

DUALITY THEORY FOR INFINITE HORIZON CONVEX MODELS*

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Often it is desirable to formulate certain decision problems without specifying a cut-off date and terminal conditions (which are sometimes felt to be arbitrary). This paper examines the duality theory that goes along with the kind of open-ended convex programming models frequently encountered in mathematical economics and operations research. Under a set of general axioms, duality conditions necessary and sufficient for infinite horizon optimality are derived. The proof emphasizes the close connection between duality theory for infinite horizon convex models and dynamic programming. Dual prices with the required properties are inductively constructed in each period as supports to the state evaluation function.

1. Introduction

An important subclass of convex programming models of special interest to mathematical economists and operations researchers can be characterized by the following Markovian property: the choice of options available at any particular time depends only on the values of the state variables at that time. In other words, all of the influence of past history on the present is summarized by current state variable levels.

With such programming models, it is often not clear how to appropriately fashion an "end" to the underlying economic process. For concreteness, this dilemma is illustrated by means of the standard model of optimal economic growth¹ (although all remarks could be given a more general character). To maximize "utility" ("gain") on an arbitrary finite interval one must first be able to evaluate the capital stock (state variables) at the end of that interval. Since the worth of capital is defined by the utility of consumption to which it gives rise, precise evaluation of this sort must await the solution of an analogous problem on a second interval. Repeated application of this reasoning leads to an infinite regress. The only way out of this regress would seem to be in recognizing that the future does not have a definite and foreseeable end, and consequently optimization must be undertaken over an infinite horizon.

Paradoxically, it is often easier to analytically solve an optimization problem modeled on an infinite interval of time than on a finite interval with arbitrary end conditions. In the infinite interval case some sort of a turnpike theorem describing limiting steady state behavior can often be demonstrated. This is typically of considerable aid in characterizing an optimal solution.

Unfortunately, infinite horizon convex programming models with a free endpoint introduce some new difficulties which are not present in their finite-dimensional counterparts. For example, the very notion of an "optimal solution" for the infinite horizon case is somewhat vague and must be carefully defined. For this purpose we use a "classical" generalization of the usual finite-dimensional criterion, based on comparing convergent infinite sums. The main difficulty in the infinite horizon case concerns the existence and form of strong (necessary and sufficient) duality relations. Duality is of course extremely useful, even essential, for characterizing an optimal solution. For finite-dimensional convex programming models, as is well known, strong

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¹ See for example [1]-[6].

duality relations can be derived. Expressed with the aid of efficiency prices, the duality theorem in the finite Markovian case takes the form of an intertemporal profit maximization condition between periods plus a specific type of transversality condition on stocks left over after the last period. As we shall see, analogous necessary and sufficient conditions can be derived for the infinite-horizon case, with a transversality condition in the limit at infinity playing a key role.²

Our proof of the existence of dual prices with the required characteristics is by induction, using the functional equation approach of dynamic programming. The argument is direct and relatively simple. It is based on the key property of efficiency prices that in any period they form a supporting hyperplane to that period's state evaluation function.

2. Definitions and Assumptions

In what follows the index t , a positive integer, will denote the period of time from instant $t - 1$ to t . The phase vector x_{t-1} is an n -component vector denoting the state of the system during period t . At the beginning, x_0 is considered given and denoted by \bar{x}_0 . In many economic applications, x_{t-1} is understood as a vector whose i th component represents the amount of capital of type i available for use at time $t - 1$ and throughout the t th period. The "gain" in period t is denoted u_t . By "gain" might be understood "utility," "profit," "income," etc. depending on the specific features of the problem under consideration. Gains in each period are expressed in comparable units. In other words, all gains are measured as payout values discounted back to the first period. This is important because economic performance will be evaluated by the *sum* of single period gains.

The amount of gain u_t attainable in period t naturally depends on the initial and terminal states for that period, x_{t-1} and x_t . The $2n + 1$ -dimensional set of "transition possibilities" for period t , denoted Q_t , consists of all realizable triples of the form (x_{t-1}, u_t, x_t) . In other words a transition which yields gain u_t can be made from state x_{t-1} at the beginning of period t to state x_t at the end of that period if and only if

$$(1) \quad (x_{t-1}, u_t, x_t) \in Q_t.$$

A program $\{u_t, x_t\}$ is called *feasible* if for each t it satisfies (1) and

$$(2) \quad x_0 = \bar{x}_0.$$

It is supposed that for all t the set Q_t obeys the following stipulations.³

^o1 If $(x, u, y) \in Q_t$, then $x \geq 0, y \geq 0$.

^o2 If $(x, u, y) \in Q_t$ and if $x' \geq x$, then $(x', u, y) \in Q_t$.

² Roughly speaking, it has been more or less well known that intertemporal profit maximization plus the transversality condition at infinity are sufficient for an optimum. It is also more or less well known that intertemporal profit maximization is a necessary condition, but not sufficient by itself. At the time this paper was written I thought it contained the first rigorous proof of the necessity of the transversality condition at infinity under fairly general conditions. Professor Menahem Yaari subsequently pointed out to me that B. Peleg had obtained essentially the same result in [3]. The proofs are very different. Peleg applies the Hahn-Banach theorem directly to an ingeniously constructed sequence of infinite programs, deriving in the limit a representation for the separating linear functional which allows the interpretation of a price system with the required properties. In the present paper dual prices are inductively constructed in each period using a straightforward argument based on dynamic programming and the theory of convex sets.

³ The standard vector notation is employed whereby $x > y$ means that each component of x is $>$ each component of y , $x \geq y$ means that each component of x is \geq each component of y , and $x \geq y$ means $x \geq y$ and $x \neq y$.

°3 If $(x, u, y) \in Q_t$ and $(x', u', y') \in Q_t$, then $(\lambda x + (1 - \lambda)x', \lambda u + (1 - \lambda)u', \lambda y + (1 - \lambda)y') \in Q_t$ for all $\lambda, 0 \leq \lambda \leq 1$.

°4 $(0, 0, 0) \in Q_t$.

These four conditions are reasonably standard. The first requires that the state variables (capital) be nonnegative. The second is a free disposal type proposition. Condition °3 is the usual convexity assumption. The fourth condition is a "nothing ventured nothing gained" statement that it is possible to start with no capital, do nothing, gain nothing, and end up once again with no capital.⁴ Note that from °2 and °4 it follows that $(x, 0, 0) \in Q_t$ for all $x \geq 0$.

A fifth condition will guarantee that a strictly positive state vector is always "reachable" at any time.⁵ This kind of a productivity stipulation is needed as a means of insuring sufficient "nonemptiness" in the production sets so that meaningful dual prices can be formed.

°5 For each t there exists an $\hat{x}_t > 0$ with corresponding $\{x_\tau^t, u_\tau^t\}_{1 \leq \tau \leq t}$ satisfying

$$(x_{\tau-1}^t, u_\tau^t, x_\tau^t) \in Q_t, \quad 1 \leq \tau \leq t, \quad x_0^t = \bar{x}_0, \quad x_t^t = \hat{x}_t.$$

Let S be the class of all summable infinite sequences ($\{s_i\} \in S$ iff $\sum_{i=1}^\infty s_i$ converges). A program $\{u_i, x_i\}$ is said to be allowable if it is feasible (satisfies (1), (2)) and if $\{u_i\} \in S$. The effect of limiting attention to programs with summable gains is to introduce a complete preference ordering on programs. A program $\{u_i^*, x_i^*\}$ is called optimal if it is allowable and if for any other allowable program $\{\tilde{u}_i, \tilde{x}_i\}$,

$$\sum_{i=1}^\infty u_i^* \geq \sum_{i=1}^\infty \tilde{u}_i.$$

It is typically more difficult to prove that in theory an optimal program must exist for an infinite-horizon model than it is for its finite horizon counterpart.⁶ Nevertheless, it seems to be empirically true that the definition of optimality used here is broad enough so that optimal programs turn up for a great many infinite-horizon models of interest.

3. Duality Theory

Under the five axioms listed in the last section, the following duality theorem holds.

THEOREM. *For the allowable program $\{u_i^*, x_i^*\}$ to be optimal, it is necessary and sufficient that there exists a sequence of nonnegative n -dimensional price row vectors $\{p_i\}$ satisfying*

1° for $t = 1$,

$$u_1^* + p_1 x_1^* \geq u + p_1 y \quad \{u, y / (\bar{x}_0, u, y) \in Q_1\},$$

2° for $t \geq 2$,

$$u_t^* + p_t x_t^* - p_{t-1} x_{t-1}^* \geq u + p_t y - p_{t-1} x \quad \{x, u, y / (x, u, y) \in Q_t\},$$

3° for $t = \infty$,

$$\lim_{t \rightarrow \infty} p_t x_t^* = 0.$$

⁴ In cases where the level of gain is arbitrary, condition °4 amounts to a normalization convention under which zero gain is always attainable. For the utility function in the theory of optimal growth, this means shifting its level so that zero utility is an absolute floor.

⁵ Actually °5 could be replaced by slightly weaker conditions. But the resulting trivial gain in generality would not be worth the cost of increased notational complexity.

⁶ Roughly speaking, the inherent discount rate used to discount (undiscounted) payouts into (discounted) gains must be at least as high as the inherent potential growth rate of (undiscounted) payouts. Otherwise the sum of one-period gains (each one of which is normalized so that zero gain is always attainable in any state) may not converge.

Conditions °1 and °2 have the obvious interpretation that an optimal transition maximizes imputed profits.⁷ The transversality condition °3 is a special feature of infinite-horizon programming models with a free endpoint. It can be interpreted as saying that in an optimal program the present value of “left over” capital must eventually go to zero. This limitation on the rate at which capital ought to be accumulated can be very important in problem solving applications. It prevents the kind of nonoptimality which results from piling up too much capital in the limit as time goes to infinity.

4. Proof of the Duality Theorem

Sufficiency

Let $\{\tilde{u}_t, \tilde{x}_t\}$ be any allowable program. Using °1 and °2,

$$\begin{aligned} \sum_{t=1}^T u_t^* - \sum_{t=1}^T \tilde{u}_t &= (u_1^* + p_1 x_1^*) - (\tilde{u}_1 + p_1 \tilde{x}_1) + \sum_{t=2}^T [(u_t^* + p_t x_t^* - p_{t-1} x_{t-1}^*) \\ &\quad - (\tilde{u}_t + p_t \tilde{x}_t - p_{t-1} \tilde{x}_{t-1})] + p_T (\tilde{x}_T - x_T^*) \\ &\geq p_T (\tilde{x}_T - x_T^*). \end{aligned}$$

Passing to the limit as $T \rightarrow \infty$, from °3 and the fact that $p_T \tilde{x}_T \geq 0$ it follows that

$$\sum_{t=1}^{\infty} u_t^* \geq \sum_{t=1}^{\infty} \tilde{u}_t.$$

Necessity

We first demonstrate the following basic separation lemma of convex programming theory.

LEMMA 1. Let v represent an l -dimensional vector and w an m -dimensional vector. Suppose M is an $(l + m)$ -dimensional convex set with the property that

$$(3) \quad (v, w) \in M \Rightarrow w \geq 0.$$

Assume that there exist vector pairs $(\bar{v}, 0) \in M$ and $(\bar{v}, \bar{w}) \in M$ with $\bar{w} > 0$. Let $f(w)$ be a concave nondecreasing function of w defined for all $w \geq 0$ and let c be an l -dimensional row vector. Suppose there is a pair $(v^*, w^*) \in M$ satisfying

$$(4) \quad cv^* + f(w^*) = \max_{(v, w) \in M} [cv + f(w)].$$

Then there exists an m -dimensional price row vector $\pi \geq 0$ satisfying

$$(5) \quad cv^* + \pi w^* = \max_{(v, w) \in M} [cv + \pi w],$$

$$(6) \quad f(w^*) - \pi w^* = \max_{w \geq 0} [f(w) - \pi w].$$

PROOF. Introducing the variable z , in the $(1 + l + m)$ -dimensional space define sets A and B as follows:

$$(7) \quad A \equiv \{(z, v, w) / (v, w) \in M, z > z^*\},$$

$$(8) \quad B \equiv \{(z, v, w) / w \geq 0, z \leq cv + f(w)\},$$

where

$$z^* \equiv cv^* + f(w^*).$$

⁷ There are various other equivalent ways of writing the optimality conditions 1°, 2°. For example, they could be expressed as a discrete version of the so-called “maximum principle” [6]. The present formulation is chosen because it seems the simplest.

Using the concavity of $f(w)$, the convexity of M , and (4), it is easily verified that the sets A and B are convex and disjoint. Applying the separation theorem, there is a nonzero row vector (λ, θ, ψ) and a scalar γ such that

$$(9) \quad (z, v, w) \in A \Rightarrow \lambda z + \theta v + \psi w \leq \gamma,$$

$$(10) \quad (z, v, w) \in B \Rightarrow \lambda z + \theta v + \psi w \geq \gamma.$$

Since $(z, v^*, w^*) \in A$ for all $z > z^*$ and $(z^*, v^*, w^*) \in B$, (9) and (10) yield

$$(11) \quad \gamma = \lambda z^* + \theta v^* + \psi w^*.$$

It follows from (8) that (10) cannot hold unless $\lambda \leq 0$, $\theta = -\lambda c$, and, by monotonicity of $f(w)$, $\psi \geq 0$.

Suppose that $\lambda = 0$, implying $\theta = 0$. From the nontriviality of (λ, θ, ψ) , $\psi \neq 0$. From (7)-(10) and (3),

$$(v, w) \in M \Rightarrow \psi w = \gamma.$$

Alternatively substituting (\bar{v}, \bar{w}) and $(\bar{v}, 0)$ in the above condition yields a contradiction with $\psi \geq 0$, forcing the conclusion $\lambda < 0$.

Dividing (9), (10), (11) by $-\lambda > 0$ and defining $\pi \equiv -\psi/\lambda$,

$$(12) \quad (z, v, w) \in A \Rightarrow -z + cv + \pi w \leq -z^* + cv^* + \pi w^*,$$

$$(13) \quad (z, v, w) \in B \Rightarrow -z + cv + \pi w \geq -z^* + cv^* + \pi w^*.$$

Checking the definitions of A and B in (7), (8), conditions (12), (13) become

$$(14) \quad (v, w) \in M \Rightarrow cv + \pi w \leq cv^* + \pi w^*,$$

$$(15) \quad w \geq 0 \Rightarrow -f(w) + \pi w \geq -f(w^*) + \pi w^*.$$

Since (14) is equivalent to (5), and (15) to (6), the lemma has been proved.

Proceeding with the main body of the proof, consider for each t the following improper function which maps the n -dimensional nonnegative orthant E_+^n into the extended real halfline $[0, +\infty]$:

$$(16) \quad \varphi_t(x) \equiv \sup \sum_{\tau=t+1}^{\infty} u_{\tau},$$

subject to

$$(17) \quad \{u_{\tau}\}_{\tau \geq t+1} \in S,$$

$$(18) \quad (x_{\tau-1}, u_{\tau}, x_{\tau}) \in Q_{\tau}, \quad \tau \geq t+1,$$

$$(19) \quad x_t = x.$$

For each $x \in E_+^n$, $\varphi_t(x)$ is well defined and nonnegative (although it might be infinite) because from \circ^2 and \circ^4 the following is a solution of (17)-(19):

$$x_t = x, \quad x_{\tau} = 0, \quad u_{\tau} = 0 \quad \text{for } \tau \geq t+1.$$

The function $\varphi_t(x)$ is sometimes called a "state valuation function" because it gives the value of an optimal program starting at time t with initial endowment x .

LEMMA 2. For all t , $\varphi_t(x)$ is a nonnegative concave function, nondecreasing in x and satisfying for each $x \geq 0$ the following functional equation of dynamic programming

$$(20) \quad \varphi_t(x) = \sup_{\{u, y\}/(x, u, y) \in Q_{t+1}} [u + \varphi_{t+1}(y)].$$

PROOF. That $\varphi_t(x)$ is nonnegative has already been noted. We now show that $\varphi_t(x)$ is a proper (finite) function defined on E_+^n .

If $x \in E_+^n, y \in E_+^n$, and $\varphi_t(x) = \infty$, then $\varphi_t(\lambda x + (1 - \lambda)y) = \infty$ for all $\lambda \in (0, 1)$. This follows directly from the definition of $\varphi_t(x)$ (16)–(19), from nonnegativity of $\varphi_t(y)$, and from the convexity of production possibilities ^o3.

Suppose there is an $x \in E_+^n$ with $\varphi_t(x) = \infty$. Since $\hat{x}_t > 0$, there exists a $\mu, 0 < \mu < 1$, such that $\hat{x}_t \geq \mu x$. Then $\hat{x}_t = \mu x + (1 - \mu)y$, where $y \equiv (\hat{x}_t - \mu x)/(1 - \mu) \geq 0$. From the remarks of the previous paragraph $\varphi_t(\hat{x}_t) = \infty$. Since \hat{x}_t is attainable in period t (cf. ^o5), the above equation contradicts optimality of the program $\{u_i^*, x_i^*\}$. Thus $\varphi_t(x) < \infty$ for all $x \geq 0$.

That $\varphi_t(x)$ must be a concave function is easy to verify using ³ and the definition (16)–(19). From ^o2, $\varphi_t(x)$ is nondecreasing in $x \geq 0$.

The state valuation functions $\{\varphi_t(x)\}$ must satisfy (20) for all $x \geq 0$ because otherwise there would be an immediate contradiction with the definition (16)–(19). This concludes the proof of Lemma 2.

For $t = 1$ the definition of an optimal program and of $\varphi_1(x)$ implies

$$(21) \quad u_1^* + \varphi_1(x_1^*) \geq u + \varphi_1(y) \quad \{u, y/(\bar{x}_0, u, y) \in Q_1\}.$$

Apply Lemma 1, taking $l = 1, m = n, v \equiv u, w \equiv y, M \equiv \{(u, y)/(\bar{x}_0, u, y) \in Q_1\}, \bar{w} \equiv \hat{x}_1, f(w) \equiv \varphi_1(y), c \equiv 1$, and noting from (21) that (4) is satisfied. There must exist a nonnegative price row vector $p_1 (= \pi)$ satisfying

$$(22) \quad u_1^* + p_1 x_1^* \geq u + p_1 y \quad \{u, y/(\bar{x}_0, u, y) \in Q_1\},$$

$$(23) \quad \varphi_1(x_1^*) - p_1 x_1^* \geq \varphi_1(x) - p_1 x \quad \{x/x \geq 0\}.$$

Now let t be an arbitrary positive integer. Suppose there is a price vector $p_t \geq 0$ supporting $\varphi_t(x)$ at $x = x_t^*$, i.e.,

$$(24) \quad \varphi_t(x_t^*) - p_t x_t^* \geq \varphi_t(x) - p_t x \quad \{x/x \geq 0\}.$$

For the induction step it will be necessary to prove that there exists a dual price vector $p_{t+1} \geq 0$ simultaneously satisfying

$$(25) \quad u_{t+1}^* + p_{t+1} x_{t+1}^* - p_t x_t^* \geq u + p_{t+1} y - p_t x \quad \{x, u, y/(x, u, y) \in Q_{t+1}\},$$

$$(26) \quad \varphi_{t+1}(x_{t+1}^*) - p_{t+1} x_{t+1}^* \geq \varphi_{t+1}(x) - p_{t+1} x \quad \{x/x \geq 0\}.$$

From (23), condition (24) holds for $t = 1$. Since (22) has already been demonstrated, ¹, ² will have been proved if for arbitrary t we can show (using (24)) how to construct a vector $p_{t+1} \geq 0$ with the desired properties (25), (26).

By (20) we know that for any $x \geq 0$

$$(27) \quad \varphi_t(x) \geq u + \varphi_{t+1}(y) \quad \{u, y/(x, u, y) \in Q_{t+1}\}.$$

From the definition of an optimal program and of $\varphi_t(x)$ it is clear that

$$(28) \quad \varphi_t(x_t^*) = u_{t+1}^* + \varphi_{t+1}(x_{t+1}^*).$$

Substituting from (27) and (28) into (24),

$$(29) \quad u_{t+1}^* + \varphi_{t+1}^*(x_{t+1}^*) - p_t x_t^* \geq u + \varphi_{t+1}(y) - p_t x \quad \{x, u, y/(x, u, y) \in Q_{t+1}\}.$$

Once again apply Lemma 1, this time taking $l = n + 1, m = n, v \equiv (x, u), w \equiv y$,

$M \equiv Q_{t+1}$, $\bar{w} \equiv \hat{x}_{t+1}$, $f(w) \equiv \varphi_{t+1}(y)$, $c \equiv (-p_t, 1)$ and identifying (29) with (4). Conditions (25), (26) drop out immediately after setting $p_{t+1} = \pi \geq 0$. This completes the induction step and proves 1°, 2°.

To verify the transversality condition, set $x = 0$ in (24), yielding

$$\varphi_t(x_t^*) - \varphi_t(0) \geq p_t x_t^*.$$

From $\varphi_t(x_t^*) \geq \varphi_t(0) \geq 0$, and $p_t x_t^* \geq 0$, and

$$\lim_{t \rightarrow \infty} \varphi_t(x_t^*) = \lim_{t \rightarrow \infty} \sum_{\tau=t+1}^{\infty} u_{\tau}^* = 0,$$

condition 3° directly follows. This concludes the proof of the theorem.

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