On the Meaning of Comparative Factor Productivity

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Total factor productivity is

$$\frac{Y}{f(K, L)}$$

(output, aggregated arithmetically, divided by combined inputs, typically aggregated geometrically corresponding to a Cobb-Douglas production function).

In such a situation the factor productivity of $a$ relative to $\beta$ might be defined as

$$\frac{Y_a f(K_a, L_a)}{Y_\beta f(K_\beta, L_\beta)}$$

(1)

(Naturally the definition is entirely from the viewpoint of $a$, but some asymmetry of that sort is unavoidable; I want to sidestep the issue in this chapter.)

Formula (1) is typically interpreted as some measure of the relative power of the technology of economy $a$ to produce $\beta$ output mix with $\beta$ factor endowments. If the technology of $a$ were hypothetically applied to produce $\beta$ factor endowments, (1) represents how much proportionately more (or less) output could be expected.

We know a fair bit about when such an interpretation of (1) makes sense, at least in simple situations. Indeed, a good part of Bergson's 1975 article is devoted to this issue. What I would like to do here is push the analysis a bit harder to see how far it can be made to go. Nothing revolutionary, or for that matter entirely satisfactory (or even exciting) seems to emerge. But I think it is worthwhile to probe this issue further in a formal theoretical direction because it gives some idea of what we are implicitly assuming when we work with formulas like (1) in a general context.

Any analysis must begin somewhere, and I will start with the following assumptions. Let there be $n$ commodities in $a$, produced by an input-output technology with variable techniques. If one unit of commodity $i$ is produced by technique $j$ (sometimes denoted $j_i$; when ambiguity must be avoided), it requires $a_{kj}$ units of commodity $k$ ($k = 1, \ldots, n$) and $c_i$ units of "composite factor." Composite factor is an amalgam of capital and labor whose substitution possibilities are given by the function $F_i(K, L)$. In other words, if one unit of commodity $i$ is to be produced by technique $j$, we must have a combination of capital $K$ and labor $L$ satisfying $F_i(K, L) \geq c_i$. The equation $F_i(K, L) = \text{constant}$ merely expresses how capital and labor substitute for each other in producing commodity $i$ with technique $j$.

We are thus making the special assumption that a technique specifies as fixed coefficients the flow of input materials but leaves some flexibility in the...
combination of capital and labor required to process the inputs. This is a 
restriction, though perhaps a fair one. Within each (different) technique for 
producing a commodity, capital and labor can substitute (differently for 
each technique) in processing materials, but the materials themselves are 
relatively fixed in proportions (different for each technique). There are 
extceptions; however, this formulation seems a fair generalization, and it will 
give a lot of analytic power.

All production is constant returns to scale (including the functions \( F_i^j(\cdot) \)).

As postulated, economy \( \alpha \) is in market equilibrium with total supplies of 
capital \( K^\alpha \) and labor \( L^\alpha \). Let the equilibrium wage rate be \( w \) and the capital 
rental be \( r \). The competitive price of commodity \( i \) is \( p_i \). The following general 
equilibrium pricing equations must hold:

\[
\begin{align*}
  p_i &= \min_j \left\{ wL_i^j + rK_i^j + \sum_{k=1}^n p_k^j a_{ki} \right\} \\
  F_i^j(K_i^j, L_i^j) &= c_i^j.
\end{align*}
\]

(2)

The equation states that production is to be done at least cost, and price 
equals cost. Given \( w \) and \( r \), equation (2) will have a solution \( j(i, K_i^{(0)}, L_i^{(0)}), p_i \) 
for each \( i \). Let the corresponding input-output matrix be \( A = (a_{ki}^{(0)}) \) with 
composite factor row vector \( \epsilon = (c_i^{(0)}) \).

On the demand side, suppose final net output is the column vector \( D^\alpha \).
The corresponding vector of gross sectoral output levels is

\[
X^\alpha = (I - A)^{-1}D^\alpha.
\]

(3)

The final equilibrium condition is then

\[
\begin{align*}
\sum_i X_i^\alpha L_i^{(0)} &= L^\alpha, \\
\sum_i X_i^\alpha K_i^{(0)} &= K^\alpha.
\end{align*}
\]

(4)

(5)

For the purpose of using \( \alpha \) data to evaluate \( \beta \), the only relevant thing about 
economy \( \beta \) is the net final output vector it produces, \( D^\beta \), and its endowments of 
capital \( K^\beta \) and labor \( L^\beta \).

Now let us define the coefficient of comparative factor productivity as the 
proportion of output that could be produced by \( \alpha \) technology given \( \beta \) factor 
endowments. The coefficient of comparative factor productivity \( \mu \) is then the 
solution (the maximum value of \( \lambda \)) of the problem:

\[
\begin{align*}
\max_{\lambda} & \quad \sum_{i} X_i^\beta - \sum_{k=1}^n a_{ki}^\beta X_k^\beta \geq \lambda D_i^\beta, \\
F_i^j(K_i^j, L_i^j) & \geq c_i^j.
\end{align*}
\]

(6)

(7)

\[\sum_{i, j} K_i^j X_i^j \leq K^\beta, \]

\[
\sum_{i, j} L_i^j X_i^j \leq L^\beta, \]

\[
X_i^\beta, K_i^\beta, L_i^\beta \geq 0.
\]

(8)

(9)

(10)

In this problem, \( X_i^j \) is the amount of commodity \( i \) produced by technique 
\( j \), and \( K_i^j \) is the amount of capital (labor) employed per unit output in 
technique \( j \) of sector \( i \).

For the most general case it is difficult to characterize the solution of (6) 
through (10) and state with precision what \( \mu \) depends upon. Fortunately, it is 
possible to prove an "approximation theorem." If the sectoral production 
functions \( F_i^j \) are close to being identical to each other (up to a scaling factor), 
in some sense we are in a situation where an aggregate production function 
"almost exists." Then it seems reasonable that the standard aggregate factor 
productivity calculation will be a good approximation to \( \mu \).

Suppose, then, there is a production function \( f(K, L) \) homogeneous of 
degree one, such that

\[
\| F_i^j(K, L) - f(K, L) \| < \varepsilon
\]

(11)

for some \( \varepsilon > 0 \), all \( i, j, K, L \). As usual, \( \| \cdot \| \) denotes any legitimate distance norm. If \( \varepsilon \) is small, \( f \) is a good approximation to each of the sectoral 
production functions.

Let \( Y^\alpha \) be the national product of economy \( \alpha \) \( (Y^\alpha = pD^\alpha) \). Let \( Y^\beta \) be 
the national product of economy \( \beta \) measured in prices of \( \alpha \) \( (Y^\beta = pD^\beta) \). Consider 
using \( f \) as an aggregate production function in the surrogate factor productivity 
comparison,

\[
\hat{\mu} = \frac{Y^\beta / (K^\beta, L^\beta)}{Y^\alpha / (K^\alpha, L^\alpha)}.
\]

(12)

The following result is not difficult to show (the proof is omitted).

\[
\lim_{\varepsilon \to 0} \hat{\mu}(\varepsilon) = \mu.
\]

(13)

Thus the aggregate approximation of comparative factor productivity 
(12) becomes the rigorous definition (6) in the limit as the aggregate 
production function \( f \) becomes a perfect approximation to each technique's 
production function \( F_i^j \). If each technique's production function is close to 
Cobb-Douglas, with similar capital and labor elasticities, there is a rigorous 
justification for aggregating price weighted outputs arithmetically and inputs geometrically.
In summary, the economy can have a very complicated input-output structure, but so long as the substitution possibilities between capital and labor are similar for each technique, use of an aggregate production function and price weighted outputs will yield a serviceable approximation to a true measure of comparative factor productivity.

References


