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On the welfare significance of national product under interest-rate uncertainty

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Abstract

This paper examines the effects of uncertainty on the relationship between comprehensive NNP and sustainable-equivalent consumption. It is shown that the classical identity continues to hold, in expectations, when the discount factor is a stochastic diffusion process taking the form of geometric Brownian motion with drift. Such a result may be useful as a point of departure for classifying the effects that various forms of uncertainty have on the welfare significance of national product. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

This paper attempts to connect the concept of 'sustainable-equivalent consumption' with the concept of 'comprehensive NNP' when there is uncertainty in the relevant interest rate. Since notions of 'sustainable-equivalent consumption' and of 'comprehensive NNP' are both open to varying interpretations, it is appropriate for the paper to begin with a verbal description of the intended conceptual entity for each idea.

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In this paper, I will define 'sustainable-equivalent consumption' to be the hypothetical annuity-equivalent constant level of consumption that would yield the same welfare as the economy actually has the potential to deliver – when discounted at the intertemporal tradeoff weights implicit in the economy's own competitive equilibrium rate of return on consumption.

'Comprehensive NNP' is typically intended to be a generalized national accounting concept that corrects for possible environmental deterioration by subtracting off from GNP not just depreciation of capital, but also appropriately calculated depletion of environmental assets. In this paper, 'comprehensive NNP' is interpreted to stand conceptually for the most inclusive possible measure of net national product, including net investments not only in traditional 'produced means of production' like equipment and structures, but also in pools of natural resources, non-standard produced means of production like human capital, and environmental assets more generally – all evaluated at their respective competitive or efficiency prices.

Now it turns out that there is a rather remarkable theoretical relationship between the two seemingly distinct concepts of 'comprehensive NNP' and 'sustainable-equivalent consumption' defined above. In the classical case of a time-autonomous technology with constant real interest rate, 'comprehensive NNP' exactly equals 'sustainable-equivalent consumption'.[1] Thus, a theoretically appealing measure of *actual present* economic activity exactly forecasts a theoretically appealing index of *potential future* power to consume.

While this sharp result can be very useful as a conceptual guide for indicating how to think about the welfare relationship between future consumption possibilities and current national income accounting measures, its practical applicability is somewhat limited by the classical assumptions of the model. A recent line of research has shown what happens when some of these limiting restrictions are loosened.[2] Thus far, however, the literature has ignored the connection between 'sustainable-equivalent consumption' and 'comprehensive NNP' in the presence of uncertainty, with the single exception of an article by Aronsson and Löfgren (1995). They show that the Hamilton–Jacobi–Bellman equation implies a mathematical relationship between the present value of expected future utility and the generalized stochastic Hamiltonian. While this relationship yields some broad insights, in the general case it is too complicated to serve as a basis for connecting an actual measurable index of future consumption with an actual measurable index of present income.

The present paper attempts to take a first step in the direction of deriving a simple expression for dealing with a 'base-case' form of uncertainty by having

[1] This is a rephrasing of the basic result in Weitzman (1976).

[2] See, for example, Kemp and Long (1982), Löfgren (1992), Aronsson and Löfgren (1993), Asheim (1996), Hartwick (1995), Nordhaus (1995), Weitzman (1997), Weitzman and Löfgren (1997) and Asheim (1996).

the discount factor evolve as a stochastic diffusion process. The central result of the paper is that a probabilistic version of the classical identity continues to hold when the discount factor follows a geometric random walk with drift. For this special form of a stochastic process, current 'comprehensive NNP' at any time always equals 'expected sustainable-equivalent consumption' over the uncertain future, viewed from that time forward.

Taken by itself, the main result might appear as a narrow, somewhat technical statement. But the statement has some significance. It provides some handle on an important issue concerning how to think about the influence of uncertainty upon the standard welfare interpretation of NNP. Unfortunately, even the simplest model of optimal growth under uncertainty introduces new complexities and technical details, which I have endeavored to keep at a minimum here by choosing the most basic formulation that yet makes the major points.[3]

The paper concludes by using the main result as a natural reference point for classifying how the connection between 'comprehensive NNP' and 'expected sustainable-equivalent consumption' is influenced by other, more general, specifications of uncertainty about the future.

2. The model

To force the problem into an analytically tractable mold, the usual abstractions are made.

First of all, it is assumed that, in effect, there is just one composite consumption good. It might be calculated as an index number with given price weights, or as a multiple of some fixed basket of goods, or, most generally, as a cardinal-utility-like aggregator function. The important thing is that the consumption level in period t can always be registered unambiguously by the single number $C(t)$. Thus, the paper effectively assumes away all of the problems that might be associated with constructing an 'ideal measure' of the standard of living akin to a utility function.[4] Purging consumption of the index number problem will allow us to focus more sharply on the general meaning and significance of combining it with net investment when there is uncertainty in the interest rate.

The notion of 'capital' used here is meant to be quite a bit more general than the traditional produced means of production like equipment and structures. Most immediately, pools of natural resources are considered to be capital.

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[3] Even so, the paper requires some previous exposure to stochastic diffusion processes in optimal growth models, since otherwise this article would, of necessity, become a book.

[4] Nordhaus (1995), in his section entitled 'What is Consumption?', contains a relevant discussion of the basic issues involved. See also the appropriate part of the report on how to calculate the cost of living by the 'Boskin Commission' contained in Boskin et al. (1996).

Human capital should also be included, to the extent we know how to measure it. Under a very broad interpretation, environmental assets generally might be treated as a form of capital.[5]

Suppose that altogether there are n capital goods, including stocks of natural resources. The stock of capital of type j ($1 \leq j \leq n$) in existence at time t is denoted $K_j(t)$, and its corresponding net investment flow is $I_j(t) = \dot{K}_j(t)$. The n-vector $\boldsymbol{K} = \{K_j\}$ denotes all capital stocks, while $\boldsymbol{I} = \{I_j\}$ stands for the corresponding n-vector of net investments. Note that the net investment flow of a non-renewable natural capital like proved oil reserves would be negative if the overall extraction rate exceeds the discovery and development of new fields. Generally speaking, investment in environmental capital should be viewed as negative whenever the underlying asset is being depleted more rapidly than it is being replaced.

The relationship between C, $\boldsymbol{K}$, and $\boldsymbol{I}$ is presumed given by

$$C = \Phi(\boldsymbol{K}; \boldsymbol{I}), \tag{1}$$

where $\Phi(\boldsymbol{K}; \boldsymbol{I})$ is a strictly-concave twice-continuously-differentiable function defined over all $\boldsymbol{K}$ and $\boldsymbol{I}$, with

$$\frac{\partial \Phi}{\partial K_j} > 0 \quad \forall j \tag{2}$$

and

$$\frac{\partial \Phi}{\partial I_j} < 0 \quad \forall j. \tag{3}$$

The 'reduced-form' expression (1) is employed here, along with very strong smoothness and strict-convexity assumptions, in order to guarantee interior solutions – which makes the corresponding duality theory much simpler. In effect, the function $\Phi(\boldsymbol{K}; \boldsymbol{I})$ is defined over *all* $\boldsymbol{K}$ and $\boldsymbol{I}$ – implicitly restricting the domain to disallow negative arguments by the contrivance of having the corresponding value of Eq. (1) approach (rapidly) to minus infinity.

Let $A(t)$ be the relevant discount factor applicable to time t. More specifically, $-\dot{A}(t)/A(t)$ is the instantaneous own rate of return at time t on the numeraire consumption good $C(t)$ – meaning it represents how much *extra* consumption could be enjoyed next period by abstaining from one unit of consumption this period.

Discount-factor uncertainty is introduced as follows. Suppose that $A(t)$ evolves as a continuous-time stochastic diffusion process[6] of the standard

[5] Mäler (1991) includes a discussion of some of the relevant issues here.

[6] See, for example, Dixit and Pindyck (1994) and the further references cited there. The present paper assumes some basic familiarity with continuous-time stochastic diffusion processes.

Itô form:

$$\mathrm{d}A = \alpha(\boldsymbol{K}, A)\,\mathrm{d}t + \beta(\boldsymbol{K}, A)\,\mathrm{d}Z\,, \tag{4}$$

where $\mathrm{d}Z$ is the increment of a Wiener process, while the instantaneous drift rate $\alpha(\boldsymbol{K}, A)$ and the instantaneous variance rate $\beta(\boldsymbol{K}, A)$ are known functions of the state variables $\boldsymbol{K}$ and A.

The model-building 'workhorse' of deterministic exponential discounting at a constant rate can now be seen as a 'base case' of Eq. (4) where $\alpha(\boldsymbol{K}, A) = -rA$ and $\beta(\boldsymbol{K}, A) = 0$, with r being a positive constant representing the exogenously given own rate of return to consumption. At the end of the day, the only analytically tractable formulation that can be fully analyzed in this paper will be an extension of the deterministic 'workhorse' specification to cover the case of multiplicative uncertainty with a constant geometric variance: $\alpha(\boldsymbol{K}, A) = -rA$, $\beta(\boldsymbol{K}, A) = \sigma A$, where the variance parameter σ is any non-negative constant. However, at this stage of the exposition it is conceptually advantageous, for several reasons, to frame the theoretical structure around the Itô-process foundation represented by the more general form of Eq. (4).

Let P_j stand for the price of investment good j relative to a normalized price of one for the single consumption good. Let $\boldsymbol{P}$ be the corresponding n-vector of all normalized investment-good prices.

At any time t, the state of the economy is described by $\boldsymbol{K}(t)$, $A(t)$.

Let $\mathrm{E}_t[\,\cdot\,]$ stand for the future-looking expectation operator defined over the stochastic diffusion process (4) from time t forward, given all relevant information available at that time.

Wherever it is otherwise clear from the context, my symbolic notation will drop explicit dependence on time for ease of exposition.

A *dynamic stochastic competitive equilibrium* in this model is a set of functions $C^*(\boldsymbol{K}, A)$, $\boldsymbol{I}^*(\boldsymbol{K}, A)$, $\boldsymbol{P}^*(\boldsymbol{K}, A)$, defined over all $\boldsymbol{K}$, A, and a trajectory of state variables $\boldsymbol{K}(t)$, $A(t)$ realized for all times $t \geq 0$, which are simultaneous solutions of the following system:

$$C^*(\boldsymbol{K}, A) = \Phi(\boldsymbol{K}; \boldsymbol{I}^*(\boldsymbol{K}, A)), \tag{5}$$

$$\dot{K}_j = I_j^*(\boldsymbol{K}, A) \quad \forall j, \tag{6}$$

$$P_j^*(\boldsymbol{K}, A) = \frac{\partial \Phi}{\partial I_j}(\boldsymbol{K}; \boldsymbol{I}^*(\boldsymbol{K}, A)) \quad \forall j, \tag{7}$$

$$\mathrm{d}A = \alpha(\boldsymbol{K}, A)\,\mathrm{d}t + \beta(\boldsymbol{K}, A)\,\mathrm{d}Z, \tag{8}$$

$$\mathrm{E}[\dot{P}_j^*(\boldsymbol{K}, A)] + P_j^*(\boldsymbol{K}, A)\frac{\mathrm{E}[\dot{A}]}{A} = -\frac{\partial \Phi}{\partial K_j}(\boldsymbol{K}; \boldsymbol{I}^*(\boldsymbol{K}, A)) \quad \forall j, \tag{9}$$

$$\lim_{t \to \infty} \mathrm{E}_t[A\, P_j^*\, K_j^*] = 0 \quad \forall j, \tag{10}$$

and which satisfy the initial conditions

$$K(0) = K_0, \quad \text{given} \tag{11}$$

and

$$A(0) = A_0, \quad \text{given.} \tag{12}$$

Temporarily setting aside issues of existence and uniqueness, it should be fairly clear that the equation system (5)–(12) represents a relatively straightforward extension of the standard dynamic competitive equilibrium conditions from deterministic capital theory to a situation where the discount factor $A(t)$ is an Itô process of the form given by Eq. (4). Because the presence of uncertainty necessitates that expectations be taken over all possible realizations of the white-noise random variable $\mathrm{d}Z$, and because the actually realized trajectories of the state variables cannot be known in advance, it is required to define a stochastic equilibrium in terms of $C^*(\cdot)$, $\boldsymbol{I}^*(\cdot)$, $\boldsymbol{P}^*(\cdot)$ expressed as contingent functions of the state variables $\boldsymbol{K}$ and A.[7]

A stochastic equilibrium is, essentially, an equilibrium in commonly shared 'rational' expectations over an infinite horizon. Enormous informational requirements are thereby implied. (In effect, the situation is '*as if*' infinitely-long-lived agents possess infinitely long price lists.) This formulation is used not so much because it is literally believable, as because no better alternative description is available.

Eqs. (5) and (6) are self-evident consistency conditions. Eq. (7) just states that at all times the perfectly competitive prices of investment goods are exactly equal to the corresponding marginal rates of transformation.

The condition given by Eq. (8) represents the equation of motion for the underlying continuous-time stochastic diffusion process that ultimately drives all of the uncertainty in the system.

Recalling that, at time t, $\boldsymbol{P}_j^*(\boldsymbol{K}(t), A(t))$ is the relative price of investment good j, while $-\dot{A}(t)/A(t)$ is the instantaneous own rate of return on the numeraire consumption good, Eq. (9) then represents the appropriate rational-expectations stochastic version of the basic intertemporal efficiency condition of a perfectly competitive capital market. If Eq. (9) did *not* hold in some rational-expectations equilibrium configuration of the economy, then a trader could expect, on average, to make positive pure profits by reallocating capital stocks,

[7] Some of this is explained in Stokey and Lucas (1989), which, while it does not explicitly cover continuous-time stochastic diffusion processes, contains a very useful descriptive overview and a rigorous treatment of the nature and properties of a dynamic stochastic competitive equilibrium in the analogous discrete-time case.

thereby changing the initial configuration and proving that the economy could not have been in a competitive equilibrium in the first place.[8]

Eq. (10) is the standard infinite-horizon transversality condition in expectations. If Eq. (10) failed to hold, the expected present discounted value of some capital stocks would not go to zero in the limit and a trader could expect, on average, to make increased profits by reallocating more efficiently such capital stocks over time and across sectors.

Eqs. (11) and (12) are merely initial consistency conditions.

Given that a dynamic stochastic competitive equilibrium exists, the value of 'Comprehensive NNP' (at time zero) is then defined to be

$$Y^* \equiv C^*(\boldsymbol{K}(0), A(0)) + \boldsymbol{P}^*(\boldsymbol{K}(0), A(0)) \cdot \boldsymbol{I}^*(\boldsymbol{K}(0), A(0)). \tag{13}$$

Having defined rigorously the concept of 'Comprehensive NNP' in this model, we turn now to the companion task of defining rigorously the concept of 'expected sustainable-equivalent consumption'.

3. Expected sustainable equivalence

For any given investment-plan function $\boldsymbol{I}_\theta(\boldsymbol{K}, A)$, let $C_\theta(t)$ stand, heuristically, for a feasible value of consumption at time t with its (implicit) corresponding probability of being realized. Intuitively, 'expected sustainable-equivalent consumption' is intended to be the largest possible hypothetically constant 'sustainable-equivalent' consumption level $\bar{C}$ that satisfies the condition

$$\mathrm{E}\left[\int_0^\infty \bar{C} A(t)\,\mathrm{d}t\right] = \mathrm{E}\left[\int_0^\infty C_\theta(t)\, A(t)\,\mathrm{d}t\right]. \tag{14}$$

Somewhat more formally, consider $\boldsymbol{I}_\theta$ to be an n-vector of control variables whose relevant control set Θ is the collection of all possible piecewise-continuous functions of the form $\{\boldsymbol{I}_\theta(\boldsymbol{K}, A)\}$. Then 'expected sustainable-equivalent consumption', denoted $\bar{C}^*$, is defined to be the maximized value – over all possible piecewise-continuous functional specifications (i.e., all θ belonging to the class Θ) – of the following stochastic optimal control problem:

$$\bar{C}^* \equiv \underset{\theta \in \Theta}{\text{maximum of}} \tag{15}$$

[8] Note here that the no-arbitrage equilibrium conditions (7) and (9) are written in 'marginal' form, while the *interpretation* justifying them is implicitly told in the form of a finite-difference variational inequality. These two approaches are essentially equivalent here, due to the assumed smoothly strong concavity of Eq. (1), which forces an interior solution.

$$\frac{\mathrm{E}\left[\int_0^{\infty} \Phi(\boldsymbol{K}; \boldsymbol{I}_\theta)\, A(t)\, \mathrm{d}t\right]}{\mathrm{E}\left[\int_0^{\infty} A(t)\, \mathrm{d}t\right]} \tag{16}$$

subject to

$$\dot{\boldsymbol{K}} = \boldsymbol{I}_\theta\,, \tag{17}$$

$$\mathrm{d}A = \alpha(\boldsymbol{K}, A)\,\mathrm{d}t + \beta(\boldsymbol{K}, A)\,\mathrm{d}Z \tag{18}$$

and satisfying the initial conditions

$$\boldsymbol{K}(0) = \boldsymbol{K}_0, \quad \text{given}\,, \tag{19}$$

$$A(0) = A_0, \quad \text{given}\,. \tag{20}$$

For now it is simply assumed that the stochastic optimal control problem given by Eqs. (15)–(20) is well defined, so that a solution value $\bar{C}^*$ exists.

The primary purpose of the paper is to analyze the relationship between $\bar{C}^*$ and Y^*. To accomplish this aim, a considerably more analytically tractable structure must be imposed on the coefficients of the stochastic diffusion equation (18).

4. The main result

Having formally defined 'comprehensive NNP' to be the variable Y^* satisfying Eq. (13) with the auxiliary conditions (5)–(12), and having formally defined 'expected sustainable-equivalent consumption' $\bar{C}^*$ to be the solution of Eqs. (15)–(20), we are finally ready to pose rigorously the natural question which this paper is leading up to. What is the connection here between 'comprehensive NNP' and 'expected sustainable-equivalent consumption'? To answer this question sharply, we restrict the underlying stochastic diffusion process (4), which defines the evolution of the discount factor, to be a geometric random walk with drift.

The following theorem is the principal result of the paper.

Theorem. Assume that the stochastic diffusion process (4) *takes the form of geometric Brownian motion with drift*

$$\alpha(\boldsymbol{K}, A) = -rA\,, \tag{21}$$

$$\beta(\boldsymbol{K}, A) = \sigma A\,, \tag{22}$$

where r and σ are given positive constants.

Further assume that the following stochastic calculus-of-variations problem is well defined for all possible initial values of K and A:[9]

$$Maximize\ \mathrm{E}\left[\int_0^\infty \Phi(\boldsymbol{K};\dot{\boldsymbol{K}})\,A(t)\,\mathrm{d}t\right] \tag{23}$$

subject to Eqs. (4), (21) *and* (22), *and the initial conditions*

$$\boldsymbol{K}(0) = \boldsymbol{K}\,, \tag{24}$$

$$A(0) = A\,. \tag{25}$$

Then there exists a unique a.e. solution of Eqs. (5)–(13), *and the problem given by Eqs.* (15)–(20) *is well defined, and*

$$Y^* = \bar{C}^*\,. \tag{26}$$

Proof. Because $\Phi(\boldsymbol{K};\boldsymbol{I}(\equiv\dot{\boldsymbol{K}}))$ is an everywhere-defined strictly concave smoothly differentiable function, it follows that the solution of Eqs. (23)–(25) is interior and a.e. unique; also, the necessary and sufficient first-order optimality conditions are the stochastic Euler equations

$$A\frac{\partial\Phi}{\partial\boldsymbol{K}} = \mathrm{E}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(A\frac{\partial\Phi}{\partial\boldsymbol{I}}\right)\right] \tag{27}$$

holding a.e. along an optimal trajectory, simultaneously with the transversality condition

$$\lim_{t\to\infty}\ \mathrm{E}_t\left[A\frac{\partial\Phi}{\partial\boldsymbol{I}}\boldsymbol{K}\right] = 0\,. \tag{28}$$

The existence of an a.e. unique solution of Eqs. (5)–(13) follows as a translation into general-equilibrium notation of the above duality conditions (27) and (28) for the problem given by Eqs. (23)–(25).[10]

[9] For the sake of conciseness, throughout the proof, I employ liberally various conventionally accepted shorthand notations that are not weighed down by a truly rigorous mathematical statement of all aspects of this well-studied kind of problem. A mathematically rigorous general treatment of the continuous-time stochastic optimal control problem is contained in Fleming and Rishel (1975), or Krylov (1980).

[10] The intuitively plausible statements and interpretations presented to this point in the 'proof' are actually not rigorously proved here. This omission is merely to save on space, as a fully rigorous proof is surprisingly messy. A rigorous treatment of a closely related problem yielding the analogue of Eqs. (27) and (28) is contained in Krylov (1980) or Fleming and Rishel (1975). Stokey and Lucas (1989) contains a rigorous treatment of the analogous isomorphism between duality conditions and competitive stochastic equilibrium for the analogous discrete-time version.

For the meaningfully formulated problem given by Eqs. (23)–(25), define the dynamic-programming state-evaluation function

$$V(\boldsymbol{K}, A) \tag{29}$$

to be the value of the maximized objective (23) as a function of the initial conditions (24) and (25).

We turn then to analyzing the value function given by Eq. (29). Again using the fact that $\Phi(\boldsymbol{K}; I(=\dot{\boldsymbol{K}}))$ is an everywhere-defined strictly concave smoothly differentiable function, and the assumption that Eqs. (4), (21)–(25) describe a meaningful stochastic optimization problem for all possible initial conditions, we conclude that $V(\boldsymbol{K}, A)$ is everywhere-well-defined, strictly concave in $\boldsymbol{K}$, and a.e. differentiable.

With the specification given by Eqs. (4), (21) and (22), a direct calculation shows that

$$\mathrm{E}\left[\int_0^\infty A(t)\,\mathrm{d}t\right] = \frac{A(0)}{r}. \tag{30}$$

Since the problems given by Eqs. (23)–(25) and Eqs. (15)–(20) differ only by a multiplicative positive constant in the objective function, it follows that, from the same initial conditions, the a.e. unique solution of the well-defined problem given by Eqs. (23)–(25) must be a.e. identical to the solution of problem given by Eqs. (15)–(20) and, hence, the latter problem is well defined. Furthermore, Eq. (30) implies that the two essentially identical problems share the following relationship between the optimized values of their respective objective functions:

$$V(\boldsymbol{K}(0), A(0)) = \frac{A(0)}{r}\,\bar{C}^*. \tag{31}$$

Now the particular linear-multiplicative structure of Eqs. (4), (21) and (22), as it appears in the stochastic optimal control problem given by Eqs. (23)–(25), implies (using a constructive argument based upon the definition of Eq. (29)) for any positive numbers μ, μ', A, A' and for all $\boldsymbol{K}$ it must be true that

$$V(\boldsymbol{K}, \mu A) \geq \mu V(\boldsymbol{K}, A), \tag{32}$$

$$V(\boldsymbol{K}, \mu' A') \geq \mu' V(\boldsymbol{K}, A'). \tag{33}$$

Then picking $\mu' = 1/\mu$, $A' = \mu A$, it follows from Eqs. (32) and (33) that for all μ, A, $\boldsymbol{K}$ the following equation must hold:

$$V(\boldsymbol{K}, \mu A) = \mu V(\boldsymbol{K}, A). \tag{34}$$

Condition (34) implies that the value function here must be of the form

$$V(\boldsymbol{K}, A) = AF(\boldsymbol{K}), \tag{35}$$

where the function $F(\boldsymbol{K})$ is everywhere-defined, strictly concave, and a.e. differentiable.

The Hamilton–Jacobi–Bellman partial differential equation of optimal stochastic control theory[11] applied to the problem defined by Eqs. (4), (23)–(25) and (29) is

$$-\frac{\partial V}{\partial A}\alpha = \max_{\boldsymbol{I}}\left[A\,\Phi(\boldsymbol{K};\boldsymbol{I}) + \frac{\partial V}{\partial \boldsymbol{K}}\cdot\boldsymbol{I}\right] + \frac{1}{2}\beta^2\frac{\partial^2 V}{\partial A^2}. \tag{36}$$

Plugging Eq. (35) into Eq. (36), taking the appropriate partial derivatives, and making use of the special structure (21) and (22) yields, after some manipulation, the equation

$$rF = \Phi - \frac{\partial \Phi}{\partial \boldsymbol{I}}\dot{\boldsymbol{K}}. \tag{37}$$

Now substitute Eqs. (31) and (35), and Eqs. (5)–(7) and (13) into Eq. (37). Collecting terms, we have then shown that, at time $t = 0$, Eq. (37) becomes the following condition:

$$\bar{C}^* = Y^*. \tag{38}$$

This concludes the proof. □

5. Discussion

The main result (Eq. (38)) means that the presence of uncertainty per se does not undo the connection between a theoretically appealing index of current actual economic activity and a theoretically appealing index of future potential to consume. The classical identity between 'comprehensive NNP' and 'sustainable-equivalent consumption', appropriately defined, continues to hold in expectations when the discount factor is a stochastic diffusion process taking the form of geometric Brownian motion with drift.

Heuristically speaking, what drives this result is a critical assumption that the only way uncertainty enters the production process is by multiplicative shifts of ex-post consumption that essentially leave unaltered the relevant tradeoffs in the ex-ante expected future economy. While a characterization like this may make the main point seem deceptively simple, the result is, however, far from trivial to formulate or to prove, as the paper itself bears witness.

Such a result leads naturally to two further questions:

1. How appropriate is the stochastic process (4), (21) and (22) as an approximate description of real-world rates of return to consumption?

[11] See, e.g., Kamien and Schwartz (1991, Eq. (18), p. 268) or, more rigorously, Fleming and Rishel (1975, Eq. (2.5), p. 154) or Krylov (1980, Eq. (6), p. 4).

2. What happens to the relation between ‘comprehensive NNP’ and ‘expected sustainable-equivalent consumption’ for other specifications of uncertainty?

In this concluding section I try briefly to address each question in turn.

The ‘own rate of return to consumption’ is a conceptual measure of how much *extra* consumption could be enjoyed next year by postponing until then the enjoyment of a unit of consumption from this year, all other things being equal. The economic entity corresponding most closely to this concept is, arguably, the annual after-tax real return on capital – because it defines (approximately) the relevant intertemporal consumption tradeoff faced by the average citizen in deciding how much to save.

Without having to rely on formal econometric testing, I think it can be stated as a stylized fact that the rate of return to capital in the advanced industrial economies has been essentially trendless over the very long run. If forced to name a figure, the round number of about 5% per annum could be given as a plausible candidate for the average annual after-tax real return on capital in the US over the last century or so.[12] During some periods this representative real interest rate might be measured a bit lower, say 3%, while in some other, equally likely, periods it might be measured higher, say 7%. But, on average, over many decades, it seems like not a bad approximation to say that the own rate of return to consumption is essentially trendless. This sounds not unlike the specification (4), (21) and (22), approximated in discrete-time periods of length Δt as

$$r_t = r + \varepsilon_t\,, \tag{39}$$

where $r_t \equiv -(\Delta A/\Delta t)/A$ is the finite-difference measured real interest rate observed over period t, and $\varepsilon_t \equiv -\sigma(\Delta Z/\Delta t)$ is an i.i.d. zero-mean normally distributed random variable.

As for what happens to the relationship between ‘comprehensive NNP’ and ‘expected sustainable-equivalent consumption’ under other, more general, forms of uncertainty, this appears to be too mathematically intractable a problem to yield any kind of neat analytical solution. However, a useful reduced-form summary characterization can be given here.

The formulation (4), (21) and (22) of uncertainty in this paper represents, in effect, a specification of purely multiplicative welfare shocks that are

[12] Of course, it depends on exactly what is being measured, but this round number of 5% could be justified by reference to, e.g., Nordhaus (1995), or Jorgenson (1994). I realize that there is an extensive literature on the appropriate social discount rate, which represents a set of issues I am sidestepping here. If the appropriate discount rate is different from the competitive own rate of return on consumption, then in principle the formulas of this paper could all be redone using corresponding shadow or efficiency prices – although I would hate to be the one who has to make such recalculations in practice.

'unbiased' in the technical sense that they do not have any curvature influence on the value function

$$\frac{\partial^2 V}{\partial A^2} = 0\,. \tag{40}$$

It is, so to speak, because condition (40) holds in the fundamental dynamic programming equation (36) that the classical identity between 'comprehensive NNP' and 'expected sustainable-equivalent consumption' is preserved in the presence of uncertainty.

Any 'biased' specification having a curvature effect of the form

$$\frac{\partial^2 V}{\partial A^2} < 0 \tag{41}$$

would cause 'comprehensive NNP' to be an overestimate of 'expected sustainable-equivalent consumption'.

The opposite 'bias' in specification, namely

$$\frac{\partial^2 V}{\partial A^2} > 0 \tag{42}$$

would make 'comprehensive NNP' be an underestimate of 'expected sustainable-equivalent consumption'.

Thus, even if the specification (4), (21) and (22) is held to be an inappropriate description for some settings, it may still be found useful as a classificatory device. The analysis of this paper might then be seen as representing a natural first step toward understanding in a more general situation the issue addressed here – how uncertainty about the future affects the relationship between Green-NNP-like comprehensive indices of present economic activity and sustainability-like measures of future power to consume.

References

Aronsson, T., Löfgren, K.-G., 1993. Welfare measurement of technological and environmental externalities in the Ramsey growth model. Natural Resources Modeling 7 (1), 1–14.

Aronsson, T., Löfgren, K.-G., 1995. National product related welfare measures in the presence of technological change, externalities and uncertainty. Environment and Resource Economics 5, 321–332.

Asheim, G.B., 1996. The Weitzman foundation of NNP with non-constant interest rates. Scandinavian Journal of Economics, forthcoming.

Boskin, M. et al., 1996. Toward measuring the cost of living. Mimeo, Washington, DC.

Dixit, A.K., Pindyck, R.S., 1994. Investment under Uncertainty. Princeton University Press, Princeton, NJ.

Fleming, W.H., Rishel, R.W., 1975. Deterministic and Stochastic Optimal Control. Springer, New York.

Hartwick, J.M., 1995. Constant consumption paths in open economies with exhaustible resources. Review of International Economics 3, 275–283.

Jorgenson, D.W., 1994. Investment and economic growth. The Simon Kuznets Lectures, Yale University, 11–13 November 1994.

Kamien, M.I., Schwartz, N.L., 1991. Dynamic Optimization. 2nd ed., North-Holland, New York.

Kemp, M.C., Long, N.V., 1982. On the evaluation of social income in a dynamic economy. In: Feiwel, G.R. (Ed.), Samuelson and Neoclassical Economics, Chapter 12. Kluwer-Nijhoff, Dordrecht.

Krylov, N.V., 1980. Controlled Diffusion Processes. Springer, New York.

Löfgren, K.-G., 1992. Comment on C.R. Hulten 'Accounting for the wealth of nations: The net versus gross output controversy and its ramifications'. Scandinavian Journal of Economics 94 (Suppl.), 25–28.

Mäler, K.-G., 1991. National accounts and environmental resources. Environmental and Resource Economics 1, 1–15.

Nordhaus, W.D., 1995. How should we measure sustainable income? Cowles Foundation discussion paper no. 1101, Yale University, New Haven, CT.

Stokey, N.L., Lucas, R.E., 1989. Recursive Models in Economic Dynamics. Harvard University Press, Cambridge, MA.

Weitzman, M.L., 1976. On the welfare significance of national product in a dynamic economy. Quarterly Journal of Economics 90, 156–162.

Weitzman, M.L., 1997. Sustainability and technical progress. Scandinavian Journal of Economics 99, 1–13.

Weitzman, M.L., Löfgren, K.-G., 1997. On the welfare significance of Green accounting as taught by parable. Journal of Environmental Economics and Management 32, 139–153.