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10. Trees vs. Fish, or Discrete vs. Continuous Harvesting

Martin L. Weitzman

1. KARL-GUSTAF LÖFGREN AND THE HARVESTING OF RENEWABLE NATURAL RESOURCES

Karl-Gustaf Löfgren has been one of the early pioneers in developing and applying dynamic economic tools (in particular, optimal control theory) to the analysis of how best to develop and harvest natural resources. At the same time, he has maintained a special interest in forestry economics. The topic of this chapter may thus be relevant for a celebration of his sixtieth birthday.

I have a suspicion that Kalle was, at a young age, attracted to the interface between human beings and the natural environment that surrounds and nurtures them. That is to say, I am guessing that early on he was relatively most interested in the part of environmental economics having to do with the combining of economics with 'nature', as opposed to, say, the combining of economics with pollution–health issues, which might be called 'Environmental Protection Agency-type' environmental economics. What then could be a more 'natural' (no pun intended) field for Kalle to specialize in than the harvesting of renewable natural resources.

This is not to say that Kalle has not done outstanding work in lots of other areas of economics. It is just that I think that the 'love of his life' has been in this area of how to balance human interests and the interests of 'nature'.

When Kalle first entered the field, back in the 1960s, the economic harvesting of natural resources was still in its murky infancy. The basic ideas were 'out there', for sure, but they were far from being in the nicely-packaged reduced form we now know and teach to students. Every model seemed special, and disconnected from every other. It was not clear what were the basic unifying underlying principles. Were they just specific particularly-exotic examples of capital theory, or was there some deeper connection with the rest of dynamic economics?

I think it is fair to say that Pontryagin's maximum principle, which was just then beginning to be applied to economics, and which Kalle latched on to very early, forced us economists to 'see' the capital-theoretic unity of all such natural resource problems. First of all, just using the maximum principle made us put all dynamic problems into a canonical form that was almost automatically a useful way of seeing the underlying unity. More importantly, the maximum principle itself is a set of duality conditions with a natural, and very important, economic interpretation centered on the co-state variables, which are competitive-like prices to us. The maximum principle has a direct economic interpretation as describing a dynamic competitive equilibrium, while other forms of dynamic optimality conditions (for example, Euler-type equations) essentially must be transformed into a maximum principle-like form to give them economic meaning.

Thus, by using the maximum principle, we economists were led to a rich understanding of the connections between the optimal regulation of a renewable fishery resource, the optimal extraction of an exhaustible mineral resource, and the neoclassical theory of optimal growth – to name just three famous models that thereby became interconnected. However, one famous and very important model that we natural-resource economists knew and loved remained somehow outside this maximum principle-contained orbit of (almost) all other dynamic resource allocation models. This was the famous Faustmann–Wicksell model of optimal forestry rotation. I would think that Kalle is perhaps the world's expert on the history of economic thought surrounding the Faustmann–Wicksell model (or models), since he has been especially concerned with the theory (and practice) of forestry economics throughout his career and has written on the intellectual history of just such a class of aging and growth models.

Forestry models seemed somehow 'different' from the other models of natural resource harvesting or extraction. The forestry models focus sharply on the age structure of a cohort, and are essentially discrete. The 'harvesting' of a tree or forest is the discrete act of cutting it down and bringing it to market. The 'renewal' of a tree or forest is the discrete act of planting seedlings. This seems very different from the continuous harvesting and renewal that characterizes, say, the classical model of the fishery.

There is an air of intellectual disappointment in not being able to combine fishery and forestry models under some unifying umbrella. At least

this was the case for me. Kalle, I believe, may have also been intellectually puzzled about this seeming dichotomy between the continual harvesting and renewal of the fishery and the discontinuous harvesting and renewal of the forest. Why should these two core models of the economics of renewable resources seem so different in structure?

What I want to show in this chapter is that there exists a way to connect the two models by turning the classical Faustmann–Wicksell forestry model into an equivalent continuous-harvesting version. We will then be able to see how the maximum principle applied to this equivalent continuous-harvesting version of the forestry rotation problem is just another form of the famous Faustmann–Wicksell first-order conditions telling us when to cut down the tree.

In the next section, we recapitulate the problem of the sole owner of the fishery as an optimal control problem that is linear in net investment, and hence supports a most rapid approach bang-bang solution. Then, the more interesting and novel part, which follows in the third section, shows that the classical forestry problem is also an optimal control problem that is linear in net investment, and hence this problem also supports a most rapid approach bang-bang solution. In this way, we show that the mathematical structure of these two famous problems of the harvesting of natural resources is essentially isomorphic. Both are linear in investment optimal control problems whose solution is the most rapid approach to their respective stationary states.

2. OPTIMAL MANAGEMENT OF THE FISHERY

The classical dynamic economic problem of optimal fishery management is typically presented as if seen through the eyes of a fictitious 'sole owner', who may be conceptualized as being either a private firm or a government regulatory agency. The sole owner is assumed to be seeking a harvesting policy that maximizes net present discounted profits.

The problem here is which form to choose for the harvesting flow rate $\{h(t)\}$ to

$$\text{maximize} \int_0^\infty \pi(x)h(t)\,\mathrm{e}^{-\rho t}\,\mathrm{d}t \tag{1}$$

subject to

$$\dot{x}(t) = F(x(t)) - h(t), \tag{2}$$

and

$$\underline{h} \leq h(t) \leq \bar{h}, \tag{3}$$

and with the given initial condition

$$x(0) = x_0. \tag{4}$$

For this model, $x(t)$ represents the stock of fish at time t, and $h(t)$ is the harvest flow taken at time t. In condition (3), $\hat{h}$ is some more or less arbitrary upper bound on harvesting; the lower bound $\underline{h}$ is perhaps somewhat less arbitrary because $\underline{h} \equiv 0$, at least, has a natural interpretation. (The upper and lower bounds are needed to make sense of the problem for technical reasons, so in a way it does not matter what they are.) The function $F(x)$ represents the net biological increase of the fish population, in the absence of any harvesting. The function $\pi(x)$ gives the net profits per fish caught when the stock of fish is x.

In the fisheries literature it is standard to take as unit profit the difference between price and catch cost, so that

$$\pi(x) = P - c(x), \tag{5}$$

where P represents the exogenously-given price of fish and $c(x)$ represents per unit 'locating and harvesting cost' as a function of fish density x. A reader typically sees the form of the right-hand side of (5), rather than our more concise notation $\pi(x)$.

To reduce the problem of the sole owner of the fishery to a canonical form, it is useful to reformulate it in terms of net investment. In this situation, net investment is the natural biological increment of the fish population *minus* the amount of fish being caught or harvested. (It is perhaps not yet entirely clear why we might want to take a problem out of the form in which it naturally suggests itself and recast it in the form of a prototype economic problem where net investment is considered to be the control variable – the reason is that this canonical form always permits the solution to be understood quickly, easily, and in the most economically intuitive way.)

With the change of variables $K \equiv x$ and $I \equiv F(x) - h$, and specifying $m(K) \equiv F(K) - \hat{h}$ and $M(K) \equiv F(K) - \underline{h}$, the optimal fishery harvesting problem is a prototype-economic problem with gain function

$$G(K, I) = \pi(K)[F(K) - I]. \tag{6}$$

The stationary rate of return on capital is defined to be

$$R(K) \equiv \frac{G_1(K, 0)}{-G_2(K, 0)}. \tag{7}$$

It can readily be shown that net investment should be positive if $R(K) > \rho$, negative if $R(K) < \rho$, and zero if $R(K) = \rho$. Once the stationary rate of return on capital has been calculated, the qualitative direction of investment (positive, negative, or zero) is determined. The only remaining question is how fast to go to a stationary state. For the linear in investment renewable resource problems under investigation here, the answer is 'as fast as possible'.

From applying formulae (7) to (6) (and remembering to evaluate at $I = 0$), the stationary rate of return on capital for this model of optimal fishery management is

$$R(K) = F'(K) + F(K)\frac{\pi'(K)}{\pi(K)}. \tag{8}$$

Equation (8) can be interpreted as saying that the stationary rate of return $R(K)$ consists of two terms representing the two economic effects that come from having a higher amount of fish capital here. The first effect, $F'(K)$, represents the increment of new fish population that comes with a higher parent fish stock. The second term on the right-hand side of (8) represents the additional profit from the lower unit harvesting cost that attends a larger fish population, since it is easier to locate and catch fish when there are more of them.

Let $\hat{K}$ represent the stationary solution where

$$R(\hat{K}) = \rho. \tag{9}$$

As is well known for a problem linear in net investment, the optimal policy is a most rapid approach to the stationary solution $\hat{K}$.

This description of the management of the fishery is familiar, because we are (by now) accustomed to seeing the classical fishery model as a linear in investment optimal control problem with a bang-bang solution. What is less familiar, and less obvious, is that the optimal forest rotation problem is also a linear in investment optimal control problem with a bang-bang solution. Hence, the mathematical structure of the two renewable resource harvesting problems is essentially the same.

3. OPTIMAL TREE HARVESTING AS A CONTROL PROBLEM

Another model whose gain function is linear in investment is the *optimal tree harvesting model*. This problem can be posed and solved directly, without invoking optimal control theory – so a formulation in terms of optimal control theory serves more to enrich an intuitive understanding of the maximum principle (as capital theory) than to serve as a mechanism for actually solving a problem that could not otherwise be solved.

Suppose that, when it is cut down and brought to market, a tree of age T yields a net value given by the function

$$F(T). \tag{10}$$

Frequently in the forestry literature, $F(T)$ is specified in the form

$$F(T) = Pf(T) - c + v, \tag{11}$$

where P is the given market price of wood and $f(T)$ is (in forestry terminology) the 'merchantable volume' of wood yielded by a tree of age T. The parameter c represents the total economic cost of cutting down the tree, processing it for sale, and bringing the wood to market. (In the forestry literature, the expression $Pf(T) - c$ is called the *net stumpage value* of the tree.) The parameter v stands for the *opportunity value* (in lumbering terminology the *land expectation* or *site value*) of the land being freed for its best subsequent economic use after the tree is felled – which 'best subsequent economic use' might well be the replanting of a sapling to start the tree-growing cycle anew.

The famous *Wicksell problem* of capital theory is to choose the time of cutting T to

$$\text{maximize } e^{-\rho T} F(T). \tag{12}$$

It might seem perverse to force such a direct statement as (12) into the seemingly more arcane form of an optimal control problem. However, an optimal control formulation will serve to reinforce economic intuition and to highlight quite dramatically the underlying unity of *all* time and capital problems. In particular, it will allow us to see sharply the relationship between the two most famous models of renewable resources – optimal harvesting of the fishery and optimal harvesting of the forest.

In the optimal control version of the Wicksell problem, the 'capital stock' is the *age of the tree* (more precisely, it is the tree of that age). The corresponding 'investment' here means allowing the tree to grow older by a year.

Suppose we fancifully imagined that 'the forest' could be continuously harvested in the spirit of 'the fishery'. For this fishery-like forest, the 'harvest-flow' generalization of the Wicksell problem in capital theory is

to control the 'investment rate' $\{I(t)\}$ to

$$\text{maximize } \int_0^\infty \rho F(K(t))[1 - I(t)]\, e^{-\rho t}\, dt \tag{13}$$

subject to

$$\dot{K}(t) = I(t), \tag{14}$$

and

$$0 \le I(t) \le 1, \tag{15}$$

and with the given initial condition

$$K(0) = 0. \tag{16}$$

The original Wicksell formulation in effect limits the investment $I(t)$ to be a step function, which takes on value one when the tree is growing (or until it is cut), and takes on value zero thereafter. As we will see, the above 'harvest-flow' generalization yields the Wicksell solution anyway. For now it suffices to note that the Wicksell problem is a special case of (13)–(16); therefore, if the optimal solution of (13)–(16) is a step function, as will turn out to be the case, then it must also represent the solution of the more restricted Wicksell problem (12).

It is useful to pose the Wicksell model formally as an optimal control model of capital accumulation because it highlights the underlying connection between growth and aging processes where *capital is time* (aging of wine is another well-known example) and the bulk of *all other* capital-theoretic models that can be formulated as simple optimal control problems where capital is *not* time. Posing the problem this way allows us to see rigorously what we otherwise can only intuit in models of tree cutting, wine aging, animal raising, and many other problems of growth and aging – the important idea that in many situations *age is capital*, but that otherwise the same general principles of capital theory apply.

So, for this Wicksell problem, let us identify 'capital' with 'age'. Applying definition (12) to the gain function

$$G(K, I) = \rho F(K)[1 - I], \tag{17}$$

which appears in (13), the stationary rate of return on capital in the optimal tree cutting problem is

$$R(K) = \frac{F'(K)}{F(K)}. \tag{18}$$

From the general consideration that the gain function of the Wicksell problem is linear in investment, we know that the optimal solution involves a most rapid approach to the stationary state $\hat{K}$ where $R(\hat{K}) = \rho$, which by (19) is equivalent to the condition

$$\frac{F'(\hat{K})}{F(\hat{K})} = \rho. \tag{19}$$

Let us see what is happening specifically in this particular optimal control problem by formally applying the maximum principle. The Hamiltonian here is

$$H = \rho F(K)[1 - I] + pI. \tag{20}$$

where p stands for the marginal value of letting a tree of age K grow for one more year.

The next step is to calculate the maximum value of the Hamiltonian over all feasible values of I. This part is easy because we are maximizing a linear function over the unit interval. With $\tilde{I}(p)$ denoting the Hamiltonian-maximizing value of investment as a function of its price, from (20) there are three possibilities:

$$p > \rho F(K) \Rightarrow \tilde{I}(p) = 1 \Rightarrow \tilde{H}(K, p) = p, \tag{21}$$

or

$$p < \rho F(K) \Rightarrow \tilde{I}(p) = 0 \Rightarrow \tilde{H}(K, p) = F(K), \tag{22}$$

or, the case of an indeterminate solution where $\tilde{I}(p)$ can be *any* feasible value,

$$p = \rho F(K) \Rightarrow 0 \leq \tilde{I}(p) \leq 1 \Rightarrow \tilde{H}(K, p) = F(K). \tag{23}$$

It is now not difficult to guess at the form of an optimal policy. Just from glancing at (21), (22), and (23), an intuitive chain of reasoning is that

$$K < \hat{K} \Rightarrow \frac{F'(K)}{F(K)} > \rho \Rightarrow p > \rho F(K) \Rightarrow \tilde{I}(p) = 1 \Rightarrow \tilde{H}(K, p) = p, \tag{24}$$

in which case we have

$$-\frac{\partial \tilde{H}}{\partial K} = 0, \tag{25}$$

and therefore the dual differential equation condition here becomes

$$\dot{p}(t) = \rho p(t), \tag{26}$$

with the 'terminal condition'

$$p(\hat{K}) = \rho F(\hat{K}). \tag{27}$$

Combining (26) with (27), yields

$$p(t) = \rho F(\hat{K})\, e^{\rho(t-\hat{K})}. \tag{28}$$

By the optimality of $\hat{K}$ for the problem (12), we must then have for $K(t) < \hat{K}$ that

$$F(K(t))\, e^{-\rho t} < F(\hat{K})\, e^{-\rho \hat{K}}. \tag{29}$$

Combining (29) with (28), we obtain the basic result that for $K(t) < \hat{K}$

$$p(t) > \rho F(K(t)). \tag{30}$$

From (30) we can say that the signal *not* to cut down the tree is that the shadow indirect value of allowing the tree to grow exceeds the direct value of harvesting it. (It is never optimal to allow a tree to grow to an age T where $F'(T)/F(T) < \rho$, but if we acquired such an 'economically overripe tree' having $T > K'$ from a nonprofit-maximizing owner, the signal to cut it down immediately would be that the shadow indirect value of allowing the tree to grow is less than the direct value of harvesting it.)

We now make some important observations about the role of the hitherto obscure parameter ν, which stands for the *opportunity value* of the land being freed for its best subsequent economic use after the tree is felled. Suppose that, instead of being concerned about the fate of an individual tree, which is the Wicksell problem, we are interested in the infinite horizon optimal rotation of a one-tree lot (or, more realistically, of a woodlot consisting of a stand of cohort trees). In this case, the opportunity value ν of the land being freed for its best subsequent economic use after the tree is felled is the present discounted value of a infinite-horizon rotation policy beginning with the replanting of a sapling to start the tree-growing cycle anew.

Suppose the parameter c now includes all costs of replanting (as well as logging, processing, and transportation costs). The competitive market value ν of the land (in forestry terminology the land expectation or the site value), right after it has been cleared and a new sapling has just been replanted, satisfies in competitive equilibrium the recursive equation

$$\nu = e^{-\rho \hat{K}}[Pf(\hat{K}) - c - \nu], \tag{31}$$

which is equivalent, after rearrangement, to

$$\nu = \frac{e^{-\rho\hat{K}}[Pf(\hat{K}) - c]}{1 - e^{-\rho\hat{K}}}. \tag{32}$$

If $\hat{K}$ is the optimal age to cut down a tree given the competitive market value ν of the site, then it seems plausible that $\hat{K}$ is chosen to maximize present site value, so that

$$\nu = \text{maximum}_K \frac{e^{-\rho K}[Pf(K) - c]}{1 - e^{-\rho K}}, \tag{33}$$

which yields the first-order condition

$$\frac{Pf'(\hat{K})}{Pf(\hat{K}) - c} = \frac{\rho}{1 - e^{-\rho\hat{K}}}. \tag{34}$$

Equation (34) is the famous *Faustmann formula* for the optimal rotation length $\hat{K}$.

Rewriting the optimization problem (33) in the equivalent form of an infinite geometric series, we have

$$\nu = \text{maximum}_K \sum_{j=1}^{\infty} e^{-jK}[Pf(K) - c]. \tag{35}$$

In effect, Equation (35) defines the *Faustmann model* of optimal forest rotation, whose solution satisfies the Faustman formula (34).

In the forestry literature, the Faustmann model and the Faustmann formula are typically contrasted with the Wicksell model and the Wicksell formula. In a serious sense, this is a false dichotomy, as Kalle has known and emphasized in his writings. The Wicksell model ostensibly takes the site value as exogenously given, often as zero, although there is evidence that Wicksell himself understood that it would be fallacious to perform comparative statics when treating ν as if it were constant. The *Faustmann and Wicksell models are identical* when proper account is taken of the market site value of forest land. We showed above that the Wicksell model with competitive market site value yields the Faustmann solution; the converse can readily be shown by substituting the Faustman formula for site value into the Wicksell formulation and confirming directly that the Wicksell-optimal cutting time satisfying condition (19) is exactly the Faustmann-optimal cutting time satisfying condition (34). The two models represent two equivalent ways of looking at optimal forestry management. The Wicksell approach emphasizes how to think about harvesting an individual tree. The Faustmann approach emphasizes how to think about the harvesting cycle of an ongoing stand of trees. So long as the opportunity value of the wood-lot is properly assessed and included, both models yield identical conclusions. It is essentially a case of looking at two sides of a single problem that more properly should be called 'the' *Faustmann–Wicksell model* of forestry management.

4. SUMMARY AND CONCLUSION

What has been shown here is that the Faustmann–Wicksell model of optimal tree harvesting has essentially the same form as the standard model of the sole owner of the fishery. Both can be seen as optimal control models linear in net investment, and both have the same form of most rapid approach to their respective stationary solution.

Of course the Faustmann–Wicksell model can be developed without optimal control theory. But applying the maximum principle to the forestry rotation problem allows us to see it as a harvesting problem of the same generic form as the standard fishery model. Thus, the two most famous models in the economics of renewable resources – the fishery and the forest – are essentially two forms of the same underlying optimal control problem. The same capital theoretic principle of most rapid approach to the stationary state underlies their solution, since both models can be expressed as optimal control problems with an objective function that is linear in net investment.

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