

Computation of Convex Hull Prices in Electricity Markets with Non-Convexities using Dantzig-Wolfe Decomposition

Panagiotis Andrianesis, Michael C. Caramanis, and William W. Hogan

Abstract—The presence of non-convexities in electricity markets has been an active research area for about two decades. The — inevitable under current marginal cost pricing — problem of guaranteeing that no truthful-bidding market participant incurs losses in the day-ahead (DA) market is addressed in current practice through make-whole payments *a.k.a.* uplift. Alternative pricing rules have been studied to deal with this problem. Among them, Convex Hull (CH) prices associated with minimum uplift have attracted significant attention. Several US Independent System Operators (ISOs) have considered CH prices but resorted to approximations, mainly because determining exact CH prices is computationally challenging, while providing little intuition about the price formation rational. In this paper, we describe CH price estimation problem by relying on Dantzig-Wolfe decomposition and Column Generation. Moreover, the approach provides intuition on the underlying price formation rational. A test bed of stylized examples elucidate an exposition of the intuition in the CH price formation. In addition, a realistic ISO dataset is used to suggest scalability and validate the proof-of-concept.

Index Terms—Convex-Hull Pricing, Dantzig-Wolfe Decomposition, Column Generation, Electricity Market Non-Convexities.

I. INTRODUCTION

MARGINAL cost pricing based on spot pricing under convexity assumptions [1], [2] has been the standard practice in organized electricity markets. However, in the presence of non-convexities (mainly due to unit commitment costs and technical constraints, e.g., minimum output requirements), marginal cost pricing from a restricted convex subproblem cannot guarantee support of the solution where market participants recover their as-bid production costs. There may be no market-clearing prices for the economically efficient solution. This problem has been typically dealt with the provision of “uplift” side-payments to market participants, to make them whole. Over the last two decades, motivated mainly by the electricity market paradigm, pricing in markets with non-convexities has attracted significant attention. Approaches have ranged from standard marginal cost pricing with recovery mechanisms — e.g., [3]–[5], to mechanism designs that “minimize”, internalize or, sometimes, even eliminate uplifts — e.g., [6]–[9]. A critical review of pricing rules employed in markets with non-convex costs is provided in [9].

Recently, FERC (Federal Energy Regulatory Commission) initiated a discussion on price formation [10], with several US Independent System Operators (ISOs) exploring Convex Hull (CH) prices, — first suggested in [7] — and their approximations, often called Extended Locational Marginal Prices (ELMPs) [10]–[14]. CH pricing was introduced in [7] as the CH of the aggregate cost function, i.e., the convex function that is closest to approximating the aggregate cost function from below. It is equivalently derived by Lagrangian dualization or CH relaxation, i.e., considering the CH of individual components, as pointed out already in [7] — see also the interesting discussion in [15].

Interestingly, up until the early 2000s, the Unit Commitment (UC) problem itself was traditionally solved using Lagrangian Relaxation (LR), first by vertically integrated utilities, and later by ISOs that “inherited” LR solvers — possibly the only commercial option at the time — in the first market implementations. Taking advantage of Mixed Integer Linear Programming (MILP) solver advances, in 2005, PJM replaced LR with MILP achieving cost savings of about \$500M/year; by 2017, all US ISOs switched to MILP with estimated cost savings of more than \$2B/year [16]. Currently, the Day-Ahead (DA) market is based on a sequence of Security Constrained UC (SCUC) and Security Constrained Economic Dispatch (SCED), followed after closure of the DA Market by an ISO executed Reliability Unit Commitment (RUC) process. DA prices are finally obtained by a pricing run with fixed unit commitments, either before (e.g., PJM) or after (e.g., NYISO) the RUC process. Some ISOs proceeded to ELMPs, by relaxing integrality constraints for fast-start units and using ELMPs as a proxy to CH prices [17]. Integer Relaxation (IR) of the MILP UC problem is considered in [15] as a simple method that produces good approximations — sometimes even exact — to the (equivalent) Lagrangian Dual (LD) and CH relaxations, which, although referred to as “an ideal solutions,” can remain computationally prohibitive.

Several works have attempted to overcome the computational difficulties by relying on two main approaches. An early approach applied sub-gradient methods [18]–[21], whereas a later and methods focused on identifying the CH of individual generators through convex primal formulations [22] — also including an AC Optimal Power Flow setting [23], extended formulations [24]–[26], a network reformulation [27], and Benders decomposition leveraging advances in thermal generator CH formulations [28]. Despite the aforementioned efforts, two important barriers to CH price implementation remain: (i)

P. Andrianesis and M. C. Caramanis are with Boston University, Boston, MA: panosa@bu.edu, mcaraman@bu.edu. W. W. Hogan is with Harvard University, Cambridge, MA, william_hogan@hks.harvard.edu. Research partially supported by ARPA-E PERFORM Grant.

computational challenges, and (ii) opacity of their properties [13]. An insightful work on the latter, [29], uses representative stylized examples to illustrate some arguably counter intuitive CH price properties.

Our aim here is to address the computational challenge while providing intuition for the CH price formation. Our approach is inspired by related experience in solving large-scale optimization problems in other application domains, in particular crew scheduling [30]. The Operations Research theory underpinnings date back to the 60's and the seminal work of Dantzig and Wolfe [31] on Generalized Linear Programming (LP), *a.k.a.* Dantzig-Wolfe (D-W) decomposition or Column Generation (CG). D-W decomposition can be also viewed as a problem characterization rendering the problem suitable to a CG algorithm. The relationship of Generalized LP as a solution method for the LD dates also back to the 70's — see e.g., [32], [33]. Around the same time, a CG process was also proposed for approximating competitive equilibria in a piecewise linear economy [34]. Later, around the 90's, the work of [35] on “branch-and-price” — among others [36] — positioned CG as a powerful solution method for huge integer problems, with two common illustrative applications: the generalized assignment problem and crew scheduling. Since then, CG has been successfully applied to large scale integer programming [37], and is possibly the only commercially available option for scheduling problems in the airline industry.

Our analysis draws from CH price “first principles” [7], and our main focus has two elements. First, we provide a D-W characterization of the UC problem, without changing any of the original functions or constraints, whose LP relaxation is equivalent to the solution of the LD of the original MILP formulation; we then present a CG algorithm to solve the LP relaxation, and derive the CH prices. Second, we illustrate the applicability of the approach to (i) stylized examples in [29] that provide intuition into the CH price formation, (ii) a more detailed ramp-constrained example from [26], to show that our approach can handle features that could not be practically addressed without reformulation of the UC problem, and (iii) a large-scale ISO dataset [38] with about 1000 generators, thus illustrating the potential scalability.

The remainder of the paper is organized as follows. In Section II, we define CH prices for a stylized UC problem. In Section III, we present a D-W reformulation of the problem and sketch the CG solution algorithm. In Section IV, we detail the application to stylized examples, and in Section V, we demonstrate the computational tractability on realistic test cases. Lastly, Section VI concludes and provides directions for further research.

II. DESCRIPTION OF CH PRICES

To fix ideas, but without loss of generality, we assume that $x_{i,t}$ refers to power output (energy for a dispatch interval) and $y_{i,t}$ refers to the discrete variables (e.g., the on/off status) of unit i , at time period t . Let \mathcal{I} denote the set of generation units, and $\mathcal{T} = \{1, 2, \dots, T\}$ the set of time periods, where T is the length of the optimization horizon. In what follows, we refer

to t as an hour for simplicity. For brevity, and occasionally with some abuse of notation, we use $\mathbf{x}_i, \mathbf{y}_i$, to denote vectors of unit i comprising the respective variables, $x_{i,t}, y_{i,t}, \forall t \in \mathcal{T}$; at a more abstract level, we use \mathbf{x} and \mathbf{y} as vectors comprising the respective variables, $x_{i,t}$ and $y_{i,t}, \forall i \in \mathcal{I}$, and $\forall t \in \mathcal{T}$. Let also D_t denote the demand for energy at hour t . A basic UC problem is formulated as follows:

$$\min_{\mathbf{x}, \mathbf{y}} f(\mathbf{x}, \mathbf{y}) = \sum_{i \in \mathcal{I}} f_i(\mathbf{x}_i, \mathbf{y}_i), \quad (1a)$$

$$\text{subject to:} \quad \sum_{i \in \mathcal{I}} x_{i,t} = D_t, \quad \forall t \in \mathcal{T}, \quad (1b)$$

$$(\mathbf{x}_i, \mathbf{y}_i) \in \mathcal{Z}_i, \quad \forall i \in \mathcal{I}. \quad (1c)$$

The objective function (1a) minimizes the aggregate commitment and dispatch costs for all units, with $f_i(\mathbf{x}_i, \mathbf{y}_i)$ denoting the respective cost for unit i over the entire optimization horizon. Constraints (1b) represent the power balance for all hours. Constraints (1c) represent all unit specific constraints, with \mathcal{Z}_i denoting the set of constraints for unit i for the entire horizon. We use \mathcal{Z}_i to ease the exposition for reasons that will soon become apparent.

Constraints (1b) can be straightforwardly extended to include all system constraints (e.g., power balance, reserve requirements, transmission constraints, etc.). Second, although throughout the paper we use for simplicity the term UC, we essentially refer to a UC and Economic Dispatch problem in the broad sense. Our analysis can also cater to current ISO needs and practices (e.g., SCUC, SCED, RUC). Third, we make no assumptions on constraints (1c). These include standard unit specific constraints (capacity, ramps, etc.), as well as integrality constraints for \mathbf{y} variables. Fourth, constraints (1c) define a *feasible schedule* for unit i . It may not be evident at this point, but this last remark is arguably the most important and key to our analysis.

Standard marginal cost pricing relates to the dual variable of constraint (1b), after fixing discrete variables \mathbf{y} to their optimal values. Enforcing such explicit equality constraints is used in [3] to price integralities, under standard marginal cost pricing, and immediately derive the make-whole payments, in what has been called the “IP pricing;” — see also [9]. ELMPs are currently derived by relaxing integer variables of only the fast-start units. In general, the IR of the MILP UC problem (1) yields less tight convex approximations. Revisiting [7], there are two equivalent methods to derive CH prices. The first method prices out complicating system constraints through Lagrangian dualization. Dualizing constraint (1b), with λ_t the respective dual and $\boldsymbol{\lambda}$ an appropriate vector, CH prices are obtained by the solution of the LD:

$$\max_{\boldsymbol{\lambda}} q(\boldsymbol{\lambda}), \quad (2a)$$

where $q(\boldsymbol{\lambda})$ is the dual function of the Lagrangian, $\mathcal{L}(\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda})$,

$$q(\boldsymbol{\lambda}) = \inf_{(\mathbf{x}_i, \mathbf{y}_i) \in \mathcal{Z}_i, \forall i \in \mathcal{I}} \mathcal{L}(\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda}), \quad (2b)$$

$$\mathcal{L}(\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda}) = \sum_{i \in \mathcal{I}} f_i(\mathbf{x}_i, \mathbf{y}_i) - \sum_{t \in \mathcal{T}} \lambda_t \left(\sum_{i \in \mathcal{I}} x_{i,t} - D_t \right). \quad (2c)$$

The second method is based on describing the CH of individual components usually employing tight/extended formulations. However, regardless of the derivation method, computational intractability persists, in particular for general UC settings and unit characteristics.

CH prices are also often called “minimum uplift” prices. This characterization essentially refers to the minimization of the total Lost Opportunity Costs (LOCs) in addition to what is referred to as “Product Revenue Shortfall” (PRS) in [29]. For brevity, we focus our discussion on LOCs, and refer the reader to [29] for PRS. Notably, “uplift minimization” is directly related to the duality gap between (1) and (2).

Compensating for LOCs, even for units that are offline as a result of the UC solution, is a means to deter inefficient self-scheduling. More specifically, given prices λ , let us denote the profit of unit i by $\phi_i(\mathbf{x}_i, \mathbf{y}_i; \lambda)$, where:

$$\phi_i(\mathbf{x}_i, \mathbf{y}_i; \lambda) = \sum_{t \in \mathcal{T}} \lambda_t x_{i,t} - f_i(\mathbf{x}_i, \mathbf{y}_i). \quad (3)$$

Unit i , following the solution of (1), assuming a market schedule $(\mathbf{x}_i^M, \mathbf{y}_i^M)$ and prices λ , would gain $\phi_i(\mathbf{x}_i^M, \mathbf{y}_i^M; \lambda)$. However, given prices λ , had the unit optimally self-scheduled, i.e., had the unit solved the following profit maximization problem:

$$(\mathbf{x}_i^S, \mathbf{y}_i^S; \lambda) \in \underset{(\mathbf{x}_i, \mathbf{y}_i) \in \mathcal{Z}_i}{\operatorname{argmax}} \phi_i(\mathbf{x}_i, \mathbf{y}_i; \lambda), \quad (4)$$

the tentative profit would have been $\phi_i(\mathbf{x}_i^S, \mathbf{y}_i^S; \lambda)$. Hence, the LOC for unit i , LOC_i , is given by:

$$\text{LOC}_i = \phi_i(\mathbf{x}_i^S, \mathbf{y}_i^S; \lambda) - \phi_i(\mathbf{x}_i^M, \mathbf{y}_i^M; \lambda). \quad (5)$$

CH prices are complemented by minimal LOCs according to (5) plus the PRS, to support the market solution by making the unit indifferent between following the market schedule $(\mathbf{x}_i^M, \mathbf{y}_i^M)$ and self-scheduling $(\mathbf{x}_i^S, \mathbf{y}_i^S)$.

III. D-W CHARACTERIZATION AND CG ALGORITHM

Consider an equivalent formulation of the UC problem (1), using variable z_i to describe a feasible schedule of unit i , for the entire optimization horizon, i.e., $z_i := (\mathbf{x}_i, \mathbf{y}_i) \in \mathcal{Z}_i$. In a parenthesis, we note that stylized formulations of the UC problem would describe it as follows:

$$\min_{\mathbf{x}} f(\mathbf{x}) = \sum_{i \in \mathcal{I}} f_i(\mathbf{x}_i), \text{ subject to: (1b), } \mathbf{x}_i \in \mathcal{Z}_i, \forall i \in \mathcal{I},$$

considering \mathbf{x}_i as a feasible schedule, internalizing the discrete (status) variables (\mathbf{y}_i) in the feasibility constraint set \mathcal{Z}_i , and defining the cost function $f_i(\cdot)$ so as to account for both dispatch and commitment costs. Indeed, for a certain schedule, the unit status and cost can be considered as endogenously determined — see for instance [39]–[41] that relate to an older work [42] for such formulations — among others. The main reason for this parenthesis is that the above formulation — which is equivalent to (1), provides a natural interpretation of the proposed D-W characterization of the convex hull.

Suppose we make the assumption that the sets of feasible schedules are finite but exponentially large. Such an assumption can be thought of as the result of the numeric (decimal) precision in the offered quantities, which implicitly

discretizes the problem. We note, however, that although this assumption eases the description, it is not formally required for our analysis; it suffices that the feasible sets are bounded [33]. Hence, let $n_i \in \mathcal{N}_i$ be a counter of unit i feasible schedules, which are contained in the set \mathcal{N}_i . Considering the UC formulation (1), each variable $z_i^{[n_i]}$ corresponds to a specific feasible schedule, denoted by $(\hat{\mathbf{x}}_i^{[n_i]}, \hat{\mathbf{y}}_i^{[n_i]})$ and has a cost denoted by $\hat{c}_i^{[n_i]} = f_i(\hat{\mathbf{x}}_i^{[n_i]}, \hat{\mathbf{y}}_i^{[n_i]})$. Equivalently, a feasible schedule is also determined by $\hat{\mathbf{x}}_i^{[n_i]}$ and $\hat{c}_i^{[n_i]}$. We will use the two representations interchangeably to facilitate the exposition.

Following the above, the UC problem can be expressed as:

$$\min_{\mathbf{z}} g(\mathbf{z}) = \sum_{i \in \mathcal{I}, n_i \in \mathcal{N}_i} \hat{c}_i^{[n_i]} z_i^{[n_i]}, \quad (6a)$$

$$\text{subject to: } \sum_{i \in \mathcal{I}, n_i \in \mathcal{N}_i} \hat{x}_{i,t}^{[n_i]} z_i^{[n_i]} = D_t, \forall t \in \mathcal{T}, \quad (6b)$$

$$\sum_{n_i \in \mathcal{N}_i} z_i^{[n_i]} = 1, \quad \forall i \in \mathcal{I}, \quad (6c)$$

$$z_i^{[n_i]} \in \{0, 1\}, \quad \forall i \in \mathcal{I}, n_i \in \mathcal{N}_i. \quad (6d)$$

Solving (6), accounting for all feasible schedules, is equivalent to solving (1). Objective function (6a) is equivalent to (1a). Constraint (6b) represents the power balance and is equivalent to (1b). Of note that $\hat{c}_i^{[n_i]}$ and $\hat{x}_{i,t}^{[n_i]}$ are parameters. Constraint (6c) requires to select exactly one feasible schedule represented by the binary variable $z_i^{[n_i]}$. Equivalently, problem (6) can be viewed as a set partitioning problem.

At a first glance, the effort of brute-force solving (6) would seem hopeless, as it would require not only an exhaustive enumeration of feasible schedules, but also the solution of a huge integer problem — (6) is an integer linear programming problem. That said, we naturally associate (6) with D-W decomposition and CG — see e.g., [35]. However, solving a large UC problem with CG does not yet seem promising — see e.g., a recent work [43]. Fortunately, we remind the reader that our goal is not to solve this integer problem, but the LD (2). With this in mind, we recall a result which dates back to the 70’s [32], that the LP relaxation of (6), solves the LD (2); this is also clearly stated in [33], that Generalized LP — *a.k.a.* D-W decomposition — solves the dual. Indeed, we know that D-W decomposition or CG is essentially a cutting plane method applied to the LD [44], which we find as particularly instructive in our problem setting, hopefully enhancing the intuition into the formation of CH prices.

Many approaches to CH pricing involve restrictions on the form of the objective function [15] or reformulations of the UC problem such as dynamic programming characterizations [45], [46]. The D-W approach starts with a natural formulation, which does not require any change in the UC functions or constraints. This allows a wide degree of flexibility in defining the UC problem. This representation allows us to characterize the full CH [47]. Essentially, the same formulation is applied in crew scheduling problems [35]. If we had to deal with the full listing of all the feasible points, the computational problem would be overwhelming. But the D-W method provides a natural CG technique that uses sub-problems to produce what

is in practice a relatively small number of feasible points adequate to characterize the solution of the CH relaxation. The intuition is that the same approach would work for the UC problem, without requiring any reformulations of the UC model, but with the computational feasibility observed in the related very large crew scheduling problems.

The solution of the LP relaxation of (6), employing a CG algorithm, proceeds as follows. First, consider a subset of feasible schedules for each unit i , say $\mathcal{N}_i^{(k)} \subset \mathcal{N}_i$, where k is an iteration counter, i.e., $\mathcal{N}_i^{(k)}$ contains all feasible schedules of unit i available at iteration k . Using these schedules, a Restricted Master Problem (RMP) at iteration k , $\text{RMP}^{(k)}$, is formulated as follows:

$$\text{RMP}^{(k)} : \quad \min_{\mathbf{z}} g^{(k)}(\mathbf{z}) = \sum_{i \in \mathcal{I}, n_i \in \mathcal{N}_i^{(k)}} \hat{c}_i^{[n_i]} z_i^{[n_i]}, \quad (7a)$$

$$\text{subject to:} \quad \sum_{i \in \mathcal{I}, n_i \in \mathcal{N}_i^{(k)}} \hat{x}_{i,t}^{[n_i]} z_i^{[n_i]} = D_t, \forall t \in \mathcal{T}, \rightarrow \lambda_t^{(k)}, \quad (7b)$$

$$\sum_{n_i \in \mathcal{N}_i^{(k)}} z_i^{[n_i]} = 1, \forall i \in \mathcal{I}, \rightarrow \pi_i^{(k)}, \quad (7c)$$

with $z_i^{[n_i]} \geq 0$, $\forall i \in \mathcal{I}$, $n_i \in \mathcal{N}_i^{(k)}$, and $\lambda_t^{(k)}$, $\pi_i^{(k)}$, the duals of constraints (7b), (7c), respectively. Note that limiting the upper bound to $z_i^{[n_i]} \leq 1$ is enforced by (7c). Then, CG “generates” new columns (feasible schedules) and adds them to the RMP, as long as they have a negative reduced cost. The reduced cost $rc(\cdot)$ of feasible schedule $(\mathbf{x}_i, \mathbf{y}_i)$ is given by:

$$rc_i(\mathbf{x}_i, \mathbf{y}_i) = f_i(\mathbf{x}_i, \mathbf{y}_i) - \sum_{t \in \mathcal{T}} \lambda_t^{(k)} x_{i,t} - \pi_i^{(k)}. \quad (8)$$

Hence, at iteration k , feasible schedules with potential negative reduced cost can be obtained by the solution of the following sub-problem, for each unit i :

$$\text{Sub}_i^{(k)} : \min_{\mathbf{x}_i, \mathbf{y}_i} h_i^{(k)}(\mathbf{x}_i, \mathbf{y}_i) = f_i(\mathbf{x}_i, \mathbf{y}_i) - \sum_{t \in \mathcal{T}} \lambda_t^{(k)} x_{i,t}, \quad (9a)$$

$$\text{subject to:} \quad (\mathbf{x}_i, \mathbf{y}_i) \in \mathcal{Z}_i, \quad (9b)$$

where we dropped the constant term $-\pi_i^{(k)}$ from the objective function. If the schedule obtained by the solution of (9) has a negative reduced cost, which is calculated using (8) — essentially by adding back the term $-\pi_i^{(k)}$ to the objective function value $h_i^{(k)}$, then a new column is added to the RMP corresponding to this schedule. The RMP is solved again, and the algorithm terminates when no new feasible schedule with negative reduced cost can be found.

This is a standard CG algorithm. We only highlight some interesting computational features. For the algorithm to proceed, we need to find at least one column with a negative reduced cost to update the duals. That said, sub-problem (9) does not need to be solved to optimality in intermediate iterations, but only at the end, to guarantee termination. Evidently, all sub-problems can be solved in parallel, or we may tailor the number of sub-problems solved at each iteration to the available computational resources. Obviously, we will not always be able to find negative reduced cost schedules at each iteration for all units; this has been the reason that we did not

use superscript (k) instead of $[n_i]$ for the feasible schedule. Assuming full parallelization, the computational time of each iteration is dominated by the solution of the RMP, which is an LP problem. The complexity of the unit specific constraints appears only in the sub-problems, not the RMP. Using standard UC formulations, the sub-problem is a small MILP, usually very easy to solve. Note that we are not obliged to solve the profit maximization problem with a MILP formulation; we only need to find a negative reduced cost feasible schedule regardless of the method. Nevertheless, a MILP formulation would definitely facilitate the implementation in current ISO Market Management Systems.

We further elaborate on the intuition that the CG algorithm provides. Evidently, the RMP selects fractional feasible schedules, in fact a convex combination of them — see constraint (7c) that is often called a “convexity constraint” for a reason. CG terminates by “shaping” the CH yielding the CH prices. Indeed, had we asked a similar question as in [48] — “since the CH is our ultimate desire, why don’t we just shape it?” — the CG algorithm would have been a natural choice. It is also evident that the sub-problem (9) is a profit maximization problem, given prices $\lambda^{(k)}$ — compare with (3) and (4). The reduced cost calculation has a natural interpretation, in the spirit of [49], which reveals a key property of CH prices and associated LOCs with respect to self-scheduling. When unit i “is notified” tentative prices $\lambda^{(k)}$ at iteration k , the unit “fictitiously” self-schedules, solving (9) and calculating the tentative profit, $\phi_i(\mathbf{x}_i^S, \mathbf{y}_i^S; \lambda^{(k)})$, which equals $-h_i^{(k)}$, where with some abuse of notation $h_i^{(k)}$ denotes the optimal value of (9). The dual $-\pi_i^{(k)}$ represents the tentative profit of unit i , as calculated by the RMP, i.e., given the prices $\lambda^{(k)}$. If the unit can do better by self-scheduling, i.e., if $\phi_i(\mathbf{x}_i^S, \mathbf{y}_i^S; \lambda^{(k)}) > -\pi_i^{(k)}$, then self-scheduling results in a negative reduced cost — see (8). Practically, the algorithm terminates when the profit-maximization problem yields a feasible schedule that already exists in the RMP for all units, implying that for the CH prices, aggregate LOCs are minimized, and the optimal solution of the LD is reached.

IV. ILLUSTRATION ON STYLIZED EXAMPLES

This section employs the stylized examples from [29], as a useful exercise, which aims at (i) illustrating the CG algorithm, and (ii) providing intuition on the formation of CH prices viewed through the lens of D-W decomposition. We refer the reader to [29] for a detailed discussion of the CH price properties that each example illustrates.

A. Example 1: 2-Gen, 1-Hour [29, Ex. 1]

Two generators, G1 and G2, serve a 35 MW load, in a single period. G1 is online, with technical minimum and maximum 10 and 50 MW, respectively, and energy offer \$50/MWh. G2 can be either offline or online and submits a block offer of 50 MW at \$10/MWh. The MILP formulation is as follows:

$$\text{MILP:} \quad \min_{x_1, x_2, y_2} f = 50x_1 + 10x_2, \quad (10a)$$

$$\text{subject to:} \quad x_1 + x_2 = 35, \quad (10b)$$

$$10 \leq x_1 \leq 50, \quad x_2 = 50y_2, \quad y_2 \in \{0, 1\}. \quad (10c)$$

This example has only one feasible solution with G1 providing 35 MW, and MILP objective function value $f^* = \$1750$. The CH price is \$10/MWh, and G1 has \$1000 LOC.

Next, we solve this example using CG. Consider trivial initial schedules: $z_1^{[1]} : \hat{x}_1^{[1]} = 10$, $\hat{c}_1^{[1]} = 500$, and $z_2^{[1]} : \hat{x}_2^{[1]} = 0$, $\hat{c}_2^{[1]} = 0$. Assuming a slack (deficit) variable s for the energy balance constraint, with penalty \$1000/MWh, the initial RMP is as follows:

$$\text{RMP}^{(1)} : \min_{z_1^{[1]}, z_2^{[1]}, s} g^{(1)} = 500z_1^{[1]} + 0z_2^{[1]} + 1000s, \quad (11a)$$

$$\text{subject to:} \quad 10z_1^{[1]} + 0z_2^{[1]} + s = 35, \rightarrow \lambda^{(1)}, \quad (11b)$$

$$z_1^{[1]} = 1, \rightarrow \pi_1^{(1)}, \quad z_2^{[1]} = 1, \rightarrow \pi_2^{(1)}, \quad (11c)$$

with $z_1^{[1]}, z_2^{[1]}, s \geq 0$. The solution of RMP⁽¹⁾ yields duals $\lambda^{(1)} = 1000$, $\pi_1^{(1)} = -9500$, and $\pi_2^{(1)} = 0$. For these values,

$$\text{Sub}_1^{(1)} : \min_{10 \leq x_1 \leq 50} h_1^{(1)} = 50x_1 - 1000x_1 = -950x_1,$$

yields $x_1 = 50$, $h_1^{(1)} = -47500$, with $rc_1^{(1)} = h_1^{(1)} - \pi_1^{(1)} = -38000 < 0$, hence $z_1^{[2]} : \hat{x}_1^{[2]} = 50$, $\hat{c}_1^{[2]} = 2500$, whereas

$$\text{Sub}_2^{(1)} : \min_{x_2=50y_2, y_2 \in \{0, 1\}} h_2^{(1)} = 10x_2 - 1000x_2 = -990x_2,$$

yields $x_2 = 50$, $y_2 = 1$, $h_2^{(1)} = -49500$, with $rc_2^{(1)} = h_2^{(1)} - \pi_2^{(1)} = -49500 < 0$, hence $z_2^{[2]} : \hat{x}_2^{[2]} = 50$, $\hat{c}_2^{[2]} = 500$. The RMP, after adding these two new columns becomes:

$$\text{RMP}^{(2)} : \min_{z, s} g^{(2)} = 500z_1^{[1]} + 2500z_1^{[2]} + 500z_2^{[2]} + 1000s,$$

$$\text{subject to:} \quad 10z_1^{[1]} + 50z_1^{[2]} + 50z_2^{[2]} + s = 35, \rightarrow \lambda^{(2)},$$

$$z_1^{[1]} + z_1^{[2]} = 1, \rightarrow \pi_1^{(2)}, \quad z_2^{[1]} + z_2^{[2]} = 1, \rightarrow \pi_2^{(2)},$$

with $z_1^{[1]}, z_2^{[1]}, z_1^{[2]}, z_2^{[2]}, s \geq 0$. The solution of RMP⁽²⁾ now yields $\lambda^{(2)} = 10$, $\pi_1^{(2)} = 400$, and $\pi_2^{(2)} = 0$. For these values, Sub₁⁽²⁾ yields $x_1 = 10$, with $h_1^{(2)} = 400$, and $rc_1^{(2)} = h_1^{(2)} - \pi_1^{(2)} = 0$ — note that a schedule with $x_1 = 10$ already exists in the RMP — whereas Sub₂⁽²⁾ yields $h_2^{(2)} = 0$, for any feasible schedule, and $rc_2^{(2)} = h_2^{(2)} - \pi_2^{(2)} = 0$. Since no feasible schedule with negative reduced cost is found, CG terminates.

The CH price is $\lambda^{(2)} = \$10/\text{MWh}$. The solution of RMP⁽²⁾ is $z_1^{[1]} = 1$, $z_2^{[1]} = 0.5$, $z_2^{[2]} = 0.5$, and the value of the objective function is $g^{(2)} = g^* = \$750$. Hence, the duality gap between the MIP and the RMP is $f^* - g^* = \$1750 - \$750 = \$1000$, which represents the LOC to be paid to G1 (uplift). Note that the maximum profit of G1 at the CH price would be derived by $x_1 = 10$, instead of $x_1 = 35$, which is the MILP dispatched quantity. Notably, this quantity was the optimal solution for the G1 sub-problem when the algorithm terminated. On the other hand, G2 has no LOC, since CH price equals its cost.

B. Example 2: Example 1 with Start-up Cost [29, Ex. 3]

This example adds to Example 1 a \$100 start-up cost to G2, i.e., $f_2(x_2, y_2) = 10x_2 + 100y_2$, yielding a CH price equal to \$12/MWh that corresponds to the average cost of G2. Still, the only feasible solution is to dispatch G1 at 35 MW, and the MILP objective function value is $f^* = \$1750$. Assuming same initial schedules, hence same RMP⁽¹⁾ duals, we get $z_1^{[2]} : \hat{x}_1^{[2]} = 50$, $\hat{c}_1^{[2]} = 2500$, and $z_2^{[2]} : \hat{x}_2^{[2]} = 50$, $\hat{c}_2^{[2]} = 600$. RMP⁽²⁾ yields duals $\lambda^{(2)} = 12$, $\pi_1^{(2)} = 380$, and $\pi_2^{(2)} = 0$, for which no negative reduced cost schedule can be found. The solution of RMP⁽²⁾ (z variables) is the same with Example 1, but the value of the objective function is now $g^{(2)} = g^* = \$800$. Hence, the duality gap is \$950, also representing the G1 LOC.

C. Example 3: 2-Gen, 2-Hour (linked) [29, Ex. 4]

This example considers 2 hours, with load 45 and 80 MW. G1 remains the same and should be online during both hours. G2 has a 25 MW minimum, a 35 MW maximum, an energy offer of \$100/MWh, and should be either online or offline during both hours. The MILP problem is formulated as follows:

$$\text{MILP:} \min_{x, y} f = 50x_{1,1} + 50x_{1,2} + 100x_{2,1} + 100x_{2,2}, \quad (12a)$$

$$\text{subject to:} \quad x_{1,1} + x_{2,1} = 45, \quad x_{1,2} + x_{2,2} = 80, \quad (12b)$$

$$10 \leq x_{1,1} \leq 50, \quad 10 \leq x_{1,2} \leq 50, \quad (12c)$$

$$25y_2 \leq x_{2,1} \leq 35y_2, \quad 25y_2 \leq x_{2,2} \leq 35y_2, \quad (12d)$$

$$y_2 \in \{0, 1\}, \quad (12e)$$

yielding $x_{1,1} = 20$, $x_{1,2} = 50$, $x_{2,1} = 25$, $x_{2,2} = 30$, $y_2 = 1$, and MILP objective function value is $f^* = \$9000$.

Skipping the first CG steps, the last RMP is as follows:

$$\text{RMP}^{(3)} : \min_{z, s} g^{(3)} = 1000z_1^{[1]} + 0z_2^{[1]} + 5000z_1^{[2]} + 7000z_2^{[2]} \\ + 3000z_1^{[3]} + 6000z_2^{[3]} + 1000s_1 + 1000s_2, \quad \text{subject to:}$$

$$10z_1^{[1]} + 50z_1^{[2]} + 35z_2^{[2]} + 10z_1^{[3]} + 25z_2^{[3]} + s_1 = 45, \rightarrow \lambda_1^{(3)},$$

$$10z_1^{[1]} + 50z_1^{[2]} + 35z_2^{[2]} + 50z_1^{[3]} + 25z_2^{[3]} + s_2 = 80, \rightarrow \lambda_2^{(3)},$$

$$z_1^{[1]} + z_1^{[2]} + z_1^{[3]} = 1, \rightarrow \pi_1^{(3)}, \quad z_2^{[1]} + z_2^{[2]} + z_2^{[3]} = 1, \rightarrow \pi_2^{(3)},$$

with $z_1^{[1]}, z_2^{[1]}, z_1^{[2]}, z_2^{[2]}, z_1^{[3]}, z_2^{[3]}, s_1, s_2 \geq 0$. RMP⁽³⁾ yields duals $\lambda_1^{(3)} = 50$, $\lambda_2^{(3)} = 135.714$, $\pi_1^{(3)} = 4285.714$, and $\pi_2^{(3)} = 0$, for which CG terminates. The solution of RMP⁽³⁾ is $z_1^{[2]} = 0.339$, $z_1^{[3]} = 0.661$, and $z_2^{[1]} = 0.143$, $z_2^{[3]} = 0.857$, and the value of the objective function is $g^{(3)} = g^* = \$8821.429$. Hence, the duality gap is \$178.571, representing G2 LOC.

D. Example 4: Example 3 with non-linked Hours [29, Ex. 5]

This example considers the previous setting but as two single-period problems. Equivalently, we can write the MILP formulation, using (12) and replacing (12d)–(12e) with: $25y_{2,1} \leq x_{2,1} \leq 35y_{2,1}$, $25y_{2,2} \leq x_{2,2} \leq 35y_{2,2}$, and $y_{2,1}, y_{2,2} \in \{0, 1\}$. CG terminates with RMP⁽³⁾ yielding duals $\lambda_1^{(3)} = 50$, and $\lambda_2^{(3)} = 100$, $\pi_1^{(3)} = -2500$, and $\pi_2^{(3)} = 0$, and

a value of the objective function $g^{(3)} = g^* = \$7750$. Hence the duality gap is \$1250, representing G2 LOC. We leave this example as an exercise to the interested reader.

V. MORE REALISTIC TEST CASES

In this section, we proceed to more realistic test cases. In Subsection V-A, we review an example provided in [26], including the “problematic” ramp constraints. In Subsection V-B, we test a 24-hour UC formulation on an ISO-sized FERC dataset [38] with about 1000 generators.

A. Ramp Constrained Example [26, Ex. 2]

This example includes 2 generators and a 3-hour horizon with load 95, 100, and 130 MW. G1 has a maximum of 100 MW, and energy offer \$10/MWh. G2 has a 20 MW minimum, 35 MW maximum, energy offer \$50/MWh, start-up cost \$1000, no-load cost \$30, ramp rate 5 MW/hour, start-up rate 22.5 MW/hour, shut-down rate 35 MW/hour, minimum up/down times of 1 hour, and initially it is offline. The UC 3-binary formulation is provided below [26].

$$\min_{\mathbf{p}, \mathbf{u}, \mathbf{v}, \mathbf{w}} f = \sum_{t=1}^3 (10p_{1,t} + 30u_{2,t} + 50p_{2,t} + 1000v_{2,t}), \quad (13a)$$

$$\text{subject to:} \quad p_{1,t} + p_{2,t} = D_t, \quad 1 \leq t \leq 3, \quad (13b)$$

$$0 \leq p_{1,t} \leq 100, \quad 1 \leq t \leq 3, \quad (13c)$$

$$20u_{2,t} \leq p_{2,t} \leq 35u_{2,t}, \quad 1 \leq t \leq 3, \quad (13d)$$

$$p_{2,t} - p_{2,t-1} \leq 5u_{2,t-1} + 22.5v_{2,t}, \quad 1 \leq t \leq 3, \quad (13e)$$

$$p_{2,t-1} - p_{2,t} \leq 5u_{2,t} + 35w_{2,t}, \quad 2 \leq t \leq 3, \quad (13f)$$

$$u_{2,t} - u_{2,t-1} = v_{2,t} - w_{2,t}, \quad 1 \leq t \leq 3, \quad (13g)$$

$$v_{2,t} \leq u_{2,t}, \quad v_{2,t} \leq 1 - u_{2,t-1}, \quad 1 \leq t \leq 3, \quad (13h)$$

with $p_{1,t}, p_{2,t} \geq 0$, and $u_{2,t}, v_{2,t}, w_{2,t} \in \{0, 1\}$, $\forall t$, representing the status, start-up and shut-down variables, respectively. The optimal schedule is: $p_{1,1} = 75$, $p_{1,2} = 75$, $p_{1,3} = 100$, and $p_{2,1} = 20$, $p_{2,2} = 25$, $p_{2,3} = 30$, with $f^* = 7340$.

The interesting feature of this example is related to the ramping constraints, which has been a challenge for the application of CH pricing. The IR of (13) yields a vector of prices $\lambda^{\text{IR}} = (10, 10, 182.701)$, which deviate from the CH prices, which we show to be $\lambda^{\text{CH}} = (10, 10, 276)$. We note that [26] reports Average Incremental Prices (AIC) prices obtained by an extended formulation that yields $\lambda^{\text{AIC}} = (10, 10, 146.333)$, while also noting that a numerical approximation using the 3-binary formulation would result in a very high price at hour 3, namely (10, 10, 1161). Apparently, in this example, hour 3 is of interest, and obviously, the IR (whose objective function value is \$6464.55) is less tight compared to the LD (whose value is shown to be \$6975).

Applying CG, we obtain: (1) Initial columns: $z_1^{[1]} : \hat{\mathbf{p}}_1^{[1]} = (0, 0, 0)$, $\hat{c}_1^{[1]} = 0$; $z_2^{[1]} : \hat{\mathbf{p}}_2^{[1]} = (0, 0, 0)$, $\hat{c}_2^{[1]} = 0$; RMP⁽¹⁾ duals: $\lambda^{(1)} = (1000, 1000, 1000)$, $\pi_1^{(1)} = 0$, $\pi_2^{(1)} = 0$. (2) New columns: $z_1^{[2]} : \hat{\mathbf{p}}_1^{[2]} = (100, 100, 100)$, $\hat{c}_1^{[2]} = 3000$; $z_2^{[2]} : \hat{\mathbf{p}}_2^{[2]} = (22.5, 27.5, 32.5)$, $\hat{c}_2^{[2]} = 5215$; RMP⁽²⁾ duals: $\lambda^{(2)} = (-970, 0, 1000)$, $\pi_1^{(2)} = 0$, $\pi_2^{(2)} = -5460$. (3) New columns: $z_1^{[3]} : \hat{\mathbf{p}}_1^{[3]} = (0, 0, 100)$, $\hat{c}_1^{[3]} = 1000$; $z_2^{[3]} :$

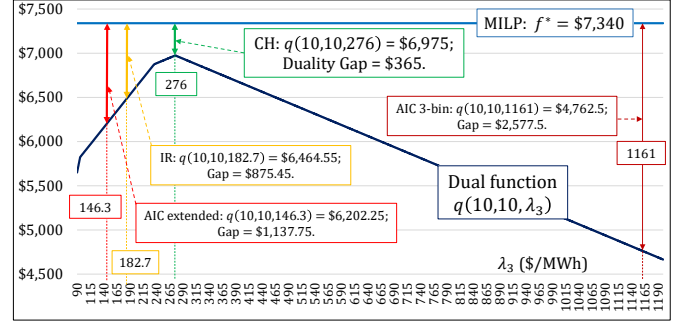


Fig. 1. Evaluation of $q(\lambda)$, for $\lambda = (10, 10, \lambda_3)$, $90 \leq \lambda_3 \leq 1200$.

$\hat{\mathbf{p}}_2^{[3]} = (0, 22.5, 27.5)$, $\hat{c}_2^{[3]} = 3560$; RMP⁽³⁾ duals: $\lambda^{(3)} = (45.832, -25.832, 150.589)$, $\pi_1^{(3)} = -14060$, $\pi_2^{(3)} = 0$. (4) New columns: $z_1^{[4]} : \hat{\mathbf{p}}_1^{[4]} = (100, 0, 100)$, $\hat{c}_1^{[4]} = 2000$; $z_2^{[4]} : \hat{\mathbf{p}}_2^{[4]} = (0, 0, 22.5)$, $\hat{c}_2^{[4]} = 2155$. RMP⁽⁴⁾ duals: $\lambda^{(4)} = (10, 10, 276)$, $\pi_1^{(4)} = -26600$, $\pi_2^{(4)} = -4255$. CG terminates with $z_1^{[2]} = 0.75$, $z_1^{[3]} = 0.163$, $z_1^{[4]} = 0.088$, and $z_2^{[2]} = 0.5$, $z_2^{[3]} = 0.5$; objective function value: $g^{(4)} = g^* = \$6975$.

In Fig. 1, we illustrate the evaluation of the dual function $q(\lambda)$ for $\lambda = (10, 10, \lambda_3)$, with λ_3 (\$/MWh) ranging from 90 — the marginal cost reported in [26] — to 1200, so as to include the high price of 1161. The CH duality gap is \$365, and corresponds to LOC of G2. Notably, the IR gap is \$875.45, whereas the extended formulation AIC gap is \$1137.75.

B. FERC Dataset Example

In this subsection, we test our approach on the PJM-like FERC “summer” dataset [38]. UC formulations have been extensively studied — see e.g., [50] for an overview of MILP formulations, and our goal in this last example is not to exhaustively include all features, but to provide evidence on the scalability to real-sized problems. In fact, as it has already become evident, a major strength of CG is that it can easily accommodate any unit model, and all system constraints currently present in electricity markets.

The employed UC MILP formulation is provided below. Unless, otherwise mentioned, $i \in \mathcal{I}$, $t \in \mathcal{T}$, and $b \in \{1, \dots, B\}$, where B is the number of block offer steps.

$$\min_{\mathbf{p}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{s}} f = \sum_{i,t} \left[C_i^{NL} u_{i,t} + C_i^{SU} v_{i,t} + C_i^{SD} w_{i,t} + \sum_b (C_{i,b}^p p_{i,t,b}) + C_i^r r_{i,t} \right] + \sum_t (M^p s_t^p + M^r s_t^r), \quad (14a)$$

$$\text{subject to:} \quad \sum_i p_{i,t} + s_t^p = D_t^p, \quad \forall t, \quad (14b)$$

$$\sum_i r_{i,t} + s_t^r \geq D_t^r, \quad \forall t, \quad (14c)$$

$$p_{i,t,b} \leq \bar{P}_{i,b} u_{i,t}, \quad \forall i, t, b, \quad p_{i,t} = \sum_b p_{i,t,b}, \quad \forall i, t, \quad (14d)$$

$$\underline{P}_i u_{i,t} \leq p_{i,t} \leq \bar{P}_i u_{i,t} - r_{i,t}, \quad r_{i,t} \leq \bar{R}_i, \quad \forall i, t, \quad (14e)$$

$$p_{i,t} - p_{i,t-1} \leq R_i^U u_{i,t-1} + R_i^{SU} v_{i,t}, \quad \forall i, t, \quad (14f)$$

$$p_{i,t-1} - p_{i,t} \leq R_i^D u_{i,t} + R_i^{SD} w_{i,t}, \quad \forall i, t, \quad (14g)$$

$$u_{i,t} - u_{i,t-1} = v_{i,t} - w_{i,t}, \quad \forall i, t, \quad (14h)$$

$$\sum_{t'=t-MUT_i+1}^t v_{i,t'} \leq u_{i,t}, \quad \forall i, t = [MUT_i, T], \quad (14i)$$

$$\sum_{t'=t-MDT_i+1}^t w_{i,t'} \leq 1 - u_{i,t}, \quad \forall i, t = [MDT_i, T], \quad (14j)$$

$$\sum_{t'=1}^{UT_i} u_{i,t'} = UT_i, \quad \sum_{t'=1}^{DT_i} u_{i,t'} = 0, \quad \forall i, \quad (14k)$$

with $p_{i,t,b}, p_{i,t}, r_{i,t}, s_t^p, s_t^r \geq 0$, and $u_{i,t}, v_{i,t}, w_{i,t} \in \{0, 1\}$, $\forall i, t, b$. In brief, (14a) minimizes the aggregate commitment costs (no-load, C_i^{NL} , start-up, C_i^{SU} , and shut-down, C_i^{SD}), dispatch costs ($C_{i,b}^p$ is the cost of block offer b), plus the reserve cost (reserve offer C_i^r), plus a penalty cost related to power/reserve deficit slacks s_t^p, s_t^r , with penalties M^p and M^r . System constraints (14b)–(14c) include power balance (demand D_t^p) and reserve requirements (D_t^r). Unit specific constraint sets (14d)–(14k) are as follows. (14d) imposes block offer b maximum, $\bar{P}_{i,b}$, and defines $p_{i,t}$ as the sum of accepted block quantities $p_{i,t,b}$. (14e) imposes minimum and maximum capacity limits for power, $\underline{P}_i, \bar{P}_i$, and maximum reserve capability \bar{R}_i . (14f)–(14g) impose ramp up/down limits R_i^U, R_i^D , including start-up/shut-down, R_i^{SU}, R_i^{SD} . (14h) defines commitment variables, (14i)–(14j) impose the minimum up/down time limits, MUT_i, MDT_i , whereas (14k) enforces initial up/down times; UT_i and DT_i are the number of hours the unit should remain online or offline, respectively, depending on initial conditions (truncated at T).

The RMP at iteration k , $\text{RMP}^{(k)}$, is formulated as:

$$\min_{\mathbf{z}, \mathbf{s}} g^{(k)} = \sum_{i, n_i \in \mathcal{N}_i^{(k)}} \hat{c}_i^{[n_i]} z_i^{[n_i]} + \sum_t (M^p s_t^p + M^r s_t^r),$$

$$\text{subject to: } \sum_{i, n_i \in \mathcal{N}_i^{(k)}} \hat{p}_{i,t}^{[n_i]} z_i^{[n_i]} + s_t^p = D_t^p, \quad \forall t \rightarrow \lambda_t^p(k),$$

$$\sum_{i, n_i \in \mathcal{N}_i^{(k)}} \hat{r}_{i,t}^{[n_i]} z_i^{[n_i]} + s_t^r \geq D_t^r, \quad \forall t \rightarrow \lambda_t^r(k),$$

$$\sum_{n_i \in \mathcal{N}_i^{(k)}} z_i^{[n_i]} = 1, \quad \forall i \rightarrow \pi_i^{(k)},$$

with $z_i^{[n_i]} \geq 0, \forall i, n_i \in \mathcal{N}_i^{(k)}$, and $s_t^p, s_t^r \geq 0, \forall t$.

The unit i sub-problem at iteration k , $\text{Sub}_i^{(k)}$, is given by:

$$\begin{aligned} \min h_i^{(k)} = & \sum_t \left[C_i^{NL} u_{i,t} + C_i^{SU} v_{i,t} + C_i^{SD} w_{i,t} \right. \\ & \left. + \sum_b (C_{i,b}^p p_{i,t,b}) + C_i^r r_{i,t} \right] - \sum_t (\lambda_t^p(k) p_{i,t} + \lambda_t^r(k) r_{i,t}), \end{aligned}$$

subject to: unit specific constraints (14d)–(14k). The reduced cost is $rc_i^{(k)} = h_i^{(k)} - \pi_i^{(k)}$.

The dataset was adjusted to fit (14). It contains 1011 units, each with up to 10-block energy offers and zero-priced reserve offers. The 24-hour D_t^p was set equal to forecasted demand, and penalties M^p and M^r to 1000 and 900. Hot-type values were used for C_i^{SU} , missing MUT_i and MDT_i were set to 1, UT_i and DT_i to 0, and all units initially online dispatched at \underline{P}_i . R_i^U and R_i^D were reduced to half, R_i^{SU} were set equal

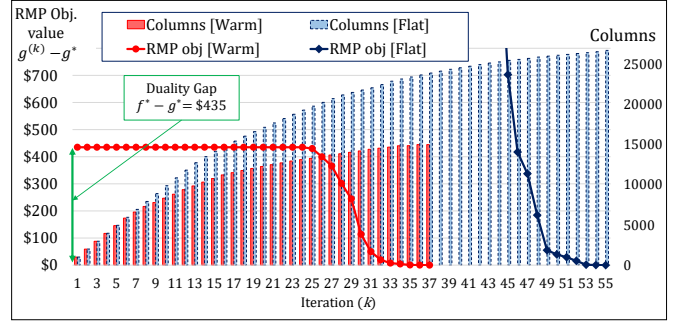


Fig. 2. CG iterations for flat and warm start. Left axis shows the RMP objective function value as the difference of $g^{(k)}$ from the optimal LD bound $g^* = \$32,169,154.7$. Right axis shows total columns at each iteration. UC MILP solved with 0.01% optimality gap. For comparison purposes, note that the objective function value of IR is $\$32,149,006.2$ (less tight bound).

to \underline{P}_i plus half the R_i^U, R_i^{SD} equal to \bar{P}_i , and \bar{R}_i equal to 30-min ramp up. CG was modeled in C, and solved on a Intel Core i7-5500U 2.4GHz with 8 GB RAM, using CPLEX 12.7.

In Fig. 2, we explore CG performance in terms of iterations, when initiated from (i) the MILP UC solution — referred to as a “Warm start,” and (ii) a “Flat start” with initial columns assuming self-scheduling at zero prices. Note that current practice normally performs the pricing run after the MILP UC solution. However, CH pricing could be run in parallel to the MILP UC problem, as it does not actually need its solution! The results indicate that CG terminates in a few tens of iterations for both cases. We do observe an anticipated plateau at the beginning of the warm start, since CG initially needs some columns to be able to “combine” them. The flat start drops rapidly (high values due to deficit variable penalties are truncated), but it takes a few more iterations to terminate. The computational times in a sequential implementation are dominated by the sub-problems; each iteration took about 14 seconds. However, we note that the RMP time was less than 1 second (usually significantly less), whereas each sub-problem was solved in the order of 10 msec. It goes without saying that parallel implementation of sub-problems would drive down each iteration to practically the time to solve the RMP, i.e., an LP problem; the estimated time for both cases assuming full parallelization would be well below 1 minute. But even with some simple partial parallelization, say e.g., solving a batch of a dozen of sub-problems, the time per iteration would be reduced by approximately a factor of 10.

VI. CONCLUSIONS AND FURTHER RESEARCH

In this work, we compute CH prices, employing D-W decomposition and CG. We balance the narrative with describing the theory, enhancing the reader’s intuition with illustrative examples, and providing indications of computational tractability and scalability to real-sized systems. The simplicity of CG, the intuitive explanation of the problem “convexification” or CH formation, and the amenable to parallelization structure using standard UC and unit specific MILP formulations reinforce the use of CG for explanation and computation. There was no use of any of the known methods that may enhance CG performance, although such strategies have been successful in

other applications. Ongoing work is directed in testing CG with actual market data on a large US ISO application.

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