Online Appendix for
"A Behavioral New Keynesian Model"

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June 10, 2019

This online appendix gives complements on the model (e.g. on variants with continuous time, fully flexible prices, with nominal illusion, another specification for the debt process). It gives also additional proofs.

12 Complements

12.1 The model in continuous time

We can write $M = 1 - \xi \Delta t$, $\beta = 1 - r\Delta t$ and $M^f = 1 - \xi^f \Delta t$. In the small time limit ($\Delta t \to 0$), $\xi, \xi^f \geq 0$ are the macro parameters of inattention. The model (27)-(28) becomes, in continuous time:

$$\dot{x}_t = \xi x_t - b_d d_t + \sigma \left(i_t - r_t - \pi_t \right), \tag{125}$$
$$\dot{\pi}_t = \left(r + \xi^f \right) \pi_t - \kappa x_t. \tag{126}$$

When $\xi = \xi^f = b_d = 0$, we recover Werning (2012)'s formulation, which has rational agents.

The Taylor criterion (35) becomes, using $\rho^f := r + \xi^f$,

$$\phi_\pi + \frac{\rho^f}{\kappa} \phi_x + \frac{\rho^f \xi}{\kappa \sigma} > 1. \tag{127}$$

Section 12.14 contains a derivation of the Phillips curve in continuous time.

12.2 Consumption and labor supply: Complements

Here I gather a few results that are useful to think about optimal consumption and labor supply with behavioral agents.

I start by recording some useful relations:

$$\hat{c}_t = \hat{\omega}_t + \hat{N}_t + \hat{y}_t^f, \tag{128}$$
$$\hat{N}_t = \frac{\hat{\omega}_t}{\phi} - \frac{\gamma}{\phi} \hat{c}_t. \tag{129}$$

The first one is the linearization of the net profits, $y_t^f = c_t - \omega_t N_t$. The second one is the labor supply condition.

12.2.1 Income effects in the static case

Consider the simpler static problem:

$$\max_{c,N} \frac{c^{1-\gamma} - 1}{1-\gamma} - \frac{N^{1+\phi}}{1+\phi} \text{ s.t. } c = wN + k,$$

around $k=0, w=1$. At that default, $c=N=1$. I next consider the impact of higher wealth, k.

First, consider the case of a fixed labor supply, e.g. we add the constraint $N=1$. Then, for function $\hat{c}(k;1)=c(k;N=1)-1$,

$$\hat{c}(k;N=1)=k. \tag{130}$$

Next, consider the case with endogenous labor supply. We have[117]

$$\hat{c}(k)=\frac{\phi}{\phi+\gamma}k, \tag{131}$$

and $\hat{N}(k)=-\frac{\gamma}{\phi+\gamma}k$. If wealth goes up by \$1, labor supply goes down, so that consumption goes up by less than \$1 (indeed, it goes up by $\frac{\phi}{\phi+\gamma}<1$). This fact will show up in the more complex intertemporal versions to which we now turn.

12.2.2 Intertemporal case

Notations. I call c_τ the planned consumption at time τ, while $\hat{y}(\boldsymbol{X}_\tau)$ and $\hat{c}(\boldsymbol{X}_\tau)$ are the aggregate income and consumption at date τ (all expressed as deviations from the steady state).

Here I state a variant of Proposition 11.2. That proposition was written as a function of planned future labor supply. Here I explicitly solve for planned future labor supply.

Proposition 12.1 (Consumption given beliefs, solving out for labor supply) *Suppose that we have*

$$\hat{y}^{BR}(\boldsymbol{X}_t,N_t)=\hat{\mathcal{Y}}^{BR}(\boldsymbol{X}_t)+w(\boldsymbol{X}_t)\left(N_t-\bar{N}\right), \tag{132}$$

where $\hat{\mathcal{Y}}^{BR}(\boldsymbol{X}_t)$ is some function that is independent of the agent's own labor supply satisfying $\hat{\mathcal{Y}}^{BR}(0)=0$; and the optimization of utility is over c_τ,N_τ. Then optimal consumption is, up to second order terms:

$$\begin{aligned} c_t &= \bar{y}+b_k k_t+\mathbb{E}_t^{BR}\left[\sum_{\tau\geq t}\frac{1}{R^{\tau-t}}\left(b_r\hat{r}^{BR}(\boldsymbol{X}_\tau)+b_k\left(\hat{\mathcal{Y}}^{BR}(\boldsymbol{X}_\tau)+\frac{\hat{w}(\boldsymbol{X}_\tau)}{\phi}\right)\right)\right], && (133)\\ b_r &:= \frac{-1}{\gamma R^2}, \qquad b_k:=\frac{r}{R}\chi, \qquad \chi:=\frac{\phi}{\phi+\gamma}. && (134)\end{aligned}$$

Note that the b_k has been multiplied by χ. Also, income has been increased by $\frac{\hat{w}(\boldsymbol{X}_\tau)}{\phi}$, because of the labor supply response. The marginal propensity to consume (MPC) out of capital is now $\frac{r}{R}\chi$, rather than $\frac{r}{R}$. This is analogous to the static MPC in (131): higher wealth k_t is spent on higher consumption and reduced labor supply.

Proof of Proposition 12.1 We start from (119), the consumption given planned future labor supply:

$$c_t=\frac{r}{R}k_t+\bar{y}+\mathbb{E}_t^{BR}\left[\sum_{\tau\geq t}\frac{1}{R^{\tau-t}}\left(b_r\hat{r}^{BR}(\boldsymbol{X}_\tau)+\frac{r}{R}\left(\hat{\mathcal{Y}}^{BR}(\boldsymbol{X}_\tau)+w(\boldsymbol{X}_\tau)\left(N_\tau-\bar{N}\right)\right)\right)\right]. \tag{135}$$

[117] Indeed, the FOC is $N^\phi(k)=c(k)^{-\gamma}$, so that (at $k=0$), $\phi N_k=-\gamma c_k$. The budget constraint $c(k)-N(k)=k$ implies $c_k-N_k=1$, so $\left(1+\frac{\gamma}{\phi}\right)c_k=1$.

Now, the labor supply relation (129) gives:

$$N_\tau - \bar{N} = \frac{\hat{\omega}\left(\boldsymbol{X}_\tau\right)}{\phi} - \frac{\gamma}{\phi}\hat{c}_\tau, \tag{136}$$

where $\hat{c}_\tau$ is consumption that the consumer plans, at time t, to enjoy at time $\tau \geq t$. Because under the agent's subjective model the Euler equation holds, we have

$$\hat{c}_\tau = \hat{c}_t + \frac{1}{\gamma R}\mathbb{E}_t^{BR}\left(\hat{r}^{BR}\left(\boldsymbol{X}_t\right) + \cdots + \hat{r}^{BR}\left(\boldsymbol{X}_{\tau-1}\right)\right). \tag{137}$$

So,

$$\begin{aligned}
\hat{c}_t - \frac{r}{R}k_t &= \mathbb{E}_t^{BR}\left[\sum_{\tau\geq t}\frac{1}{R^{\tau-t}}\left(b_r\hat{r}^{BR}\left(\boldsymbol{X}_\tau\right) + \frac{r}{R}\left(\hat{\mathcal{Y}}^{BR}\left(\boldsymbol{X}_\tau\right) + \frac{\hat{\omega}\left(\boldsymbol{X}_\tau\right)}{\phi} - \frac{\gamma}{\phi}\hat{c}_\tau\right)\right)\right] \\
&= \mathbb{E}_t^{BR}\left[\sum_{\tau\geq t}\frac{1}{R^{\tau-t}}\left(b_r\hat{r}^{BR}\left(\boldsymbol{X}_\tau\right) + \frac{r}{R}\left(\hat{\mathcal{Y}}^{BR}\left(\boldsymbol{X}_\tau\right) + \frac{\hat{\omega}\left(\boldsymbol{X}_\tau\right)}{\phi}\right)\right)\right] - \frac{\gamma}{\phi}\left(A+B\right),
\end{aligned}$$

with

$$\begin{aligned}
A &= \sum_{\tau\geq t}\frac{1}{R^{\tau-t}}\frac{r}{R}\hat{c}_t = \hat{c}_t, \\
B &= \frac{r}{R}\frac{1}{\gamma R}\mathbb{E}_t^{BR}\sum_{\tau\geq t+1}\frac{1}{R^{\tau-t}}\left(\hat{r}^{BR}\left(\boldsymbol{X}_t\right) + \cdots + \hat{r}^{BR}\left(\boldsymbol{X}_{\tau-1}\right)\right) = \frac{r}{\gamma R^2}\mathbb{E}_t^{BR}\sum_{\tau\geq t}\frac{1}{R^{\tau-t}}\left(\sum_{k\geq 1}\frac{1}{R^k}\right)\hat{r}^{BR}\left(\boldsymbol{X}_\tau\right) \\
&= \frac{r}{\gamma R^2}\mathbb{E}_t^{BR}\sum_{\tau\geq t}\frac{1}{R^{\tau-t}}\frac{1}{r}\hat{r}^{BR}\left(\boldsymbol{X}_\tau\right) = -b_r\mathbb{E}_t^{BR}\sum_{\tau\geq t}\frac{1}{R^{\tau-t}}\hat{r}^{BR}\left(\boldsymbol{X}_\tau\right).
\end{aligned}$$

Hence,

$$\hat{c}_t - \frac{r}{R}k_t = \mathbb{E}_t^{BR}\left[\sum_{\tau\geq t}\frac{1}{R^{\tau-t}}\left(\left(1+\frac{\gamma}{\phi}\right)b_r\hat{r}^{BR}\left(\boldsymbol{X}_\tau\right) + \frac{r}{R}\left(\hat{\mathcal{Y}}^{BR}\left(\boldsymbol{X}_\tau\right) + \frac{\hat{\omega}\left(\boldsymbol{X}_\tau\right)}{\phi}\right)\right)\right] - \frac{\gamma}{\phi}\hat{c}_t,$$

i.e.

$$\left(1+\frac{\gamma}{\phi}\right)\hat{c}_t = \frac{r}{R}k_t + \mathbb{E}_t^{BR}\left[\sum_{\tau\geq t}\frac{1}{R^{\tau-t}}\left(\left(1+\frac{\gamma}{\phi}\right)b_r\hat{r}^{BR}\left(\boldsymbol{X}_\tau\right) + \frac{r}{R}\left(\hat{\mathcal{Y}}^{BR}\left(\boldsymbol{X}_\tau\right) + \frac{\hat{\omega}\left(\boldsymbol{X}_\tau\right)}{\phi}\right)\right)\right].$$

This gives the announced result. $\square$

12.2.3 Using general equilibrium considerations

Let us consider the general equilibrium, including income potential from other transfers $\mathcal{T}^{BR}(\boldsymbol{X}_\tau)$ (which come from fiscal policy, which could also be misperceived,[118] or some other source).

$$\hat{y}(\boldsymbol{X}_\tau, N_\tau) = m_y \hat{c}(\boldsymbol{X}_\tau) + \omega(\boldsymbol{X}_\tau)(N_t - N(\boldsymbol{X}_\tau)) + \mathcal{T}^{BR}(\boldsymbol{X}_\tau). \tag{138}$$

The resulting consumption function is as follows.

Proposition 12.2 (Consumption with active fiscal policy) *Consider an agent maximizing over* (c_τ, N_τ) *utility* $U = \mathbb{E}_t^{BR} \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau, N_\tau)$ *subject to the law of motion for wealth* (50), *and with extra transfers* $\mathcal{T}^{BR}(\boldsymbol{X}_\tau)$ *that do not depend on the agent's own actions. Up to second order terms (and for small wealth* k_t*), consumption is:*

$$\hat{c}_t = b_k k_t + \mathbb{E}_t^{BR}\left[\sum_{\tau \geq t} \frac{1}{R^{\tau-t}}\left(b_r \hat{r}^{BR}(\boldsymbol{X}_\tau) + b_Y \hat{c}(\boldsymbol{X}_\tau) + b_k \mathcal{T}^{BR}(\boldsymbol{X}_\tau)\right)\right], \tag{139}$$

with $b_r = \frac{-1}{\gamma R^2}, b_k = \frac{r}{R}\chi$ *with* $\chi = \frac{\phi}{\gamma+\phi}$, *and* $b_Y = \frac{r}{R} m_Y$ *with* $m_Y = \frac{\phi m_y + \gamma}{\phi + \gamma}$. *As usual, the chosen labor supply is given by* $N_t^{\phi} = \omega(\boldsymbol{X}_t) c_t^{-\gamma}$.

So, the situation is a little subtle, as the coefficient on future aggregate consumption is $\frac{r}{R} m_Y$ (so it is equal to $\frac{r}{R}$ if $m_y = 1$), while the coefficient on future transfers and wealth is $\frac{r}{R}\chi$. This is because $\hat{c}(\boldsymbol{X}_\tau)$ implicitly incorporates the effects of future labor supply.

Proof of Proposition 12.2 Income (138) fits in framework (132) by defining $\hat{\mathcal{Y}}^{BR}(\boldsymbol{X}_\tau) := m_y \hat{c}(\boldsymbol{X}_\tau) - \omega(\boldsymbol{X}_\tau)\hat{N}(\boldsymbol{X}_\tau) + \mathcal{T}^{BR}(\boldsymbol{X}_\tau)$ – i.e., linearizing,

$$\hat{\mathcal{Y}}^{BR}(\boldsymbol{X}_\tau) = m_y \hat{c}(\boldsymbol{X}_\tau) - \hat{N}(\boldsymbol{X}_\tau) + \mathcal{T}^{BR}(\boldsymbol{X}_\tau).$$

Using (133), we have, with $b_k := \frac{r}{R}\chi$,

$$\begin{aligned}
\hat{c}_t - b_k k_t &= \mathbb{E}_t^{BR}\left[\sum_{\tau \geq t} \frac{1}{R^{\tau-t}}\left(b_r \hat{r}^{BR}(\boldsymbol{X}_\tau) + b_k\left(m_y \hat{c}(\boldsymbol{X}_\tau) - \hat{N}(\boldsymbol{X}_\tau) + \frac{\hat{\omega}(\boldsymbol{X}_\tau)}{\phi}\right) + b_k \mathcal{T}^{BR}(\boldsymbol{X}_\tau)\right)\right], \\
&= \mathbb{E}_t^{BR}\left[\sum_{\tau \geq t} \frac{1}{R^{\tau-t}}\left(b_r \hat{r}^{BR}(\boldsymbol{X}_\tau) + b_k\left(m_y + \frac{\gamma}{\phi}\right)\hat{c}(\boldsymbol{X}_\tau) + b_k \mathcal{T}^{BR}(\boldsymbol{X}_\tau)\right)\right], \text{ using (129)} \\
&= \mathbb{E}_t^{BR}\left[\sum_{\tau \geq t} \frac{1}{R^{\tau-t}}\left(b_r \hat{r}^{BR}(\boldsymbol{X}_\tau) + \frac{r}{R}\left(\frac{\phi m_y + \gamma}{\phi + \gamma}\right)\hat{c}(\boldsymbol{X}_\tau) + b_k \mathcal{T}^{BR}(\boldsymbol{X}_\tau)\right)\right].
\end{aligned}$$

□

[118] The 2018 NBER WP version of this paper uses

$$\mathcal{T}^{BR}(\boldsymbol{Z}_\tau, \boldsymbol{Z}_t^d) = (1 - m_y)\mathcal{T}(\boldsymbol{Z}_t^d) + m_y \mathcal{T}(\boldsymbol{Z}_\tau) = \mathcal{T}(\boldsymbol{Z}_t^d) + m_y \mathcal{T}(\boldsymbol{Z}_\tau - \boldsymbol{Z}_t^d)$$

which leads to

$$\mathbb{E}_t^{BR}\left[\mathcal{T}^{BR}(\boldsymbol{X}_\tau)\right] = -\frac{r}{R} B_t + m_y \bar{m}^{\tau-t} \mathbb{E}_t\left[d_\tau - r\sum_{u=t}^{\tau-1} d_u\right].$$

12.3 A variant with high marginal propensity to consume

In the preceding model, the current MPC out of current income is low. Here is a simple variant with high MPC. The upshot is that this does not change the form of the macroeconomic behavior, but it does change the microeconomic behavior.

Consider an agent perceiving at date t her date τ income. Here, we generalize the basic setup (51) to:

$$\hat{y}^{BR}\left(N_\tau, \boldsymbol{X}_\tau; \boldsymbol{X}_t\right) = m_y \hat{y}\left(\boldsymbol{X}_\tau\right) + \omega\left(\boldsymbol{X}_\tau\right)\left(N_\tau - N\left(\boldsymbol{X}_\tau\right)\right) + m_{y0}\hat{y}\left(\boldsymbol{X}_t\right). \tag{140}$$

The new term is the last one, $m_{y0}\hat{y}\left(\boldsymbol{X}_t\right)$. It means that the agent anchors her perceptions of future income on current income, with a weight $m_{y0} \in [0,1]$. The rational case corresponds to $(m_{y0}, m_y) = (0,1)$. If the agent imagines that future income will be exactly current income (i.e., complete extrapolation) then $(m_{y0}, m_y) = (1,0)$.[119]

Intuitively, if $m_{y0} > 0$ then the agent "overreacts" to current income. That generates a high MPC out of current income. The next Proposition makes this precise.

Proposition 12.3 (IS curve, anchoring on the present) *Suppose that agents anchor their projection of future income on current income, with a weight m_{y0}. Then, the IS curve of the basic behavioral setup (Proposition 6.3) still holds, but replacing in M and σ the parameters (m_Y, m_r) by $(m_Y', m_r') = \frac{1}{1-\chi m_{y0}}(m_Y, m_r)$, where $\chi = \frac{\phi}{\gamma+\phi}$. The values of M and σ increase with the anchoring m_{y0}.*

At the same time, the marginal propensity to consume out of a 0-persistence shock to current income is

$$MPC = \chi m_{y0} + \chi \frac{r}{R} m_y. \tag{141}$$

Proof of Proposition 12.3 The general equilibrium result comes straight from Proposition 12.2. We just set the perceived extra transfer to $\mathcal{T}^{BR}\left(\boldsymbol{X}_\tau\right) := m_{y0}\hat{y}\left(\boldsymbol{X}_t\right)$. The consumption policy becomes (we take the case with zero wealth), with $b_r = \frac{-1}{\gamma R^2}$, $b_k = \frac{r}{R}\chi$ and $\chi = \frac{\phi}{\gamma+\phi}$:

$$\begin{aligned}
\hat{c}_t &= \mathbb{E}_t^{BR}\left[\sum_{\tau\geq t}\frac{1}{R^{\tau-t}}\left(b_r\hat{r}^{BR}\left(\boldsymbol{X}_\tau\right) + \frac{r}{R}m_Y\hat{c}\left(\boldsymbol{X}_\tau\right) + b_k m_{y0}\hat{y}\left(\boldsymbol{X}_t\right)\right)\right], \\
&= \chi m_{y0}\hat{y}\left(\boldsymbol{X}_t\right) + \mathbb{E}_t^{BR}\left[\sum_{\tau\geq t}\frac{1}{R^{\tau-t}}\left(b_r\hat{r}^{BR}\left(\boldsymbol{X}_\tau\right) + \frac{r}{R}m_Y\hat{y}\left(\boldsymbol{X}_\tau\right)\right)\right].
\end{aligned}$$

This gives the MPC. Taking into account that income equals consumption, so that $\hat{c}_t = \hat{y}\left(\boldsymbol{X}_t\right)$, we have:

$$\begin{aligned}
\hat{c}_t &= \frac{1}{1-\chi m_{y0}}\mathbb{E}_t^{BR}\left[\sum_{\tau\geq t}\frac{1}{R^{\tau-t}}\left(b_r\hat{r}^{BR}\left(\boldsymbol{X}_\tau\right) + \frac{r}{R}m_Y\hat{y}\left(\boldsymbol{X}_\tau\right)\right)\right] \\
&= \frac{1}{1-\chi m_{y0}}\mathbb{E}_t\left[\sum_{\tau\geq t}\frac{\bar{m}^{\tau-t}}{R^{\tau-t}}\left(b_r m_r\hat{r}\left(\boldsymbol{X}_\tau\right) + \frac{r}{R}m_Y\hat{y}\left(\boldsymbol{X}_\tau\right)\right)\right].
\end{aligned}$$

This is the expression (53) of the main text, but replacing (m_Y, m_r) by $(m_Y', m_r') = \frac{1}{1-\chi m_{y0}}(m_Y, m_r)$.

[119]Formally, at time t, the agent maximizes utility U subject to the perceived law of motion of wealth as in the main text in Section 2.1, but using (140) for perceived income.

$\square$

There is a minor surprise: when agents anchor more in the present (with a higher m_{y0}), then the IS curve becomes *more* forward looking. The reason is a GE effect. Because present consumption reacts more to current income (higher m_{y0}), GE effects are stronger, and in particular, the impact of future disturbances are amplified.

The *macro* model has the same form as in the baseline version, with a slightly modified values for M and σ. However, the *micro* behavior is very different. Indeed, the MPC out of a 0-persistence innovation to labor income is (141), so quantitatively it is close to χm_{y0}. Estimates from the tax rebate and other literatures point to $m_{y0}\chi \simeq 0.3$ (for example, see Johnson et al. (2006)). Quantitatively, that makes fairly little difference to the value of M. Still, now the GE channel for the transmission of monetary policy is strong, much like in Kaplan et al. (2018).

Discussion Through the mechanism highlighted in this section, we capture features as in Kaplan et al. (2018), where a high MPC is important for the transmission of monetary policy. The impact of interest rates comes in part via intertemporal substitution, and in good part via the GE effect on aggregate income – but for that we need a high MPC.

Here, behavioral economics allows us quite easily to capture a high MPC out of current income. What to make of that ease?

The favorable interpretation is that part of the art of macroeconomics is to find useful metaphors, and this behavioral metaphor is quite useful as it is tractable, intuitive (one understands clearly the worldview of the agent), and quite possibly true to the first order. Certainly, it is much simpler than tracking the heterogeneity among credit-constrained agents as in Kaplan et al. (2018). The unfavorable interpretation is that behavioral economics is too free. My own sense is that we are in a period of explorations of theoretical possibilities, and that this is a good thing. One can at least discipline behavioral models with microeconomic data (see e.g. Taubinsky and Rees-Jones (2017) for the measurement of attention to taxes), and exploring those behavioral models is a fruitful enterprise. Those simple-to-use behavioral models might even become the models of choice to a variety of issues whenever there is non-standard behavior, with the understanding that they might be a metaphor for more complex mechanisms, such as those coming from credit constraints.

12.4 A simpler model of decision

In Section 12.2 we handled future labor supply. That was a bit complicated. So I wish to record here a potentially useful variant, that forgoes the need to think about future labor supply and wages. In the basic decision problem, $\max_{(c_\tau,N_\tau)_{\tau\geq t}} U$ subject to (8) and (50), we add the constraint that

$$N_\tau = N\left(\boldsymbol{X}_\tau\right) \text{ for } \tau > t. \tag{142}$$

This means that to simulate his future labor supply, the agent just imagine he'll do like the rest of the agents. However, we allow the agent to actively think of the optimization as on today's labor supply. This is a very close variant of Woodford (2013), who makes the assumption, verbatim: "I further assume that households have no choice but to supply the hours of work that are demanded by firms, at a wage that is fixed by a union that bargains on behalf of the households. A household then has a single decision each period, which is the amount to spend on consumption."

Proposition 12.4 (Consumption with active fiscal policy) *Consider an agent maximizing utility $U = \mathbb{E}_t^{BR} \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau, N_\tau)$ subject to the law of motion for wealth (6), and the perception (142), and with active fiscal policy, as in the setup of Sections 5.1 and 11.1. Up to second order terms (and for small wealth k_t), consumption is:*

$$\hat{c}_t = b_k k_t + \mathbb{E}_t^{BR}\left[\sum_{\tau \geq t} \frac{1}{R^{\tau-t}} \left(b_r \hat{r}^{BR}(\boldsymbol{X}_\tau) + b_k \hat{c}(\boldsymbol{X}_\tau) + b_k \mathcal{T}^{BR}(\boldsymbol{X}_\tau)\right)\right], \tag{143}$$

where $b_r = \frac{-1}{\gamma R^2}, b_k = \frac{r}{R}$. As usual, the chosen labor supply is given by $N_t^\phi = \omega(\boldsymbol{X}_t)\, c_t^{-\gamma}$.

This model is rather easier to handle, as it obviates the problem of choosing labor supply at all future dates.[120] Then, the proof is much simpler. The result is the same, but we replace $\chi = \frac{\phi}{\phi+\gamma}$ by 1, so that $b_k = \frac{r}{R}$.

12.5 Cognitive discounting: Link with some empirical evidence on expectations

This paper is not the proper place to assess the respective merits of models of expectations formation, which would require a full paper. The limited goal of the present section is to give pointers for such a future assessment.

I offer here a brief discussion of how extant evidence relates to cognitive discounting. Before this, let us note that most evidence is on professional forecasters, whereas the model is about regular consumers and workers: hence, matching professional forecasters is not the main metric. Still, it is useful to think about that.

Three facts are salient in the empirical evidence: (a) looking at the aggregate forecast, past revisions predict future forecast errors (Coibion and Gorodnichenko (2015)), pointing to under-reaction to news; (b) there is evidence of slow incorporation of information (Coibion and Gorodnichenko (2012)); (c) at the individual level, forecasters appear to over-react to news (Bordalo et al. (2018)).

Cognitive discounting generates (a) well, as we shall see, if it is viewed as a theory of aggregate forecasting behavior. However, it does not generate (b). That fact is captured, for example, by behavioral models of slow incorporation of news (Gabaix and Laibson (2002); Mankiw and Reis (2002)). It would be easy to mix cognitive discounting with slow incorporation of information.

Cognitive discounting without embellishments would also not generate (c), as it is more of a model of under-reaction. However, that would be easy to amend: one would assume that forecasters receive noisy signals with noise σ_ε, and they think that their signals are more precise than they truly are: as Bordalo et al. (2018) show, this can account for the individual-level overreaction.

Models of learning with noisy signals (e.g. Angeletos and Huo (2019)) can also get (a) and (b), and also generate no under-reaction or over-reaction in individual forecasts, since individual forecasters act rationally given their information set. If these models had to match strict over-reaction as in (c), they could be amended in the way indicated above (by assuming overconfidence about the precision of private signals), as also discussed by Angeletos and Huo (2019).

[120] Note that I do not allow the agent to think that his future consumption is equal to future aggregate consumption, as I want the agent to feel that spending more today will have an impact in the future.

Momentum in revision and cognitive discounting Here I detail how cognitive discounting might explain (a), when viewed as a theory of the aggregate forecasting agent. Coibion and Gorodnichenko (2015) run the regression:

$$\underbrace{x_{t+h} - F_t x_{t+h}}_{\text{Forecast error}} = c + \beta(\underbrace{F_t x_{t+h} - F_{t-1} x_{t+h}}_{\text{Forecast revision}}) + error_t,$$

and find that forecast revision predicts ex-post forecast errors (when averaging across agents), i.e. $\hat{\beta} > 0$.

I consider the univariate case, and show how cognitive discounting framework maps to this result. The true dynamics is $x_{t+1} = \Gamma x_t + \varepsilon_{t+1}$, but agent perceives instead $x_{t+1} = \bar{m}(\Gamma x_t + \varepsilon_{t+1})$. The subjectively expected value at time t of the future variable x_{t+h} is

$$F_t x_{t+h} = \mathbb{E}_t^{BR} x_{t+h} = (\bar{m}\Gamma)^h x_t. \tag{144}$$

Hence forecast revision for x_{t+h} between date $t-1$ and date t are:

$$\begin{aligned}\text{Forecast revision}_h &= \mathbb{E}_t^{BR} x_{t+h} - \mathbb{E}_{t-1}^{BR} x_{t+h} = (\bar{m}\Gamma)^h x_t - (\bar{m}\Gamma)^{h+1} x_{t-1} \\ &= (\bar{m}\Gamma)^h \Gamma(1-\bar{m}) x_{t-1} + (\bar{m}\Gamma)^h \varepsilon_t.\end{aligned}$$

We note that when $\bar{m} = 1$, then forecast revision is exactly in the rational case: $\Gamma^h \varepsilon_t$, i.e. one iterates the shock forward to revise forecast.

Now we turn to the ex-post forecast error:

$$\begin{aligned}\text{Ex-post forecast error} &= x_{t+h} - \mathbb{E}_t^{BR} x_{t+h} = \Gamma^h x_t - \mathbb{E}_t^{BR} x_{t+h} + FE_{t,t+h}^{rat} \\ &= \Gamma^h(1-\bar{m}^h)\Gamma x_{t-1} + \Gamma^h(1-\bar{m}^h)\varepsilon_t + FE_{t,t+h}^{rat},\end{aligned}$$

where $FE_{t,t+h}^{rat} = \sum_{j=0}^{h-1} \Gamma^j \varepsilon_{t+h-j}$ is the forecast error in the rational case.

We can see already that if $\bar{m} < 1$, forecast revisions predict ex-post forecast errors, as found in Coibion and Gorodnichenko (2015).

For a sharper mapping, we can show that an unconditional regression of ex-post forecast error on forecast revision for forecast horizon h will have a β bounded below by

$$\beta_h \geq \underline{\beta}_h \equiv \frac{1-\bar{m}^h}{\bar{m}^h},$$

which is quite similar to the $\beta_{CG} = \frac{\lambda}{1-\lambda}$ in their theory framework, where λ is the level of information rigidity. If we set $\lambda = 1 - \bar{m}^h$, i.e. my inattention to future variable is the source of rigidity, then we map our theory to their empirical finding. Empirically, they find $\lambda \simeq 0.23$ at the one-period horizon[121], which would give $\bar{m} \simeq 0.73$ – much in line with the calibration. For other horizons, the standard errors become big, but they find $\lambda \in [0.3, 1]$.

In conclusion, applying the cognitive discounting model to the professional forecasters of Coibion and Gorodnichenko (2015), one estimates $\bar{m} \simeq 0.73$. Of course, the model is not about professional forecasters, but about the average consumer, which might have a lower $\bar{m}$. I conclude that use the Coibion and Gorodnichenko (2015), coupled the cognitive discounting parameterization, would be a

[121]See their Figure 1, $\beta = \frac{\lambda}{1-\lambda} = 0.3$, which gives $\lambda = 0.23$.

fruitful path of future research – potentially estimating the whole "term structure of attention" with some m_r and $\bar{m}$.

12.6 Nominal illusion

Let us here explore nominal illusion: the consumer perceives future inflation as

$$\pi^{BR}(\boldsymbol{X}_t) = m_\pi^c \pi(\boldsymbol{X}_t), \tag{145}$$

where $m_\pi^c \in [0,1]$ is the consumer's attention to inflation. This makes it so that the perceived interest rate is:

$$\hat{r}^{BR}(\boldsymbol{X}_t) = m_r\left(i_t - m_\pi^c \mathbb{E}_t\left[\pi\left(\boldsymbol{X}_{t+1}\right)\right] - \bar{r}\right). \tag{146}$$

12.6.1 Impact in main model

In the IS curve (27), that will lead to replacing $\mathbb{E}_t\pi_{t+1}$ by $m_\pi^c\mathbb{E}_t\pi_{t+1}$. The Taylor criterion becomes (35): the equilibrium is determinate iff[122]

$$\phi_\pi + \frac{\left(1-\beta M^f\right)}{\kappa}\phi_x + \frac{\left(1-\beta M^f\right)(1-M)}{\kappa\sigma} > m_\pi^c. \tag{147}$$

Again, bounded rationality makes it easier to satisfy the Taylor criterion.

In the basic model, instead of formulation (a) $i_t - \mathbb{E}_t\pi_{t+1} - r_t^n$, to be very strict the IS curve of Proposition 2.5 should have (b) $i_t - \bar{m}\mathbb{E}_t\pi_{t+1} - r_t^n$. Formulation (a), however, can easily be justified, by assuming the consumer faces a market for savings with real interest rates, and can invest at a guaranteed real rate. Formulation (b) is the natural one if the consumer only has access to a nominal market with no special advice on how to handle the real rate. The economics is anyways almost the same. Then, one just uses the analysis of the present subsection, with $m_\pi^c = \bar{m}$. The formulation adopted in this paper is cleaner intellectually, as it allows to separate the issues of nominal illusion (discussed in the present subsection) from general cognitive discounting.

12.6.2 The economy with fully flexible prices

What happens if the economy has fully flexible prices? To study this, I revisit Galí (2015, Chapter 2.4), with behavioral agents.

I say that the consumer suffers from nominal illusion, i.e. perceives inflation as 145, so that (lightening up the notation by replacing m_π^c by m_π) the perceived interest rate is:

$$\hat{r}^{BR}(\boldsymbol{X}_t) = \bar{r} + m_r\left(i_t - m_\pi \mathbb{E}_t\left[\pi\left(\boldsymbol{X}_{t+1}\right)\right] - \bar{r}\right).$$

I suppose that the central bank follows a Taylor rule

$$i_t = j_t + \phi_\pi \pi_t,$$

with $\phi_\pi \geq 0$.

[122]This is a good classroom exercise, so I leave that as an exercise to the reader (the proof is available upon request; hint: it can be done with almost no calculations, simply relabeling the right variables).

In a model with flexible prices and no capital, the output gap is always $x_t = 0$. The behavioral IS curve still imposes:

$$r^n(\boldsymbol{X}_t) = \hat{r}^{BR}(\boldsymbol{X}_t).$$

Take for simplicity an economy with constant $r_t^n = r^n = j_t$. Then, we have:

$$\phi_\pi \pi_t = m_\pi \mathbb{E}_t[\pi_{t+1}]. \tag{148}$$

When is the equilibrium determinate?

Proposition 12.5 Determinacy in the flexible price economy) *Take the flexible price economy, when the consumer has pays only an attention m_π to inflation (with $m_\pi = 1$ in the rational case). We have determinacy if and only if:*

$$\phi_\pi > m_\pi. \tag{149}$$

Proof. This is just because $\phi_\pi \pi_t = m_\pi \mathbb{E}_t[\pi_{t+1}]$, and we have determinacy iff $\phi_\pi > m_\pi$.$\square$

Hence, we see a similar weakening of the Taylor criterion, from bounded rationality.

12.7 Complements on cognitive discounting: when there are non-trivial deterministic trends

In the basic framework, we dealt with variables with zero trend growth rate. Here I present the more general version where the macro state vector includes potentially trending variables, for instance productivity.

I call the macro state vector $\boldsymbol{S}_t = (\boldsymbol{X}_t, \boldsymbol{K}_t)$. The potentially trending variables are gathered as a vector $\boldsymbol{K}_t$ (I use this letter to evoke "capital" variables, that can trend without bounds), and with more conventional stationary variables, gathered in $\boldsymbol{X}_t$. For instance, $\boldsymbol{K}_t$ might contain the log price level, of a deterministic trend for productivity. Call $\bar{\boldsymbol{X}}$ the mean of stationary variables. I suppose that $\bar{\boldsymbol{X}} = \boldsymbol{G}^{\boldsymbol{X}}(\bar{\boldsymbol{X}}, \boldsymbol{K}, 0)$, for all $\boldsymbol{K}$, so that indeed $\bar{\boldsymbol{X}}$ is a stationary mean.

How would an agent simulate the future? I propose the following model.

At a given time s, the agent simulates the future as follows.

Step 1 (simulate the trend of non-stationary variables): the agent initializes $\boldsymbol{S}_{*,s} = (\bar{\boldsymbol{X}}, \boldsymbol{K}_s)$; and for $t \geq s$, she simulates the process:

$$\boldsymbol{S}_{*,t+1} = \boldsymbol{G}^{\boldsymbol{S}}(\boldsymbol{S}_{*,t}, 0). \tag{150}$$

This gives the "non-stochastic trend" in the economy:

Step 2 (simulate the deviations from the trend found in step 1): the agent initializes $\boldsymbol{S}_s$ at its true value; and then she simulates the whole economy, as in:

$$\boldsymbol{S}_{t+1} = (1 - \bar{m})\boldsymbol{S}_{*,t+1} + \bar{m}\boldsymbol{G}^{\boldsymbol{S}}(\boldsymbol{S}_t, \boldsymbol{\epsilon}_{t+1}). \tag{151}$$

That is, the agent only partially sees the deviations of the economy from its trend.

In Step 1, the simulation handles the basic non-stationarity of the variables. In Step 2, the simulation anchors in the trend value $\boldsymbol{S}_{*,t}$, and enriches it partially to handle the dynamics.

To build intuition, let us take some examples. First, if there is no macro-capital variable, we just generalized the baselines procedure. Indeed, take the case without any capital variable, so $\boldsymbol{S}_t = \boldsymbol{X}_t$.

Then, step 1 just generates $\boldsymbol{S}_{*t} = \bar{\boldsymbol{X}}$, and step 2 generates what we had above (see (8)), $\boldsymbol{X}_{t+1} = (1-\bar{m})\,\bar{\boldsymbol{X}} + \bar{m}\boldsymbol{G}^{\boldsymbol{X}}\left(\boldsymbol{X}_t, \boldsymbol{\epsilon}_{t+1}\right)$.

Next, enrich the case with $\boldsymbol{K}_{t+1} = \boldsymbol{K}_t + g + b\boldsymbol{X}_t$, where g is some trend growth rate. For instance, $\boldsymbol{K}_t$ could be the permanent part of log productivity. Then, step 1 gives $\boldsymbol{K}_{*t} = \boldsymbol{K}_0 + gt$ and $\boldsymbol{X}_{*t} = \bar{\boldsymbol{X}}$. That is, the simulation sees the baseline. Next, Step 2 gives deviations from that benchmark. Then, under the BR simulation, $\mathbb{E}_t^{BR}\left[\hat{\boldsymbol{S}}_{t+k}\right] = \bar{m}^k \mathbb{E}_t^{BR}\left[\hat{\boldsymbol{S}}_{t+k}\right]$, where $\hat{\boldsymbol{S}}_\tau := \boldsymbol{S}_\tau - (0, \boldsymbol{K}_t + g(\tau - t))$ is the deviation from the baseline. This phenomenon is general, as the next Proposition records.

Proposition 12.6 *Suppose that we have a system:*

$$\boldsymbol{K}_{t+1} = b^{\boldsymbol{K}}_{\boldsymbol{K}}\boldsymbol{K}_t + b^{\boldsymbol{K}}_{\boldsymbol{X}}\boldsymbol{X}_t + b^{\boldsymbol{K}} + \varepsilon^{\boldsymbol{K}}_{t+1}, \tag{152}$$

$$\boldsymbol{X}_{t+1} = b^{\boldsymbol{X}}_{\boldsymbol{X}}\boldsymbol{X}_t + b^{\boldsymbol{X}} + \varepsilon^{\boldsymbol{X}}_{t+1}, \tag{153}$$

where $\varepsilon^{\boldsymbol{K}}, \varepsilon^{\boldsymbol{X}}$ are mean-zero variables independent across periods. The mean of $\boldsymbol{X}_t$ satisfies:

$$\boldsymbol{X}_* = b^{\boldsymbol{X}}_{\boldsymbol{X}}\boldsymbol{X}_* + b^{\boldsymbol{X}}. \tag{154}$$

The above procedure gives for the trend:

$$\boldsymbol{K}_{*,t+1} = b^{\boldsymbol{K}}_{\boldsymbol{K}}\boldsymbol{K}_{*,t} + b^{\boldsymbol{K}}_{\boldsymbol{X}}\boldsymbol{X}_* + b^{\boldsymbol{K}}, \tag{155}$$

and, calling $\hat{\boldsymbol{S}}_t := \boldsymbol{S}_t - \boldsymbol{S}_{*t}$, we have:

$$\mathbb{E}^{BR}\left[\hat{\boldsymbol{S}}_t\right] = \bar{m}^t \mathbb{E}\left[\hat{\boldsymbol{S}}_t\right]. \tag{156}$$

Proof We note that $b^{\boldsymbol{X}}_{\boldsymbol{K}} = 0$, meaning that the long-run trend doesn't affect the short-run variables. We can rewrite the system (152)-(153) as

$$\boldsymbol{S}_{t+1} = b^{\boldsymbol{S}}_{\boldsymbol{S}}\boldsymbol{S}_t + b^{\boldsymbol{S}}.$$

Step 1 directly gives, we have (155). In other terms, it gives:

$$\boldsymbol{S}_{*,t+1} = b^{\boldsymbol{S}}_{\boldsymbol{S}}\boldsymbol{S}_{*t} + b^{\boldsymbol{S}} + \boldsymbol{\varepsilon}^{\boldsymbol{S}}. \tag{157}$$

Step 2. $\hat{\boldsymbol{S}}_t := \boldsymbol{S}_t - \boldsymbol{S}_{*t}$, so that $\hat{K}_t := K_t - K_{*t}$. Step 2 is here:

$$\boldsymbol{S}_{t+1} = (1-\bar{m})\left(b^{\boldsymbol{S}}_{\boldsymbol{S}}\boldsymbol{S}_{*t} + b^{\boldsymbol{S}}\right) + \bar{m}\left(b^{\boldsymbol{S}}_{\boldsymbol{S}}\boldsymbol{S}_t + b^{\boldsymbol{S}} + \boldsymbol{\varepsilon}^{\boldsymbol{S}}_{t+1}\right) = b^{\boldsymbol{S}}_{\boldsymbol{S}}\boldsymbol{S}_{*t} + b^{\boldsymbol{S}} + \bar{m}b^{\boldsymbol{S}}_{\boldsymbol{S}}\hat{\boldsymbol{S}}_t + \bar{m}\boldsymbol{\varepsilon}^{\boldsymbol{S}}_{t+1}. \tag{158}$$

Subtracting (157) from this gives:

$$\hat{\boldsymbol{S}}_{t+1} = b^{\boldsymbol{S}}_{\boldsymbol{S}}\bar{m}\hat{\boldsymbol{S}}_t + \bar{m}\varepsilon^{\boldsymbol{S}}_{t+1}, \tag{159}$$

so that

$$\mathbb{E}^{BR}\left[\hat{\boldsymbol{S}}_t\right] = \bar{m}^t\left(b^{\boldsymbol{S}}_{\boldsymbol{S}}\right)^t\hat{\boldsymbol{S}}_0.$$

As the rational case corresponds to the special case where $\bar{m} = 1$, we have $\mathbb{E}\left[\hat{\boldsymbol{S}}_t\right] = \left(b_S^S\right)^t \hat{\boldsymbol{S}}_0$. Hence, we have $\mathbb{E}^{BR}\left[\hat{\boldsymbol{S}}_t\right] = \bar{m}^t \mathbb{E}\left[\hat{\boldsymbol{S}}_t\right]$.$\square$

12.8 Another formulation for the impact of fiscal deficits

The formulation in the paper generates a random walk behavior for the public debt (if deficits have mean 0): if there are not future deficits, B_t is constant. Here I study another formulation, where public debt mean-reverts to a fixed value. We shall see that the economics is quite similar.

Debt is $B_t = B_* + \hat{B}_t$, where B_* is the steady-state level of debt. Transfers are

$$\mathcal{T}\left(\hat{B}_t\right) = -\frac{r}{R} B_* + \hat{\mathcal{T}}\left(\hat{B}_t\right),$$

where $-\frac{r}{R} B_*$ is the payment of the permanent part of the debt, and $\hat{\mathcal{T}}\left(\hat{B}_t\right)$ is the payment of its temporary part. For instance, we could have $\hat{\mathcal{T}}\left(\hat{B}_t\right) = -\psi_B \hat{B}_t$. Debt $\hat{B}_t$ is part of the state vector $\boldsymbol{X}_t$, so is seen only with cognitive discounting, and for simplicity we assume perceptions are otherwise correct, i.e. $\hat{\mathcal{T}}^{BR}\left(\hat{B}_t\right) = \hat{\mathcal{T}}\left(\hat{B}_t\right)$.

Proposition 12.7 (Discounted Euler equation with sensitivity to budget deficits, in alternative formulation) *In the alternative formulation above, we have the following variant for Proposition 5.1 on the impact of public debt. Because agents are not Ricardian, a temporary increase $\hat{B}_t$ of public debt increases economic activity. The IS curve (23) becomes:*

$$x_t = M\mathbb{E}_t\left[x_{t+1}\right] + \tilde{b}_d\left(d_t + \left(1 - \frac{r}{R}\right)\hat{B}_t\right) - \sigma\left(i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^{n0}\right), \tag{160}$$

where r_t^{n0} is the "pure" natural rate with zero deficits (derived in (22)), d_t is the budget deficit and $\tilde{b}_d = \frac{r}{R - r m_Y}\frac{\phi}{\phi+\gamma}\left(1 - \bar{m}\right)$ is the sensitivity to temporary debt increases. When agents are rational, $\tilde{b}_d = 0$, but with behavioral agents, $b_d > 0$. We can equivalently write this equation by saying that the behavioral IS curve (24) *holds, but with the following modified natural rate, which captures the stimulative action of deficits:*

$$r_t^n = r_t^{n0} + \frac{\tilde{b}_d}{\sigma}\left(d_t + \left(1 - \frac{r}{R}\right)\hat{B}_t\right). \tag{161}$$

Hence, this formulation is close in spirit to that of the main text. The main difference is the now the "default" perception of debt (when simulating the future at time t) is B_* (the steady state level of debt), rather than B_t. Hence, the stimulative impact now also includes the temporary deviations from steady state debt, $B_t - B_*$. The "impact of the debt" can also be written:

$$d_t + \left(1 - \frac{r}{R}\right)\hat{B}_t = \hat{\mathcal{T}}\left(\hat{B}_t\right) + \hat{B}_t, \tag{162}$$

as $d_t = \hat{\mathcal{T}}_t + \frac{r}{R}\hat{B}_t$.

A simple application is the following. When we have $\hat{\mathcal{T}}\left(\hat{B}_t\right) = -\psi_B \hat{B}_t$, the impact on the debt is:

$$\tilde{b}_d\left(d_t + \left(1 - \frac{r}{R}\right)\hat{B}_t\right) = \tilde{b}_d\left(1 - \psi_B\right)\hat{B}_t. \tag{163}$$

For instance, the natural rate becomes:

$$r_t^n = r_t^{n0} + \frac{\tilde{b}_d}{\sigma} (1 - \psi_B) \hat{B}_t. \tag{164}$$

So, temporarily high debt has a stimulative effects on the natural interest rate, as it makes agents feel richer.

Proof of Proposition 12.7 We apply Proposition 12.2 (setting interest rate deviations to 0, as they are orthogonal to the rest of the discussion), which gives:

$$\hat{c}_t = b_k k_t + \mathbb{E}_t^{BR} \left[\sum_{\tau \geq t} \frac{1}{R^{\tau - t}} \left(b_Y \hat{c} (\boldsymbol{X}_\tau) + b_k \mathcal{T} \left(\hat{B} (\boldsymbol{X}_\tau) \right) \right) \right], \tag{165}$$

with $b_Y = \frac{r}{R} m_Y$. Now, as $k_t = B_t = B_* + \hat{B}(\boldsymbol{X}_t)$, the terms in B_* cancel out in (165), as

$$B_* + \sum_{\tau \geq t} \frac{1}{R^{\tau - t}} \left(-\frac{r}{R} B_* \right) = 0.$$

This is a form of Ricardian equivalence: agents do not react to the "permanent" part of debt. Only the deviations of debt from the baseline B_* matter. Hence, the outcome will be the same as if $B_* = 0$, and we will assume that $B_* = 0$ in what follows.

Using the language of forward operators ($F y_t := y_{t+1}$), we can rewrite (165) as:

$$\hat{c}_t = b_k \hat{B}_t + (1 - \beta \bar{m} F)^{-1} \left(b_Y \hat{c}_t + b_k \hat{\mathcal{T}}_t \right),$$

i.e.,

$$(1 - \beta \bar{m} F) \hat{c}_t = b_Y \hat{c}_t + b_k D_t$$

with

$$D_t := (1 - \beta \bar{m} F) \hat{B}_t + \hat{\mathcal{T}}_t = \hat{\mathcal{T}}_t + \hat{B}_t - \beta \bar{m} \hat{B}_{t+1},$$

and given $\hat{B}_{t+1} = R \left(\hat{B}_t + \hat{\mathcal{T}}_t \right)$, we have, using

$$d_t := \mathcal{T}_t + \frac{r}{R} B_t = \hat{\mathcal{T}}_t + \frac{r}{R} \hat{B}_t \tag{166}$$

$$D_t = (1 - \bar{m}) \left(\hat{\mathcal{T}}_t + \hat{B}_t \right) = (1 - \bar{m}) \left(d_t + \left(1 - \frac{r}{R} \right) \hat{B}_t \right).$$

We have derived

$$\hat{c}_t = M \mathbb{E}_t \left[\hat{c}_{t+1} \right] + \tilde{b}_d \left(d_t + \left(1 - \frac{r}{R} \right) \hat{B}_t \right),$$

with $\tilde{b}_d = \frac{b_k (1 - \bar{m})}{1 - b_Y}$, i.e.

$$\tilde{b}_d = \frac{r}{R - r m_Y} \frac{\phi}{\phi + \gamma} (1 - \bar{m}). \tag{167}$$

Reintegrating interest rates,

$$\hat{c}_t = M \mathbb{E}_t \left[\hat{c}_{t+1} \right] + \tilde{b}_d \left(d_t + \left(1 - \frac{r}{R} \right) \hat{B}_t \right) - \sigma \left(i_t - \mathbb{E}_t \left[\pi_{t+1} \right] - r_t^{n0} \right).$$

The rest of the proof is as in the proof of Proposition 5.1. □

12.9 The ex ante benefits of the possibility of future fiscal policy

Here I investigate further optimal policy at the ZLB. I suppose that we have a "crisis period" between times T_1 and T_2: $I = (T_1, T_2)$. During that period, we have a negative "pure" (pre-government deficits) natural rate ($r_t^{n0} < 0$), so that the ZLB binds. But $r_t^{n0} > 0$ outside that period. The next proposition details how with fiscal policy and behavioral agents, the first best can be restored.

Proposition 12.8 (Optimal mix of fiscal and monetary policy in a ZLB environment). *The following monetary and fiscal policies yield the first best ($x_t = \pi_t = 0$) at all dates:* During the crisis *($t \in (T_1, T_2)$), use fiscal policy*

$$d_t = -\frac{\sigma r_t^{n0}}{b_d},$$

i.e. run a deficit with low interest rates, $i_t = 0$. After the crisis *($t \geq T_2$), pay back the accumulated debt by running a government fiscal surplus and keeping the economy afloat with low rates, e.g. $d_t = R^{-1}\left(B_{T_2} - B_0\right)\left(1 - \rho_d\right)\rho_d^{t-T_2} < 0$ for some $\rho_d \in (0,1)$, and adjust $i_t = r_t^n \equiv r_t^{n0} + \frac{b_d d_t}{\sigma}$ to ensure full macro stabilization, $x_t = \pi_t = 0$.* Before the crisis *($t < T_1$), set $i_t = d_t = 0$.*

Proof. The proof is simply by examination of the basic equations of the NK model, (27)-(28). We adjust the instruments so that $x_t = \pi_t = 0$ at all dates. Note that there are multiple ways to soak up the debt after the crisis, so that $d_t = R^{-1}\left(B_{T_2} - B_0\right)\left(1 - \rho_d\right)\rho_d^{t-T_2}$ is simply indicative. □

The ex-ante preventive benefits of potential ex-post fiscal policy. Proposition 12.8 shows that "the possibility of fiscal policy as ex-post cure produces ex-ante benefits". Imagine that fiscal policy is not available. Then, the economy is depressed at the ZLB during (T_1, T_2). However, it is also depressed before: because the IS curve is forward looking, output threatens to be depressed before T_1, and that can put the economy to the ZLB at a time T_0 before T_1.[123] Hence, the threat of a ZLB-depression in (T_1, T_2) creates an earlier recession at (T_0, T_2) with $T_0 < T_1$. Intuitively, agents feel "if something happens, monetary policy will be impotent, so large dangers loom". However, if the government has fiscal policy in its arsenal, the agents feel "worse case, the government will use fiscal policy, so there is no real threat", and there is no recession in (T_0, T_1). Hence, there is a possibility of fiscal policy as an ex-post cure to produce ex-ante benefits.

In general, monetary and fiscal policies are substitutes (d_t and i_t enter symmetrically in (27)), so a great number of policies achieve the first best. However, fiscal policy d_t helps monetary policy if there is a constraint (e.g. at the ZLB), so the possibility of future fiscal policy is a complement to the monetary policy (as it relieves the ZLB).[124]

[123] Future negative output gaps will creates a low output gap at times 0, 1, say, and so low that a central bank would need negative rates to fight those gaps.

[124] This "second instrument" could be very useful even in normal times, in a richer model with capital. Suppose that consumers get too optimistic about the future: the central bank should raise the interest rate. But then, that depresses investment. We do not get the first best any more, without a second instrument.

12.10 Fiscal policy with government spending

I detail a variant of the model where the government can consume an amount G_t of goods. I call $g_t = \frac{G_t}{\bar{y}}$ the size of government spending as a fraction of steady state output. The following generalizes the basic behavioral IS curve. It extends in a behavioral context previous analyzes of government spending (Eggertsson (2011); Woodford (2011)).

Proposition 12.9 (Model with government consumption). *Given government consumption and deficits, the basic two-equation behavioral New Keynesian model of Proposition 2.5 still holds, except that in the IS curve the natural rate of interest given by*

$$r_t^n = r_t^{n0} + \frac{b_g}{\sigma} \left(g_t - M\mathbb{E}_t \left[g_{t+1} \right] \right) + \frac{b_d}{\sigma} d_t, \tag{168}$$

where r_t^{n0} is the "pure" natural rate of interest that prevails without fiscal policy, $b_g = \frac{\phi}{\phi+\gamma}$, and b_d is given in Proposition 5.1. The corresponding natural rate of consumption (i.e., the consumption level that would prevail if prices were flexible) is:

$$\hat{c}_t^n = -b_g g_t + \frac{1+\phi}{\gamma+\phi} \zeta_t. \tag{169}$$

Proof of Proposition 12.9. In the proof, we consider the case with zero deficit – as deficits enter linearly and are treated in Proposition 2.5. The aggregate resource condition is $Y_t = c_t + G_t = e^{\zeta_t} N_t$ (up to second order terms due to price dispersion), i.e.

$$\hat{c}_t + g_t = \zeta_t + \hat{N}_t. \tag{170}$$

The first order condition for labor supply is still

$$\hat{\omega}_t = \phi \hat{N}_t + \gamma \hat{c}_t,$$

so

$$\hat{\omega}_t = \left(\phi + \gamma \right) \hat{c}_t + \phi \left(g_t - \zeta_t \right). \tag{171}$$

In the frictionless economy we have $\hat{\omega}_t = \zeta_t$, so the natural rate of consumption $\hat{c}_t^n$ satisfies

$$\zeta_t = \left(\phi + \gamma \right) \hat{c}_t^n + \phi \left(g_t - \zeta_t \right), \tag{172}$$

i.e. (169).

In the natural (i.e., flexible-price) economy, we still have (21):

$$\hat{c}_t^n = M\mathbb{E}_t \left[\hat{c}_{t+1}^n \right] - \sigma \hat{r}_t^n,$$

so that the natural rate of interest is:

$$\hat{r}_t^n = r_t^{n0} + \frac{b_g}{\sigma} \left(g_t - M\mathbb{E}_t \left[g_{t+1} \right] \right),$$

where r_t^{n0} is the"pure" natural rate before government intervention, as in (22). The general case with deficits enters additively, as we consider a linearization of the economy. Hence the natural rate of interest is changed. But the IS curve otherwise does not change.

The Phillips curve also does not change. The proof is as at the end of the proof of Proposition 2.5 in Section 11.2. Taking (172) minus (171) gives, using $x_t = \hat{c}_t - \hat{c}_t^n$,

$$\mu_t := \zeta_t - \hat{\omega}_t = (\phi + \gamma)(\hat{c}_t^n - \hat{c}_t) = -(\phi + \gamma)x_t,$$

like in the basic model without fiscal policy. Hence, the Phillips curve does not change. $\square$

The next proposition calculates the corresponding increase in GDP.

Proposition 12.10 (Impact of government spending with passive monetary policy). *Suppose that at time 0, the government spends g_0, financed by a deficit d_0 at time 0, and the central bank does not adjust the interest rate i_0. Then, consumption changes by*

$$\hat{c}_0 = b_d d_0,$$

and GDP changes by:

$$\hat{Y}_0 = g_0 + b_d d_0. \tag{173}$$

Hence, as long as the government spending is deficit-financed ($d_0 = g_0$), we have

$$\hat{Y}_0 = (1 + b_d)\, g_0,$$

and the fiscal multiplier is $1 + b_d$.

Proof of Proposition 12.10. By linearity, we can suppose that we start from the steady state (so $i_0 = r_0^{n0} = \bar{r}$). At time $t \geq 1$, the economy will be fully at the steady state, with no deficit. Hence, $x_t = \pi_t = 0$ for $t \geq 1$. At time $t = 0$, we have (using $i_0 = r_0^{n0}$, and (168))

$$\begin{aligned} x_0 &= M\mathbb{E}_0[x_1] - \sigma(i_0 - \mathbb{E}_0[\pi_1] - r_0^n) = -\sigma(i_0 - r_0^n) \\ &= \sigma\left(\frac{b_g}{\sigma}(g_0 - M\mathbb{E}_0[g_1]) + \frac{b_d}{\sigma}d_0\right) \\ x_0 &= b_g g_0 + b_d d_0. \end{aligned}$$

As $\hat{c}_0^n = -b_g g_0$,

$$\hat{c}_0 = \hat{c}_0^n + x_0 = b_d d_0,$$

and the GDP change is

$$\hat{Y}_0 = g_0 + \hat{c}_0 = g_0 + b_d d_0.$$

$\square$

12.11 Losses from inattention

Here I show the derivation of (74). The derivation is close to the derivation of Lemma 2 in Gabaix (2014). The formulation a little more general than in Gabaix (2014), as I do not assume that $v(a, x, m)$ has the form $v^r(a, m_1 x_1, ..., m_n x_n)$.

Call $a^r(\boldsymbol{S}_t) = a(1, \boldsymbol{S}_t)$ the rational response and $\breve{a}(m, \boldsymbol{S}_t) \coloneqq a(m, \boldsymbol{S}_t) - a(1, \boldsymbol{S}_t)$ the "mistake" in action due to inattention. The losses from attention m (rather than full attention, which would be $m = 1$) are (calling $\varepsilon^2 = \mathbb{E}\left[\|\boldsymbol{S}_t\|^2\right]$, and in the limit of small ε):

$$
\begin{aligned}
L(\boldsymbol{S}_t, m) &= v(a(m, \boldsymbol{S}_t), \boldsymbol{S}_t, 1) - v(a(1, \boldsymbol{S}_t), \boldsymbol{S}_t, 1) \\
&= v(a(1, \boldsymbol{S}_t) + \breve{a}(m, \boldsymbol{S}_t), \boldsymbol{S}_t, 1) - v(a(1, \boldsymbol{S}_t), \boldsymbol{S}_t, 1) \\
&= \frac{1}{2}\left[\breve{a}(m, \boldsymbol{S}_t)' v_{aa}(a(1, \boldsymbol{S}_t), \boldsymbol{S}_t, 1)\, \breve{a}(m, \boldsymbol{S}_t)\right] + o\left(\varepsilon^2\right) \\
&= \frac{1}{2}\left[\breve{a}(m, \boldsymbol{S}_t)' v_{aa}\left(a\left(m^d, 0\right), 0, m^d\right) \breve{a}(m, \boldsymbol{S}_t)\right] + o\left(\varepsilon^2\right) \qquad (174)
\end{aligned}
$$

where the last equality comes from the fact that

$$
v_{aa}(a(1, \boldsymbol{S}_t), \boldsymbol{S}_t, 1) = v_{aa}\left(a\left(m^d, 0\right), 0, m^d\right) + O(\varepsilon).
$$

Let us first take the case where inattention enters linearly, i.e. when $a(m, \boldsymbol{S}_t) = a^r(m\boldsymbol{S}_t) + O(\varepsilon^2)$ where $a^r(\boldsymbol{S}_t) = a(1, \boldsymbol{S}_t)$ is the rational response. This is the simpler case, and covers the situation with m_r, m_y, m_π^f, m_x^f (and it was the case in Gabaix (2014)). We have:

$$
a(m, \boldsymbol{S}_t) = a^r(m\boldsymbol{S}_t) + O\left(\varepsilon^2\right) = a^r(0) + \partial a^r \cdot m\boldsymbol{S}_t + O\left(\varepsilon^2\right)
$$

where ∂a^r is the derivative of $a^r(\boldsymbol{S}_t)$ at 0, so

$$
\breve{a}(m, \boldsymbol{S}_t) = \partial a^r (m - 1)\, \boldsymbol{S}_t + O\left(\varepsilon^2\right)
$$

and as $a_{m,\boldsymbol{S}_t}\left(m^d, 0\right) = \partial a^r$,

$$
\breve{a}(m, \boldsymbol{S}_t) = a_{m,\boldsymbol{S}_t}\left(m^d, 0\right)(m - 1)\, \boldsymbol{S}_t + O\left(\varepsilon^2\right)
$$

and

$$
L(\boldsymbol{S}_t, m) = \left[\frac{1}{2}\boldsymbol{S}_t a'_{m,\boldsymbol{S}}\left(m^d, 0\right) v_{aa}\left(a\left(m^d, 0\right), 0, m^d\right) a_{m,\boldsymbol{S}}\left(m^d, 0\right) \boldsymbol{S}_t\right](1 - m)^2 + o\left(\varepsilon^2\right).
$$

which gives 74, as the agent takes this leading quadratic approximation of her utility losses when choosing optimal attention m.

Let us next take the more complex nonlinear case where $a(m, \boldsymbol{S}_t) = a^r(H(m)\, \boldsymbol{S}_t) + O(\varepsilon^2)$ for a non-linear function $H(m)$, such that $H(0) = 0$ and $H(1) = 1$. This is for instance the case when considering $\bar{m}$, where we have a non-linear response (see (223)). The same algebra shows:

$$
a(m, \boldsymbol{S}_t) = a^r(H(m)\, \boldsymbol{S}_t) + O\left(\varepsilon^2\right) = a^r(0) + \partial a^r \cdot H(m)\, \boldsymbol{S}_t + O\left(\varepsilon^2\right)
$$

so

$$
\breve{a}(m, \boldsymbol{S}_t) = \partial a^r \cdot (H(m) - 1)\, \boldsymbol{S}_t + O\left(\varepsilon^2\right)
$$

As, evaluating derivatives at $\boldsymbol{S}_t = 0$,

$$
a_{m,\boldsymbol{S}_t}(m, 0) = \partial a^r \cdot H'(m)
$$

we also have

$$\breve{a}\left(m, \boldsymbol{S}_t\right) = a_{m,\boldsymbol{S}_t}\left(m^d, 0\right) \frac{H\left(m\right)-1}{H'\left(m^d\right)} \boldsymbol{S}_t + O\left(\varepsilon^2\right) = a_{m,\boldsymbol{S}_t}\left(m^d, 0\right) J\left(m\right)\left(m-1\right) \boldsymbol{S}_t + O\left(\varepsilon^2\right)$$

where

$$J\left(m\right) \coloneqq \frac{H\left(m\right)-1}{\left(m-1\right) H'\left(m^d\right)} \tag{175}$$

and

$$L\left(\boldsymbol{S}_t, m\right) = \left[\frac{1}{2} \boldsymbol{S} a'_{m,\boldsymbol{S}}\left(m^d, 0\right) v_{aa}\left(a\left(m^d, 0\right), 0, m^d\right) a_{m,\boldsymbol{S}}\left(m^d, 0\right) \boldsymbol{S}\right] J\left(m\right)^2 \left(1-m\right)^2 + o\left(\varepsilon^2\right). \tag{176}$$

When $H\left(m\right)$ is linear, $J\left(m\right)=1$, but otherwise it's a bit different from 1. Hence, to be formal, we assume that the agent also does a Taylor expansion of the losses in m, which implies that the agent replaces $J\left(m\right)$ by 1.

12.12 A one-factor economy

We consider a one-factor economy, that will be useful for Proposition 9.2 and 9.3. The primitive is the log TFP level ζ_t, which follows an AR(1) with autocorrelation ρ. As a result, all variables are proportional to ζ_t. Proposition 12.11 records their values. The Taylor rule is assume to be: $i_t = \phi_\pi \pi_t + \phi_x x_t + \bar{r}$.

Proposition 12.11 *In the one-factor economy where all shocks come from TFP and the central banks follow a Taylor rule, we have (with $\hat{r}_t = r_t - \bar{r}$, $\hat{c}_t = c_t - \bar{c}$):*

$$\left(\hat{r}_t^n, x_t, \pi_t, \hat{r}_t, \hat{c}_t, \hat{y}_t, \mu_t\right) = \left(b_\zeta^{r^n}, b_\zeta^x, b_\zeta^\pi, b_\zeta^r, b_\zeta^c, b_\zeta^y, b_\zeta^\mu\right) \zeta_t, \tag{177}$$

with

$$b_\zeta^{r^n} = -\frac{1+\phi}{\sigma\left(\gamma+\phi\right)}\left(1-\rho M\right), \tag{178}$$

$$b_\zeta^x = \frac{\sigma}{1-M\rho+\left(\phi_\pi-\rho\right)\frac{\kappa\sigma}{1-\rho\beta^f}+\sigma\phi_x} b_\zeta^{r^n}, \tag{179}$$

$$b_\zeta^\pi = \frac{\kappa}{1-\rho\beta^f} b_\zeta^x, \tag{180}$$

$$b_\zeta^r = \left(\phi_\pi-\rho\right) b_\zeta^\pi + \phi_x b_\zeta^x, \tag{181}$$

$$b_\zeta^c = b_\zeta^y = b_\zeta^x + \frac{1+\phi}{\gamma+\phi}. \tag{182}$$

$$b_\zeta^\mu = -\left(\phi+\gamma\right) b_\zeta^x \tag{183}$$

Proof of Proposition 12.11 First, (22) gives

$$\hat{r}_t^n = \frac{-1}{\sigma}\frac{1+\phi}{\gamma+\phi}\left(1-\rho M\right)\zeta_t,$$

hence the value of $b_\zeta^{r^n}$. Next, by definition of x_t, $\hat{c}_t = x_t + \hat{c}_t^n$ and using (20), $\hat{c}_t^n = \frac{1+\phi}{\gamma+\phi}\zeta_t$, which gives the value of b_ζ^c. Next, as $\hat{y}_t = \hat{c}_t$, $b_\zeta^y = b_\zeta^c$. Also, b_ζ^r comes from the fact that:

$$\hat{r}_t = i_t - \mathbb{E}_t\left[\pi_{t+1}\right] = \phi_\pi \pi_t + \phi_x x_t - \rho \pi_t.$$

Next, in our AR(1) world, the 2-equation model of Proposition 2.5 reads (with $\beta^f := \beta M^f$):

$$\begin{aligned}
(1 - M\rho)\, x_t &= -\sigma\left(\phi_\pi \pi_t + \phi_x x_t - \rho\pi_t - r_t^n\right), \\
\left(1 - \rho\beta^f\right)\pi_t &= \kappa x_t,
\end{aligned}$$

which solves as:

$$x_t = \frac{1}{\frac{1-M\rho}{\sigma} + \phi_x + \left(\phi_\pi - \rho\right)\frac{\kappa}{1-\rho\beta^f}} r_t^n,$$

which gives the value of b_ζ^x, and then $\pi_t = \frac{\kappa}{\left(1-\rho\beta^f\right)} x_t$ gives the value of b_ζ^π. Finally, we have $\mu_t = -\left(\gamma + \phi\right) x_t$ (see 116), which gives $b_\zeta^\mu = -\left(\phi + \gamma\right) b_\zeta^x$.

12.13 Complements to the 2-period Model

This section gives complements to the 2-period model of Section 10.

Discounted Euler equation in the 2-period model We will see that the consumer satisfies a discounted Euler equation. Call $R = 1/\beta$ the steady state interest rate, so that $R_0 = R + \hat{r}_0$ and the perceived interest rate is: $R_0 = R + m_r \hat{r}_0$. Rewrite (82) as

$$c_0 = b\left(c_0 + \frac{c_1^d + \bar{m}\hat{c}_1}{R + m_r \hat{r}_0}\right),$$

where $c_0^d = b\left(c_0^d + \frac{c_1^d}{R}\right)$. Then, we have:

$$\hat{c}_0 = b\left(\hat{c}_0 + \frac{\bar{m}\hat{c}_1 - \frac{m_r}{R}\hat{r}_0}{R}\right),$$

i.e.

$$\hat{c}_0 = \frac{b}{1-b}\frac{1}{R}\left(\bar{m}\hat{c}_1 - \frac{m_r}{R}\hat{r}_0\right).$$

In the rational model, we have $c_0 = \frac{b}{1-b}\frac{1}{R}c_1$ and $c_0 = c_1 = 1$. Hence, $\frac{b}{1-b}\frac{1}{R} = 1$. We obtain:

$$\hat{c}_0 = \bar{m}\mathbb{E}_0\left[\hat{c}_1\right] - \frac{m_r}{R}\hat{r}_0. \tag{184}$$

This is a "discounted Euler equation" (with discount factor $\bar{m}$), i.e. instead of the rational Euler equation, $\hat{c}_0 = \mathbb{E}\left[\hat{c}_1\right] - \hat{r}_0$. The same factor m gives power to fiscal policy, and yields a discounted Euler equation.

Derivation of (85). Call k_1 the wealth at the beginning of period 1 (before receiving labor income and profit), and $\mathcal{T}_1$ the transfer received from the government, and I_1 the profit income from the oligopolistic firms (so that $\omega_1 N_1 + I_1 = c_1$ when aggregating). The rational value function at time 1 is, given the labor supply fixed at 1 at $t = 1$:

$$V^r(k_1, \mathcal{T}_1) = \max_{c_1} u(c_1, 1) \text{ s.t. } c_1 \le \omega_1 + I_1 + k_1 + \mathcal{T}_1.$$

The decision at time 0 is

$$\operatorname*{smax}_{c_0, N_0 | \bar{m}} u(c_0, N_0) + \beta V^r \left(R_0 (\omega_0 N_0 + I_0 + \mathcal{T}_0 - c_0), \bar{m}\mathcal{T}_1\right),,$$

where $\bar{m}$ is optimized upon in the sparse max. Taking here the $\bar{m}$ as given, then the decision is simply:

$$\max_{c_0, N_0} u(c_0, N_0) + \beta V^r \left(R_0 (\omega_0 N_0 + I_0 + \mathcal{T}_0 - c_0), \bar{m}\mathcal{T}_1\right).$$

The first order conditions are:

$$\begin{aligned} u_{c_0} &= \beta R_0 V_{k_1}, \\ u_{N_0} &= -\omega_0 \beta R_0 V_{k_1}, \end{aligned}$$

so that the intra-period labor supply condition $\omega_0 u_{c_0} + u_{N_0} = 0$ holds. Given that $V_{k_1} = u_{c_1}$, we obtain

$$u_{c_0}(c_0, N_0) = \beta R_0 u_{c_1}(c_1, N_1).$$

Now, we have $V^r_{k_1} = u'(c_1) = u'(k_1 + y_1)$ with $y_1 = \omega_1 N_1 + I_1 + \bar{m}\mathcal{T}_1$, so

$$\frac{1}{c_0} = \frac{\beta R_0}{c_1},$$

with $c_1 = y_1 + R(y_0 - c_0)$ i.e. $c_0 + \frac{c_1}{R} = y_0 + \frac{y_1}{R}$, and with the Euler equation $c_1 = \beta R_0 c_0$:

$$c_0 = \frac{1}{1+\beta}\left(y_0 + \frac{y_1^s}{R}\right) = b\left(y_0 + \frac{y_1 + \bar{m}\hat{y}_1}{R}\right).$$

12.14 Derivation of the Phillips curve in continuous time

Here I show the derivation of the Phillips curve (28) in continuous time. In exploring variants of the NK model, I found it quicker to use this continuous-time derivation than the discrete time version (the 2-period model is also useful for basic conceptual issues).

I use notations from Section 12.1. The Calvo reset probability per unit of time is λdt (i.e. $\theta = 1 - \lambda \Delta t$). I follow the derivation of Proposition 2.5. I use the notations:

$$\delta := r + \lambda, \qquad \alpha = \delta + \xi. \tag{185}$$

The discrete-time $\bar{m}^t$ becomes $e^{-\xi t}$, where $\xi \ge 0$ is the amount of cognitive discounting (rationality corresponds to $\xi = 0$).

If a firm can reset its price at time 0, it sets it to the value in (26):

$$p_0^* - p_0 = \mathbb{E}\left[\int_0^\infty \delta e^{-(\delta+\xi)t}\left(-m_x^f \mu_t + m_\pi^f \int_0^t \pi_s ds\right) dt\right],$$

and using

$$\int_{t=0}^{\infty} e^{-\alpha t} \left(\int_{s=0}^{t} \pi_s ds \right) dt = \int_{s=0}^{\infty} \pi_s ds \left(\int_{t=s}^{\infty} e^{-\alpha t} dt \right) = \int_{s=0}^{\infty} \frac{e^{-\alpha s}}{\alpha} \pi_s ds$$

we have

$$p_0^* - p_0 = \mathbb{E} \left[\int_0^{\infty} e^{-\alpha t} \left(m'_{f\pi} \pi_t - \mu'_t \right) dt \right], \tag{186}$$

with

$$m'_{f\pi} := \frac{\delta}{\alpha} m_{\pi}^{f}, \qquad \mu'_t := \delta m_x^f \mu_t.$$

Inflation at time 0 is $\pi_0 = \dot{p}_0 = \lambda \left(p_0^* - p_0 \right)$. Hence, we have:

$$\pi_0 = \lambda \mathbb{E} \left[\int_0^{\infty} e^{-\alpha t} \left(m'_{f\pi} \pi_t - \mu'_t \right) dt \right]. \tag{187}$$

To solve this, it is useful to use the differentiation operator, $D = \frac{d}{dt}$. With this notation, for a function f (sufficiently regular), the Taylor expansion formula can be written as:

$$f(t + \tau) = \sum_{k=0}^{\infty} f^{(k)}(t) \frac{\tau^k}{k!} = \sum_{k=0}^{\infty} \left(D^k \frac{\tau^k}{k!} \right) f = e^{\tau D} f,$$

i.e.

$$f(t + \tau) = e^{\tau D} f(t). \tag{188}$$

Hence, we have (formally at least):

$$\int_0^{\infty} e^{-\alpha \tau} f(\tau) d\tau = \int_0^{\infty} e^{-\alpha \tau} e^{\tau D} f(0) d\tau = \frac{1}{\alpha - D} f(0). \tag{189}$$

Hence (187) can be rewritten (dropping the expectations for ease of notation):

$$\pi_t = \frac{\lambda}{\alpha - D} \left(m'_{f\pi} \pi_t - \mu'_t \right), \tag{190}$$

and multiplying by $\alpha - D$,

$$(\alpha - D) \pi_t = \lambda \left(m'_{f\pi} \pi_t - \mu'_t \right).$$

We recall that $\mu_t = -(\gamma + \phi) x_t$ (see (116)), so that

$$(r + \lambda + \xi - D) \pi = \lambda \frac{\delta}{\alpha} m_{\pi}^f \pi_t + \kappa x_t, \tag{191}$$

with $\kappa = \lambda \delta m_x^f (\gamma + \phi)$, i.e.

$$\kappa = \bar{\kappa} m_x^f, \tag{192}$$

$$\bar{\kappa} = \lambda (r + \lambda) (\gamma + \phi). \tag{193}$$

This gives the continuous-time version of the Phillips curve in the basic model, equation (28).

$$\left(r+\xi^{f}\right)\pi_t-\dot{\pi}_t=\kappa x_t, \tag{194}$$

with $\xi^f=\xi+\lambda-\lambda\frac{\delta}{\alpha}m_\pi^f$, i.e.

$$\xi^{f}=\xi+\lambda\left(1-\frac{r+\lambda}{r+\lambda+\xi}m_\pi^f\right). \tag{195}$$

12.15 A dynamic programming formulation

Here I show another proof for Proof of Proposition 6.2. It is a bit less intuitive, but may be handy to automatize when considering medium-scale extensions of this model.

In the perceived model, the value function is:[125]

$$V\left(k,\boldsymbol{X}\right)=\max_{c,N}u\left(c,N\right)+\beta\mathbb{E}V\left(R\left(\boldsymbol{X}\right)\left(k+\bar{y}+m_y\hat{y}\left(\boldsymbol{X}\right)+w\left(\boldsymbol{X}\right)\left(N-N\left(\boldsymbol{X}\right)\right)-c\right),\bar{m}\left(\boldsymbol{\Gamma X}+\varepsilon\right)\right), \tag{196}$$

and optimal consumption satisfies $u_c\left(c\left(k,\boldsymbol{X}\right),N\right)=V_k\left(k,\boldsymbol{X}\right)$ (independently of N because utility is separable), so that $c_{\boldsymbol{X}}=\frac{V_{k\boldsymbol{X}}}{u_{cc}}$ and (using the fact that we linearize around $\bar{c}=\bar{N}=1$):

$$c_{\boldsymbol{X}}=-\frac{V_{k\boldsymbol{X}}}{\gamma}, \tag{197}$$

which gives $\hat{c}_t=c_{\boldsymbol{X}}\boldsymbol{X}_t$. Hence, to derive consumption, we simply need to calculate $V_{k\boldsymbol{X}}$.

To calculate $V_{k\boldsymbol{X}}$, I use the general procedure outlined in Gabaix (2016), Section 10.1 — but the present derivation is self-contained. Call $a=(c,N)$ the action, and define:

$$K\left(k,\boldsymbol{X},a\right)=k+\bar{y}+m_y\hat{y}\left(\boldsymbol{X}\right)+w\left(\boldsymbol{X}\right)\left(N-N\left(\boldsymbol{X}\right)\right)-c, \tag{198}$$

so that, taking the deterministic limit:

$$V\left(k,\boldsymbol{X}\right)=\max_{a}u\left(a\right)+\beta V\left(R\left(\boldsymbol{X}\right)K\left(k,\boldsymbol{X},a\right),\bar{m}\boldsymbol{\Gamma X}\right). \tag{199}$$

Behavior at the default, steady state model. I call the default model the model at the steady state ($\boldsymbol{X}=0$), with steady state values for income, wage and interest rate, and only private wealth k_t potentially variable (but close to the steady state value, which is 0). At the default (with constant interest and income), the optimal policy is $c\left(k\right)=\bar{y}+b_k k$, $b_k=\chi\frac{r}{R}$, $\chi=\frac{\phi}{\phi+\gamma}$, and $N\left(k\right)=1-\frac{r}{R}\left(1-\chi\right)k$ (linearizing for small k). This is the permanent-income analog of Section 12.2.1, when the agent consumes a fraction $\frac{r}{R}$ of his wealth every period on higher consumption and leisure. So, using $c^{-\gamma}=V_k$, we have:$V_k\left(k\right)=\left(1+b_k k\right)^{-\gamma}$so at $k=0$

$$V_k=1,\qquad V_{kk}=-\frac{\gamma\phi}{\phi+\gamma}\frac{r}{R},$$

$$N_k=-\frac{r}{R}\frac{\gamma}{\phi+\gamma},\qquad c_k=\frac{\phi}{\phi+\gamma}\frac{r}{R}.$$

[125]Here I use the notation $R\left(\boldsymbol{X}\right)=1+\bar{r}+m_r\hat{r}\left(\boldsymbol{X}\right)$ for the perceived gross interest rate. To lighten up the notation, I use $R\left(\boldsymbol{X}\right)$, rather than the slightly more precise $R^{BR}\left(\boldsymbol{X}\right)$.

We use the notation D_k for the total derivative with respect to k: for a function $f(k,a)$,

$$D_k f(k, a(k)) = \partial_k f + (\partial_a f)\,\partial_k a(k). \tag{200}$$

We do a few more preparatory calculations. As at the default policy preserves capital,

$$K(k, 0, a(k,0)) = \beta k, \tag{201}$$

so that

$$D_k K(k, 0, a(k,0)) = \beta. \tag{202}$$

Also (198) gives:

$$D_k K_{\boldsymbol{X}} = w_{\boldsymbol{X}} N_k = -\frac{r}{R}\frac{\gamma}{\phi+\gamma} w_{\boldsymbol{X}}. \tag{203}$$

We also calculate, using the first order condition $N_{\boldsymbol{X}} = \frac{w_{\boldsymbol{X}}}{\phi} - \frac{\gamma}{\phi} c_{\boldsymbol{X}}$ and $c = y$ in equilibrium:

$$K_{\boldsymbol{X}} = m_y \hat{y}_{\boldsymbol{X}} - N_{\boldsymbol{X}} = m_y \hat{y}_{\boldsymbol{X}} + \frac{\gamma}{\phi}\hat{c}_{\boldsymbol{X}} - \frac{w_{\boldsymbol{X}}}{\phi} = \left(m_y + \frac{\gamma}{\phi}\right)\hat{y}_{\boldsymbol{X}} - \frac{w_{\boldsymbol{X}}}{\phi}.$$

We next proceed the to main derivation. We first differentiate (199) w.r.t. $\boldsymbol{X}$, using the envelope theorem:

$$V_{\boldsymbol{X}}(k, \boldsymbol{X}) = \beta V_k \cdot \left(R(\boldsymbol{X}) K_{\boldsymbol{X}}(k, \boldsymbol{X}, a(k, \boldsymbol{X})) + R_{\boldsymbol{X}} K(k, \boldsymbol{X}, a(k, \boldsymbol{X})), \bar{m}\boldsymbol{\Gamma}\boldsymbol{X}\right) + \beta V_{\boldsymbol{X}} \bar{m}\boldsymbol{\Gamma}.$$

Next, we totally differentiate w.r.t k, and evaluate all derivatives at $(k, \boldsymbol{X}) = (0,0)$, using $R D_k K = 1$,

$$V_{k\boldsymbol{X}} = \beta V_{kk}\left[R K_{\boldsymbol{X}} + R_{\boldsymbol{X}} K\right] + \beta V_k \left[R D_k K_{\boldsymbol{X}} + R_{\boldsymbol{X}} D_k K\right] + \beta V_{k\boldsymbol{X}} \bar{m}\boldsymbol{\Gamma}.$$

This gives:

$$\begin{aligned}
(1-\beta\bar{m}\boldsymbol{\Gamma}) V_{k\boldsymbol{X}} &= V_{kk}\left[\beta R K_{\boldsymbol{X}} + \beta R_{\boldsymbol{X}} K\right] + V_k\left[\beta R D_k K_{\boldsymbol{X}} + \beta R_{\boldsymbol{X}} D_k K\right] \\
&= -\frac{\gamma\phi}{\phi+\gamma}\frac{r}{R}\left[\left(m_y + \frac{\gamma}{\phi}\right)\hat{y}_{\boldsymbol{X}} - \frac{w_{\boldsymbol{X}}}{\phi}\right] - \frac{r}{R}\frac{\gamma}{\phi+\gamma} w_{\boldsymbol{X}} + \beta R_{\boldsymbol{X}} \beta \\
&= -m_Y \hat{y}_{\boldsymbol{X}} \gamma \frac{r}{R} + \beta^2 R_{\boldsymbol{X}},
\end{aligned}$$

with $m_Y := \frac{\phi m_y + \gamma}{\phi+\gamma}$; and

$$V_{k\boldsymbol{X}} = (1-\beta\bar{m}\boldsymbol{\Gamma})^{-1}\left(-\gamma\frac{r}{R} m_Y \hat{y}_{\boldsymbol{X}} + \beta^2 R_{\boldsymbol{X}}\right). \tag{204}$$

Now, we use (197), $c_{\boldsymbol{X}} = -\frac{V_{k\boldsymbol{X}}}{\gamma}$, which gives:

$$c_{\boldsymbol{X}} = (1-\beta\bar{m}\boldsymbol{\Gamma})^{-1}\left[\frac{r}{R} m_Y \hat{y}_{\boldsymbol{X}} - \frac{\beta^2}{\gamma} R_{\boldsymbol{X}}\right]. \tag{205}$$

We are now almost done. Let us observe that

$$(1-\beta\bar{m}\boldsymbol{\Gamma})^{-1}\boldsymbol{X}_t = \sum_{\tau\geq t}(\beta\bar{m}\boldsymbol{\Gamma})^{\tau-t}\boldsymbol{X}_t = \sum_{\tau\geq t}(\beta\bar{m})^{\tau-t}\,\mathbb{E}_t \boldsymbol{X}_\tau.$$

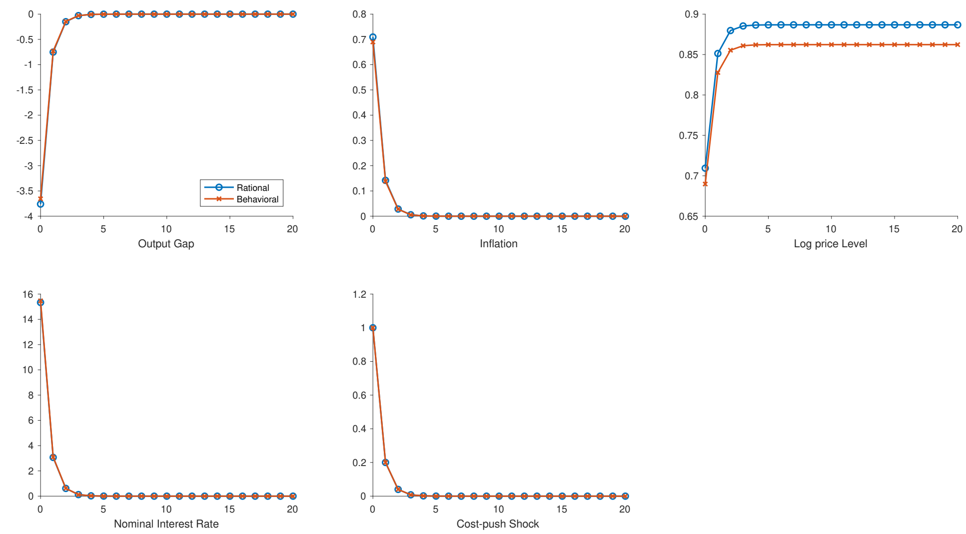

Figure 7: This figure shows the optimal interest rate policy in response to a cost-push shock (ν_t), when the central bank follows the optimal discretionary strategy. The behavior is very similar in the two cases, as the central bank does not rely on future commitments for its optimal policy. This illustrates Proposition 4.3. Units are percentage points. The cost-push shock follows an AR(1) process with autocorrelation $\rho_\nu = 0.2$.

Given that

$$\hat{c}_t = c_{\boldsymbol{X}} \boldsymbol{X}_t, \qquad \hat{y}\left(\boldsymbol{X}_\tau\right) = y_{\boldsymbol{X}} \boldsymbol{X}_\tau, \qquad \hat{r}\left(\boldsymbol{X}_\tau\right) = r_{\boldsymbol{X}} \boldsymbol{X}_\tau,$$

this equivalently expresses:

$$\hat{c}_t = \mathbb{E}_t \sum_{\tau \geq t} (\beta \bar{m})^{\tau - t} \left[\frac{r}{R} m_Y \hat{y}_{\boldsymbol{X}} - \frac{\beta^2}{\gamma} R_{\boldsymbol{X}} \right] \boldsymbol{X}_\tau \tag{206}$$

$$= \mathbb{E}_t \left[\sum_{\tau \geq t} (\beta \bar{m})^{\tau - t} \left(\frac{r}{R} m_Y \hat{y}\left(\boldsymbol{X}_\tau\right) - \frac{\beta^2}{\gamma} m_r \hat{r}\left(\boldsymbol{X}_\tau\right) \right) \right]. \tag{207}$$

which is the statement of Proposition 6.2.

12.16 Optimal no-commitment policy: a graphic illustration

Figure 7 illustrates the optimal policy of Proposition 4.3. It is the analogue of Figure 3, which illustrated the commitment case.

12.17 Robustness check on parameters

Figure 8 shows how Figure 1 changes with other parameters of the literature. The plots are similar, but the output gap at the ZLB is even more strongly negative, as those other parameterizations have somewhat higher values of $\kappa\sigma$.[126]

[126] Note that Werning (2015) parameter's is $\bar{\kappa} = 0.5$ year^{-2} in continuous time, it is $\kappa = \bar{\kappa}\frac{1}{4^2}$ in discrete time with quarterly units.

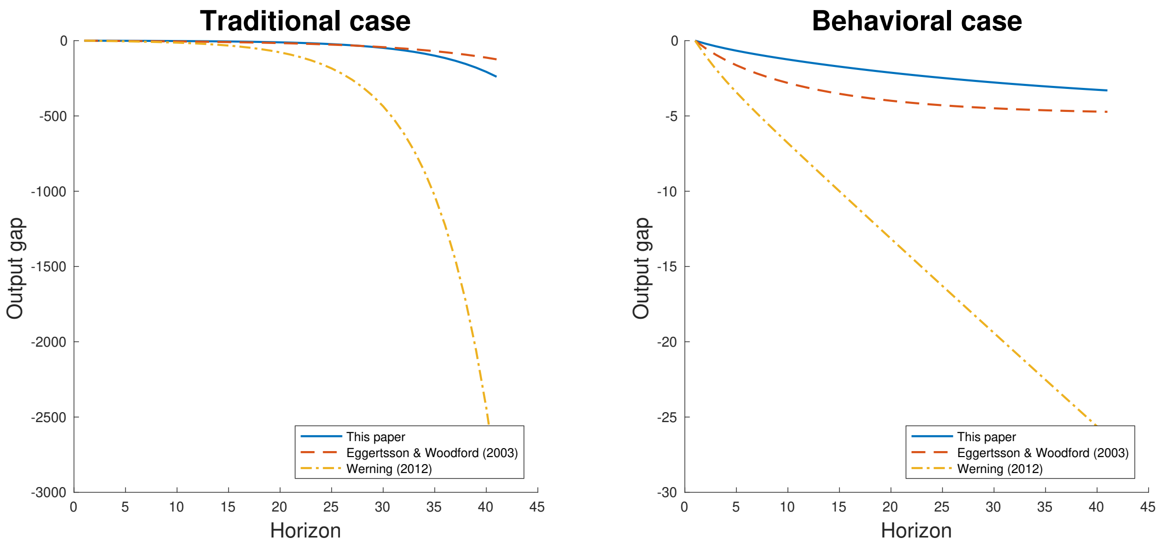

Figure 8: Cost of ZLB under various calibrations. This paper (connected line) uses $\kappa = 0.11$, $\sigma = 0.20$, $\beta = 0.99$ for quarterly time units. Eggertsson and Woodford (2003) (dashed line) uses $\kappa = 0.02$, $\sigma = 0.5$, $\beta = 0.99$. Werning (2015) (dashed and dotted) uses $\kappa = 0.0312$, $\sigma = 1$, $\beta = 0.99$. Units are percentage points.

13 Further proofs

Proof of Lemma 4.1 The proof mimics the ones in Woodford (2003b) and Galí (2015). We have

$$W = -\frac{1}{2} u_c c \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[(\gamma + \phi) x_t^2 + \varepsilon var_i (p_t (i)) \right],$$

where $var_i (p_t (i))$ is the dispersion of prices at time t. As in Woodford (2003, Chapter 6),

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t var_i (p_t (i)) &= \frac{\theta}{(1-\theta)(1-\beta\theta)} \sum_{t=0}^{\infty} \beta^t \pi_t^2 + \frac{\theta}{1-\beta\theta} v_{-1} \\ &= \frac{\gamma+\phi}{\bar{\kappa}} \sum_{t=0}^{\infty} \beta^t \pi_t^2 + \frac{\theta}{1-\beta\theta} v_{-1}, \end{aligned}$$

using (118), and calling $v_{-1} := var_i (p_{-1} (i))$.

Hence,

$$\begin{aligned} W &= -\frac{1}{2} u_c c \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[(\gamma + \phi) x_t^2 + \varepsilon \frac{\gamma+\phi}{\bar{\kappa}} \pi_t^2 \right] - \frac{1}{2} u_c c \varepsilon \frac{\theta}{1-\beta\theta} v_{-1} \\ &= -\frac{1}{2} u_c c (\gamma+\phi) \frac{\varepsilon}{\bar{\kappa}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\pi_t^2 + \frac{\bar{\kappa}}{\varepsilon} x_t^2 \right) + W_- \\ &= -\frac{1}{2} K \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\pi_t^2 + \vartheta x_t^2 \right] + W_-, \end{aligned}$$

with $K := u_c c\left(\gamma+\phi\right)\frac{\varepsilon}{\bar{\kappa}}$,$\vartheta := \frac{\bar{\kappa}}{\varepsilon}$,and

$$W_{-} \quad := \quad -\frac{1}{2}u_c c\varepsilon\frac{\theta}{1-\beta\theta}var_i\left(p_{-1}\left(i\right)\right). \tag{208}$$

Proof of Proposition 4.3: Complements Here is the derivation of i_t. Substitute (43) into the Phillips curve:

$$\pi_t = \beta M^f \mathbb{E}_t\pi_{t+1} + \kappa\left(-\frac{\kappa}{\vartheta}\right)\pi_t + \nu_t \Rightarrow \pi_t = \frac{\beta M^f\vartheta}{\vartheta+\kappa^2}\mathbb{E}_t\pi_{t+1} + \frac{\vartheta}{\vartheta+\kappa^2}\nu_t.$$

Iterating forward:

$$\begin{aligned}\pi_t &= \sum_{\tau=t}^{\infty}\left(\frac{\beta M^f\vartheta}{\vartheta+\kappa^2}\right)^{\tau-t}\frac{\vartheta}{\vartheta+\kappa^2}\mathbb{E}_t\nu_\tau = \sum_{\tau=t}^{\infty}\left(\frac{\beta M^f\vartheta\rho_\nu}{\vartheta+\kappa^2}\right)^{\tau-t}\frac{\vartheta}{\vartheta+\kappa^2}\nu_t = \frac{\vartheta}{\vartheta+\kappa^2}\frac{1}{1-\frac{\beta M^f\vartheta\rho_\nu}{\vartheta+\kappa^2}}\nu_t \\ &= \frac{\vartheta}{\vartheta+\kappa^2-\beta M^f\vartheta\rho_\nu}\nu_t = \vartheta\Phi\nu_t\end{aligned}$$

for $\Phi := (\vartheta+\kappa^2-\beta M^f\vartheta\rho_\nu)^{-1}$. It quickly follows that $x_t = -\kappa\Phi\nu_t$.

Plug these expressions for x_t and π_t into the Behavioral IS curve, we can solve for the nominal interest rate:[127]

$$i_t = \frac{x_t - M\mathbb{E}_t x_{t+1}}{-\sigma} + \mathbb{E}_t\pi_{t+1} + r_t^n = \frac{-\kappa\Phi\nu_t + M\kappa\Phi\mathbb{E}_t\nu_{t+1}}{-\sigma} + \vartheta\Phi\mathbb{E}_t\nu_{t+1} + r_t^n.$$

Again, $\mathbb{E}_t\nu_{t+1} = \rho_\nu\nu_t$. Simplifying the expression gives us

$$i_t = (\kappa\sigma^{-1}(1-M\rho_\nu) + \vartheta\rho_\nu)\Phi\nu_t + r_t^n.$$

Proof of Proposition 5.1 To lighten up the proof, we take the case with no deviation of the interest rate. The general case is the same, as all things enter additively. I give the proof in the case $m_y = 1$ (so that $m_Y = 1$) – the 2018 NBER Working paper version of this paper has the general case.

We start from Proposition 12.2, which gives optimal consumption with fiscal policy.[128] We have, with $b_k = \frac{r}{R}\chi$, $b_Y = \frac{r}{R}$,

$$\begin{aligned}\hat{c}_t &= b_k k_t + \mathbb{E}_t^{BR}\left[\sum_{\tau\geq t}\frac{1}{R^{\tau-t}}\left(b_k\mathcal{T}\left(\boldsymbol{X}_\tau\right) + b_Y\hat{c}\left(\boldsymbol{X}_\tau\right)\right)\right] \\ &= b_k B_t + b_k\sum_{\tau\geq t}\frac{\mathbb{E}_t^{BR}\left[\mathcal{T}\left(\boldsymbol{X}_\tau\right)\right]}{R^{\tau-t}} + F_t,\end{aligned}$$

with

$$F_t := b_Y\mathbb{E}_t^{BR}\sum_{\tau\geq t}\frac{1}{R^{\tau-t}}\hat{c}\left(\boldsymbol{X}_\tau\right) = b_Y\mathbb{E}_t\sum_{\tau\geq t}\frac{\bar{m}^{\tau-t}}{R^{\tau-t}}\hat{c}\left(\boldsymbol{X}_\tau\right). \tag{209}$$

[127]Take for simplicity $r_t^n = 0$.

[128]Note that this uses some notations from Section 6.1, but the logic is not circular of course.

We also use (104)

$$\mathbb{E}_t^{BR}\left[\mathcal{T}\left(\boldsymbol{X}_\tau\right)\right] = -\frac{r}{R}B_t + \bar{m}^{\tau-t}\mathbb{E}_t\left[d_\tau - r\sum_{u=t}^{\tau-1} d_u\right],$$

so we have:

$$\hat{c}_t = b_k B_t + b_k \sum_{\tau\geq t}\frac{1}{R^{\tau-t}}\left(-\frac{r}{R}B_t + \mathbb{E}_t\left[\bar{m}^{\tau-t}\left(d_\tau - r\sum_{u=t}^{\tau-1} d_u\right)\right]\right) + F_t \tag{210}$$

$$= \mathbb{E}_t\left[\sum_{\tau\geq t}\frac{\bar{m}^{\tau-t}}{R^{\tau-t}}b_k\left(d\left(\boldsymbol{X}_\tau\right) - r\sum_{u=t}^{\tau-1} d\left(\boldsymbol{X}_u\right)\right)\right] + F_t. \tag{211}$$

We see that the impact of B_t cancels out, a form of partial Ricardian equivalence. Old debt (B_t) does not make the agent feel richer. But a new deficit today ($d\left(\boldsymbol{X}_t\right)$) does. Let us calculate the cumulative impact of a deficit in (211)

$$J := 1 - r\sum_{k\geq 1}\frac{\bar{m}^k}{R^k} = 1 - r\frac{\frac{\bar{m}}{R}}{1-\frac{\bar{m}}{R}} = 1 - \frac{r\bar{m}}{R-\bar{m}} = \frac{R\left(1-\bar{m}\right)}{R-\bar{m}}.$$

So,

$$\begin{aligned}
\hat{c}_t &= \mathbb{E}_t\left[\sum_{\tau\geq t}\frac{\bar{m}^{\tau-t}}{R^{\tau-t}}b_k J d\left(\boldsymbol{X}_\tau\right)\right] + F_t \\
&= \mathbb{E}_t\left[\sum_{\tau\geq t}\frac{\bar{m}^{\tau-t}}{R^{\tau-t}}\left(b_k J d\left(\boldsymbol{X}_\tau\right) + \tilde{b}_Y\hat{c}\left(\boldsymbol{X}_\tau\right)\right)\right]
\end{aligned}$$

We now solve for equilibrium consumption, like in the derivation of Proposition 2.3:

$$\hat{c}_t = \frac{r}{R}\hat{c}_t + b_k J d_t + \frac{\bar{m}}{R}\mathbb{E}_t[\hat{c}_{t+1}].$$

So, multiplying by R and gathering the $\hat{c}_t$ terms:

$$\hat{c}_t = R b_k J d_t + \bar{m}\mathbb{E}_t[\hat{c}_{t+1}] = b_d d_t + \bar{m}\mathbb{E}_t[\hat{c}_{t+1}],$$

with $b_d = R b_k J$, i.e.

$$b_d = \frac{\phi r R\left(1-\bar{m}\right)}{\left(\phi+\gamma\right)\left(R-\bar{m}\right)}. \tag{212}$$

Re-integrate the interest rate terms, we have:

$$\hat{c}_t = M\mathbb{E}_t\left[\hat{c}_{t+1}\right] + b_d d_t - \sigma\hat{r}_t.$$

The rest is as in the proof of Proposition 2.3. We have:

$$\hat{c}_t^n = M\mathbb{E}_t\left[\hat{c}_{t+1}^n\right] - \sigma\hat{r}_t^{n0},$$

so, with the output gap: $x_t = \hat{c}_t - \hat{c}_t^n$, we have:

$$x_t = M\mathbb{E}_t\left[x_{t+1}\right] + b_d d_t - \sigma\left(\hat{r}_t - \hat{r}_t^{n0}\right).$$

□

Proof of Proposition 6.2: Complements Recall that we have: $c_0 = \mu\Omega$ with

$$\mu = \frac{1}{\sum_{t\geq 0}\beta^{t\psi}q_t^{1-\psi}}, \qquad \Omega = k_0 + \sum_{t\geq 0} q_t y_t^{BR},$$

with $q_t = 1/\prod_{\tau=0}^{t-1}\left(1 + r_\tau^{BR}\right)$. To lighten up the notation, I drop here the BR superscripts—still remembering that we reason in the space of perceived interest rate and incomes.

I linearize around the steady state, which has $q_t = \beta^t$ and $1 = \beta R = \beta\left(1 + \bar{r}\right)$. This gives, at the steady state, $\mu = 1 - \beta = \frac{\bar{r}}{R} = b_y$, and $\Omega = k_0 + \frac{\bar{y}}{1-\beta}$, so $c_0 = \mu\Omega = \frac{r}{R}k + \bar{y}$.

The impact of a change dy_τ is easy to derive:

$$dc_0 = \mu d\Omega = \mu\frac{dy_\tau}{R^\tau} = b_y\frac{dy_\tau}{R^\tau}.$$

This implies:

$$\frac{\partial c_0}{\partial y_\tau} = b_y\frac{1}{R^\tau}.$$

The impact of an interest rate is more delicate. Consider a change change dr_τ, for just one date τ. It creates a bond price change $dq_t = \frac{-1}{R^{t+1}}dr_\tau 1_{t>\tau}$, so that

$$\sum_{t\geq 0} dq_t = \sum_{t\geq 0}\frac{-1}{R^{t+1}}dr_\tau 1_{t>\tau} = \sum_{t\geq\tau+1}\frac{-1}{R^{t+1}}dr_\tau = \frac{-1}{rR^{\tau+1}}dr_\tau.$$

This gives

$$\begin{aligned}\frac{d\mu}{\mu} &= -\mu\left(1-\psi\right)\sum_{t\geq 0}\beta^{t\psi}q_t^{-\psi}dq_t = -\frac{r}{R}\left(1-\psi\right)\sum_{t\geq 0}dq_t \\ &= \left(1-\psi\right)\frac{r}{R}\frac{1}{rR^{\tau+1}}dr_\tau = \left(1-\psi\right)\frac{dr_\tau}{R^{\tau+2}}.\end{aligned}$$

Also,

$$d\Omega = \bar{y}\sum_{t\geq 0}dq_t = \frac{-\bar{y}}{rR^{\tau+1}}dr_\tau.$$

Recalling that $c_0 = \mu\Omega$:

$$\begin{aligned}dc_0 &= \mu\Omega\frac{d\mu}{\mu} + \mu d\Omega = c_0\left(1-\psi\right)\frac{dr_\tau}{R^{\tau+2}} + \frac{r}{R}\frac{-\bar{y}}{rR^{\tau+1}}dr_\tau \\ &= \left(-\psi c_0 + c_0 - \bar{y}\right)\frac{dr_\tau}{R^{\tau+2}} = \left(-\psi c_0 + \frac{rk_0}{R}\right)\frac{dr_\tau}{R^{\tau+2}} \\ &= \frac{b_r}{R^\tau}dr_\tau,\end{aligned}$$

with $b_r = \frac{\frac{\bar{r}}{R}k_0 - \psi c_0}{R^2}$. In the main text, we linearize around $c_0 = \bar{c} = 1$, $k_0 = 0$, so $b_r = \frac{-1}{\gamma R^2}$. We just showed that:

$$\frac{\partial c_0}{\partial r_\tau} = b_r \frac{1}{R^\tau}.$$

Proof of Equation (58) If the firm were free to choose its real (log) price q_{it} freely, it would choose price q_{it}^* maximizing (13), i.e. $e^{q_{it}^*} = \frac{1-\tau_f}{1-\frac{1}{\varepsilon}} MC_t$. The subsidy $\tau_f = \frac{1}{\varepsilon}$ was chosen to eliminate the monopoly distortion on average.

The FOC for the (subjectively) optimal flexible price is $q_i^{*,BR}(\boldsymbol{X}_\tau) := \operatorname{argmax}_{q_i} v^{BR}(q_i, \boldsymbol{X}_\tau)$. For firms facing the Calvo pricing friction, we have, much as in the traditional model, that the price is the weighted average of future optimal prices:[129]

$$q_{it} = (1-\beta\theta) \sum_{\tau \geq t} (\beta\theta)^{\tau-t} \mathbb{E}_t^{BR} \left[q_i^{*,BR}(\boldsymbol{X}_\tau) \right], \tag{213}$$

which is a behavioral counterpart to Galí's (G11).

Given the behavioral perceptions in (56), we have, i.e., linearizing:

$$q_i^{*,BR}(\boldsymbol{X}_\tau) = m_\pi^f \Pi(\boldsymbol{X}_\tau) - m_x^f \mu(\boldsymbol{X}_\tau). \tag{214}$$

Now, by the now usual cognitive discounting (11), we have:

$$\mathbb{E}_t^{BR}[\Pi(\boldsymbol{X}_\tau)] = \bar{m}^{\tau-t}\mathbb{E}_t[\Pi(\boldsymbol{X}_\tau)], \qquad \mathbb{E}_t^{BR}[\mu(\boldsymbol{X}_\tau)] = \bar{m}^{\tau-t}\mathbb{E}_t[\mu(\boldsymbol{X}_\tau)].$$

So, we have the following counterpart to the equation right before (G16):

$$\begin{aligned} q_{it} &= (1-\beta\theta) \sum_{\tau \geq t} (\beta\theta)^{\tau-t} \mathbb{E}_t^{BR} \left[m_\pi^f \Pi(\boldsymbol{X}_\tau) - m_x^f \mu(\boldsymbol{X}_\tau) \right] \\ &= (1-\beta\theta) \sum_{\tau \geq t} (\beta\theta)^{\tau-t} \bar{m}^{\tau-t} \mathbb{E}_t \left[m_\pi^f \Pi_\tau - m_x^f \mu_\tau \right]. \end{aligned}$$

Proof of Proposition 6.6 The state vector is $\boldsymbol{z}_t = \left(x_t, \pi_t, \pi_t^d\right)'$. We can write the system of Proposition 6.5, together with the Taylor rule, as $\mathbb{E}_t \boldsymbol{z}_{t+1} = \boldsymbol{B}\boldsymbol{z}_t + \tilde{\boldsymbol{b}}\left(a_t, \bar{\pi}_t^{CB}\right)'$, for a matrix $\boldsymbol{B}$ and coefficient $\tilde{\boldsymbol{b}}$, where as before $a_t = j_t - r_t^n$. Calculations show that:

$$\boldsymbol{B} = \begin{pmatrix} \frac{1+\sigma\phi_x}{M} + \frac{\sigma\kappa}{M\beta^f} & \frac{\sigma}{M}(\phi_\pi - \frac{1}{\beta^f} - (1-\zeta)\eta) & -\frac{\sigma}{M}(1-\eta-\frac{1}{\beta^f}) \\ -\frac{\kappa}{\beta^f} & \frac{1}{\beta^f} + (1-\zeta)\eta & 1-\eta-\frac{1}{\beta^f} \\ 0 & (1-\zeta)\eta & 1-\eta \end{pmatrix}. \tag{215}$$

To study equilibrium multiplicity, we dispense with the forcing term $\tilde{\boldsymbol{b}}\left(a_t, \bar{\pi}_t^{CB}\right)'$: indeed, the difference between two candidate equilibria will satisfy $\mathbb{E}_t \boldsymbol{z}_{t+1} = \boldsymbol{B}\boldsymbol{z}_t$. Hence, we study the system $\mathbb{E}_t \boldsymbol{z}_{t+1} = \boldsymbol{B}\boldsymbol{z}_t$,

[129] The proof is as in the traditional model: the FOC of problem (16) is $\mathbb{E}^{BR}\sum_{\tau\geq t}(\beta\theta)^{\tau-t} v_{q_i}^{BR}(q_{it}, \boldsymbol{X}_\tau) = 0$ and linearizing around $q_i^{*,BR}(\boldsymbol{X}_\tau)$, the FOC is $\mathbb{E}^{BR}\sum_{\tau\geq t}(\beta\theta)^{\tau-t} v_{q_iq_i}^{BR}\left(q_i^{*,BR}(\boldsymbol{X}_\tau), \boldsymbol{X}_\tau\right) \cdot \left(q_{it} - q_i^{*,BR}(\boldsymbol{X}_\tau)\right) = 0$. Taking the Taylor expansion around 0 disturbances so $q_i^{*,BR}(\boldsymbol{X}_\tau)$ close to 0, the terms $v_{q_iq_i}^{BR}\left(q_i^{*,BR}(\boldsymbol{X}_\tau), \boldsymbol{X}_\tau\right)$ are approximately constant and equal to $v_{q_iq_i}^{BR}(0,0)$ up to first order terms, and the FOC is (up to second order terms) $\mathbb{E}^{BR}\sum_{\tau\geq t}(\beta\theta)^{\tau-t}\left(q_{it} - q_i^{*,BR}(\boldsymbol{X}_\tau)\right) = 0$, which gives (213).

Consider also the characteristic polynomial of $\boldsymbol{B}$, $\Phi(\Lambda) := \det(\Lambda \boldsymbol{I} - \boldsymbol{B})$ (with $\boldsymbol{I}$ the identity matrix), which factorizes as $\Phi(\Lambda) = \prod_{i=1}^{3} (\Lambda - \Lambda_i)$, where the Λ_i's are the eigenvalues of $\boldsymbol{B}$.

Inflation π_t^d is a predetermined variable, not a jump variable. Hence, for the Blanchard and Kahn (1980) determinacy, $\boldsymbol{B}$ needs to have 1 eigenvalue less than 1 in modulus (corresponding to the predetermined variable π_t^d), and 2 greater than 1 (corresponding to the free variables x_t, π_t). This implies that a necessary condition is $\Phi(1) > 0$. We can calculate this term:

$$\frac{M\beta^f}{\kappa\sigma\eta}\Phi(1) = \phi_\pi - 1 + \zeta \frac{(1-\beta M^f)(1+\sigma\phi_x - M)}{\kappa\sigma} > 0, \tag{216}$$

which is equivalent to the behavioral Taylor criterion, (63). This, however, is not sufficient. The sufficient conditions are the "auxiliary Routh-Hurwitz" conditions, to which I now turn.

"Auxiliary Routh-Hurwitz" conditions for determinacy To derive sufficiency conditions, consider a Möbius transformation of the characteristic polynomial:

$$\Psi(\lambda) := (\lambda - 1)^3 \Phi\left(\frac{\lambda+1}{\lambda-1}\right). \tag{217}$$

There is a one-to-one mapping from any (non-unitary) root of $\Psi(\cdot)$ to a root of $\Phi(\cdot)$ by construction: $\lambda \mapsto \Lambda(\lambda) = \frac{\lambda+1}{\lambda-1}$. It is easy to show that $Re(\lambda) < 0$ if and only if $|\Lambda(\lambda)| < 1$. Thus, the conditions for B to have exactly two eigenvalues Λ outside the unit circle is the same as the conditions for $\Psi(\cdot)$ to have exactly two roots λ with positive real parts. We next use the Routh-Hurwitz theory, which has been developed to handle that case.

We write $\Phi(\Lambda)$ as

$$\Phi(\Lambda) = \sum_{i=0}^{3} a_i \Lambda^i$$

with

$$\begin{aligned}
a_3 &= 1, \\
a_2 &= -\left(1 - \eta\zeta + \frac{1}{\beta^f}\right) - C_1 - C_2 < 0, \\
a_1 &= \frac{1}{\beta^f}(1 - \eta\zeta) + \left(\frac{1}{\beta^f} + 1 - \eta\zeta\right) C_1 + (\phi_\pi + 1 - \eta)\, C_2 > 0, \\
a_0 &= -\frac{1-\eta\zeta}{\beta^f} C_1 - (1-\eta)\phi_\pi C_2 < 0,
\end{aligned}$$

with $C_1 \equiv \frac{1+\sigma\phi_x}{M}$ and $C_2 \equiv \frac{\kappa\sigma}{M\beta^f}$ are defined for convenience.

We can then rewrite $\Psi(\lambda)$ as

$$\Psi(\lambda) = \sum_{i=0}^{3} b_i \lambda^i,$$

where

$$
\begin{aligned}
b_3 &= a_3 + a_2 + a_1 + a_0, \\
b_2 &= 3a_3 + a_2 - a_1 - 3a_0, \\
b_1 &= 3a_3 - a_2 - a_1 + 3a_0, \\
b_0 &= a_3 - a_2 + a_1 - a_0.
\end{aligned}
$$

The criterion $b_3 > 0$ is exactly the Taylor criterion in the text. Also, by inspection, $b_0 > 0$ (since $a_3, a_1 > 0$ while $a_2, a_0 < 0$). We assume that ϕ_π, ϕ_x are nonnegative.

Applying the Routh-Hurwitz determinacy criterion for polynomial, $\Psi(\lambda)$ has exactly two roots with positive real parts if and only if when going through the sequence

$$
b_3 \to b_2 \to b_1' := \frac{b_2 b_1 - b_3 b_0}{b_2} \to b_0
$$

signs change exactly twice (see for example Meinsma (1995)). Given b_3 and b_0 are positive, this is possible if and only if (b_2, b_1') are not both positive, i.e. if and only if b_2 and $b_1' := b_2 b_1 - b_3 b_0$ are not both positive (i.e., Not($b_2 > 0$ and $b_1' > 0$)). Thus we have proven the following.

Proposition 13.1 (Equilibrium determinacy with behavioral agents – with backward looking terms) *Assume that ϕ_π, ϕ_x are nonnegative. A necessary and sufficient condition for equilibrium determinacy is that the Taylor criterion (63) in the text holds, and that the following "auxiliary Routh-Hurwitz condition" holds:*

$$
b_2 \text{ and } b_1' := b_2 b_1 - b_3 b_0 \text{ are not both positive.} \tag{218}
$$

I conducted some numerical explorations, making sure that the main Taylor criterion was verified. Then, the auxiliary Routh-Hurwitz condition (218) was always verified. Without claiming that it is actually always verified, it seems that the "hard" economic essence is in the Taylor criterion of the main text, while auxiliary Routh-Hurwitz condition (218) is a much less demanding condition.

Proof of Proposition 9.2 Call $\delta c_t = \hat{c}(m) - \hat{c}(1)$, the difference between the actual consumption $\hat{c}(m)$ given inattention m, and the ideal consumption with full attention (which would have $m = 1$). As the agent pays full attention to the wage, we can write the first order condition for labor supply at decision time as $\hat{N}_t(m) = \frac{-\gamma}{\phi}\hat{c}_t(m) + \frac{1}{\phi}\hat{\omega}_t$, which gives $\delta N_t = \frac{-\gamma}{\phi}\delta c_t$. So, with $a_t = (c_t, N_t)$, we have the following expression:

$$
(\delta a)' u_{aa} \delta a = u_{cc} (\delta c)^2 + u_{NN} (\delta N)^2 = -\left(\gamma + \phi \left(\frac{-\gamma}{\phi}\right)^2\right) (\delta c)^2 = -\frac{\gamma(\gamma + \phi)}{\phi} (\delta c)^2 . \tag{219}
$$

This represents the leading Taylor expansion term of the utility losses from a suboptimal action driven by inattention, times two.

Endogenizing $\bar{m}$ Consumption is (see (53)):

$$
\hat{c}_t(\bar{m}) = \mathbb{E}_t \sum_{\tau \geq 0} \beta^\tau \bar{m}^\tau f_{t+\tau}, \qquad f_t := \tilde{b}_y \hat{y}_t + m_r b_r \hat{r}_t,
$$

with $\tilde{b}_y = m_Y \frac{r}{R} = \frac{\phi m_y^d + \gamma}{\phi + \gamma} \frac{r}{R}$. This gives the following marginal impact of attention $\bar{m}$ on consumption:

$$c_{\bar{m}} := \frac{\partial \hat{c}_t(\bar{m})}{\partial \bar{m}} = \mathbb{E}_t \left[\sum_{\tau \geq 0} \tau \beta^\tau \bar{m}^{\tau-1} f_{t+\tau} \right]. \tag{220}$$

I consider the limit of small time intervals. The reason is in (70), we obtain $v_{aa} = u_{aa}$ in the limit of small time intervals (this is quite well-known; see e.g. Gabaix (2016), footnote 38). Hence, the prefactor for the losses from inattention is:

$$\Lambda = \frac{\gamma (\gamma + \phi)}{\phi} \mathbb{E} \left[c_{\bar{m}}^2 \right]. \tag{221}$$

I now consider the case where all fluctuations are driven by productivity ζ_t, as in the setup of Section 12.12. So, f_t is an AR(1) with autocorrelation ρ, and

$$c_{\bar{m}} = \frac{1}{\bar{m}} \sum_{\tau \geq 0} \tau \beta^\tau \bar{m}^\tau \rho^\tau f_t = \frac{1}{\bar{m}} \frac{\beta \rho \bar{m}}{(1 - \beta \rho \bar{m})^2} f_t$$

where I used:

$$\sum_{\tau \geq 0} \alpha^\tau \tau = \frac{\alpha}{(1-\alpha)^2} \text{ for } |\alpha| < 1. \tag{222}$$

Hence,

$$c_{\bar{m}} = \frac{\beta \rho}{(1 - \beta \rho \bar{m})^2} f_t, \tag{223}$$

As all is by productivity, then $\hat{y}_t = b_\zeta^y \zeta_t$ and $\hat{r}_t = b_\zeta^r \zeta_t$, so

$$f_t = \left(\tilde{b}_y b_\zeta^y + m_r b_r b_\zeta^r \right) \zeta_t,$$

so that $c_{\bar{m}} = c_{\bar{m},\zeta} \zeta_t$ with

$$c_{\bar{m},\zeta} = \frac{\beta \rho}{(1 - \beta \rho \bar{m})^2} \left(\tilde{b}_y b_\zeta^y + m_r b_r b_\zeta^r \right). \tag{224}$$

So, the endogenization of $\bar{m}$ follows from Proposition 9.1, with

$$\lambda \sigma_S^2 = \Lambda = \frac{\gamma (\gamma + \phi)}{\phi} \mathbb{E} \left[c_{\bar{m}}^2 \right] = \frac{\gamma (\gamma + \phi)}{\phi} c_{\bar{m},\zeta}^2 \sigma_\zeta^2. \tag{225}$$

Endogenizing attention to interest rate and income. Let us now endogenize attention to the interest rate. The marginal impact of attention to the interest rate on consumption is: (e.g. starting from (53) or (133))

$$c_{m_r} = \mathbb{E}_t \left[\sum_{\tau \geq 0} \beta^\tau \bar{m}^\tau b_r \hat{r}_{t+\tau} \right],$$

and in the AR(1) case,

$$c_{m_r} = \mathbb{E}_t \left[\sum_{\tau \geq 0} \beta^\tau \bar{m}^\tau b_r b_\zeta^r \zeta_{t+\tau} \right] = \sum_{\tau \geq 0} \beta^\tau \bar{m}^\tau b_r b_\zeta^r \rho^\tau \zeta_t,$$

so that $c_{m_r} = c_{m_r,\zeta}\zeta_t$ with

$$c_{m_r,\zeta} = \frac{1}{1-\beta\bar{m}\rho} b_r b_\zeta^r. \tag{226}$$

So, the announced endogenization of m_r follows from Proposition 9.1.

The endogenization of attention to income is very similar. We have (e.g. starting from (53) or (133))

$$c_{m_y} = \mathbb{E}_t\left[\sum_{\tau\geq 0}\beta^\tau \bar{m}^\tau b_y \hat{y}_{t+\tau}\right],$$

with $b_y = \frac{r}{R}\frac{\phi}{\phi+\gamma}$, and in the AR(1) case,

$$c_{m_y} = \mathbb{E}_t\left[\sum_{\tau\geq 0}\beta^\tau \bar{m}^\tau b_y b_\zeta^y \zeta_{t+\tau}\right] = \mathbb{E}_t\left[\sum_{\tau\geq 0}\beta^\tau \bar{m}^\tau b_y b_\zeta^y \rho^\tau \zeta_t\right],$$

so that $c_{m_y} = c_{m_y,\zeta}\zeta_t$ with

$$c_{m_y,\zeta} = \frac{1}{1-\beta\bar{m}\rho} b_y b_\zeta^y. \tag{227}$$

So, the announced endogenization of m_y follows from Proposition 9.1.

Proof of Proposition 9.3 Let us do some calculations first, using the notations of Section 2.4. At the steady state (which has $q = \mu = 0$, $c = 1$), differentiating (13), and under the optimal subsidy $\tau = \frac{1}{\varepsilon}$, we have:

$$v_{qq} = v_{qq}^{BR} = 1-\varepsilon. \tag{228}$$

Next, as

$$V^{BR}(q_{it}, \boldsymbol{X}_\tau) := \mathbb{E}_t^{BR}\sum_{\tau=t}^{\infty}(\beta\theta)^{\tau-t} v^{BR}(q_{it}, \boldsymbol{X}_\tau)$$

we have, again at the steady state:

$$V_{qq}^{BR} = \frac{1-\varepsilon}{1-\beta\theta}. \tag{229}$$

Endogenization $\bar{m}$ for firms Equation (26) gives:

$$q_{\bar{m}} := \frac{\partial q_{it}(\bar{m})}{\partial \bar{m}} = (1-\beta\theta)\frac{1}{\bar{m}}\sum_{k=0}^{\infty}(\beta\theta\bar{m})^k k\mathbb{E}_t\left[m_\pi^f(\pi_{t+1}+...+\pi_{t+k}) - m_x^f \mu_{t+k}\right]. \tag{230}$$

In total, the expected losses from inattention are: $\frac{1}{2}\Lambda^f(1-m)^2 = \frac{1}{2}V_{qq}\mathbb{E}\left[q_{\bar{m}}^2\right](1-m)^2$, i.e.

$$\Lambda^f = \frac{1-\varepsilon}{1-\beta\theta}\mathbb{E}\left[q_{\bar{m}}^2\right].$$

Now, let us go to the case of a one-factor model driven by ζ_t, the productivity shock, which follows an AR(1) with coefficient ρ, and the central bank follows a Taylor rule. Then, we have $\pi_t = b_\zeta^\pi \zeta_t$ and $\mu_t = b_\zeta^\mu \zeta_t$ in equilibrium. This implies that

$$q_{\bar{m}} = q_{\bar{m},\zeta}\zeta_t \tag{231}$$

for a coefficient

$$
\begin{aligned}
q_{\bar{m},\zeta} &= (1-\beta\theta)\frac{1}{\bar{m}}\sum_{k=0}^{\infty}(\beta\theta\bar{m})^{k}\,k\left(m_{\pi}^{f}\left(\rho+\cdots+\rho^{k}\right)b_{\zeta}^{\pi}-m_{x}^{f}b_{\zeta}^{\mu}\right)\\
&= (1-\beta\theta)\frac{1}{\bar{m}}\sum_{k=0}^{\infty}(\beta\theta\bar{m})^{k}\,k\left(m_{\pi}^{f}\frac{1-\rho^{k}}{1-\rho}\rho b_{\zeta}^{\pi}-m_{x}^{f}b_{\zeta}^{\mu}\right),
\end{aligned}
$$

so, using again (222),

$$
q_{\bar{m},\zeta}=(1-\beta\theta)\frac{1}{\bar{m}}\left[\frac{\rho}{1-\rho}\left(\frac{\beta\theta\bar{m}}{(1-\beta\theta\bar{m})^{2}}-\frac{\beta\theta\bar{m}\rho}{(1-\beta\theta\bar{m}\rho)^{2}}\right)m_{\pi}^{f}b_{\zeta}^{\pi}-\frac{\beta\theta\bar{m}}{(1-\beta\theta\bar{m})^{2}}m_{x}^{f}b_{\zeta}^{\mu}\right] \tag{232}
$$

evaluated at $\bar{m}=\bar{m}^{d}$.

So using Proposition 9.1, we have

$$
\bar{m}^{f}=\mathcal{A}\left(\frac{\lambda^{f}\sigma_{\zeta}^{2}}{\mathcal{K}^{f}},\bar{m}^{d}\right)
$$

with

$$
\lambda^{f}=\frac{\varepsilon-1}{1-\beta\theta}q_{\bar{m},\zeta}^{2}. \tag{233}
$$

Endogenization m_{π}^{f} and m_{x}^{f} The same reasoning holds for other attention factors. Equation (26) gives:

$$
q_{m_{x}^{f}}=(1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta\bar{m})^{k}\,\mathbb{E}_{t}\left[-\mu_{t+k}\right], \tag{234}
$$

so that in the AR(1) case, $q_{m_{x}^{f}}=q_{m_{x}^{f},\zeta}\zeta_{t}$ with:

$$
q_{m_{x}^{f},\zeta}=-\frac{1-\beta\theta}{1-\beta\theta\bar{m}\rho}b_{\zeta}^{\mu}. \tag{235}
$$

Likewise, in the AR(1) case:

$$
\begin{aligned}
q_{m_{\pi}^{f}} &= (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta\bar{m})^{k}\,\mathbb{E}_{t}\left[\pi_{t+1}+...+\pi_{t+k}\right]\\
&= (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta\bar{m})^{k}\left(\rho+\cdots+\rho^{k}\right)b_{\zeta}^{\pi}\zeta_{t}\\
&= (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta\bar{m})^{k}\rho\frac{1-\rho^{k}}{1-\rho}b_{\zeta}^{\pi}\zeta_{t}\\
&= q_{m_{\pi}^{f},\zeta}\zeta_{t},\\
q_{m_{\pi}^{f},\zeta} &= (1-\beta\theta)\frac{\rho}{1-\rho}\left(\frac{1}{1-\beta\theta\bar{m}}-\frac{1}{1-\beta\theta\bar{m}\rho}\right)b_{\zeta}^{\pi},
\end{aligned}
$$

hence

$$
q_{m_{\pi}^{f},\zeta}=\frac{(1-\beta\theta)\,\beta\theta\bar{m}\rho}{(1-\beta\theta\bar{m})(1-\beta\theta\bar{m}\rho)}b_{\zeta}^{\pi}. \tag{236}
$$

where again expressions are evaluated at $\bar{m} = \bar{m}^d$.

References for the Online Appendix

Gjerrit Meinsma. 1995. "Elementary proof of the Routh-Hurwitz test", *Systems & Control Letters* 25(4): 237 - 242.