primary factor, this result is the famous "non-substitution theorem" first proved by Samuelson and outlined in Ch. VII of [9].

now consider *any* $d \geqq 0$, because $(I - P^*)^{-1} d \geqq 0$ (i.e., P^* is productive). The conclusion follows by applying Corollary 1 to the theorem of the alternative (see Appendix 2).

3. The Dual Problem

The dual problem is to find a non-negative n component row vector $\pi = (\pi_1, \cdots, \pi_n)$ which

$$\text{(4)} \qquad \text{maximizes} \quad \pi d$$
$$\text{subject to} \quad \pi(U_i - P_i^{k_i}) \leqq c_i^{k_i} \qquad i = 1, \cdots, n; k_i = 1, \cdots, m_i.$$

An equivalent way of writing this constraint is

$$\text{(5)} \qquad \pi_i \leqq c_i^{k_i} + \pi P_i^{k_i} \qquad i = 1, \cdots, n; k_i = 1, \cdots, m_i.$$

In words, the dual problem is to ascertain production prices which maximize the value of final output without permitting any departmental profits.

Suppose initially that $d > 0$. Let P^* be an optimal technology matrix. Let the vector of direct exogenous costs associated with the optimal technology be $c^* = (c_1^{k_1*}, \cdots, c_n^{k_n*})$. The theorem of the alternative (see Appendix 2) tells us that *any* optimal dual variables $\pi^* = (\pi_1^*, \cdots, \pi_n^*)$ must satisfy

$$\pi^*(I - P^*) = c^*$$

or, equivalently,

$$\text{(6)} \qquad \pi^* = c^* + \pi^* P^*.$$

The solution to the dual problem is unique because $(I - P^*)$ is non-singular and the basis is non-degenerate. This is in contrast with the primal which can have alternative optimal solutions.[8] By Theorem 1, the basic variables in an optimal primal solution are the same for all $d \geqq 0$. Hence, the dual solution to (6) is optimal for all $d \geqq 0$. We have proved the following.

Theorem 2: Under the assumptions of the model, there is a unique optimal solution to the dual problem in the sense that there exists one and only one dual solution which is optimal for *all* $d \geqq 0$.

We refer to this solution as "the" dual solution, the sense in which it is unique being understood implicitly.[9]

An interpretation of π^* is as follows. Since the dual vector is optimal for all

[8] For example, any feasible solution is optimal if all direct costs are zero. In general, multiple primal solutions are possible whenever some commodities can be produced with the same direct and indirect costs in more than one way.

[9] We could strengthen this result so that the dual solution would have to be truly unique if we knew beforehand that every department would have to obtain a positive quantity of the commodity over which it takes charge. This would be the case, e.g., if all possible technology matrices were indecomposable and $d \geq 0$. In general, the price of a non-obtained commodity need not be unique; it may only be required to lie within a certain range. If a good is produced or procured, however, its optimal price is unique.

$d \geqq 0$, we let $d_i = 1$ and $d_j = 0, j \neq i$. By the duality theorem,[10] $\pi^* d = \pi_i^* = c^* X^*$, where $X^* = (X_1^{k_1^*}, \cdots, X_n^{k_n^*})$ is a basic optimal primal solution. This shows that π_i^* can be interpreted as the total annual exogenous cost of obtaining a unit of commodity i by the best known method of production. The following corollary has been proved.

Corollary: π^* is a minimal cost vector in the sense that not even one component may have a lower value with any other technology.

At this point, we introduce and motivate one further assumption, called the assumption of no free goods. We suppose that the problem is one dealing only with *economic* commodities—i.e., those having a non-zero cost of procurement or manufacture in any technology. Equivalently, we assume that for any i, some commodity with positive exogenous cost is always involved in producing commodity i, either directly or indirectly. This is not a genuine restriction because if there exists a method of obtaining unlimited free quantities of some commodity, it is redundant to include that commodity in the optimization problem.

4. A Derived Set of Extremal Equations

Since π^* is feasible and optimal, it follows from (5) and (6) that

$$\pi_i^* = \min \{(c_i^{k_i} + \pi^* P_i^{k_i}) \mid 1 \leqq k_i \leqq m_i\} \qquad i = 1, \cdots, n. \tag{7}$$

The existence of this set of extremal equations is a feature of this particular model which is not shared by linear programming models in general. We stress this point because it will enable us to bring to bear on the solution of (7) some powerful techniques of applied mathematics which have no direct relation to linear programming.

An economic interpretation of (7) is that an optimal set of prices has the property that each department charges at cost for the commodity it produces, using the cheapest method of production at that set of prices.

From (1) and (7), each department would determine unit costs for a given activity by spreading out over its gross annual output the sum of direct costs, plus total intermediate materials costs, plus total amortization payments imputed to departmental fixed capital.

Lemma 2: Let P be any technology matrix with associated direct cost vector c. Under the assumption of no free goods, if there exists a non-negative price vector σ, such that $\sigma \geqq c + \sigma P$, then the technology represented by P is productive.

Proof: By induction,

$$\sigma \geqq c + \sigma P \geqq c + (c + \sigma P)P \geqq \cdots \geqq c(\textstyle\sum_{t=0}^{\gamma} P^t) + \sigma P^{\gamma+1} \geqq \cdots \tag{8}$$

The assumption of no free goods means that, either directly or indirectly, *some* commodity with positive exogeneous cost is always involved in producing each commodity. Mathematically, for each $i = 1, \cdots, n$ there exists a non-negative integer $j(i)$ such that when P is raised to the jth power, the ith component of cP^j is positive. Let $\gamma = \max \{j(i) \mid i = 1, \cdots, n\}$. Define

[10] See, e.g., [3] p. 129.

$$Q = \sum_{t=0}^{\gamma} P^t.$$

Then $cQ > 0$. From (8),

$$\sigma \geqq cQ + \sigma P^{\gamma+1} \tag{9}$$

By induction on (9),

$$\sigma \geqq cQ\left(\sum_{\tau=0}^{\delta} (P^{\gamma+1})^\tau\right) + \sigma(P^{\gamma+1})^{\delta+1} \qquad \delta = 0, 1, \cdots \tag{10}$$

Because $cQ > 0$, (10) implies that

$$\lim_{\tau\to\infty} (P^{\gamma+1})^\tau = 0.$$

In turn, this implies that

$$\lim_{\tau\to\infty} P^\tau = 0,$$

which is equivalent to saying that the technology matrix P is productive (see Appendix 1).

The following theorem establishes a formal equivalence between the solution of the extremal equations (7) and the solution of the dual linear programming problem (4), (5).

Theorem 3: Under the assumptions of strong feasibility and no free goods, the dual solution is the unique non-negative solution[11] of (7).

Proof: An optimal solution exists and is unique by Theorem 2. We have shown that the optimal solution satisfies (7). By the definition of feasibility, it is non-negative. It remains to show that any non-negative solution to (7), π^*, is optimal for any $d \geqq 0$. Clearly π^* satisfies (5) and is therefore feasible. For each $i = 1, \cdots, n$, we choose k_i^* as that index which minimizes the right hand side of (7). Let P^* be the technology matrix $(P_1^{k_1^*}, \cdots, P_n^{k_n^*})$. Since π^* is a solution to (7), it must satisfy (6). The premises of Lemma 2 are fulfilled; it follows that the technology represented by P^* is productive. This is equivalent to saying that the primal basis is feasible. Invoking the theorem of the alternative, we conclude that

$$\{U_i - P_i^{k_i^*}\} \qquad i = 1, \cdots, n$$

is an optimal primal basis and π^* is the optimal dual price vector.

5. Solving the Extremal Equations

The equations (7) are of a form that readily lends itself to many methods of solution.[12]

[11] Actually, by our assumption of no free goods, we know that the dual solution must be strictly positive.

[12] In Bellman's terminology, (7) would be an example of the "prototype equation of dynamic programming;" [1], p. 78. It should be emphasized that in general the solution to a linear programming problem cannot be reduced to the solution of a vector functional equation. The special structure of the problem under investigation allows us to perform such a reduction in this particular case.

1. The Simplex Method of Linear Programming

One way of solving the primal problem is to use the simplex method. At any stage in the simplex calculations, suppose that we have a technology matrix P with associated cost vector c. We solve

$$\pi = c + \pi P \tag{11}$$

for the dual price vector $\pi = (\pi_1, \cdots, \pi_n)$.

The next step is to determine whether π, defined by (11), satisfies the dual inequality conditions (5). This is equivalent to determining whether π solves the system of extremal equations (7). If the dual inequalities are satisfied, the technology matrix is optimal. If not, there are integers r and s for which

$$\pi_r > c_r^s + \pi P_r^s \tag{12}$$

(i.e.,—the shadow price of commodity r is greater than its total cost of production using technique s, the sum of c_r^s, the direct cost, plus πP_r^s, the sum of indirect materials cost and amortization charges on fixed capital). Inserting $(U_r - P_r^s)$ into the basis will lower the price vector. By the simplex rules, $(U_r - P_r^{k_r})$ must leave the basis. The simplex method as applied to our problem is essentially a procedure for testing whether a technology is optimal, and if not, modifying exactly one activity to obtain a new technology that is better than the old one.

One disappointment with the managerial analogue of the simplex method is that price formation must be centralized (an inverse matrix must be centrally calculated). Another disappointment is that only one change at a time is allowed. If activity replacement occurs in a department operating at a positive level, total exogenous cost is diminished at each iteration,[13] but this process may take too long. Should significant innovations occur in several activities for several different departments, the manager would like to know that the *simultaneous* introduction of all profitable activities will lower real costs. We seek a generalization of the simplex method that will allow for the simultaneous entry of all lower cost activities, and yet preserve the desirable comparative statics property that total costs are lowered at each step. Such a method will be called "activity iteration."

2. Activity Iteration

Activity iteration is the process of forming a new technology matrix by replacing the current activity of at least one department by an activity of that department for which (12) holds. As usual, π, defined by (11), is the dual price vector. If (12) does not hold for any department, then activity iteration is said to have

[13] Strictly speaking, this phrase should read "*when the proper adjustments have been made*, real exogenous costs are diminished." In practice, securing a "proper adjustment" may be no small matter, especially if significant capital stock changes are involved. Since this paper is concerned only with long-run comparative statics situations, here and elsewhere we will only consider and compare equilibrium states, assuming always that a proper adjustment mechanism is at work.

"converged" on this technology. In this case each department finds that it is not possible to further reduce costs by switching activities.

The definition of activity iteration requires only that at least one change be made. It does not prescribe that only the integer set $\{h_i\}$ that fulfills

$$\min\{(c_i^{k_i} + \pi P_i^{k_i}) \mid 1 \leqq k_i \leqq m_i\} = c_i^{h_i} + \pi P_i^{h_i} \quad i = 1, \cdots, n \tag{13}$$

must be chosen (i.e.,—that at each stage only the cheapest activities must always be used by each department).[14] However, we would strongly expect that choosing h_i to be department i's next activity will result in the quickest convergence in most cases.[15]

Theorem 4: We assume that activity iteration starts with a productive technology.

(a) Each application of activity iteration results in a productive technology.

(b) Each application of activity iteration results in a technology that can produce every commodity at no more cost than the previous technology and which can produce at least one commodity at less cost.[16]

(c) If activity iteration converges, the technology on which it has converged is optimal and the associated dual price vector is optimal.

(d) Activity iteration will converge in a finite number of iterations.

Proof: Supposing first that convergence has not occurred, we replace by lower cost activities a non-empty subset of the current activities of P for which (12) holds, no more than one substitution per department being permitted. Let P', with associated c', be the technology matrix that results from this change. From the theorem of Appendix 1, productivity of P is equivalent to existence and non-negativity of $(I - P)^{-1}$. Therefore, $\pi = c(I - P)^{-1} \geqq 0$. From (12),

$$\pi \geq c' + \pi P'. \tag{14}$$

The proof of (a) follows immediately from Lemma 2.

[14] Employing the choice (13) is an example of an approach to solving extremal equations, analogous to Newton's method, which is discussed by Bellman under the title "approximation in policy space" in [1]. R. Howard has applied Bellman's methods to Markov chain policy problems, and in this context he dubs the technique "policy iteration." There is an interesting isomorphism between Howard's Markov chain model with time discounting, discussed in Chapter 7 of [7], and our own input-output system. Guy de Ghellinck should be credited with first pointing out the connection between policy iteration and linear programming, [6]. E. Malinvaud discusses a variant of this approach in Chapter III of [11].

[15] In the case of completely separable blocks of departments, the assertion is true for departments belonging to different blocks. If some blocks are "prior" to others, it will always be better to make a change in the more basic blocks irrespective of whether or not changes have been made in the less basic blocks, [2]. If we assume that the coefficients of the various technologies which are candidates for the next technology to be chosen at any stage in the activity iteration procedure obey the same probability distribution, a probabilistic variant of this assertion can be proved, which can be made more strong according as the technologies are assumed to be more acyclic.

[16] This statement can be strengthened if we are willing to make additional assumptions about the structure of the technology. For example, if all possible technologies are indecomposable, each application of activity iteration will cause *all* costs to diminish.

By induction, equation (14) yields the following string of inequalities:

$$\pi \geq c' + \pi P' \geqq c' + (c' + \pi P')P' \geqq \cdots$$
$$\geqq c'(\textstyle\sum_{j=0}^{\gamma} (P')^j) + \pi(P')^{\gamma+1} \quad \gamma = 0, 1, \cdots$$

From part (a), P' is productive. Hence,[17]

$$\lim_{\gamma\to\infty} c'(\textstyle\sum_{j=0}^{\gamma} (P')^j) + \pi(P')^{\gamma+1} = c'(I - P')^{-1}.$$

By definition $\pi' = c'(I - P')^{-1}$. We have shown that $\pi \geq \pi'$, which proves part (b).

Suppose now that activity iteration has converged on technology P^*. From part (a), P^* is productive; i.e.,—the primal basis is feasible for all $d \geqq 0$. If (12) does not hold for any activities of any departments, the dual inequalities (5) are satisfied. By the theorem of the alternative, the dual price vector is optimal and the primal basis is optimal. This proves (c).

From (b), π strictly declines at each iteration. We cannot return to an "old" technology, because the prices would have to be identical. The conclusion is that no cycling can occur. Since there are only a finite number of different technologies, the process must end. This proves (d).

The advantage of activity iteration is that it utilizes a kind of a simultaneity property with respect to decision making on the basis of internal prices. This is in contrast with the simplex one step at a time approach. From a computational point of view, this means that convergence is likely to occur much more quickly. Institutionally, it means that a system described by this linear production model can enjoy a greater measure of departmental autonomy; this is because decision making on technological questions can occur simultaneously with advantageous overall results when equilibrium has been restored. The institutional analogue of the simplex method would require that a new set of prices be ascertained after only one decision. Of course, calculating the new prices may be easier with the simplex method, because in general only one pivot operation is necessary to obtain a new inverse matrix from the old one. Both the simplex method and activity iteration are decentralized planning schemes in that the center possesses incomplete information at any given time and exerts influence on the various departments only via the issuance of decision-making parameters. Both converge monotonically because real costs are lowered, in a comparative statics sense, after each iteration.

Computationally and institutionally, the activity iteration method leaves one undesirable feature. It would be of great benefit to be able to form new prices without having to invert a technology matrix at each stage. Such a method is presented in the next section.

3. *Successive Approximations*

As usual, we assume no free goods and strong feasibility. Let π^* be the unique non-negative solution to (7), whose existence is guaranteed by Theorem 3. We

[17] See Appendix 1.

start off with an initial non-negative price vector $\pi(0)$ and calculate $\pi(t)$ recursively as follows.

$$\pi_i(t) = \min\{(c_i^{k_i} + \pi(t-1)P_i^{k_i}) \mid 1 \leqq k_i \leqq m_i\} \qquad i = 1, \cdots, n;\ t = 1, 2, \cdots \tag{15}$$

Theorem 5: The difference equation (15) converges[18] to the optimal price vector π^*.

Proof: Theorem 3 implies the existence of an optimal productive technology matrix P^* with associated c^*, such that $\pi^* = c^* + \pi^* P^*$.

From (15),

$$\pi(t) \leqq c^* + \pi(t-1)P^* \qquad t = 1, 2, \cdots \tag{16}$$

Subtracting $\pi^* = c^* + \pi^* P^*$ from both sides of (16),

$$\pi(t) - \pi^* \leqq (\pi(t-1) - \pi^*)P^* \qquad t = 1, 2, \cdots \tag{17}$$

By induction on (17),

$$\pi(t) - \pi^* \leqq (\pi(0) - \pi^*)(P^*)^t \qquad t = 1, 2, \cdots$$

This provides an upper limit on the sequence $\{\pi(t)\}$.

To obtain a lower bound on $\{\pi(t)\}$, we study the sequence $\{\eta(t)\}$:

$$\eta_i(t) = \min\{(c_i^{k_i} + \eta(t-1)P_i^{k_i}) \mid 1 \leqq k_i \leqq m_i\} \qquad i = 1, \cdots, n;\ t = 1, 2, \cdots \tag{18}$$

where $\eta(0) = 0$. It is obvious that if $\eta(t) \leqq \pi(t)$, then $\eta(t+1) \leqq \pi(t+1)$. Since $\eta(0) \leqq \pi(0)$, it follows by induction that $\eta(t) \leqq \pi(t)$ for all t.

Suppose now that $\eta(t-1) \leqq \eta(t) \leqq \pi^*$. Then,

$$c_i^{k_i} + \eta(t-1)P_i^{k_i} \leqq c_i^{k_i} + \eta(t)P_i^{k_i} \leqq c_i^{k_i} + \pi^* P_i^{k_i} \quad i = 1, \cdots, n; k_i = 1, \cdots, m_i.$$

This implies that for $i = 1, \cdots, n$

$$\min\{(c_i^{k_i} + \eta(t-1)P_i^{k_i}) \mid 1 \leqq k_i \leqq m_i\}$$
$$\leqq \min\{(c_i^{k_i} + \eta(t)P_i^{k_i}) \mid 1 \leqq k_i \leqq m_i\} \leqq \min\{(c_i^{k_i} + \pi^* P_i^{k_i}) \mid 1 \leqq k_i \leqq m_i\},$$

i.e., $\eta(t) \leqq \eta(t+1) \leqq \pi^*$. Since $\eta(0) \leqq \eta(1) \leqq \pi^*$, we have proved by induction that $\eta(t) \leqq \eta(t+1) \leqq \pi^*$ for all t. Because $\{\eta(t)\}$ is a bounded monotonic sequence, it must converge. The convergent sequence (18) must satisfy (7) in the limit. By the uniqueness part of Theorem 3, $\{\eta(t)\}$ converges to π^*.

Collecting our results,

[18] M. Morishima, working on the problem of dynamic adjustments in a generalized, dynamized Leontief system operating under perfect competition with a fixed interest rate, has proved an analogous result. His Theorem 2, [13], p. 100, states that a long-run set of prices is globally stable under a profit maximizing adjustment mechanism which disregards capital gains and losses. This adjustment mechanism can be viewed as a sequence of successive approximations iterations. I am indebted to Professor Robert M. Solow for pointing this out to me.

$$\eta(t) \leqq \pi(t) \leqq \pi^* + (\pi(0) - \pi^*)(P^*)^t \qquad t = 0, 1, \cdots$$

Because P^* is productive, we know that $\lim_{t\to\infty} (P^*)^t = 0$. $\{\pi(t)\}$ is trapped between two sequences, both of which converge to π^*, and the proof is concluded.

The iteration sequence (15) amounts to having each department charge for its product on the next round the cost of the cheapest production method as determined by this round's prices. On the next round, new prices are issued by the departments, and each department again selects a cheapest method of production on the basis of these prices and charges at cost. Such a response would be guaranteed by a regime of perfect competitors who extrapolate past prices into the future.

An advantage of this method of internal pricing is that the center does not have to figure out and set prices—that is automatically done by the departments concerned. Institutionally, this economizes on the exchange of information. Computationally it means that no inverse matrices have to be computed or stored.

The disadvantage of this technique is that, heuristically speaking, decisions are made too quickly, before prices are given a chance to settle down. Even if successive approximations costs have been lowered, there is no way of guaranteeing that at each iteration the total exogenous cost has diminished. Any given iteration may produce an increase in the total exogenous cost, even though "eventually" this method of iteration will converge to a least cost technology. The managerial implications of this property would be unfavorable to the prospect of price guidance in a decentralized framework.

Computationally, successive approximations methods are undesirable for the following reason. Experience shows that large problems often take so long to solve that they must be interrupted before convergence has occurred. Successive approximations, unlike the simplex method or activity iteration, does not converge with the property that feasibility is maintained while costs are monotonically decreased. Thus, short of inspection, there is no way of inferring whether or not the technology at any iteration is superior to the initial one, or even if it is productive.

We have seen that both activity iteration and successive approximations have their own advantages and disadvantages. In the next section we present a "combined method," which incorporates the most desirable features of each.

4. *The Combined Method*

Let π^0 be the price vector of a given non-optimal productive technology P^0 with associated c^0, such that $\pi^0 = c^0 + \pi^0 P^0$. We use the activity iteration method to choose the next technology matrix, call it P', with associated c'. By Theorem 4 (a), P' is productive. Prices are formed iteratively for this new technology as follows.

$$\begin{aligned} \pi(t) &= c' + \pi(t-1)P' \qquad t = 1, 2, \cdots \\ \pi(0) &= \pi^0. \end{aligned} \tag{19}$$

It is easily seen by successive substitution in (19) until $\pi(t)$ is expressed only

in terms of $\pi(0)$, P', and c', that this difference equation reduces to

$$\pi(t) = c'(\textstyle\sum_{i=0}^{t-1} (P')^i) + \pi(0)(P')^t$$

which, because P' is productive, converges to the π' satisfying[19] $\pi' = c'(I - P')^{-1}$, or

$$\pi' = c' + \pi' P'. \tag{20}$$

The institutional analogue of (19) is that each department charges at cost with respect to the activity it is currently using, revising the price of its output in lieu of changes in input prices. In practice, only a few iterations may be necessary to attain decent near-convergence, especially if the technology changes have not been profound, because in that case π' and π^0 are nearly equal. Computationally, the convergence to π' is geometric, and there is no accumulated roundoff error.[20]

When $\pi(t)$ has nearly converged, activity iteration proceeds and a new technology is selected. Institutionally, department heads would likely introduce only those activities showing substantial savings. This kind of "damped response" would insure that the conclusions to Theorem 4 could be invoked even if the prices were only "fairly near" to being the solution to (20).[21] In this manner, the processes of price iteration and activity change could proceed simultaneously with price iteration "leading" activity iteration.

The likeness of the combined method to the successive approximations algorithm is the common use of a price iteration scheme as opposed to some sort of centralized equation solving. The difference is that in the combined method, price iteration is used to attain a near solution to the dual prices for a given technology, and then technological decisions are made. As was pointed out, it might not be necessary to uphold this distinction rigorously in practice if the timing of an introduction were made dependent upon the magnitude of the proposed cost saving changes. This is probably a superior approach in a "real world" situation where innovation and technical change are continuously occurring.

The advantage of the combined method is that the benefits of activity iteration are enjoyed without having to centralize price formation.

[19] This procedure can be viewed as a kind of dual analogue to the "material balances" method of ascertaining the gross output necessary to sustain a given final demand. The method of "material balances" is described in [12], Appendix A, p. 335.

[20] See [8] p. 71, corollary 4.1.4, and [4] p. 264. The superiority of obtaining π' by the iterative scheme (19), instead of by direct matrix inversion is made apparent if a simple time estimate based upon the expected number of arithmetic operations is made for each technique. The only advantage of direct matrix inversion is for systems where n is small, or where convergence of (19) is extremely lengthy. The latter situation corresponds to a near-singular $(I - P)$ matrix, in which case direct inversion is also made difficult, and roundoff error is excessively large.

[21] These notions could be formalized and proved. That this and other variants of the combined method will (eventually) converge to the optimum price vector can be shown by an argument similar to the one used in proving Theorem 5.

6. Some Extensions

The assumption that each department work with only a finite number of activities is equivalent to postulating that the $n - 1$ dimensional set of input requirements per unit output, the departmental "production set," is the convex hull of those finite activities, hence a convex polyhedron. We could consider more general production sets, required only to be convex and closed. This includes, e.g., the case of a smooth departmental production function homogeneous of degree one. Such a problem intrinsically possesses greater mathematical intricacy than the one solved in this paper (effort must be expended, e.g., to demonstrate that the minimization implied by equation (13), now over an infinite set, is well defined). However, no unexpected or substantive differences emerge and no further insight is otherwise gained. One necessary modification is that while activity iteration still converges monotonically, it may no longer do so in a finite number of iterations.

In general, modifying the assumption of fixed final demand results in an economic system whose properties are similar to those of the simple production system. Consider, as a first example, the case of an economic system which seeks to maximize profits given elastic final demand schedules for each commodity. An optimal basis is still a technology. The optimal technology and price vector are obtained in the same manner as if this were a simple production system. Final output is determined as follows. If any profit can be made on a commodity, it is produced at that quantity which equates marginal cost (equals average cost in a linear system) with marginal revenue. Otherwise, it is not produced for final demand. In this economic system, activity iteration possesses the desirable property of monotonically increasing profits at each iteration. A second example, in which the optimal technology is also the same as would be obtained in a simple production system, is the case of maximizing a differentiable concave "utility" function of final output subject now to the additional constraint of a fixed budget. As well as not exceeding in total cost the fixed budget, commodities produced for final demand must equate between them the ratio of marginal utility to marginal cost. In this system monotonic convergence means monotonically increasing utility with each iteration.

Modifying the assumption of fixed purchasing costs could seriously affect our conclusions. For example, if inelastic supply schedules are specified, a least cost basis no longer need be independent of final demand. A critical problem now becomes the pricing of scarce resources; this necessitates paying careful attention to supply and demand conditions throughout the system and weakens our results on decentralization. The algorithms discussed in this paper may still be useful, but they now must be either supplemented with a tâtonnement system for pricing scarce resources, or else embedded as a subroutine in a decomposition algorithm.

Appendix 1

Some Properties of Productive Non-Negative Matrices

A technology is represented by an n by n non-negative technology matrix P. Consider the following four statements.

(i) There exists a positive final demand vector d' which can be produced by P.
(ii) P is productive.
(iii) $(I - P)^{-1}$ exists, is non-negative in each entry, and can be represented by the matrix power series $\sum_{j=0}^{\infty} P^j$.
(iv) $\lim_{j\to\infty} P^j = 0$.

Theorem: The statements (i), (ii), (iii), (iv) are equivalent in the sense that any one implies all the others.

Proof: Clearly (ii) implies (i).

From (i) there exist positive n-vectors d' and X' such that

$$(I - P)X' \geqq d'$$

(X' must be positive because d' is and $P \geqq 0$).

This implies that

$$X' > PX'$$

which in turn implies the existence of a number λ, $0 < \lambda < 1$, such that

$$\lambda X' > PX'$$

By induction on j,

$$\lambda^j X' > P^j X' \qquad \text{for } j = 1, 2, \cdots$$

Thus,

$$\lim_{j\to\infty} P^j X' = 0$$

which implies, since $X' > 0$, that

$$\lim_{j\to\infty} P^j = 0.$$

This demonstrates that (iv) can be derived from (i).

Consider $A_j = \sum_{i=0}^{j-1} P^i \geqq 0$. Because

$$(I - P)A_j = I - P^j,$$

it follows that

$$\lim_{j\to\infty} (I - P)A_j = I$$

if and only if

$$\lim_{j\to\infty} P^j = 0,$$

if and only if $(I - P)^{-1} = \lim_{j\to\infty} A_j$, proving that (iii) and (iv) are equivalent.

Suppose $(I - P)^{-1}$ exists and is non-negative in each entry. Consider *any* final demand vector $d \geqq 0$.

Define $X = (I - P)^{-1}d$.

(a) X fulfills the equation $(I - P)X = d$
(b) $X \geqq 0$.

The technology matrix P is capable of producing *any* non-negative final demand, and hence is productive. Thus (iii) implies (ii), and the proof of the Theorem is concluded.

Appendix 2

The Theorem of the Alternative and a Corollary

The theorem of the alternative can be stated in the following form for a linear programming problem and its dual, both expressed in inequality format.[22]

A necessary and sufficient condition for optimality in feasible basic primal and dual systems is that if the kth variable is basic in the primal system, the kth relation of its dual is an equality.

Corollary 1: In a feasible linear programming problem, let the right-hand side constraints be changed. If the optimal basis to the original problem is still feasible (although generally with basis vectors at different levels), this basis is still optimal.

Proof: Let the dual variables in the changed problem have the same values they had before the change (this set of dual variables is still feasible). The primal basis is still feasible by hypothesis. The conclusion follows by invoking necessity of the theorem of the alternative for the old problem and then sufficiency for the changed problem.

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[22] See, for example, [3], p. 136.