VARIABLE RARE DISASTERS: AN EXACTLY SOLVED FRAMEWORK FOR TEN PUZZLES IN MACRO-FINANCE*

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This article incorporates a time-varying severity of disasters into the hypothesis proposed by Rietz (1988) and Barro (2006) that risk premia result from the possibility of rare large disasters. During a disaster an asset's fundamental value falls by a time-varying amount. This in turn generates time-varying risk premia and, thus, volatile asset prices and return predictability. Using the recent technique of linearity-generating processes, the model is tractable and all prices are exactly solved in closed form. In this article's framework, the following empirical regularities can be understood quantitatively: (i) equity premium puzzle; (ii) risk-free rate puzzle; (iii) excess volatility puzzle; (iv) predictability of aggregate stock market returns with price-dividend ratios; (v) often greater explanatory power of characteristics than covariances for asset returns; (vi) upward-sloping nominal yield curve; (vii) predictability of future bond excess returns and long-term rates via the slope of the yield curve; (viii) corporate bond spread puzzle; (ix) high price of deep out-of-the-money puts; and (x) high put prices being followed by high stock returns. The calibration passes a variance bound test, as normal-times market volatility is consistent with the wide dispersion of disaster outcomes in the historical record. The model extends to a setting with many factors and to Epstein-Zin preferences. *JEL* Codes: E43, E44, G12.

I. INTRODUCTION

There has been a revival of a hypothesis proposed by Rietz (1988) that the possibility of rare disasters, such as economic depressions or wars, is a major determinant of asset risk premia. Indeed, Barro (2006) has shown that internationally, disasters have been sufficiently frequent and large to make Rietz's proposal viable and account for the high risk premium on equities. Additionally, the recent economic crisis has given disaster risk a renewed salience.

The rare disaster hypothesis is almost always formulated with constant severity of disasters. This is useful for thinking

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about averages but cannot account for some key features of asset markets, such as volatile price-dividend ratios for stocks, volatile bond risk premia, and return predictability. In this article, I formulate a variable-severity version of the rare disasters hypothesis and investigate the impact of time-varying disaster severity on the prices of stocks and bonds as well as on the predictability of their returns.[1]

I show that many asset puzzles can be qualitatively understood using this model. I then demonstrate that a parsimonious calibration allows one to understand the puzzles quantitatively, provided that real and nominal variables are sufficiently sensitive to disasters (which I argue is plausible below).

The proposed framework allows for a very tractable model of stocks and bonds in which all prices are in closed form. In this setting, the following patterns are not puzzles but emerge naturally when the present model has just two shocks: a real one for stocks and a nominal one for bonds.[2]

I.A. *Stock Market: Puzzles about the Aggregates*

1. Equity premium puzzle: The standard consumption-based model with reasonable relative risk aversion (less than 10) predicts a too low equity premium (Mehra and Prescott 1985).
2. Risk-free rate puzzle: Increasing risk aversion leads to a too high risk-free rate in the standard model (Weil 1989).[3]
3. Excess volatility puzzle: Stock prices seem more volatile than warranted by a model with a constant discount rate (Shiller 1981).
4. Aggregate return predictability: Future aggregate stock market returns are partly predicted by price/dividend (P/D) and similar ratios (Campbell and Shiller 1998).

I.B. *Stock Market: Puzzles about the Cross-Section of Stocks*

5. Characteristics vs. covariances puzzle: Stock characteristics (e.g., the P/D ratio) often predict future returns as well

1. A later companion paper, Farhi and Gabaix (2011), studies exchange rates. A brief introduction is Gabaix (2008), but almost all results appear here for the first time.

2. I mention just a few references, but most puzzles have been documented by numerous authors.

3. For this and the above puzzle, the article simply imports from Rietz (1988), Longstaff and Piazzesi (2004), and Barro (2006).

as or better than covariances with risk factors (Daniel and Titman 1997).

I.C. Nominal Bond Puzzles

6. Yield curve slope puzzle: The nominal yield curve slopes up on average. The premium of long-term yields over short-term yields is too high to be explained by a traditional RBC model. This is the bond version of the equity premium puzzle (Campbell 2003).
7. Long-term bond return predictability: A high slope of the yield curve predicts high excess returns on long-term bonds (Macaulay 1938; Fama and Bliss 1987; Campbell and Shiller 1991).
8. Credit spread puzzle: Corporate bond spreads are seemingly higher than warranted by historical default rates (Almeida and Philippon 2007).

I.D. Options Puzzles

9. Deep out-of-the-money puts have higher prices than predicted by the Black-Scholes model (Jackwerth and Rubinstein 1996).
10. When prices of puts on the stock market index are high, so are its future returns (Bollerslev, Tauchen, and Zhou 2009).

To understand the economics of the model, first consider bonds. Consistent with the empirical evidence reviewed shortly, a disaster leads on average to a positive jump in inflation in the model. This has a greater detrimental impact on long-term bonds, so they command a high risk premium relative to short-term bonds. This explains the upward slope of the nominal yield curve. Next, suppose that the size of the expected jump in inflation itself varies. Then, the slope of the yield curve will vary and predict excess bond returns. A high slope will mean-revert and, thus, predicts a drop in the long rate and high returns on long-term bonds. This mechanism accounts for many stylized facts on bonds.

The same mechanism is at work for stocks. Suppose that a disaster reduces the fundamental value of a stock by a time-varying amount. This yields a time-varying risk premium that generates a time-varying price-dividend ratio and the "excess volatility" of stock prices. It also makes stock returns predictable via measures such as the price-dividend ratio. When agents perceive the severity of disasters as low, price-dividend ratios are high and future returns are low.

The model's mechanism also impacts disaster-related assets such as corporate bonds and options. If high-quality corporate bonds default mostly during disasters, then they should command a high premium that cannot be accounted for by their behavior during normal times. The model also generates option prices with a "volatility smirk," that is, a high put price (and, thus, implied volatility) for deep out-of-the-money put options.

After laying out the framework and solving it in closed form, I calibrate it. The values for disasters are essentially taken from Barro and Ursua's (2008) analysis of many countries' disasters, defined as drops in GDP or consumption of 10% or more. The calibration yields results for stocks, bonds, and options consistent with empirical values. The volatilities of the expectation about disaster sizes are very hard to measure directly. However, the calibration generates a steady-state dispersion of anticipations that is lower than the dispersion of realized values. This is shown by "dispersion ratio tests" in the spirit of Shiller (1981), which are passed by the disaster model. By that criterion, the calibrated values in the model appear reasonable. Importantly, they generate a series of fine quantitative predictions. Hence, the model calibrates quite well.

So far, asset price movements come from changes in how badly the asset will perform if a disaster happens (i.e., movements in the asset-specific recovery). The power utility model allows us to think about that quantitatively. However, as found by previous authors (see, for instance, Barro 2009), the power utility model has one important anomalous feature: when the disaster probability goes up, even though risk premia increase, the safe rate decreases so much that asset prices tend to go up. To counteract the strong movement in the short rate, it is useful to have an Epstein-Zin model, which basically weakens this movement, as people's savings behavior is decoupled from their risk aversion. I extend the model to Epstein-Zin preferences only later in the article, as the machinery is substantially more complex. For movements in asset-specific fears, the Epstein-Zin model leads to very similar predictions. However, it makes arguably better predictions for movements in disaster probability. Hence, I recommend the basic power utility model for many asset pricing issues, such as the volatility of stocks, bonds, and the predictability of their returns, but to study the impact of movements in disaster probability, I recommend paying the somewhat higher cost of using the Epstein-Zin model.

Throughout this article, I use the class of "linearity-generating" (LG) processes (Gabaix 2009), which was motivated by the present article. That class keeps all expressions in closed form. The entire article could be rewritten with other processes (e.g., affine-yield models) albeit with considerably more complicated algebra and the need to resort to numerical solutions. The LG class and the affine class yield the same expression to a first-order approximation. The use of the LG processes should thus be viewed as a mere analytical convenience.

Relation to the Literature. A few papers address the issue of time-varying disasters. Longstaff and Piazzesi (2004) consider an economy with constant severity of disasters, but in which stock dividends are a variable, mean-reverting share of consumption. They find a high equity premium and highly volatile stock returns. Veronesi (2004) considers a model in which investors learn about a world economy that follows a Markov chain through two possible economic states, one of which may be a disaster state. His model yields GARCH effects and apparent "overreaction." Weitzman (2007) provides a Bayesian view that the main risk is model uncertainty, as the true volatility of consumption may be much higher than the sample volatility.[4] Unlike the present work, all of those papers neither consider bonds nor study return predictability.

After the present paper was circulated, Wachter (2009) proposed a different model, based on Epstein-Zin utilities, where valuation movements come solely from the stochastic probability of disasters and which analyzes stocks and the short-term rate, but not nominal bonds. The present article, in contrast, allows the stochasticity to come both from movements in the probability of disasters and from the expected recovery rate of various assets, and can work with power utility as well as Epstein-Zin utility. Importantly, it is conceived to easily handle several assets, such as nominal bonds and stocks (as in this article), stocks with different timing of cash flows (Binsbergen, Brandt, and Koijen

4. Another related literature explores the idea that fear of medium-frequency (e.g., yearly) market crashes (rather than macroeconomic disasters) is important for risk premia. Such high-frequency extreme events could be due to the trades of large funds trading under limited liquidity (Gabaix et al. 2003, 2006; Brunnermeier, Nagel, and Pedersen 2008).

forthcoming), particular corporate sectors (Ghandi and Lustig 2011), and exchange rates (Farhi and Gabaix 2011). This choice is motivated by the empirical evidence which shows that several factors are needed to explain risk premia (Fama and French 1993) across stocks and bonds. It is useful to have asset-specific shocks, as single-factor models generate perfect correlations of risk premia across assets, while empirically valuation ratios are not highly correlated across assets (see Section IV.A).

Within the class of rational, representative-agent frameworks that deliver time-varying risk premia, the variable rare disasters model may be a third workable framework, along with the external-habit model of Campbell and Cochrane (CC 1999) and the long-run risk model of Bansal and Yaron (BY 2004). These have proven to be two very useful and influential models. Still, the reader might ask: why do we need another model of time-varying risk premia? The variable rare disasters framework has several useful features, besides the obvious feature that disaster risk might be substantially crucial for financial prices.

First, as emphasized by Barro (2006), the model uses the traditional isoelastic expected utility framework like the majority of models in macroeconomic theory. CC and BY use more complex utility functions with external habit and Epstein and Zin (1989) utility, which are harder to embed in macroeconomic models. In Gabaix (2011) (see also Gourio 2011), I show how the present model (in an endowment economy) can be directly mapped into a production economy with traditional real business cycle features. Hence, the rare-disasters idea brings us closer to the long-sought unification of macroeconomics and finance. Second, the model makes different predictions for the behavior of "tail-sensitive" assets, such as deep out-of-the-money options and high-yield corporate bonds—broadly speaking, the model naturally predicts that such assets command very high premia. Third, the model is particularly tractable. Stock and bond prices have linear closed forms. As a result, asset prices and premia can be derived and analytically understood without recourse to simulations. Fourth, the model easily accounts for some facts that are hard to generate in the CC and BY models. In my proposed model, "characteristics" (such as P/D ratios) predict future stock returns better than market covariances, which is virtually impossible to generate in the CC and BY frameworks. The model also generates a low correlation between consumption growth and stock market returns, which is also hard to achieve in the CC and BY models.

There is a well-developed literature that studies jumps particularly with option pricing in mind. Using options, Liu, Pan, and Wang (2005) calibrate models with constant risk premia and uncertainty aversion, demonstrating the empirical relevance of rare events in asset pricing. Santa-Clara and Yan (2010) also use options to calibrate a model with frequent jumps. Typically, the jumps in these papers happen every few days or months and affect consumption by moderate amounts, whereas the jumps in the rare-disasters literature happen perhaps once every 50 years and are larger. The authors also do not study the impact of jumps on bonds and return predictability.

Section II presents the macroeconomic environment and the cash-flow processes for stocks and bonds. Section III derives equilibrium prices. Section IV proposes a calibration and reports the model's implications for stocks, options, and bonds. Section V discusses various extensions of the model, in particular to an Epstein-Zin economy. The Appendix contains notations and some derivations. An Online Appendix contains supplementary information and extensions.

II. Model Setup

II.A. Macroeconomic Environment

The environment follows Rietz (1988) and Barro (2006), and adds a stochastic probability and severity of disasters. There is a representative agent with utility $\mathbb{E}_0\left[\sum_{t=0}^{\infty} e^{-\rho t}\frac{C_t^{1-\gamma}-1}{1-\gamma}\right]$, where $\gamma \geq 0$ is the coefficient of relative risk aversion and $\rho > 0$ is the rate of time preference. She receives a consumption endowment C_t. At each period $t+1$, a disaster may happen with a probability p_t. If a disaster does not happen, $\frac{C_{t+1}}{C_t} = e^{g_C}$, where g_C is the normal-time growth rate of the economy. If a disaster happens, $\frac{C_{t+1}}{C_t} = e^{g_C} B_{t+1}$, where $B_{t+1} > 0$ is a random variable.[5] For instance, if $B_{t+1} = 0.8$, consumption falls by 20%. To sum up:[6]

$$(1) \qquad \frac{C_{t+1}}{C_t} = e^{g_C} \times \begin{cases} 1 & \text{if there is no disaster at } t+1 \\ B_{t+1} & \text{if there is a disaster at } t+1 \end{cases}.$$

5. Typically, extra i.i.d. noise is added, but given that it never materially affects asset prices, it is omitted here. It could be added without difficulty. Also, countercyclicality of risk premia could easily be added to the model without hurting its tractability.

6. The consumption drop is permanent. One could add mean-reversion after a disaster.

The pricing kernel is the marginal utility of consumption $M_t = e^{-\rho t} C_t^{-\gamma}$, and follows:

$$(2) \qquad \frac{M_{t+1}}{M_t} = e^{-\delta} \times \begin{cases} 1 & \text{if there is no disaster at } t+1 \\ B_{t+1}^{-\gamma} & \text{if there is a disaster at } t+1 \end{cases},$$

where $\delta = \rho + \gamma g_c$, the "Ramsey" discount rate, is the risk-free rate in an economy that would have a zero probability of disasters. The price at t of an asset yielding a stream of dividends $(D_s)_{s \geq t}$ is: $P_t = \frac{\mathbb{E}_t\left[\sum_{s \geq t} M_s D_s\right]}{M_t}$.

II.B. Setup for Stocks

I consider a typical stock i which is a claim on a stream of dividends $(D_{it})_{t \geq 0}$:[7]

$$(3) \qquad \frac{D_{i,t+1}}{D_{it}} = e^{g_{iD}} \left(1 + \varepsilon_{i.t+1}^{D}\right) \times \begin{cases} 1 & \text{if there is no disaster at } t+1 \\ F_{i,t+1} & \text{if there is a disaster at } t+1 \end{cases},$$

where $\varepsilon_{i,t+1}^{D} > -1$ is a mean-zero shock that is independent of the disaster event. It matters only for the calibration of dividend volatility. In normal times, D_{it} grows at an expected rate of g_{iD}. But if there is a disaster, the dividend of the asset is partially wiped out following Longstaff and Piazzesi (2004) and Barro (2006): the dividend is multiplied by a random variable $F_{i,t+1} \geq 0$, which is the recovery rate of the dividend. In other terms, for this individual asset i, there can be a partial "default" in a disaster, without any necessary effect on aggregate consumption and the pricing kernel. When $F_{i,t+1} = 0$, the asset is completely destroyed or expropriated. When $F_{i,t+1} = 1$, there is no dividend loss.

To model the time variation in the asset's recovery rate, I introduce the notion of "resilience" H_{it} of asset i,

$$(4) \qquad H_{it} = p_t \mathbb{E}_t^D \left[B_{t+1}^{-\gamma} F_{i,t+1} - 1\right],$$

where $\mathbb{E}^D$ (resp. $\mathbb{E}^{ND}$) is the expected value conditionally on a *d*isaster happening at $t+1$ (resp. *n*o *d*isaster).[8] In (4), p_t and $B_{t+1}^{-\gamma}$

7. There can be many stocks. The aggregate stock market is a priori not aggregate consumption, because the whole economy is not securitized in the stock market. Indeed, stock dividends are more volatile than aggregate consumption.

8. Later in the paper, when there is no ambiguity (e.g., for $\mathbb{E}\left[B_{t+1}^{-\gamma}\right]$), I will drop the D.

are economy-wide variables, whereas the resilience and recovery rate $F_{i,t+1}$ are stock-specific though typically correlated with the rest of the economy.

When the asset is expected to do well in a disaster (high $F_{i,t+1}$), H_{it} is high—investors are optimistic about the asset. In the cross-section, an asset with higher resilience H_{it} is safer than one with low resilience. As is intuitive, assets with high resilience will command low risk premia.

I specify the dynamics of H_{it} directly rather than through the individual components p_t, B_{t+1}, and $F_{i,t+1}$. I split resilience H_{it} into a constant part H_{i*} and a variable part $\widehat{H}_{it}$:

$$H_{it} = H_{i*} + \widehat{H}_{it},$$

and I postulate the following linearity-generating (Gabaix 2009) process for the variable part $\widehat{H}_{it}$:

$$\widehat{H}_{i,t+1} = \frac{1 + H_{i*}}{1 + H_{it}} e^{-\phi_H} \widehat{H}_{it} + \varepsilon^H_{i,t+1}, \tag{5}$$

where $\mathbb{E}_t \varepsilon^H_{i,t+1} = 0$ and $\varepsilon^H_{i,t+1}$, ε^D_{t+1}, and the disaster event are uncorrelated variables.

To interpret (5), observe that to the leading order, it implies that $\widehat{H}_{i,t+1} \simeq e^{-\phi_H} \widehat{H}_{it} + \varepsilon^H_{i,t+1}$ (as H_{it} hovers around H_{i*}, $\frac{1+H_{i*}}{1+H_{it}}$ is close to 1): $\widehat{H}_{it}$ mean-reverts to 0 at a speed ϕ_H, but has innovations at every period. To the leading order, the process is an autoregressive AR(1) process. However, this is a "twisted" AR(1); the "twist" term $\frac{1+H_{i*}}{1+H_{it}}$ makes prices linear in the factors and independent of the functional form of the noise.[9]

Economically, $\widehat{H}_{it}$ does not jump if there is a disaster. However, one can imagine, for instance, that resilience falls in a disaster. Such a feature could easily be added in the form of an extra negative jump in (5) in case of a disaster. Everything would go through qualitatively, though in addition, equities would be even riskier. However, to keep the model parsimonious, I shrink from postulating that extra feature.

I turn to bonds.

9. The noise ε^H_{t+1} can be heteroskedastic, but its variance need not be spelled out, as it does not enter into the prices. However, the process needs to satisfy $\frac{\widehat{H}_{it}}{(1+H_{i*})} \geq e^{-\phi_H} - 1$ for it to be stable, and also $\widehat{H}_{it} \geq -p - H_{i*}$ to ensure $F_{it} \geq 0$. Hence, the variance needs to vanish in a right neighborhood $\max\left(\left(e^{-\phi_H} - 1\right)\left(1 + H_{i*}\right), -p - H_{i*}\right)$ (see Gabaix 2009).

II.C. Setup for Bonds

The two most salient facts on nominal bonds are arguably the following. First, the nominal yield curve slopes up on average, that is, long-term rates are higher than short-term rates (e.g., Campbell 2003, Table VI). Second, there are stochastic bond risk premia. The risk premium on long-term bonds increases with the difference between the long-term rate and the short-term rate (Fama and Bliss 1987; Campbell and Shiller 1991; Cochrane and Piazzesi 2005). These facts are considered to be puzzles because they are not derived from standard macroeconomic models, which generate risk premia that are too small (Mehra and Prescott 1985).

I propose the following explanation. When a disaster occurs, inflation increases (on average). Since very short-term bills are essentially immune to inflation risk and long-term bonds lose value when inflation is higher, long-term bonds are riskier, so they yield a higher risk premium. Thus, the yield curve slopes up. Moreover, the magnitude of the surge in inflation is time-varying, which generates a time-varying bond premium. If that bond premium is mean-reverting, it generates the Fama-Bliss puzzle. Note that this explanation does not hinge on the specifics of the disaster mechanism. The advantage of the disaster framework is that it allows for formalizing and quantifying the idea in a simple way.

Several authors have models where inflation is higher in bad times, which makes the yield curve slope up. An earlier unification of several puzzles is provided by Wachter (2006), who studies a Campbell and Cochrane (1999) model with extra nominal shocks, and concludes that it explains an upward-sloping yield curve and the Campbell and Shiller (1991) findings. The Brandt and Wang (2003) study is also a Campbell and Cochrane (1999) model, but one where risk aversion depends directly on inflation. Bansal and Shaliastovich (2009) build on Bansal and Yaron (2004). In Piazzesi and Schneider (2007), inflation also rises in bad times, although in a very different model. Finally, Dai and Singleton (2002) and Duffee (2002) present econometric frameworks that deliver the Fama-Bliss and Campbell-Shiller results.

I decompose trend inflation I_t as $I_t = I_* + \widehat{I}_t$, where I_* is its constant part and $\widehat{I}_t$ is its variable part. The variable part of inflation follows the process:

$$\widehat{I}_{t+1} = \frac{1 - I_*}{1 - I_t} \cdot \left(e^{-\phi_I}\widehat{I}_t + 1_{\{\text{Disaster at } t+1\}} J_t\right) + \varepsilon^I_{t+1}, \tag{6}$$

where ε^{I}_{t+1} has mean 0 and is uncorrelated with the realization of a disaster. This equation means, first, that if there is no disaster, $\mathbb{E}_t\widehat{I}_{t+1} = \frac{1-I_*}{1-I_t}e^{-\phi_I}\widehat{I}_t \simeq e^{-\phi_I}\widehat{I}_t$, that is, inflation follows the LG-twisted autoregressive process (Gabaix 2009). Inflation mean-reverts at a rate ϕ_I, with the LG twist $\frac{1-I_*}{1-I_t}$ to ensure tractability. In addition, in case of a disaster, inflation jumps by an amount J_t, decomposed into $J_t = J_* + \widehat{J}_t$, where J_* is the baseline jump in inflation and $\widehat{J}_t$ is the mean-reverting deviation of the jump size from baseline. This jump in inflation makes long-term bonds particularly risky. It follows a twisted autoregressive process and, for simplicity, does not jump during crises:

$$\widehat{J}_{t+1} = \frac{1-I_*}{1-I_t}e^{-\phi_J}\widehat{J}_t + \varepsilon^{J}_{t+1}, \tag{7}$$

where ε^{J}_{t+1} has mean 0. ε^{J}_{t+1} is uncorrelated with disasters but can be correlated with innovations in I_t.

A few more notations are useful. I define $H_\$ = p_t\mathbb{E}_t\left[F_{\$,t+1}B^{-\gamma}_{t+1} - 1\right]$, where $F_{\$,t+1}$ is one minus the default rate on bonds (later this will be useful to differentiate government from corporate bonds). For simplicity, I assume that $H_\$$ is a constant: there will be much economics coming solely from the variations of I_t. I call π_t the variable part of the bond risk premium:

$$\pi_t \equiv \frac{p_t\mathbb{E}_t\left[B^{-\gamma}_{t+1}F_{\$,t+1}\right]}{1+H_\$}\widehat{J}_t. \tag{8}$$

The second notation is only useful when the typical jump in inflation J_* is not zero, and the reader is invited to skip it in the first reading. I parametrize J_* in terms of a variable $\kappa \leq \frac{1-e^{-\phi_I}}{2}$, called the inflation disaster risk premium:[10]

$$\frac{p_t\mathbb{E}_t\left[B^{-\gamma}_{t+1}F_{\$,t+1}\right]J_*}{1+H_\$} = (1-I_*)\,\kappa\left(1-e^{-\phi_I}-\kappa\right), \tag{9}$$

that is, in the continuous time limit: $p_t\mathbb{E}_t\left[B^{-\gamma}_{t+1}F_{\$,t+1}\right]J_* = \kappa\left(\phi_I - \kappa\right)$. A high κ means a high central jump in inflation if there is a disaster. For most of the paper it is enough to think that $J_* = \kappa = 0$.

10. Calculating bond prices in a linearity-generating process sometimes involves calculating the eigenvalues of its generator. I presolve by parameterizing J_* by κ. The upper bound on κ implicitly assumes that J_* is not too large.

II.D. Expected Returns

I conclude the presentation of the economy by stating a general lemma about the expected returns.

LEMMA 1. (Expected returns) Consider an asset i and call $r_{i,t+1}$ the asset's return. Then, the expected return of the asset at t, conditional on no disasters, is:

$$r_{it}^{e} = \frac{1}{1-p_t}\left(e^{\delta} - p_t \mathbb{E}_t^D\left[B_{t+1}^{-\gamma}\left(1+r_{i,t+1}\right)\right]\right) - 1. \tag{10}$$

In the limit of small time intervals,

$$r_{it}^{e} = \delta - p_t \mathbb{E}_t^D\left[B_{t+1}^{-\gamma}\left(1+r_{i,t+1}\right) - 1\right] = r_f - p_t \mathbb{E}_t^D\left[B_{t+1}^{-\gamma} r_{i,t+1}\right], \tag{11}$$

where r_f is the real risk-free rate in the economy:

$$r_f = \delta - p_t \mathbb{E}_t^D\left[B_{t+1}^{-\gamma} - 1\right]. \tag{12}$$

The unconditional expected return is $\left(1-p_t\right) r_{it}^{e} + p_t \mathbb{E}_t^D\left[r_{i,t+1}\right]$.

Proof. It comes from the Euler equation, $1 = \mathbb{E}_t\left[\left(1+r_{i,t+1}\right) M_{t+1}/M_t\right]$, that is:

$$1 = e^{-\delta}\{\underbrace{\left(1-p_t\right)\cdot\left(1+r_{it}^{e}\right)}_{\text{No disaster term}} + \underbrace{p_t \cdot \mathbb{E}_t^D\left[B_{t+1}^{-\gamma}\left(1+r_{i,t+1}\right)\right]}_{\text{Disaster term}}\}. \qquad \blacksquare$$

Equation (10) indicates that only the behavior in disasters (the $r_{i,t+1}$ term) creates a risk premium. It is equal to the risk-adjusted (by $B_{t+1}^{-\gamma}$) expected capital loss of the asset if there is a disaster.

The unconditional expected return on the asset (i.e., without conditioning on no disasters) in the continuous time limit is $r_{it}^{e} - p_t \mathbb{E}_t^D\left[r_{i,t+1}\right]$. Barro (2006) observes that the unconditional expected return and the expected return conditional on no disasters are very close. The possibility of disaster affects primarily the risk premium, and much less the expected loss.

III. ASSET PRICES AND RETURNS

III.A. Stocks

THEOREM 1. (Stock prices) Let $h_{i*} = \ln\left(1+H_{i*}\right)$ and define $\delta_i = \delta - g_{iD} - h_{i*}$, which will be called the stock's effective discount rate. The price of stock i is:

$$P_{it} = \frac{D_{it}}{1 - e^{-\delta_i}} \left(1 + \frac{e^{-\delta_i - h_{i*}} \widehat{H}_{it}}{1 - e^{-\delta_i - \phi_H}}\right). \tag{13}$$

In the limit of short time periods, the price is:

$$P_{it} = \frac{D_{it}}{\delta_i} \left(1 + \frac{\widehat{H}_{it}}{\delta_i + \phi_H}\right). \tag{14}$$

The next proposition links resilience H_{it} and the equity premium.

PROPOSITION 1. (Expected stock returns) The expected return on stock i, conditional on no disasters, is:

$$r^e_{it} = \delta - H_{it}. \tag{15}$$

The equity premium (conditional on no disasters) is $r^e_{it} - r_f = p_t \mathbb{E}_t \left[B^{-\gamma}_{t+1} \left(1 - F_{i,t+1}\right)\right]$, where r_f is the risk-free rate derived in (12). To obtain the unconditional values of these two quantities, subtract $p_t \mathbb{E}^D_t \left[1 - F_{i,t+1}\right]$.

Proof. If a disaster occurs, dividends are multiplied by F_{it}. As $\widehat{H}_{it}$ does not change, $1 + r_{it} = F_{it}$. So returns are, by (11), $r^e_{it} = \delta - p_t \left(\mathbb{E}_t \left[B^{-\gamma}_{t+1} F_{i,t+1}\right] - 1\right) = \delta - H_{it}$. ■

As expected, more resilient stocks (assets that do better in a disaster) have a lower ex ante risk premium (a higher H_{it}). When resilience is constant ($\widehat{H}_{it} \equiv 0$), equation (14) is Barro (2006)'s expression. The P/D ratio is increasing in the stock's resilience h_{i*}.

The key advance in Theorem 1 is that it derives the stock price with a stochastic resilience $\widehat{H}_{it}$. More resilient stocks (high $\widehat{H}_{it}$) have a lower equity premium and a higher valuation. Since resilience $\widehat{H}_{it}$ is volatile, so are price-dividend ratios, in a way that is potentially independent of innovations to dividends. Hence, the model generates a time-varying equity premium and there is "excess volatility," that is, volatility of the stocks unrelated to (normal-times) cash-flow news. As the P/D ratio is stationary, it mean-reverts. Thus, the model generates predictability in stock prices. Stocks with a high P/D ratio will have low returns, and stocks with a low P/D ratio will have high returns. Section IV.B quantifies this predictability. Proposition 11 in the Online Appendix extends equation (14) to a world that has variable expected growth rates of cash flows in addition to variable risk premia.

III.B. Nominal Government Bonds

THEOREM 2. (Bond prices) In the limit of small time intervals, the nominal short-term rate is $r_t = \delta - H_\$ + I_t$, and the price of a nominal zero-coupon bond of maturity T is:

$$(16)\ Z_{\$t}\left(T\right) = e^{-(\delta - H_\$ + I_{**})T}\left(1 - \frac{1 - e^{-\psi_I T}}{\psi_I}\left(I_t - I_{**}\right) - K_T \pi_t\right),$$

$$K_T \equiv \frac{\frac{1-e^{-\psi_I T}}{\psi_I} - \frac{1-e^{-\psi_J T}}{\psi_J}}{\psi_J - \psi_I},$$

where I_t is inflation, π_t is the bond risk premium, $I_{**} \equiv I_* + \kappa$, $\psi_I \equiv \phi_I - 2\kappa$, and $\psi_J \equiv \phi_J - \kappa$. The discrete-time expression is in (42).

Theorem 2 gives a closed-form expression for bond prices. As expected, bond prices decrease with inflation and with the bond risk premium. Indeed, expressions $\frac{1-e^{-\psi_I T}}{\psi_I}$ and K_T are non-negative and increasing in bond maturity T. The term $\frac{1-e^{-\psi_I T}}{\psi_I} I_t$ simply expresses that inflation depresses nominal bond prices and mean-reverts at a (risk-neutral) rate ψ_I. The bond risk premium π_t reduces the price all bonds of positive maturity, but not the short-term rate.

When $\kappa > 0$ (resp. $\kappa < 0$) inflation typically increases (resp. decreases) during disasters. While ϕ_I (resp. ϕ_J) is the speed of mean reversion of inflation (resp. of the bond risk premium, which is proportional to J_t) under the physical probability, ψ_I (resp. ψ_J) is the speed of mean reversion of inflation (resp. of the bond risk premium) under the risk-neutral probability.

I calculate expected bond returns, bond forward rates, and yields, again in the limit of small time intervals.

PROPOSITION 2. (Expected bond returns) Conditional on no disasters, the short-term real return on a short-term bill is $r^e_{\$t}(0) = \delta - H_\$$, and the real excess return on the bond of maturity T is:

$$(17)\qquad r^e_{\$t}\left(T\right) - r^e_{\$t}\left(0\right) = \frac{\frac{1-e^{-\psi_I T}}{\psi_I}\left(\kappa\left(\psi_I + \kappa\right) + \pi_t\right)}{1 - \frac{1-e^{-\psi_I T}}{\psi_I}\left(I_t - I_{**}\right) + K_T \pi_t}$$

$$(18)\qquad = T\left(\kappa\left(\psi_I + \kappa\right) + \pi_t\right) + O\left(T^2\right) + O\left(\pi_t, I_t, \kappa\right)^2$$

$$(19)\qquad = T p_t \mathbb{E}_t\left[B_{t+1}^{-\gamma} F_{\$,t+1}\right] J_t + O\left(T^2\right) + O\left(\pi_t, I_t, \kappa\right)^2.$$

Proof. After a disaster in the next time interval of size $dt \to 0$, inflation jumps by $dI_t = J_t$ and π_t by 0. By (16), the bond price jumps by $dZ_{\$,t}(T) = Z_{\$,t+dt}(T - dt) - Z_{\$,t}(T) = -e^{-(\delta - H_\$ + I_{**})T} \cdot \frac{1-e^{-\psi_I T}}{\psi_I} J_t + O\left(\sqrt{dt}\right)$. Lemma 1 gives the risk premia,

$$r^e_{\$t}(T) - r^e_{\$t}(0) = -p_t \mathbb{E}_t \left[B_{t+1}^{-\gamma} \frac{dZ_{\$,t}(T)}{Z_{\$t}(T)}\right] = \frac{\frac{1-e^{-\psi_I T}}{\psi_I} p_t \mathbb{E}_t \left[B_{t+1}^{-\gamma} F_{\$,t+1} J_t\right]}{1 - \frac{1-e^{-\psi_I T}}{\psi_I}(I_t - I_{**}) + K_T \pi_t},$$

and we conclude using $p_t \mathbb{E}_t \left[B_{t+1}^{-\gamma} F_{\$,t+1} J_t\right] = \kappa(\phi_I - \kappa) + \pi_t = \kappa(\psi_I + \kappa) + \pi_t$. ■

Equation (19) shows the first-order value of the bond risk premium for bonds of maturity T. It is the maturity T of the bond multiplied by an inflation premium, $p_t \mathbb{E}_t \left[B_{t+1}^{-\gamma} F_{\$,t+1}\right] J_t$. The inflation premium is equal to the risk-neutral probability of disasters (adjusting for the recovery rate), $p_t \mathbb{E}_t \left[B_{t+1}^{-\gamma} F_{\$,t+1}\right]$, times the jump in expected inflation if there is a disaster, J_t. We note that a lower recovery rate reduces risk premia, a general feature that we will explore in greater detail in Section III.D

LEMMA 2. (Bond yields and forward rates) The forward rate, $f_t(T) \equiv -\partial \ln Z_{\$t}(T) / \partial T$, is:

$$f_t(T) = \delta - H_\$ + I_{**} + \frac{e^{-\psi_I T}(I_t - I_{**}) + \frac{e^{-\psi_I T} - e^{-\psi_J T}}{\psi_J - \psi_I} \pi_t}{1 - \frac{1-e^{-\psi_I T}}{\psi_I}(I_t - I_{**}) - K_T \pi_t} \tag{20}$$

$$= \delta - H_\$ + I_{**} + e^{-\psi_I T}(I_t - I_{**}) + \frac{e^{-\psi_I T} - e^{-\psi_J T}}{\psi_J - \psi_I} \pi_t + O(I_t - I_{**}, \pi_t)^2. \tag{21}$$

The bond yield is $y_t(T) = -\frac{(\ln Z_{\$t}(T))}{T}$ with $Z_{\$t}(T)$ given by (16), and its Taylor expansion is given in (43)–(44).

The forward rate increases with inflation and the bond risk premia. The coefficient of inflation decays with the speed of mean reversion of inflation, ψ_I, under the "risk-neutral" probability. The coefficient of the bond premium, π_t, is $\frac{e^{-\psi_I T} - e^{-\psi_J T}}{\psi_J - \psi_I}$ and, thus, has value 0 at both very short and very long maturities and a positive hump shape in between. Very short-term bills, being safe, do not command a risk premium, and long-term forward rates are also essentially constant (Dybvig, Ingersoll, and Ross 1996). Therefore, the time-varying risk premium only affects intermediate maturities of forwards.

III.C. Options

Let us next study options, which offer a potential way to measure disasters. The price of a European one-period put on a stock i with strike K expressed as a ratio to the initial price is: $V_t = \mathbb{E}_t\left[\frac{M_{t+1}}{M_t}\max\left(0, K - \frac{P_{i,t+1}}{P_{it}}\right)\right]$. Recall that Theorem 1 yields $\frac{P_{it}}{D_{it}} = a + b\widehat{H}_{it}$ for two constants a and b. Hence, $\mathbb{E}_t^{ND}\left[\frac{P_{i,t+1}}{P_{it}}\right] = e^{\mu_{it}}$ with $\mu_{it} = g_{iD} + \ln \frac{a+b\frac{e^{-\phi_H}\widehat{H}_{it}}{1+e^{-h^*}\widehat{H}_{it}}}{a+b\widehat{H}_{it}}$. Therefore, I parameterize the noise according to:[11]

$$(22)\quad \frac{P_{i,t+1}}{P_{it}} = e^{\mu_{it}} \times \begin{cases} e^{\sigma u_{i,t+1} - \sigma^2/2} & \text{if there is no disaster at } t+1 \\ F_{i,t+1} & \text{if there is a disaster at } t+1 \end{cases},$$

where $u_{i,t+1}$ is a standard Gaussian variable and $F_{i,t+1}$ is as already given. This parameterization ensures that the option price has a closed form, and at the same time conforms to the essence of the underlying economics. Economically, I assume that in a disaster most of the option value comes from the disaster, not from “normal-times” volatility. In normal times, returns are log-normal. However, if there is a disaster, stochasticity comes entirely from the disaster (there is no Gaussian $u_{i,t+1}$ noise, though adding some would have little impact). The structure takes advantage of the flexibility in the modeling of the noise in $\widehat{H}_{it}$ and D_{it}. Rather than modeling them separately, I assume that their aggregate yields exactly a log-normal noise (the Online Appendix provides a way to ensure that this is possible). At the same time, (22) is consistent with the processes and prices in the remainder of the article.

PROPOSITION 3. (Put price) The value of a put with strike K (the fraction of the initial price at which the put is in the money) and a one-period maturity is $V_{it} = V_{it}^{ND} + V_{it}^{D}$ with V_{it}^{ND} and V_{it}^{D} corresponding to the events with no disasters and with disasters, respectively:

$$(23)\qquad V_{it}^{ND} = e^{-\delta+\mu_{it}}\left(1-p_t\right) V_{Put}^{BS}\left(Ke^{-\mu_{it}}, \sigma\right)$$

$$(24)\qquad V_{it}^{D} = e^{-\delta+\mu_{it}} p_t \mathbb{E}_t\left[B_{t+1}^{-\gamma}\max\left(0, Ke^{-\mu_{it}} - F_{i,t+1}\right)\right],$$

11. Recall that with LG processes many parts of the variance need not be specified to calculate stock and bond prices. So, when calculating options, one is free to choose a convenient and plausible specification of the noise.

where $V_{Put}^{BS}(K,\sigma)$ is the Black-Scholes value of a put with strike K, volatility σ, initial price 1, maturity 1, and interest rate 0.

III.D. Corporate Spread, Government Debt, and Inflation Risk

Consider the corporate spread, which is the difference between the yield on the corporate bonds issued by the safest corporations (such as AAA firms) and government bonds. The "corporate spread puzzle" is that the spread is too high compared to the historical rate of default (Almeida and Philippon 2007). It has a very natural explanation under the disaster view. It is mostly during disasters (i.e., in bad states of the world) that very safe corporations will default. Hence, the risk premia on default risk will be very high. To explore this effect quantitatively, I consider the case of constant severity of disasters. The following proposition summarizes the effects, which are analyzed quantitatively in the next section. It deals with one-period bonds; the economics would be similar for long-term bonds.

PROPOSITION 4. (Corporate bond spread, disasters, and expected inflation) Consider a corporation i; call F_i the recovery rate of its bond[12] and λ_i the default rate conditional on no disaster, then the yield on debt is $y_i = \delta + \lambda_i - p\mathbb{E}^D\left[B^{-\gamma}F_{\$}F_i\right]$. So, denoting by y_G the yield on government bonds, the corporate spread is:

$$y_i - y_G = \lambda_i + p\mathbb{E}^D\left[B^{-\gamma}F_{\$}\left(1 - F_i\right)\right]. \tag{25}$$

In particular, when inflation is expected to be high during disasters (i.e., $F_{\$}$ is low, perhaps because the current Debt/GDP ratio is high), then (i) the spread $|y_i - y_j|$ between two nominal assets i and j is low, and (ii) the yield on nominal assets is high.

Proof. The Euler equation is $1 = e^{-\delta}\left(1+y_i\right)\left[\left(1-p\right)\left(1-\lambda_i\right) + p\mathbb{E}^D\left[B^{-\gamma}F_{\$,t+1}F_i\right]\right]$, and the proposition follows from taking the limit of small time intervals. ■

12. In the assumptions of Chen, Collin-Dufresne, and Goldstein (2009) and Cremers, Driessen, and Maenhout (2008), the loss rate conditional on a default, λ^d, is the same across firms, but only their probability of defaulting in a disaster state, p_i, varies. Then $F_i = 1 - p_i\lambda^d$, which is a particular case of this article.

IV. A Quantitative Investigation

IV.A. Calibrated Parameters

I propose the following calibration of the model's parameters. I assume that time variation of disaster risk enters through the recovery rate F_{it} for stocks and through the potential jump in inflation J_t for bonds. I take the limit of small time intervals and report annualized units. The calibration's inputs are summarized in Table I, while the results from the calibration are in Tables II–VII and Figure I. Section V.C will show that in the calibration realized disaster risk varies enough compared to the volatility of resilience, so that the calibrated numbers are reasonable by that criterion.

Macroeconomy. In normal times, consumption grows at rate $g_c = 2.5\%$. To keep things parsimonious, the probability and conditional severity of macroeconomic disasters are taken to be constant over time; I discuss this assumption later and relax it in Section V.A. The disaster probability is $p = 3.63\%$, Barro and Ursua (2008)'s estimate. I take $\gamma = 4$, for which Barro and Ursua (2008)'s evaluation of the probability distribution of B_{t+1} gives $\mathbb{E}\left[B^{-\gamma}\right] = 5.29$, so that the utility-weighted mean recovery rate of consumption is $\overline{B} = \mathbb{E}\left[B^{-\gamma}\right]^{-\frac{1}{\gamma}} = 0.66$. Because of risk aversion, bad events receive a high weight: the modal loss is less severe. There is an active literature centering around the basic disaster parameters, namely, Barro and Ursua (2008) and Nakamura et al.

TABLE I
Variables Used in the Calibration

Variables	Values
Time preference, risk aversion	$\rho = 6.57\%, \gamma = 4$
Growth rate of consumption and dividends	$g = g_{iD} = 2.5\%$
Volatility of dividends	$\sigma_D = 11\%$
Probability of disaster, recovery rate of C after disaster	$p = 3.63\%, \overline{B} = 0.66$
Stocks' recovery rate: typical value, volatility, speed of mean reversion	$F_{i*} = \overline{B}, \sigma_F = 10\%, \phi_H = 13\%$
Inflation: typical value, volatility, speed of mean reversion	$I_* = 3.7\%, \sigma_I = 1.5\%, \phi_I = 18\%$
Jump in inflation: typical value, volatility, speed of mean reversion	$J_* = 2.1\%, \sigma_J = 15\%, \phi_J = 92\%$

TABLE II
SOME VARIABLES GENERATED BY THE CALIBRATION

Variables	Values
Ramsey discount rate	$\delta = 16.6\%$
Risk-adjusted probability of disaster	$pE[B_{t+1}^{-\gamma}] = 19.2\%$
Stocks: effective discount rate	$\delta_i = 5.0\%$,
Stock resilience: typical value, volatility	$H_{i*} = 9.0\%$, $\sigma_H = 1.9\%$
Stocks: equity premium, conditional on no disasters, uncond.	6.5%, 5.3%
Real short-term rate	1.0%
Resilience of one nominal dollar	$H_\$ = 16.0\%$
5-year nominal slope $y_t(5) - y_t(1)$: mean and volatility	0.57%, 0.92%
Long-run − short-run yield: typical value	$\kappa = 2.6\%$
Inflation parameters	$I_{**} = 6.3\%$, $\psi_I = 13\%$, $\psi_J = 90\%$
Bond risk premium: volatility	$\sigma_\pi = 2.9\%$

Notes. The main other objects generated by the model are in Tables III–VII and Figure I.

(2011) who find estimates consistent with the initial Barro (2006) numbers.

The key number is the risk-neutral probability of disasters, $p\mathbb{E}[B^{-\gamma}] = 19.2\%$. This high risk-neutral probability allows the model to calibrate a host of high risk premia. Following Barro and Ursua, I set the rate of time preference to match a risk-free rate of 1%, so in virtue of (12), the rate of time preference is $\rho = 6.6\%$.

Stocks. I use a growth rate of dividends $g_{iD} = g_C$, consistent with the international evidence (Campbell 2003, Table III). The volatility of the dividend is $\sigma_D = 11\%$, as in Campbell and Cochrane (1999). The speed of mean reversion of resilience ϕ_H is the speed of mean reversion of the P/D ratio. It has been carefully examined in two recent studies based on U.S. data. Lettau and van Nieuwerburgh (2008) find $\phi_H = 9.4\%$. However, they find $\phi_H = 26\%$ when allowing for a structural break in the time series, which they suggest is warranted. Cochrane (2008) finds $\phi_H = 6.1\%$ (std.err. 4.7%). I take the mean of those three estimates, which leads to $\phi_H = 13\%$. Given these ingredients, a typical volatility $\sigma_H = 1.9\%$ helps match the volatility of stock returns.[13]

13. The Online Appendix details a specific volatility process for H_{it}, which satisfies the requirement that volatility vanishes at a lower bound, see note 9.

To specify the volatility of the recovery rate F_{it}, I specify that it has a baseline value $F_{i*} = \overline{B}$ and support $F_{it} \in [F_{\min}, F_{\max}] = [0, 1]$. That is, if there is a disaster, dividends can do anything from losing all their value to losing no value at all. The process for H_{it} then implies that the corresponding average volatility for F_{it}, the expected recovery rate of stocks in a disaster, is 10%. This may be considered a high volatility. Economically, it reflects the fact that it seems easy for stock market investors to alternatively feel extreme pessimism and optimism (e.g., during the large turning points around 1980, 2001, and 2008). In any case, this perception of the risk for F_{it} is not directly observable, so the calibration does not appear to contradict any known fact about observable quantities.

The disaster model implies a high covariance of stock prices with consumption during disasters. Is that true empirically? First, it is clear that we need multicountry data, as, for instance, a purely U.S.-based sample would not represent the whole distribution of outcomes because it would contain too few disasters. Using such multicountry data, Ghosh and Julliard (2008) find a low importance of disasters. On the other hand, Barro (2009) report a high covariance between consumption and stock returns during a disaster, which warrants the basic disaster model. The methodological debate, which involves missing observations—for instance due to closed stock markets, price controls, the measurement of consumption, and the very definition of disasters—is likely to continue for years to come. My reading of Barro (2009) is that the covariance between consumption and stock returns, once we include disaster returns, is large enough to vindicate the disaster model.

Inflation and Nominal Bonds. For simplicity and parsimony, I consider the case when inflation does not burst during disasters, $F_{\$,t+1} = 1$. Bond and inflation data come from CRSP. Bond data are monthly prices of zero-coupon bonds with maturities of one to five years, from June 1952 through September 2007. In the same time sample, I estimate the inflation process as follows. First, I linearize the LG process for inflation: $I_{t+1} - I_* = e^{-\phi_I \Delta t}(I_t - I_*) + \varepsilon_{t+1}^I$. Next, it is well known that inflation contains a substantial high-frequency and transitory component, which is in part due

Fortunately, many moments (e.g., stock prices) do not depend on the detalis of that process.

to measurement error. The model accommodates this. Call $\widetilde{I}_t = I_t + \eta_t$ the measured inflation (which can be thought of as trend inflation plus mean-zero noise), while I_t is the trend inflation. I estimate inflation using the Kalman filter, with $I_{t+1} = C_1 + C_2 I_t + \varepsilon^I_{t+1}$ for the trend inflation and $\widetilde{I}_t = I_t + \eta_t$ for the noisy measurement of inflation. Estimation is at the quarterly frequency, and yields $C_2 = 0.954$ (std.err. 0.020), that is, the speed of mean reversion of inflation is $\phi_I = 0.18$ in annualized values. Also, the annualized volatility of innovations in trend inflation is $\sigma_I = 1.5\%$. I have also checked that estimating the process for I_t on the nominal short rate yields substantially the same conclusion. Finally, I set I_* at the mean inflation, 3.7% (note that the slight nonlinearity in the LG term process makes I_* differ from the mean of I_t by only a trivial amount).

To assess the process for J_t, I consider the five-year slope, $s_t = y_t(5) - y_t(1)$. Equation (44) shows that, conditional on no disasters, it follows (up to second-order terms) that $s_{t+1} = a + e^{-\phi_J \Delta t} s_t + bI_t + \varepsilon^s_{t+1}$, where Δt is the length of "a period" (e.g., a quarter means $\Delta t = \frac{1}{4}$). I estimate this process at a quarterly frequency. The coefficient on s_t is 0.78 (std.err. 0.043). This yields $\phi_J = 0.92$. The standard deviation of innovations to the slope is 0.92%.

To calibrate κ, I consider the baseline value of the yield, which from (16) is $y_t(T) = y_t(0) + \kappa + \ln \frac{1 - \frac{1-e^{-\psi_I T}}{\psi_I}\kappa}{T}$ with $\psi_I = \phi_I - 2\kappa$, and I compute the value of κ such that it ensures $y_t(5) - y_t(1) = 0.0057$, the empirical mean of the five-year slope. This gives $\kappa = 2.6\%$. By (9), this implies that in a disaster the expected jump in inflation is $J_* = 2.1\%$.

As a comparison, Barro and Ursua (2008) find a median increase of inflation during disasters of 2.4%. They find a median inflation rate of 6.6% during disasters, compared to 4.2% for long samples taken together. This is heartening, but one must keep in mind that Barro and Ursua (2008) find that the average increase in inflation during disasters is equal to 109%—because of hyperinflations, inflation is very skewed.[14] I conclude that a jump

14. There is a difference between wars and financial disasters: wars very rarely lead to deflations, but financial disasters often do, especially during the Great Depression. The inflation jump is a bit higher during wars than financial disasters, by about 1% or 4%, depending on whether one takes the median or the mean of winsorized values. It is useful to note that financial disasters in non-OECD countries are typically inflationary.

in inflation of 2.1% is consistent with the historical experience. Investors do not know ex ante if disasters will bring about inflation or deflation; on average, however, they expect more inflation.

As there is considerable variation in the actual jump in inflation, there is much room for variations in the perceived jump in inflation, $J_t = J_* + \widehat{J}_t$—something that the calibration will indeed deliver. We saw that empirically the standard deviation of the innovations to the spread is 0.92% (in annualized values), whereas in the model it is $(K_5 - K_1)\,\sigma_\pi$. Hence, we calibrate $\sigma_\pi =$ 2.9%. As a result, the standard deviation of the five-year spread is $\frac{(K_5-K_1)\sigma_\pi}{\sqrt{2\phi_J}} = 0.68\%$, while in the data it is 0.79%. Therefore, the model is reasonable in terms of observables.

An important nonobservable is the perceived jump in inflation during a disaster, J_t. Its volatility is $\sigma_J = \frac{\sigma_\pi}{p\mathbb{E}[B^{-\gamma}]} = 15.4\%$, and its population standard deviation is $\frac{\sigma_J}{\sqrt{2\phi_J}} = 11\%$. This is arguably high, although it does not violate the constraint that the actual jump in inflation should be more dispersed than its expectation (see Section V.C). One explanation is that the yield spread has some high-frequency transitory variation that leads to a very high measurement of ϕ_J; with a lower value one would obtain a considerably lower value of σ_J. Another interpretation is that the demand for bonds shifts at a high frequency (perhaps for liquidity reasons). While this is captured by the model as a change in perceived inflation risk, it could be linked to other factors. In any case, we shall see that the model does well in a series of dimensions explored in Section IV.C.

Fixed Versus Variable p_t The baseline calibration uses a fixed p_t. Let us see how things would change with a variable p_t. If only p_t varied, then the correlation between stock and bond risk premia would be perfect. Empirically, we shall see that the correlation is much closer to 0, which suggests that asset-class-specific factors drive the bulk of stock versus bond returns, rather than a common factor. This suggests that a calibration with a constant p_t is a useful first pass.

To be more quantitative, one would like long time series of $\frac{P_t}{D_t}$ and of real bond yields. To obtain such a long-term time series, I use the real short-term yield. Then, I regress $\Delta \ln \frac{P_t}{D_t} = \alpha + \beta \Delta r_{ft} + \varepsilon_t$. I observe that, in the model, p_t affects r_{ft} and $\ln \frac{P_t}{D_t}$, but that F_{it} affects only $\frac{P_t}{D_t}$. Hence, using the model, the interpretation of the R^2 is an answer to the question: how much of the variation in $\frac{P_t}{D_t}$ comes from p_t rather than F_{it}? Empirically, $R^2 = 13\%$ and $\beta = -2.1$ (std.err. 1.1). This means that, prima facie, 87% of the variation in the P/D ratio comes from the recovery rate, and 13% from changes in p_t. For the calibration's parsimony, I take $\sigma_p = 0$. Using the regression on the Cochrane and Piazzesi (CP, 2005) factor, regressing $\Delta \ln \frac{P_t}{D_t} = \alpha + \beta \Delta CP_t + \varepsilon_t$, yields an R^2 of 0.04%. This also points to a very small role for a common shock in bond versus stock premia. I note that this 13% / 87% breakdown is, of course, provisional. In addition, it is undoubtedly the case that in some episodes, variations in p_t are important (e.g., during the 2008 crisis), and then it is useful to pay the somewhat higher cost of using the Epstein-Zin model developed later.

Let me expand on the theme that the correlation between stock and nominal bond premia appears to be small. Viceira (2007) reports that the correlation between stock and bond returns is 3%. The correlation between the change in the CP factor and stock market returns is 3%, and the correlation between the level of CP and the change in stock market returns is also 3%, at monthly frequencies. In the model, this could be accounted for by setting $corr\left(\varepsilon^J_{t+1}, -\varepsilon^H_{i,t+1}\right) \simeq corr\left(\varepsilon^J_{t+1}, \varepsilon^D_{i,t+1}\right) \simeq 3\%$.

On the Degree of Parsimony of this Calibration. This article is mainly concerned with the value of stocks and government bonds. It uses two latent measures of riskiness, one for real quantities (the stock resilience H_{it}) and another for nominal quantities (the bond risk premium π_t), both of which load on just one macro shock, the disaster shock. This assumption of at least one risk premium for nominal quantities and another risk premium for real quantities is used by several authors, for example, Wachter (2006), Piazzesi and Schneider (2007), Bansal and Shaliastovich (2009), and Lettau and Wachter (2011).

My conclusion is that it is hardly possible to be more parsimonious and still account for the basic facts of asset prices. Indeed, a tempting, though ultimately inadequate, idea would be the following: nominal bonds and stocks are driven by just one factor, perhaps the disaster probability. However, there is much evidence that risk premia are driven by more than one factor (see above, and also Fama and French 1993 who find that five factors are necessary to account for stocks and bonds). Hence, the framework in this article using two time-varying risk premia (one for nominal assets, one for real assets) is, in a sense, the minimal framework to make sense of asset price puzzles on stocks and

TABLE III
SOME STOCK MARKET MOMENTS

	Data	Model
Mean P/D	23	18.2
Std. dev. $\ln P/D$	0.33	0.30
Std. dev. of stock returns	0.18	0.15

Notes. Stock market moments. The data are from Campbell (2003, Table 1 and 10)'s calculation for the United States 1891–1997.

nominal bonds. Of course, those premia ultimately compensate for just one source of risk—disaster risk.

I next turn to the return predictability generated by the model. Sometimes I use simulations, in a sample without disasters, as in most of the theory. The calibration was designed to match two-thirds of Table III, but the predictions in Figure I and Tables IV–VII are out of sample, that is, were not directly targeted in the calibration.

IV.B. Stocks: Level, Excess Volatility, Predictability

Average Levels. The equity premium (conditional on no disasters) is $r_{it}^{e} - r_f = p\mathbb{E}\left[B^{-\gamma}\right](1 - F_{i*}) = 6.5\%$. The unconditional equity premium is 5.3% (the above value minus $p(1 - F_{i*})$). So, as in Barro (2006), the excess returns of stocks mostly reflect a risk premium, not a peso problem.[15] The mean value of the P/D ratio is 18.2 (and is close to equation (14), evaluated at $\widehat{H}_{it} = 0$), in line with the empirical evidence reported in Table III. The central value of the D/P ratio is $\delta_i = 5.0\%$. [16]

"Excess" Volatility. The model generates "excess" volatility and predictability. Consider (14), $\frac{P_{it}}{D_{it}} = \frac{\frac{1+\widehat{H}_{it}}{\delta_i+\phi_H}}{\delta_i}$. As stock market resilience $\widehat{H}_{it}$ is volatile, so are stock market prices and P/D ratios. Table III reports the numbers. The standard deviation of $\ln(P/D)$ is 0.27. Volatile resilience yields a volatility of the log of the P/D ratio equal to 10%. For parsimony's sake, I assume

15. Note that this explanation for the equity premium is very different from the one proposed in Brown, Goetzmann, and Ross (1995), which centers around survivorship bias.

16. In those tables, the sample sometimes includes the Great Depression, but as shown by Campbell (2003), for the stock market moments considered, the broad facts do not depend on including the Great Depression.

that innovations to dividends and resilience are uncorrelated. The volatility of equity returns is 15%. I conclude that the model can quantitatively account for an "excess" volatility of stocks through a stochastic risk-adjusted severity of disasters. In addition, changes in the P/D ratio reflect only changes in future returns, not future dividends. This is in line with the empirical findings of Campbell and Cochrane (1999).

Predictability. Consider (14) and (15). When $\widehat{H}_{it}$ is high, (15) implies that the risk premium is low and P/D ratios (14) are high. Hence, the model generates above-average subsequent stock market returns when the market-wide P/D ratio is below average. This is the view held by many, but not all, researchers (see the discussion in Cochrane 2008). The model predicts the following magnitudes for regression coefficients.

PROPOSITION 5. (Predicting stock returns via D/P ratios) Consider the predictive regressions of the return from holding the stock from t to $t+T$, $r_{it\to t+T}$, on the initial dividend-price ratio, $\ln\left(\frac{D_{it}}{P_{it}}\right)$:

$$r_{it\to t+T} = \alpha_T + \beta_T \ln\left(\frac{D_{it}}{P_{it}}\right) + \text{noise} \tag{26}$$

$$r_{it\to t+T} = \alpha'_T + \beta'_T \left(\frac{D_{it}}{P_{it}}\right) + \text{noise.} \tag{27}$$

In the model, for small holding horizons T, the slopes are, to the leading order: $\beta_T = (\delta_i + \phi_H)\,T$ and $\beta'_T = \left(\frac{1+\phi_H}{\delta_i}\right) T$.

Proof. Proposition 1 states the expected returns over a short horizon T to be $r^e_{it\to t+T} = (\delta - H_{it})\,T$. Equation (14) implies that the right-hand side of (26) is, to the leading order, $\ln(D/P)_t = \ln\delta_i - \frac{\widehat{H}_{it}}{\delta_i+\phi_H}$. So the regression is, to a first order, $r^e_{it\to t+T} = (\delta - H_{i*} - \widehat{H}_{it})T = \alpha_T - \beta_T\frac{\widehat{H}_{it}}{\delta_i+\phi_H}$. Equating the $\widehat{H}_{it}$ terms, $\beta_T = (\delta_i + \phi_H)\,T$. The same reasoning yields β'_T. ■

The intuition for the value of β_T is as follows. First, the slope is proportional to T simply because returns over a horizon T are proportional to T. Second, when the P/D ratio is lower than the baseline by 1%, it increases returns through two channels: the dividend yield is higher by δ_i%, and mean reversion of the P/D ratio creates capital gains of ϕ%.

Using the paper's calibration of $\delta_i = 5\%$ and $\phi_H = 13\%$, Proposition 5 predicts a slope coefficient $\beta_1 = 0.18$ at a one-year horizon.

TABLE IV
PREDICTING RETURNS WITH THE DIVIDEND/PRICE RATIO

	Data			Model	
Horizon	Slope	s.e.	R^2	Slope	R^2
1 yr	0.11	(0.053)	0.04	0.17	0.06
4 yr	0.42	(0.18)	0.12	0.45	0.19
8 yr	0.85	(0.20)	0.29	0.79	0.30

Notes. Predictive regression for the expected stock return $r_{it\to t+T} = \alpha_T + \beta_T \ln\left(\frac{D_{it}}{P_{it}}\right)$, at horizon T (annual frequency), up to an 8-year horizon. The data are from Campbell (2003, Table 10 and 11B)'s calculation for the United States 1891–1997.

This prediction is in line with the careful estimates of Lettau and van Nieuwerburgh (2008) who find a β_1 value of 0.23 in their preferred specification. Also, Cochrane (2008) runs regression (27) at the annual horizon, and finds $\beta_1' = 3.8$ with a standard error of 1.6. Proposition 5 predicts $\beta_1' = 3.6$. We note that the approximation in Proposition 5, valid for "small" T, appears to be valid up to approximately a one-year holding period. Table IV reports the model predictions for large T, using simulations, and their arguably good congruence with empirical data.

I conclude that the model is successful in matching not only the level but also the variation and predictability of the stock market.

Characteristics Versus Covariances. In a rare-disaster economy, characteristics tend to predict returns better than covariances, which a strand of research argues is true (Daniel and Titman 1997). Indeed, in a sample without disasters, betas will only reflect the covariance during "normal times," but risk premia are only due to the covariance with consumption in disasters. The two can be entirely different. Hence, the "normal-times" betas can have no relation with risk premia. However, "characteristics," like the P/D ratio, imbed measures of risk premia (as in (14)). Hence, characteristics will predict returns better than covariances.

However, there could be some spurious links if stocks with low H_{i*} have higher cash-flow betas. One could conclude that a cash-flow beta commands a risk premium; however, this is not because cash-flow betas cause the latter, but simply because stocks with high cash-flow betas happen to be stocks that have a large loading on the disaster risk.

These points may help explain the somewhat contradictory findings in the debate about whether characteristics or covariances explain returns. When normal-times covariances badly measure the true risk, as is the case in a disaster model, characteristics will often predict expected returns better than covariances.[17]

IV.C. Bond Premia and Yield Curve Puzzles

Excess Returns and Time-Varying Risk Premia.

i. Bonds Carry a Time-Varying Risk Premium. Equation (18) indicates that bond premia are (to a first order) proportional to bond maturity T. This is the finding of Cochrane and Piazzesi (2005). The explanatory factor here is the inflation premium π_t, that is, compensation for a jump in inflation if a disaster happens. The model delivers this because a bond's loading of inflation risk is proportional to its maturity T.

ii. The Nominal Yield Curve Slopes Up On Average. Suppose that when a disaster happens, inflation jumps by $J_* > 0$. This leads to a positive parametrization κ of the bond premia (equation (9)). The typical nominal short-term rate (i.e., the one corresponding to $I_t = I_*$) is $y(0) = \delta - H_\$ + I_*$, while the long-term rate is $y(0) + \kappa$ (i.e., $-\lim_{T\to\infty} \ln \frac{Z_{\$t}(T)}{T}$). Hence, the long-term rate exceeds the short-term rate by $\kappa > 0$. The yield curve slopes up. Economically, this is because long-maturity bonds are more sensitive to inflation risk than short-term bonds, so they command a risk premium.

On the other hand, in the model, the yield curve on real bonds is flat: all yields are equal to r_f. The empirical evidence on real bonds is scarce and mixed (e.g., Nakamura et al. 2011). In the UK, the real yield curve has been downward sloping, but in the US, it has been upward-sloping. Hence, a flat real yield curve may be a good benchmark.

The Forward Spread Predicts Bond Excess Returns. Fama and Bliss (1987) regress short-term excess bond returns on the forward spread, that is, the forward rate minus the short-term rate:

17. A recent working paper (Koijen, Lustig, and van Nieuwerburgh 2010) brings new substance to this debate, showing that value stocks' dividends fell a lot during the Great Depression.

(28) Fama-Bliss regression: Excess return on bond of maturity $T = \alpha_T + \beta_T \cdot (f_t(T) - r_t) + \text{noise}$.

The expectation hypothesis yields constant bond premia and, thus, predicts $\beta_T = 0$. I next derive the model's prediction. As in the calibration $\frac{var(I_t)\psi_I^2}{var(\pi_t)} = 0.023$, I highlight the case where this quantity is small, which means that changes in the slope of the yield curve come from changes in the bond risk premium rather than from changes in the drift of the short-term rate.

PROPOSITION 6. (Coefficient in the Fama-Bliss regression) The slope coefficient β_T of the Fama-Bliss regression (28) is given in (45). When $\frac{var(I_t)\psi_I^2}{var(\pi_t)} \ll 1$,

$$\beta_T = 1 + \frac{\psi_J}{2} T + O\left(T^2\right). \tag{29}$$

When $var(\pi_t) = 0$ (no risk premium shocks), the expectation hypothesis holds and $\beta_T = 0$. In all cases, the slope β_T is nonnegative and eventually goes to 0: $\lim_{T\to\infty} \beta_T = 0$.

To understand the economics of the previous proposition, consider the variable part of the two sides of the Fama-Bliss regression (28). The excess return on a $T-$maturity bond is approximately $T\pi_t$ (see equation (18)), while the forward spread is $f_t(T) - r_t \simeq T\pi_t$ (see equation (21)). Both sides are proportional to $\pi_t T$. Thus, the Fama-Bliss regression (28) has a slope equal to 1, which is the leading term of (29).

This value β_T above 1 is precisely what Fama and Bliss (1987) have found, a result confirmed by Cochrane and Piazzesi (2005). This is quite heartening for the model. Table V reports the results. We also see that as maturity increases, coefficients initially rise but then fall at long horizons, as predicted by Proposition 6. Economically, most of the variations in the slope of the yield curve are due to variations in the risk premium, not due to the expected change of inflation.

The Slope of the Yield Curve Predicts Future Movements in Long Rates. Campbell and Shiller (CS, 1991) find that a high slope of the yield curve predicts that future long-term rates will fall. CS regress yield changes on the spread between the yield and the short-term rate:

TABLE V
FAMA-BLISS EXCESS RETURN REGRESSION

	Data			Model	
Maturity T	β	(std. err.)	R^2	β	R^2
2 yr	0.99	(0.33)	0.16	1.33	0.34
3 yr	1.35	(0.41)	0.17	1.71	0.23
4 yr	1.61	(0.48)	0.18	1.84	0.14
5 yr	1.27	(0.64)	0.09	1.69	0.08

Notes. The regressions are the excess returns on a zero-coupon bond of maturity T, regressed on the spread between the T forward rate and the short-term rate: $rx_{t+1}(T) = \alpha + \beta(f_t(T) - f_t(1)) + \varepsilon_{t+1}(T)$. The unit of time is one year. The empirical results are from Cochrane and Piazzesi (2005, Table II). The expectation hypothesis implies $\beta = 0$.

TABLE VI
CAMPBELL-SHILLER YIELD CHANGE REGRESSION

	Data		Model
Maturity T	β	(std. err.)	β
3 m	−0.15	(0.28)	−1.03
6 m	−0.83	(0.44)	−1.16
12 m	−1.43	(0.60)	−1.41
24 m	−1.45	(1.00)	−1.92
48 m	−2.27	(1.46)	−2.83

Notes. The regressions are the change in bond yield on the slope of the yield curve: $y_{t+1}(T-1) - y_t(T) = \alpha + \frac{\beta}{T-1}(y_t(T) - y_t(1)) + \varepsilon_{t+1}(T)$. The time unit is one month. The empirical results are from Campbell, Lo, and MacKinlay (1997, Table 10.3). The expectation hypothesis implies $\beta = 1$.

Campbell-Shiller regression:

$$\frac{y_{t+\Delta t}(T-\Delta t) - y_t(T)}{\Delta t} = a + \beta_T \cdot \frac{y_t(T) - y_t(0)}{T} + \text{noise}. \quad (30)$$

The expectation hypothesis predicts $\beta_T = 1$. However, CS find negative β_T's, with a roughly affine shape as a function of maturity (see Table VI). This empirical result is predicted by the model, as the next proposition shows. As in the calibration $\frac{var(I_t)\phi_I^2}{var(\pi_t)} = 0.045$, I highlight the case where these quantities are small.

PROPOSITION 7. (Coefficient in the Campbell-Shiller regression) The slope coefficient β_T in the Campbell and Shiller (1991) regression (30) is given by (46). When $\frac{\phi_I^2 var(I_t)}{var(\pi_t)} \ll 1$, $\kappa T \ll 1$, $\beta_T = -\left(1 + \frac{2\psi_j - \psi_I}{3} T\right) + o(T)$ when $T \to 0$, and $\beta_T = -\psi_J T + o(T)$ when $T \gg 1$.

Table VI also contains simulation results of the model's predictions. They are in line with CS's results. To understand the economics better, I use a Taylor expansion in the case where inflation is minimal. The slope of the yield curve is, to the leading order, $\frac{y_t(T)-y_t(0)}{T} = \frac{\pi_t}{2} + O(T)$. Hence, to a first-order approximation (when inflation changes are not very predictable), the slope of the yield curve reflects the bond risk premium. The change in yield is (the proof of Proposition 7 justifies this):

$$\frac{y_{t+\Delta t}(T-\Delta t) - y_t(T)}{\Delta t} \simeq -\frac{\partial y_t(T)}{\partial T} = \frac{-\pi_t}{2} + O(T).$$

Hence, the CS regression yields a coefficient of -1, to the leading order. Economically, it means that a high bond premium increases the slope of the yield curve (by $\frac{\pi_t}{2}$).

As bond maturity increases, Proposition 7 predicts that the coefficient in the CS regression becomes more and more negative. The economic reason is the following. For long maturities, yields have vanishing sensitivity to the risk premium, which the model says has the shape $y_t(T) = a + \frac{b\pi_t}{T} + o\left(\frac{1}{T}\right)$ for some constants a, b. Thus, the slope of the yield curve varies with $\frac{b\pi_t}{T^2}$, and the expected change in the yield is $\frac{-b\phi_J\pi_t}{T}$. So the slope in the CS regression (30) is $\beta_T \simeq -\phi_J T$. On the other hand, the expression for β_T shows that when the predictability due to inflation is non-negligible, the CS coefficient should go to 1 for very large maturities.

In Table VI, we see that the fit between theory and evidence is rather good. The only poor fit is for a short maturity. The CS coefficient is closer to 0 than in the model. The short-term rate has a larger predictable component at short-term horizons than in the model. For instance, this could reflect a short-term forecastability in Fed Funds rate changes. That feature could be added to the model, as in the Online Appendix. Given the small errors in fit, it is arguably better not to change the baseline model which broadly accounts for the CS finding. Economically, the CS finding reflects the existence of a stochastic one-factor bond risk premium.

A Tent-shaped Combination of Forward Rates Predicts the Bond Risk Premium. Cochrane and Piazzesi (CP, 2005) establish that (i) a parsimonious description of bond premia is given by a stochastic one-factor risk premium, (ii) (zero-coupon) bond premia are proportional to bond maturity, (iii) this risk premium is empirically well approximated by a "tent-shaped" linear combination of forward rates, and (iv) a regression of excess bond returns on five forward rates gives such a tent shape.

Equation (18) is consistent with their findings (i)–(ii): there is a single bond risk factor π_t, and the loading on it is proportional to bond maturity.[18] Economically, this is because a bond of maturity T has a sensitivity to inflation risk approximately proportional to T.

We shall see that the model delivers finding (iii) but not finding (iv). It cannot deliver (iv) because it has only two factors, so that five forwards rates in a regression would be collinear. It is conceivable that richer extensions (e.g., with five factors) of the model might deliver (iv), although at the same time it would be nice to see if (iv) replicates robustly in other countries (see Kozak 2011).

However, we shall see that the model is account for CP's finding (iii). To understand this, rewrite (21) as:

$$f_t(T) = F(T) + e^{-\psi_I T} I_t + \Lambda(T)\,\pi_t, \qquad \Lambda(T) \equiv \frac{e^{-\psi_I T} - e^{-\psi_J T}}{\psi_J - \psi_I}.$$

This model's interpretation of the CP "tent-shaped" effect is that, in the model, forward rates of maturity T necessarily have a "tent-shaped" loading $\Lambda(T)$ on the bond risk premium. To see that the model's loading $\Lambda(T)$ is tent-shaped, observe that $\Lambda(0) = \Lambda(\infty) = 0$ and $\Lambda(T) > 0$ for $T > 0$. We saw earlier (after Lemma 2) that the economic reason for this tent shape of $\Lambda(T)$ is that short-term bonds have no inflation risk premium, and long-term forwards are constant (in this model, $f_t(\infty) = \delta - H_\$ + I_{**}$), so that only intermediate maturity forwards have a loading on the bond risk premium.

To econometrically capture the bond risk premium, a tent-shaped $\sum_{T=1}^{5} w_T f_t(T)$ combination of forwards predicts the bond risk premium. The simple ($\sum_{T=1}^{5} w_T$) and maturity-weighted ($\sum_{T=1}^{5} T w_T$) sums of the weights should be roughly 0, so as to eliminate slow-moving factors such as $e^{-\psi_I T} I_t$ up to second-order terms. This reasoning leads one to ask if there is a simple combination of

18. This is not an artifact of postulating a single bond risk premium, coming from J_t. If there are K bond risk premia $\widehat{J}_{kt}$, with different speeds of mean reversion (as in the multifactor model of the Online Appendix), then, to the leading order, the risk premium on a bond of maturity T is still $T\pi_t$, with $\pi_t = p_t \mathbb{E}_t\left[B_{t+1}^{-\gamma}\right] \sum_{k=1}^{K} \widehat{J}_{kt}$.

forward rates that one might expect to robustly proxy for the risk premia. The next proposition provides an answer.[19]

PROPOSITION 8. (Estimation-free combinations of forwards to proxy for the bond risk premium) Given time horizons a and b, consider the following "estimation-free" combinations of forwards:

$$CP_t^{EF}(a,b) \equiv \frac{-f_t(a) + 2f_t(a+b) - f_t(a+2b)}{b^2}, \tag{31}$$

where $f_t(T)$ are the forwards of maturity T. Then, up to third-order terms, for small a and b, $CP_t^{EF}(a,b) = (\psi_I + \psi_J)\,\pi_t$ is proportional to the bond risk premium.

Proof. From (21), up to third-order terms, $CP_t^{EF} = (\psi_I + \psi_J)\,\pi_t$. The leading inflation term is $-\psi_I^2 I_t$, a third-order term. ■

For instance, $CP_t^{EF}(1,2) = \frac{-f_t(1)+2f_t(3)-f_t(5)}{4}$ uses the forwards up to a maturity of 5 years. Proposition 8 suggests that CP_t^{EF} could be used in practice to proxy for the bond risk premia without requiring a preliminary estimation.[20] Over the period 1964–2008, repeating the CP analysis gives an average R^2 of 28% to predict excess bond return, while the estimation-free $CP_t^{EF}(1,2)$ yields a R^2 of 23%. This is arguably a good performance, given the CP analysis uses five regressors, and the estimation-free CP_t^{EF} uses just one. In addition, the correlation between CP's variable and CP_t^{EF} is 0.89. Finally, consider a country with a short data set: the estimate of the CP coefficients will be very noisy. Researchers could thus use the estimation-free CP_t^{EF} to evaluate risk premia.

I conclude that the model can account for the CP findings (i)–(iii) and proposes new combinations of factors to predict the bond risk premium. The latter are "estimation-free" and might be useful empirically.

19. Combination (31) is not unique in the present two-factor model. However, if we add a small third persistent factor, then the combination becomes unique. It goes to (31) when the extra persistence of that factor and inflation go to 0 (see the Online Appendix). In that sense, (31) is a special combination that comes naturally out of a small perturbation of the baseline model. Lettau and Wachter (2011) proposed earlier another combination of theoretical factors to obtain the risk premium, but it is not estimation-free.

20. The CP estimate is very close to $8CP_t^{EF}(1,2) - \frac{1}{2}CP_t^{EF}(2,1)$, which is $-2f(1) + 0.5f(2) + 3f(3) + 0.5f(4) - 2f(5)$.

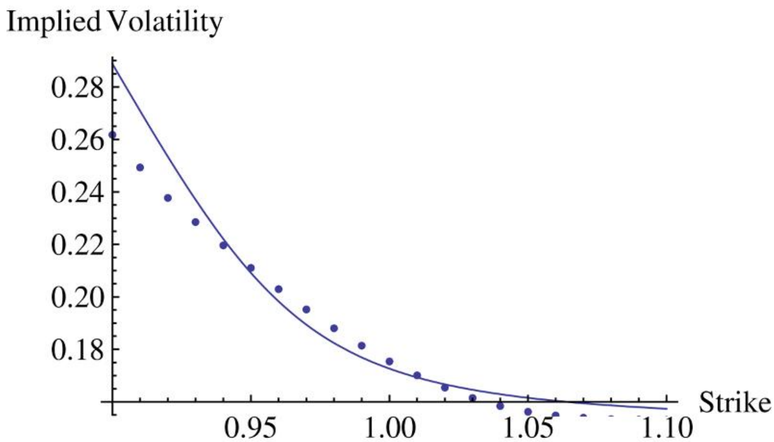

FIGURE I
Option implied volatility in the model and in the data.

This figure presents the Black-Scholes annualized implied volatility of a one-month put on the stock market. The solid line is from the model's calibration. The dots are the empirical average (January 2001 to February 2006) for the options on the S&P 500 index (Figlewski 2008). The initial value of the market is normalized to 1. The implied volatility of deep out-of-the-money puts is higher than the implied volatility of at-the-money uts, which reflects the probability of rare disasters.

IV.D. Options

I now investigate whether the model's calibration (which did not target any option-specific value) yields good values for options. I calculate the model's Black-Scholes implied volatilities of puts with a one-month maturity. I am very grateful to Stephen Figlewski for providing the empirical implied volatilities of one-month options on the S&P 500 from January 2001 to February 2006 (Figlewski 2008).

Figure I reports the implied volatilities from the data and the calibration. The correspondence is quite good, despite the fact that no extra parameter was tuned to match option prices.[21] Hence, I conclude that as a first pass and for the maturity presented here, the variable rare disasters model can yield correct option prices. Du (2011) finds other parametrizations of jumps that match option prices. Of course, a more systematic study would be desirable. Farhi et al. (2009) investigate the link between

21. I use $\sigma = 15\%$ (from Table III), and the central case $F_{i,t+1} = F_*$, which yields $H_{it} = H_{i*}$.

currency option prices and currency levels, finding support for the existence of a disaster risk premium.

Backus, Chernov, and Martin (forthcoming) study a specification that equity dividends are $D_t = C_t^{\lambda}$ for some $\lambda > 0$, and cannot fit option prices in a disaster framework with constant disaster risk. In their framework, a high λ (about 4) is necessary to attain a high volatility of equity (so that dividend volatility is λ times consumption volatility). But then, equity is immensely risky during disasters, and put prices are too high. In contrast, in the present framework, equity volatility comes from resilience volatility, and put prices can be moderate and calibrate naturally.

Proposition 3 suggests a way to extract key structural parameters of disasters from options data. Stocks with a higher put price (controlling for "normal-times" volatility) should have a higher risk premium, because they have higher future expected returns. Evaluating this prediction would be most interesting. Supportive evidence comes from Bollerslev, Tauchen, and Zhou (BTZ, 2009). They find that when put prices are high, subsequent stock market returns are high. This is qualitatively what a disaster-based model predicts.

For a more quantitative assessment, consider the "variance premium" VP_t, which is the risk-neutral expected variance minus the expected variance (conditional on no disasters). It is straightforward to derive $VP_t = p_t \mathbb{E}^D \left[B_{t+1}^{-\gamma} \left(1 - F_{i,t+1}\right)^2 \right]$, as the jump size in a disaster is $1 - F_{i,t+1}$. BTZ regress annualized stock market returns on the variance premium VP_t. Table VII reports the results.[22] In addition, the mean and standard deviation of VP_t are, respectively, 2.2% and 0.53% in the BTZ data, and 3.4% and 0.66% in the model. Hence, the model is broadly congruent with the empirical results of BTZ. It cannot account for all the patterns in the variance premium, as it is a one-factor model and the VIX index clearly exhibits some high-frequency transient dynamics. They might be accounted for by a resilience made of fast and slow components, as in the Online Appendix. Still, it is comforting to see that the model can generate some of the qualitative and quantitative empirical patterns.

22. I convert BTZ's results to annualized units, which multiplies their slopes by $\frac{100}{12}$. The model's baseline prediction (i.e., neglecting small nonlinear terms) for the slope in Table VII is $\beta = -\left(\frac{\partial H_{it}}{\partial F_{it}}\right) / \left(\frac{\partial VP_t}{\partial F_{it}}\right) = \frac{1}{(2(1-F_*))} = 1.5$. Table VII's simulations confirm the prediction.

TABLE VII
PREDICTING RETURNS WITH THE VARIANCE PREMIUM

	Data			Model	
Horizon	Slope	std. err.	R^2	Slope	R^2
1 m	3.2	(1.8)	0.01	1.6	0.006
3 m	3.9	(1.4)	0.06	1.6	0.02
6 m	2.5	(1.2)	0.05	1.6	0.03
12 m	1.0	(0.9)	0.01	1.4	0.05

Notes. Predictive regression for the expected stock return $\frac{1}{T}r_{it\to t+T} = \alpha_T + \beta_T VP_T$ at horizon T (using annualized units). VP_T is the variance risk premium. The empirical data come from Bollerslev, Tauchen, and Zhou (2009, Table II).

IV.E. Corporate Bonds

The calibration allows us to evaluate Proposition 4. The disaster risk premium is $\pi_i^D = y_i - y_G - \lambda_i$, the difference between the yield on corporate bonds and governance bonds minus the historical default rate of corporate bonds. The rare disaster model provides a macroeconomic foundation for Almeida and Philippon (2007)'s view that the corporate spread reflects the existence of bad states of the world and for reduced-form models of credit risk.

Almeida and Philippon (2007) yield an estimate of π_i^D as the difference between the risk-adjusted annualized probability of default and the historical one.[23] For instance, this implies π_i^D to be about 4.05% for a bond rated B (resp. 0.60% for a AAA bond). With $p\mathbb{E}\left[B^{-\gamma}\right] = 19.2\%$, this means that the expected loss in a disaster is $\frac{4.05\%}{0.192} = 21\%$ (resp. $\frac{0.60\%}{0.192} = 3.2\%$ for a AAA bond). This is a moderate loss. We see how easily, though, a disaster model can rationalize the corporate spread.

Moody's (1999) reports evidence on the United States. During the Great Depression, the realized loss rate was 5.3% on "investment-grade" bonds (of which AAA are the least risky) and 25% for the "speculative-grade" bonds (a category that comprises

23. I take Almeida and Philippon (2007)'s Table III, which has the 10-year risk-neutral and historical probability, and apply the transformation $-\ln\frac{(1-x)}{10}$ to obtain the annualized probability of default. I also add back the AAA-Treasury spread of 0.51% to get the actual AAA-Treasury spread. This yields the following disaster premia: 0.60% (AAA bonds), 1.11% (AA), 1.71% (A), 2.57% (BBB), 3.06% (BB), and 4.05% (B). Almeida and Philippon (2007) do not report standard errors.

B bonds).[24] This is roughly in line with the magnitude from market prices and disaster models.

Prediction (i) of Proposition 4 seems quite novel. The intuition is the following. To take an extreme example, suppose agents know that there will be hyperinflation in disasters, so that the real value of all nominal assets will be zero ($F_\$ = 0$). Then, government bonds lose all their advantage over corporate bonds, as the value of all bonds will be wiped out in a disaster. Hence, before the disaster, there should be no difference in the disaster risk premium between government bonds, AAA bonds, or any nominal bond. In less extreme scenarios, the same logic applies: high inflation risk lowers the spread between nominal bonds, because high inflation compresses the performance of all nominal bonds in the disaster state.

Prediction (i) provides an explanation for Krishnamurthy and Vissing-Jorgensen (2008)'s finding that when the Debt/GDP ratio is high, the AAA-Treasury and the BAA-AAA spreads are low: in their 1925–2005 U.S. sample, regressing the AAA-Treasury and BAA-AAA spreads on the Debt/GDP ratio yields significant coefficients of –1.5 and –1.2, respectively. The first AAA-Treasury finding can be explained by their favored interpretation of a liquidity demand for treasuries, but the BAA-AAA spread may be harder to explain by liquidity. The disaster hypothesis offers an explanation for both, hence it is complementary to the liquidity explanation. When Debt/GDP is high, the temptation to default via inflation (should a risk occur) is high,[25] thus $F_\$$ is low and so are nominal spreads.

Prediction (ii) of Proposition 4 allows one to think about the impact of the government Debt/GDP ratio. It is plausible that if the Debt/GDP ratio is high, then—once there is a disaster—the government will sacrifice monetary rectitude, so that J_t is high (this effect could be microfounded). This implies that when the Debt/GDP ratio (or the deficit/GDP) is high, then long-term rates are high and the slope of the yield curve is steep (controlling for inflation and expectations about future inflation in normal times). In addition, in the Krishnamurthy and Vissing-Jorgensen (2008) data, regressing bond rates minus the bill rate on the

24. I average over the two methodologies in the Moody's (1999) paper (Exhibits 5-11 and 5-13). I use the 1932–1935 cumulative rate. The earlier losses are much smaller.

25. Catao and Terrones (2005) find that high Debt/GDP leads, on average, to an increase in inflation.

Debt/GDP ratio yields a significant coefficient of 1.8, consistent with the disaster hypothesis: when the Debt/GDP ratio is high, the bond risk premium is high, so the slope of the yield curve is also high.

Likewise, suppose that an independent central bank has a more credible commitment not to increase inflation during disasters (J_t smaller). Then, real long-term rates (e.g., nominal rates minus expected inflation) are lower and the yield curve is less steep. This effect works in an economy where Ricardian equivalence holds. Higher deficits increase long-term rates not because they "crowd out" investment but instead because they increase the government's temptation to inflate away the debt if there is a disaster. In such a case, there is an inflation risk premium on nominal bonds.

V. Discussion and Extensions

V.A. *Epstein-Zin Preferences*

As mentioned, the power utility model has one qualitative deficiency: if the disaster probability increases, the risk-free rate decreases so much that the prices of most assets, including a consumption claim, go up.[26] To remedy this strong movement of the risk-free rate, I extend the model to the preferences introduced by Epstein and Zin (EZ, 1989), which allow one to decouple the intertemporal elasticity of substitution (IES), ψ, and risk aversion, γ.[27] This decoupling is indexed by $\chi = \frac{1-1/\psi}{1-\gamma}$, which is equal to 1 in the case of power utility preferences.[28] EZ show that the stochastic discount factor (SDF) evolves as $\frac{M_{t+1}}{M_t} = e^{-\frac{\rho}{\chi}} \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\chi\psi}} R_{c,t+1}^{1/\chi-1}$, where $R_{c,t+1} = \frac{P_{c,t+1}}{P_{ct}-C_t}$ is the gross return of a consumption claim —

26. The same anomalous effect works with a change in the recovery rate of consumption, B_{t+1}, and is fixed by Epstein-Zin preferences.

27. This effect of "bad news increase the price" is also present without any risk. In that case, the interest rate is $r_f = \rho + \frac{g_C}{\psi}$, and the stock price is $P_t = \frac{D_t}{\rho+g_C/\psi-g_D}$ by the Gordon formula. Hence, if both g_C and g_D go *down* by the same amount (bad news), the asset price *increases*, unless $\psi > 1$. That is why it is useful to have $\psi > 1$. At the same time, we need $\gamma > 1$ to have large risk premia. EZ preferences allow for $\psi > 1$ and $\gamma > 1$, whereas power utility preferences, which impose $\psi\gamma = 1$, do not.

28. This subsection complements the prior and different Epstein-Zin model of Wachter (2009), whose many results require a unit IES, and the numerical treatment in Gourio (2008a, 2008b).

the asset that gives a consumption C_t as a dividend and the price of which we call P_{Ct}.

The Appendix presents a setup in which this Epstein-Zin model is exactly solved. To make the exposition more transparent, I present here the economics of the model using a first-order approximation (in the exposition I neglect quadratic terms).[29]

The resilience of a consumption claim, $H_{ct} = p_t \mathbb{E}\left[B_{t+1}^{1-\gamma} - 1\right]$, is again assumed to follow the LG-twisted process, $\widehat{H}_{c,t+1} = \frac{1+H_{c*}}{1+H_{ct}} e^{-\phi_H} \widehat{H}_{c,t} + \varepsilon^H_{c,t+1}$ to the leading order. The main tool is the value of the SDF.

THEOREM 3. (SDF with EZ preferences) In the Epstein-Zin setup, the stochastic discount factor is given exactly by (37) and approximately by:

$$\frac{M_{t+1}}{M_t} = e^{-\delta}\left(1 + (\chi - 1) H_{ct} + \varepsilon^M_{t+1}\right) \times \begin{cases} 1 & \text{if there is no disaster at } t+1 \\ B_{t+1}^{-\gamma} & \text{if there is a disaster at } t+1 \end{cases}, \tag{32}$$

where $\delta = \rho + g_c/\psi$, $\delta_c = \delta - g_c - \chi H_{c*}$ and $\varepsilon^M_{t+1} = (1-\chi)\frac{\varepsilon^H_{c,t+1}}{\delta_c + \phi_H}$. To the leading order, the risk-free rate is $r_{ft} = \delta - p_t \mathbb{E}^D_t\left[B_{t+1}^{-\gamma}\left(1 + (\chi - 1) B_{t+1}\right) - \chi\right]$.

The key impact of disaster is in $B_{t+1}^{-\gamma}$, as in the power utility case. However, it is now modulated by the H_{ct} term, which cancels out in the power utility case, $\chi = 1$. When χ decreases (up to a point), the interest rate becomes less sensitive to p_t, as people are less willing to increase savings when disaster risk is high.

Consider a stock i. I define its *Epstein–Zin–enriched resilience* in (38), whose leading term is $H^{EZ}_{it} = H_{it} + (\chi - 1) H_{ct}$, that is,

$$H^{EZ}_{it} = p_t \mathbb{E}^D_t\left[B_{t+1}^{-\gamma}\left(F_{i,t+1} + (\chi - 1) B_{t+1}\right) - \chi\right]. \tag{33}$$

When $\chi = 1$, (33) is the earlier definition of resilience, given in (4). The EZ resilience is the power utility resilience plus an asset-independent term that reflects variation in the risk-free rate. Hence, when noise is small, introducing EZ preferences changes the risk-free rate but leaves risk premia unchanged to the

29. The Online Appendix finds that, for parameters used in the calibration, the second-order terms are very small (less than 1% of the first-order terms).

leading order (the risk premium remains $p_t \mathbb{E}_t^D \left[B_{t+1}^{-\gamma} \left(1 - F_{it}\right)\right]$).[30] This echoes Barro (2009), which find this result (EZ changes the interest rate, but not the risk premium) when disaster risk is constant over time, but finds it for a time-varying disaster risk.

I assume that the EZ-enriched resilience follows, up to second-order terms, an LG process, with $H_{it}^{EZ} = H_{i*}^{EZ} + \widehat{H}_{it}^{EZ}$ and $\mathbb{E}_t \left[\widehat{H}_{i,t+1}^{EZ} \right] = \frac{1+H_{i*}^{EZ}}{1+H_{it}^{EZ}} e^{-\phi_H} \widehat{H}_{it}^{EZ}$ (to the leading order; see (40) for all specifics). The next proposition shows that the above results on stocks (e.g., Theorem 1, Proposition 1) follow, provided one uses the enriched notion of resilience.

PROPOSITION 9. (Stock price with EZ preferences) With EZ preferences, the price of a stock i is the same exact expression (14) as in the power utility case, but with the EZ-enriched resilience:

$$\frac{P_{it}}{D_{it}} = \frac{1}{\delta_i} \left(1 + \frac{\widehat{H}_{it}^{EZ}}{\delta_i + \phi_H} \right), \qquad \delta_i \equiv \delta - g_i - H_{i*}^{EZ}. \tag{34}$$

In particular, a consumption claim has a resilience $H_{Ct}^{EZ} = \chi p_t \mathbb{E}_t^D \left[B_{t+1}^{1-\gamma} - 1 \right]$, and its price is $P_t = \frac{C_t}{\delta_c} \left(1 + \frac{\chi \widehat{H}_{Ct}}{\delta_c + \phi_c} \right)$. We see that in the case $\chi < 0$ (i.e., an IES > 1 and $\gamma > 1$), when p_t goes up, the resilience of a consumption claim and its price go down. This is the key qualitative effect from an EZ model. In contrast, the power utility model ($\chi = 1$) would lead to the opposite prediction, which is likely anomalous.

Consider now a more general stock. When $\chi < 0$, as the disaster probability goes up, the stock price goes down. This is true as long as $F_{it} \leq B_t$, that is, the asset is riskier than consumption. However, if the asset is very safe (e.g., with $F_{i,t+1} = 1$) and χ is negative enough, then if the disaster probability goes up, the asset's resilience and price go up. This corresponds to a "flight to safety."

Calibration. I follow Barro (2009) and take $\psi = 2$. This leads to a value $\chi = -\frac{1}{6}$. For σ_p, I observe that the variations in the risk-free rate are small (Campbell and Cochrane 1995 calibrate a constant risk-free rate), so I calibrate a small volatility of p_t,

30. Comovement between the SDF and resilience also affects the asset's risk premium, which is $-cov_t \left(\frac{M_{t+1}}{M_t}, r_{i,t+1} \right)$. The Appendix derives terms due to the normal-times covariance with the innovations in the SDF, but we shall see that they are quantitatively small.

$\sigma_p = 0.4\%$, which implies a volatility of the risk-free rate of $\sigma_r = 0.55\%$. To lower the number of free parameters, I assume that the movements in p_t and F_{it} are uncorrelated, and preserve the values of all parameters as in the power utility calibration in Table I, except the rate of time preference, which changes to $\rho = 4.8\%$. Further details are in the Online Appendix.

It turns out that the extra volatility coming from the variation in p_t increases the volatility of resilience only by a fraction 1%: $\sigma_H^{EZ} = \sigma_H \cdot 1.01$. To see why, consider the case of $F_{i,t+1} = B_{t+1} = \overline{B}$. Then, to the leading order, $\widehat{H}_{it}^{EZ} = \chi \overline{B}^{1-\gamma} \widehat{p}_t + p_* \overline{B}^{-\gamma} \widehat{F}_{it}$. As χ is small, the resilience is not too sensitive to deviation of the disaster probability from trend, $\widehat{p}_t$, but it remains equally sensitive as in the power utility case to the deviation of the recovery rate from trend, $\widehat{F}_{it}$. At least in this calibration, most of the volatility comes from the movement in the recovery rate, and most of the risk premia come from disaster risk (the "normal-times" volatility in the SDF is very small, $\sigma_{\varepsilon^M} = 6.4\%$).

Hence, a time-varying probability usually has little impact on the volatility of stocks and their price-earnings ratio. However, the major impact of that probability likely occurs in times of crises, that is, when it suddenly moves a lot. During such episodes, the fraction of the variance explained by p_t would be high.

For nominal bonds with Epstein-Zin utility, we can anticipate that the economics would be like in the power utility case, except for an extra variation in the risk-free rate, in a fashion very analogous to the one developed in Proposition 13 of the Online Appendix.

The present EZ model has some limitations. Indeed, Nakamura et al. (2011) find that empirically consumption drops a couple of years after the disaster strikes and then partly recovers. Then, using a numerical solution, the authors find that the impact of EZ preferences on risk premia is first order, and the partial recovery affects the relative premia of short-term versus long-term assets. The present setup, which assumes that consumption falls on impact and does not recover, misses that effect. I conjecture that the present setup could be extended so as to yield predictable consumption dynamics after the disaster and complement the available numerical treatments, and then account for some further asset pricing puzzles (e.g., for short-term versus long-term assets). Still, the benchmark of consumption dropping at the onset of the disaster is surely useful to study asset prices before the disaster, and the present model offers a reasonably transparent way (via equations (33) and (34)) to understand the economics of that benchmark.

To conclude, the main advantage of this Epstein-Zin setting is that it inverts the anomalous qualitative impact of p_t. At the same time, the quantitative impact of the Epstein-Zin setting is not very large on average, though it can be large in times of crises. Hence, I recommend using the power utility setting with time-varying recovery rates for a host of asset pricing issues (e.g., excess volatility, predictability, options, and the impact of p_t on the relative risk premia) and paying the cost of the somewhat more complex Epstein-Zin framework only when movements in p_t affect the risk-free rate too much. A simple way to think about all these issues is to use the Epstein-Zin enriched resilience (33), which incorporates much of the economics in a compact way.

V.B. Alternative Interpretation of the Model

Some derivations on stocks and bonds do not depend solely on the disaster hypothesis. On the other hand, for some predictions about “tail assets” (e.g., options and high-grade corporate bonds) the disaster model is crucial. This is formalized in the next proposition.

PROPOSITION 10. (Models generating the same stock and government bond prices as a disaster economy, but not the same option and corporate bond prices) Consider a model with stochastic discount factor $\frac{M_{t+1}}{M_t} = e^{-r_f}\left(1+\varepsilon^M_{t+1}\right)$ and a stock with dividend following $\frac{D_{i,t+1}}{D_{it}} = e^{g_{iD}}\left(1+\varepsilon^D_{i,t+1}\right)$, where all $\varepsilon^H_{i,t+1}$'s have expected value 0 at time t. Call $H_{it} = \mathbb{E}_t\left[\varepsilon^M_{t+1}\varepsilon^D_{i,t+1}\right] = H_{i*} + \widehat{H}_{i,t+1}$, so that $-H_{it}$ is the risk premium on the dividend, and assume $\widehat{H}_{i,t+1} = \frac{1+H_{i*}}{1+H_{it}} e^{-\phi_H}\widehat{H}_{it} + \varepsilon^H_{i,t+1}$, with $\varepsilon^H_{i,t+1}$ uncorrelated with the innovations to $\frac{M_{t+1}D_{i,t+1}}{(M_t D_{it})}$. Then, Theorem 1 and Proposition 1 hold, except that the equity premium is $-H_{it}$ and the interest rate is r_f. Furthermore, suppose that inflation is $I_t = I_* + \widehat{I}_t$ and follows $\widehat{I}_{t+1} = \frac{1-I_*}{1-I_t}\left(e^{-\phi_I}\widehat{I}_t + \varepsilon^I_{t+1}\right)$. Call $\mathbb{E}_t\left[\varepsilon^M_{t+1}\varepsilon^I_{t+1}\right] = \pi_* + \pi_t$, the inflation risk premium, and assume $\pi_{t+1} = \frac{1-I_*}{1-I_t}e^{-\phi_J}\pi_t + \varepsilon^\pi_{t+1}$, with $\mathbb{E}_t\left[\varepsilon^M_{t+1}\varepsilon^\pi_{t+1}\right] = 0$. Also, use the notation $\pi_* = (1-I_*)\,\kappa\left(1-e^{-\phi_I}-\kappa\right)$. Then, Theorem 2 on bond values (with $H_\$ = 0$) and Propositions 2, 6–8 on bond predictability hold (except equation (19)).

However, such a model generically has different prices for options and corporate bonds (which are more tail-sensitive).

Proof. With $\frac{Q_{t+1}}{Q_t} = 1 - I_t$, $M_t D_{it}\left(1, \widehat{H}_{it}\right)$ and $M_t Q_t\left(1, \frac{\widehat{I}_t}{1-I_*}, \frac{\pi_t}{1-I_*}\right)$ are both LG processes with the same moments as in the disaster economy. ■

Proposition 10 shows that in many models stocks and bonds will behave exactly as in a disaster economy (however, options or defaultable bonds will behave differently), so that disaster analytics shed light on many models. Rather than a disaster-based pricing kernel, the proposition studies a generic pricing kernel M_t, which could include a "behavioral" one. Relatedly, in some respects one way to model "time-varying sentiment" in a time-consistent way would be to model it as a time-varying perception of disaster risk. Note that it is hard for agents to learn a "rational" perception of disaster risk, as feedback will be scant (Chen, Joslin, and Tran 2011).

Yet disaster models make clearly distinctive predictions for tail-sensitive assets such as options and high-grade corporate bonds (and gold, which could be modeled similarly to a stock, but with a very high resilience). These assets are naturally the object of scrutiny of the growing literature that examines disaster risk empirically, and to which I now turn.

V.C. A Further Provisional Empirical Assessment

This section provides an assessment of the empirical evidence on the link between disasters and asset price movements, as worked out in the present model.

Are the Movements of Asset Prices Correlated with the Movements in Objective Disaster Risk?

i. Political Measures of Disaster Risk. A question very high on the empirical agenda is to find "objective" measures of disaster risk that ideally do not come from asset prices. A few papers attempt to do this. Using a database of 447 major international political crises during the period 1918–2006, Berkman, Jacobsen, and Lee (2011) show that high war risk leads to a drop in asset prices: returns are low when a crisis starts and are high when it ends, and crisis risk is positively correlated with the dividend yield. Other papers measure (on shorter data sets) the impact of the probability of war on asset prices. Bittlingmayer (1998) finds that political risk was an important factor of volatility between 1880 and World War II. Amihud and Wohl (2004) document the link between the probability of the second Iraq War (obtained from prediction markets) and the stock market. All in all, a growing number of studies document a link between political risk and the volatility and level of asset prices, in a way consistent with the disaster hypothesis. A fully structural empirical analysis has yet to be carried out, probably enriched with new data, but the extant evidence is encouraging.

ii. Disaster Risk Measured by Tail Behavior of Asset Prices. Alternatively, we may detect disaster risk in asset prices. Bollerslev, Tauchen, and Zhou (2009) show that when put prices are high, future stock returns are low, like in this article. In addition, the high price of put prices is consistent with disaster risk. Bollerslev and Todorov (forthcoming) find large jumps in option prices that are much harder to detect than in the physical probability. In currency markets, Farhi et al. (2009) find that when put prices on currencies are high, the return on investing in "risky" currencies (by the measure of their put prices) is high. They find that when a currency falls in value, its put prices increase, with a correlation of -0.4. This is consistent with the disaster view. Burnside et al. (2011) also calibrate that disaster risk might account for the violations of uncovered interest rate parity.

In conclusion, put prices are high and they predict future returns, as in the disaster hypothesis. Hence, the evidence is, though not systematically so, supportive of the disaster hypothesis. Finally, a recent paper by Kelly (2011) finds that a moving average of the cross-sectional tail distribution of realized stock returns may be a good proxy for aggregate tail risk, predicting equity returns.

Do High-yield Assets do Particularly Poorly During Disasters? Barro (2009) find that stocks indeed do particularly poorly during disasters. Farhi et al. (2009) report that high-yield currencies do particularly poorly during currency market crashes, consistent with rare-event risk premia. Koijen, Lustig, and van Nieuwerburgh (2010) find that during the Great Depression dividends of value stocks fell a lot more than dividends of growth stocks.[31] Ongoing work with Joachim Voth investigates Russia

31. It also finds that the Cochrane and Piazzesi (2005) factor helps predict value stock returns. This might indicate that disaster risk is important for value stocks and is higher in the cross-section of stocks when it is higher in the

and Germany around 1917, and finds that high-yield stocks did do particularly badly during disasters. Furthermore, I mentioned earlier that Moody's (1999) finds that high-yield bonds did worse than low-yield bonds during the Great Depression.

Is There Sufficient Variation of Disaster Risk Compared to that of Resilience? In the model, we need a large enough variation of resilience. One indirect test is to compare the volatility of the required resilience to the dispersion of actual outcomes in asset markets. Thus, I define and perform the disaster counterpart of the Shiller (1982) excess volatility test. Consider indeed the asset-to-disaster dispersion ratio for a variable X that pays off during disasters:

$$DR_X \equiv \frac{\text{Dispersion of prediction of } X \text{ from asset markets}}{\text{Dispersion of realized values of } X}.$$

It should be less than 1. Indeed: denote by V_X the standard deviation of the variable X, then $DR_X \equiv \frac{V_{\mathbb{E}[X|\mathcal{G}]}}{V_X} \leq 1$ for any information set $\mathcal{G}$.

To evaluate the dispersion of stock resiliences, I consider $X = B_{t+1}^{-\gamma}(1 + r_{i,t+1})$. As the calibration has p_t constant, we have $V_H = pV_{\mathbb{E}^D\left[B_{t+1}^{-\gamma}(1+r_{i,t+1})\right]}$. As (14) gives $V_{\ln \frac{P}{D}} = \frac{V_H}{\delta_i + \phi_H}$, we obtain:

$$DR_{\text{Stocks}} = \frac{p^{-1}(\delta_i + \phi_H)\, V_{\ln P/D}}{V_{B_{t+1}^{-\gamma}(1+r_{i,t+1})|\text{disaster}}}.$$

To evaluate this dispersion ratio, I use the Barro (2009) data, which report series of B_{t+1} and stock market returns during disasters. Note that they use a flexible window to circumvent a variety of econometric problems, including missing data. I find: $V_{B_{t+1}^{-\gamma}(1+r_{i,t+1})|\text{disaster}} = 5.05$. Using also $\frac{\delta_i+\phi_H}{p} V_{\ln \frac{P}{D}} = \frac{0.18}{0.0363}\, 0.33 = 1.63$, I obtain a dispersion ratio $DR_{\text{Stocks}} = 0.32$. It is less than 1, so I conclude that the stocks pass the dispersion-ratio test. This is something of a success for the disaster hypothesis. Economically, the test means that the P/D ratio is volatile, but it is less volatile than the dispersion of (marginal-utility-adjusted) actual stock returns during disasters. Of course, this test is simple and aggregate, and refining it across asset classes, say, would be a very good thing to do.

cross-section of bonds. This could be modeled as a common innovation in the risk premia of value stocks and bonds.

For inflation, I use the similar reason for the change in inflation during a disaster, $X = \Delta I_t$. As $V_\pi = pV_{B_{t+1}^{-\gamma}\Delta I_{t+1}}$, the dispersion ratio is: $DR_{\text{Inflation}} = \frac{V_\pi}{pV_{B_{t+1}^{-\gamma}(\Delta I_{t+1})|\text{disaster}}}$. The calibration gives $V_\pi = 2.1\%$, and the empirical value is $V_{B_{t+1}^{-\gamma}(\Delta I_{t+1})|\text{disaster}} = 6.36$, so that $DR_{\text{Inflation}} = 0.09$. The dispersion ratio is less than 1, consistent with the disaster hypothesis. I conclude that the rare disaster model passes the dispersion-ratio test, for both stocks and inflation. There is enough dispersion in the realized outcomes during disasters to warrant the volatility of stock and bond prices in samples without disasters.

Do Variations in Recovery Rates Implied by the Model Make Sense? For stocks, the P/D ratio reflects resilience in the model. For instance, the low P/D ratio in the 1930s and 1970s reflects the fact that agents believed that if there was going to be a disaster, the stock market would do terribly. This seems reasonable, given the recent experience. Likewise, the measured subjective risk premia for bonds rise and peak between 1975 and 1982 (Piazzesi and Schneider 2011). In terms of the model, agents reasoned: "if there is a disaster, inflation will really go up." This seems sensible given their experience with a (nondisaster) crisis and the concomitant rise in inflation. It would be insightful to have a way of formalizing that intuition and probing it, for example, via a learning model. More broadly, a learning model of how agents might form beliefs about disasters, and conditional asset behavior in disasters, would seem to be quite useful and an important new frontier for research on time-varying assessment of disaster risk.

VI. Conclusion

This article presents a tractable way to handle a time-varying severity of rare disasters, demonstrates its impact on stock and bond prices, and shows its implications for time-varying risk premia and asset predictability. Many finance puzzles can be understood through the lens of the variable rare disasters model. On the other hand, the model does suffer from several limitations and suggests several questions for future research.

First, it would be useful to empirically examine the model's joint expression of the values of stocks, bonds, and options. Here I have only examined their behavior separately, relying on robust stylized facts from many decades of research. The present study suggests specifications for joint cross-asset patterns of predictability. The multifactor extension of the model presented in the Online Appendix could be instrumental in such an endeavor.

It would be useful to understand how investors estimate disaster risk. Risk premia seem to decrease following good news for the economy (Campbell and Cochrane 1999) and for individual firms (the growth-firms effect). So it seems that updating will involve resiliences increasing following good news about the fundamental values of the economy or about individual stocks. Modeling this idea would lead to a link between recent events, risk premia, and future predictability.

This model brings us one step closer to a unified framework for various puzzles in macroeconomics and finance. A companion paper (Farhi and Gabaix 2011) suggests that various puzzles in international macroeconomics (including the forward premium puzzle and the excess volatility puzzle on exchange rates) can be accounted for in an international version of the variable rare disasters framework. Furthermore, ongoing work (Gabaix 2011; Gourio 2011) shows how to embed the rare-disasters idea in a production economy by modifying its asset pricing properties while preserving its business-cycle properties, producing realistic empirical results. Thus, variable rare disaster modeling may bring us closer to the long-sought goal of a joint, tractable framework for macroeconomics and finance.

Appendix

Notations

The article often uses a decomposition of a generic variable X_t as follows: $X_t = X_* + \widehat{X}_t$, where X_* is a constant part (or "typical value") and $\widehat{X}_t$ a variable part centered around 0. ε^X_{t+1} is an innovation to X, and σ_X, its standard deviation, is the volatility of variable X_t. The other notations are as follows.

B_{t+1} : recovery rate of consumption in a disaster
$\overline{B} = \mathbb{E}\left[B_{t+1}^{-\gamma}\right]^{-\frac{1}{\gamma}}$: risk-adjusted average B_{t+1}
β_T : slope in a predictive regression with horizon T
χ : in Epstein-Zin, $\chi = \frac{1-1/\psi}{1-\gamma}$, which is 1 in the power utility case
D_{it} : dividend of stock i
δ : "Ramsey" discount rate
δ_i : stock i's effective discount rate

$\mathbb{E}_t^D[X_{t+1}]$ (resp. $\mathbb{E}_t^{ND}[X_{t+1}]$): expected value conditional on a disaster (resp. on no disaster)
$f_t(T)$: nominal forward rate of maturity T
$F_{\$t}$: recovery rate of a nominal dollar
F_{it} : recovery rate of stock i
g_C : growth rate of consumption
g_{iD} : growth rate of stock i's dividend
γ : coefficient of relative risk aversion
H_{it} : resilience of stock i
H_{it}^{EZ} : Epstein-Zin-enriched resilience of stock i
$H_{\$}$: resilience of a nominal dollar
I_t : inflation
I_{**} : risk-adjusted central part of inflation
J_t : jump in inflation in a disaster
κ : inflation disaster risk premium
K_T : loading on bond risk premium
M_t : pricing kernel
μ_{it} : expected growth of the stock price, conditional on no disasters (only used for options)
p_t : disaster probability
P_{it} : price of stock i
π_t : variable part of the bond risk premium
ϕ_X : rate of mean reversion of variable X_t
ψ : intertemporal elasticity of substitution
ψ_X : rate of mean reversion of variable X_t under the risk-adjusted measure
r_f : risk-free rate
r_{it}^e : stock i's expected stock return, over a one-period horizon
$r_{it\to t+T}$: stock i's expected stock return, over a horizon T
$r_{\$t}^e(T)$: expected return on a nominal bond of maturity T
ρ : subjective rate of time preference
t : calendar time
T : maturity (for a bond) or horizon (for a regression)
V_X : dispersion (standard deviation of the distribution) of variable X
y_i : yield on the debt of corporation i
$y_t(T)$: nominal yield of maturity T
$Z_{\$t}(T)$: price of a nominal zero-coupon bond of maturity T

Exact Setup and Results for the Epstein-Zin Model. Here are the exact postulates for the EZ model, which lead to closed forms.

I postulate that the variable part of resilience can be written as $\widehat{H}_{ct} = k_t + \frac{1-\chi}{2} w^2(k_t)$, with a process k_t following

$$dk_t = -(\phi_c + \chi k_t) k_t dt + (\delta_c + \phi_c + \chi k_t)\left[(\chi - 1) w^2(k_t) dt + w(k_t) dz_t\right], \tag{35}$$

where z_t is a standard Brownian motion.[32] Up to second-order terms, the interpretation is $\widehat{H}_{ct} \simeq k_t$ and $dk_t \simeq -\phi_c k_t dt +$ noise, that is, $\widehat{H}_{ct}$ mean-reverts with speed ϕ_c. For tractability, the process is best expressed with the primitive k_t rather than $\widehat{H}_{ct}$. The volatility function w can be arbitrary, except that it vanishes before k_t hits $-\frac{\phi_c}{\chi}$ (as in Gabaix 2009). This postulate has two joint consequences, derived in the Online Appendix. The price of a consumption claim obtains in exact closed form:

$$P_t = \frac{C_t}{\delta_c}\left(1 + \frac{\chi k_t}{\delta_c + \phi_c}\right). \tag{36}$$

In addition, the SDF M_t follows:

$$\frac{dM_t}{M_{t^-}} = -\delta dt + (\chi - 1)\left(\left(H_{ct} + \frac{\chi w^2(k_t)}{2}\right) dt - w(k_t) dz_t\right) + \left(B_t^{-\gamma} - 1\right) d\mathcal{J}_t, \tag{37}$$

where $d\mathcal{J}_t$ is the disaster jump process, equal to 1 with probability $p_t dt$, and otherwise equal to 0. As a result, $r_{ft} = \delta - (\chi - 1)\left(H_{ct} + \frac{\chi w^2(k_t)}{2}\right) - p_t \mathbb{E}_t^D\left[B_t^{-\gamma} - 1\right]$. The interpretation for both is that in the main text, where the second-order term $\frac{\chi w^2(k_t)}{2}$ is neglected.

The Epstein-Zin enriched resilience of a stock i is defined as:

$$H_{it}^{EZ} = H_{it} + (\chi - 1)\left(H_{ct} + \frac{\chi w^2(k_t)}{2}\right) + \left\langle \frac{dM_t}{M_t}, \frac{dD_{it}}{D_{it}} \right\rangle^{ND}, \tag{38}$$

where $H_{it} = p_t \mathbb{E}_t^D\left[B_t^{-\gamma} F_{it} - 1\right]$ and $\langle dx_t, dy_t \rangle^{ND} dt := cov^{ND}(dx_t, dy_t)$ is the normal-times (i.e., conditional on No Disaster) covariance

32. The reader might ask: why those postulates? I tried to have $\frac{P_t}{C_t} = \frac{1+x_t}{\delta_c}$, with x_t a small term meanreverting at speed ϕ_c, and allowed to add second-order terms to its equation of motion. That generates movements in the SDF, which in turn generate the price of a consumption claim. To ensure self-consistency and find $\frac{P_t}{C_t} = \frac{1+x_t}{\delta_c}$ while having $M_t C_t (1, x_t)$ an LG process, there is very little freedom. Basically, one is more or less directly led to postulate (35). Hence, in the spirit of Campbell-Cochane (1995), Gabaix (2009), and this article, I (proudly) reverse-engineered second-order terms to make the first-order economics transparent.

between two processes x_t and y_t. Hence $H_{it}^{EZ} \simeq H_{it} + (\chi - 1) H_{ct}$, that is, the EZ resilience is the power utility asset-specific resilience (H_{it}) plus an asset-independent term that reflects variations in the interest rate ($(\chi - 1) H_{ct}$). I posit that it follows an LG-twisted process: $H_{it}^{EZ} = H_{i*}^{EZ} + \widehat{H}_{it}^{EZ}$, with

$$d\widehat{H}_{it}^{EZ} + \left\langle \frac{dM_t}{M_t} + \frac{dD_{it}}{D_{it}}, dN_{it}^{EZ} \right\rangle^{ND} dt = -\left(\phi_H + \widehat{H}_{it}^{EZ}\right) \widehat{H}_{it}^{EZ} dt + dN_{it}^{EZ}, \tag{39}$$

where dN_{it}^{EZ} is a mean-zero innovation to $\widehat{H}_{it}^{EZ}$. The small LG terms are largely mathematical conveniences that ensure that the model is solved in closed form, but the interpretation of the leading order is easy: $d\widehat{H}_{it}^{EZ} \simeq -\phi_H \widehat{H}_{it}^{EZ} dt$+noise, that is, $\widehat{H}_{it}^{EZ}$ mean-reverts with a speed ϕ_H.

Then, the Online Appendix proves Proposition 9. It is easy to check that the stock price is higher if the dividends covary with the price kernel in normal times, or if resilience innovations covary with it. One can easily verify that $\widehat{H}_{ct}^{EZ} = \chi k_t$, which elucidates the meaning of k_t: up to a multiplicative factor, it is the variable part of the EZ-enriched resilience of a consumption claim.

Proof of Theorem 1. Following the general procedure for LG processes, I use (2), (3), and form:

$$\frac{M_{t+1}D_{i,t+1}}{M_t D_{it}} = e^{-\delta + g_{iD}} \left(1 + \varepsilon_{t+1}^D\right) \times \begin{cases} 1 & \text{if there is no disaster at } t+1 \\ B_{t+1}^{-\gamma} F_{i,t+1} & \text{if there is a disaster at } t+1 \end{cases}.$$

As the probability of disasters at $t + 1$ is p_t, and $H_{it} \equiv p_t \left(\mathbb{E}_t \left[B_{t+1}^{-\gamma} F_{i,t+1}\right] - 1\right)$,

$$\begin{aligned} \mathbb{E}_t \left[\frac{M_{t+1}D_{i,t+1}}{M_t D_{it}}\right] &= e^{-\delta + g_{iD}} \{ \underbrace{(1 - p_t) \cdot 1}_{\text{No disaster term}} + \underbrace{p_t \cdot \mathbb{E}_t \left[B_{t+1}^{-\gamma} F_{i,t+1}\right]}_{\text{Disaster term}} \} \\ &= e^{-\delta + g_{iD}} \left(1 + H_{it}\right) = e^{-\delta + g_{iD}} \left(1 + H_{i*} + \widehat{H}_{it}\right) \\ &= e^{-\delta + g_{iD} + h_{i*}} \left(1 + e^{-h_{i*}} \widehat{H}_{it}\right) \\ &= e^{-\delta_i} \left(1 + e^{-h_{i*}} \widehat{H}_{it}\right), \end{aligned} \tag{40}$$

where I use the notations $h_{i*} = \ln(1 + H_{i*})$ and $\delta_i = \delta - g_{iD} - h_{i*}$. Next, as $\widehat{H}_{i,t+1}$ is independent of whether there is a disaster, and is uncorrelated with ε^D_{t+1},

$$\begin{aligned}
\mathbb{E}_t\left[\frac{M_{t+1}D_{i,t+1}}{M_tD_{it}}\widehat{H}_{i,t+1}\right] &= \mathbb{E}_t\left[\frac{M_{t+1}D_{i,t+1}}{M_tD_{it}}\right]\mathbb{E}_t\left[\widehat{H}_{i,t+1}\right] \\
&= e^{-\delta+g_{iD}}\left(1+H_{it}\right)\cdot\frac{1+H_{i*}}{1+H_{it}}e^{-\phi_H}\widehat{H}_{it} \qquad (41)\\
&= e^{-\delta+g_{iD}+h_{i*}-\phi_H}\widehat{H}_{it} = e^{-\delta_i-\phi_H}\widehat{H}_{it}.
\end{aligned}$$

We see that in (5) the reason for the $1{+}H_{it}$ term in the denominator was to ensure that the above expression would remain linear in $\widehat{H}_{it}$.

There are two ways to conclude. The first way (which is not entirely rigorous, but is elementary) is to look for a solution of the type $P_{it}{=}D_{it}\left(a+b\widehat{H}_{it}\right)$ for some constants a and b. The price must satisfy: $P_{it} = D_{it} + \mathbb{E}\left[\frac{M_{t+1}P_{i,t+1}}{M_t}\right]$, that is, for all $\widehat{H}_{it}$,

$$\begin{aligned}
a + b\widehat{H}_{it} &= 1 + \mathbb{E}_t\left[\frac{M_{t+1}D_{i,t+1}}{M_tD_{it}}\left(a+b\widehat{H}_{i,t+1}\right)\right] \\
&= 1 + aE_t\left[\frac{M_{t+1}D_{i,t+1}}{M_tD_{it}}\right] + bE_t\left[\frac{M_{t+1}D_{i,t+1}}{M_tD_{it}}\widehat{H}_{i,t+1}\right] \\
&= 1 + ae^{-\delta_i}\left(1+e^{-h_{i*}}\widehat{H}_{it}\right) + be^{-\delta_i-\phi}\widehat{H}_{it} = \left(1+ae^{-\delta_i}\right) \\
&\quad + \left(ae^{-\delta_i-h_{i*}} + be^{-\delta_i-\phi_H}\right)\widehat{H}_{it}.
\end{aligned}$$

Solving for a and b, we get $a = 1 + ae^{-\delta_i}$, $b = ae^{-\delta_i-h_{i*}} + be^{-\delta_i-\phi_H}$, and (13).

The second way is rigorous, and uses linearity-generating (LG) processes. A very short summary of the machinery is available on my web page, and the reader is encouraged to refer to it for the following argument and the proof of Theorem 2. Equations (40) and (41) ensure that $M_tD_{it}\left(1,\widehat{H}_{it}\right)$ is an LG process with generator $\begin{pmatrix} e^{-\delta_i} & e^{-\delta_i-h_{i*}} \\ 0 & e^{-\delta_i-\phi_H}\end{pmatrix}$. The stock price (13) comes from Theorem 2 in Gabaix (2009).

Proof of Theorem 2. The proof is simpler when $J_* = \kappa = 0$, and this is the best case to keep in mind in a first reading. I call $\rho_I = e^{-\phi_I}$ and $\rho_J = e^{-\phi_J}$, use the inflation-adjusted (i.e., real) face value of the bond, Q_t, so that $\frac{Q_{t+1}}{Q_t} = (1 - I_t)$ in normal times, and $\frac{Q_{t+1}}{Q_t} = (1 - I_t) F_{\$,t+1}$ if there is a disaster. I calculate the LG moments.

$$\begin{aligned}
\mathbb{E}_t\left[\frac{M_{t+1}Q_{t+1}}{M_tQ_t}\right] &= e^{-\delta}(1 - I_t)\left\{(1 - p_t)\cdot 1 + p_t \cdot \mathbb{E}_t\left[B_{t+1}^{-\gamma} F_{\$,t+1}\right]\right\}\\
&= e^{-\delta}(1 + H_\$)\left(1 - I_* - \widehat{I}_t\right).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}_t\left[\frac{M_{t+1}Q_{t+1}}{M_tQ_t}\widehat{I}_{t+1}\right]\\
&= e^{-\delta}(1 - I_t)\left\{(1 - p_t)\,\mathbb{E}_t^{ND}\left[\widehat{I}_{t+1}\right] + p_t \cdot \mathbb{E}_t^{D}\left[B_{t+1}^{-\gamma} F_{\$,t+1}\widehat{I}_{t+1}\right]\right\}\\
&= e^{-\delta}(1 - I_t)\frac{1 - I_*}{1 - I_t}\left\{\left(1 - p_t + p_t\mathbb{E}_t\left[B_{t+1}^{-\gamma} F_{\$,t+1}\right]\right) \times \rho_I \widehat{I}_t\right.\\
&\qquad \left. + p_t\mathbb{E}_t\left[B_{t+1}^{-\gamma} F_{\$,t+1}\right]\left(J_* + \widehat{J}_t\right)\right\}\\
&= e^{-\delta}(1 + H_\$)(1 - I_*)\left(\rho_I\widehat{I}_t + \frac{p_t\mathbb{E}_t\left[B_{t+1}^{-\gamma} F_{\$,t+1}\right]}{1 + H_\$}\left(J_* + \widehat{J}_t\right)\right)\\
&= \Psi\left(\rho_I\widehat{I}_t + (1 - I_*)\kappa(1 - \rho_I - \kappa) + \pi_t\right),\\
&\qquad \Psi \equiv e^{-\delta}(1 + H_\$)(1 - I_*)
\end{aligned}$$

using (8) and (9). This gives:

$$\begin{aligned}
\mathbb{E}_t\left[\frac{M_{t+1}Q_{t+1}}{M_tQ_t}\frac{\widehat{I}_{t+1}}{1 - I_*}\right] &= \Psi\left(\kappa(1 - \rho_I - \kappa) + \rho_I\frac{\widehat{I}_t}{1 - I_*} + \frac{\pi_t}{1 - I_*}\right)\\
\mathbb{E}_t\left[\frac{M_{t+1}Q_{t+1}}{M_tQ_t}\widehat{J}_{t+1}\right] = \mathbb{E}_t\left[\frac{M_{t+1}Q_{t+1}}{M_tQ_t}\right]\mathbb{E}_t\left[\widehat{J}_{t+1}\right] &= \Psi\frac{1 - I_*}{1 - I_t}\rho_J\widehat{J}_{t+1}\\
&= \Psi\rho_J\widehat{J}_{t+1},
\end{aligned}$$

so that, as $\frac{\pi_t}{1-I_*}$ is proportional to $\widehat{J}_t$ (equation (8)), $\mathbb{E}_t\left[\frac{M_{t+1}Q_{t+1}}{M_tQ_t}\frac{\pi_{t+1}}{1-I_*}\right] = \Psi\rho_J\frac{\pi_{t+1}}{1-I_*}$. Hence, $M_tQ_t\left(1, \frac{\widehat{I}_t}{1-I_*}, \frac{\pi_t}{1-I_*}\right)$ is an LG process with generator $\Omega = \Psi\begin{pmatrix} 1 & -1 & 0\\ \kappa(1 - \rho_I - \kappa) & \rho_I & 1\\ 0 & 0 & \rho_\pi\end{pmatrix}$. Theorem 1 in Gabaix (2009) gives the bond price, $Z_{\$t}(T) = (1,0,0)\,\Omega^T \left(1, \frac{\widehat{I}_t}{1-I_*}, \frac{\pi_t}{1-I_*}\right)'$, which concludes the derivation of (16) when $\kappa = 0$. When $\kappa \neq 0$, one more step is required. The eigenvalues of Ω are $\Psi\{1 - \kappa, \rho_I + \kappa, \rho_\pi\}$. It is convenient to factorize by $1 - \kappa$ and, hence, to define: $\widetilde{\rho}_i = \frac{\rho_I+\kappa}{1-\kappa}$ and $\widetilde{\rho}_\pi = \frac{\rho_\pi}{1-\kappa}$, which are the discrete-time analogues of the continuous-time mean-reversion speeds $\psi_I \equiv \phi_I - 2\kappa$ and $\psi_J \equiv \phi_J - \kappa$. Calculating Ω^T gives the bond price:

$$Z_{\$t}(T) = (\Psi(1-\kappa))^T \times \left\{ 1 - \frac{1}{1-\kappa} \frac{1-\widetilde{\rho}_i^T}{1-\widetilde{\rho}_i} \left(\frac{\widehat{I}_t}{1-I_*} - \kappa \right) - \frac{1}{(1-\kappa)^2} \cdot \frac{\frac{1-\widetilde{\rho}_i^T}{1-\widetilde{\rho}_i} - \frac{1-\widetilde{\rho}_\pi^T}{1-\widetilde{\rho}_\pi}}{\widetilde{\rho}_i - \widetilde{\rho}_\pi} \cdot \frac{\pi_t}{1-I_*} \right\}. \tag{42}$$

The corresponding value of the yield $y_t(T) = -\frac{\ln Z_{\$t}(T)}{T}$ is:

$$y_t(T) = \delta - H_\$ + I_{**} + \frac{1-e^{-\psi_I T}}{\psi_I T}(I_t - I_{**}) + \frac{\frac{1-e^{-\psi_I T}}{\psi_I} - \frac{1-e^{-\psi_J T}}{\psi_I}}{(\psi_J - \psi_I)T} \pi_t + O(I_t - I_{**}, \pi_t)^2 \tag{43}$$

$$= \delta - H_\$ + I_t + (\kappa(\phi_I - \kappa) + \pi_t - \phi_I(I_t - I_*)) \frac{T}{2}. \tag{44}$$

Proof of Proposition 3. We have $V_t = V_t^{ND} + V_t^D$ with:

$$V_t^{ND} = (1-p_t)\mathbb{E}_t^{ND}[e^{-\delta}\left(K - \frac{P_{i,t+1}}{P_{it}}\right)^+] = (1-p_t)e^{-\delta} \times \mathbb{E}_t\left[\left(K - e^{\mu+\sigma u_{t+1}-\sigma^2/2}\right)^+\right]$$

$$V_t^D = p_t \mathbb{E}_t^D\left[e^{-\delta}\left(B_{t+1}^{-\gamma}K - \frac{P_{i,t+1}}{P_{it}}\right)^+\right] = p_t e^{-\delta}\mathbb{E}_t\left[B_{t+1}^{-\gamma}(K - e^{\mu}F_{i,t+1})^+\right],$$

where $x^+ = \max(0, x)$. Recall that the Black-Scholes value of a put with maturity 1 is: $\mathbb{E}_t[e^{-r}(K - e^{r+\sigma u_{t+1}-\sigma^2/2})^+] = V_{Put}^{BS}(Ke^{-r}, \sigma)$. Hence, the first term is $(1-p_t)$ times:

$$e^{-\delta}\mathbb{E}_t[(K - e^{\mu+\sigma u_{t+1}-\sigma^2/2})^+] = e^{-\delta+\mu}\mathbb{E}_t[(Ke^{-\mu} - e^{\mu+\sigma u_{t+1}-\sigma^2/2})^+]$$
$$= e^{-\delta+\mu}V_{Put}^{BS}\left(Ke^{-\mu}, \sigma\right).$$

Proof of Proposition 6. The Fama-Bliss regression (28) yields $\beta_T = \frac{cov\left(r_{\$t}^e(T) - r_{\$t}^e(0), f_t(T) - f_t(0)\right)}{var(f_t(T) - f_t(0))}$. Equations (17) and (21) give $r_{\$t}^e(T) - r_{\$t}^e(0) = \frac{1-e^{-\psi_I T}}{\psi_I}\pi_t + O\left(\widehat{I}_t, \pi_t\right)^2$ and

$$f_t(T) - f_t(0) = \left(e^{-\psi_I T} - 1\right)\widehat{I}_t + \frac{e^{-\psi_I T} - e^{-\psi_J T}}{\psi_J - \psi_I}\pi_t + a_T + O\left(\widehat{I}_t, \pi_t\right)^2,$$

where a_T is a constant. So to the leading order,

$$\beta_T = \frac{\frac{e^{-\psi_I T} - e^{-\psi_J T}}{\psi_J - \psi_I} \cdot \frac{1 - e^{-\psi_I T}}{\psi_I} var(\pi_t)}{var\left(\left(e^{-\psi_I T} - 1\right)\widehat{I}_t + \frac{e^{-\psi_I T} - e^{-\psi_J T}}{\psi_J - \psi_I}\pi_t\right)}, \tag{45}$$

which implies that $\lim_{T\to\infty} \beta_T = 0$, $\lim_{T\to 0} \beta_T = \frac{var(\pi_t)}{var(\psi_I \hat{I}_t + \pi_t)}$, and (29).

Proof of Proposition 7. This proof is in the limit of $\sigma_I \to 0$, $\widehat{I}_t = 0$, $\kappa \to 0$, and $\Delta t \to 0$. Equation 43 gives: $y_t(T) = a + b(T)\pi_t$, with $b(T) = \frac{\frac{1-e^{-\psi_I T}}{\psi_I} - \frac{1-e^{-\psi_J T}}{\psi_I}}{(\psi_J - \psi_I)T} = \frac{T}{2} - \frac{\psi_I + \psi_J}{6}T^2 + O(T^3)$. Hence:

$$\begin{aligned} \frac{y_{t+\Delta t}(T - \Delta t) - y_t(T)}{\Delta t} &= \frac{\mathbb{E}_t[dy_t(T)]}{dt} - \frac{\partial y_t(T)}{\partial T} \\ &= \left(-\phi_J b(T) - b'(T)\right)\pi_t. \end{aligned}$$

As $\frac{y_t(T) - r_t}{T} = \frac{b(T)\pi_t}{T}$, $-\beta = \frac{\phi_J b(T) + b'(T)}{b(T)/T}$, that is,

$$\beta = -\frac{Tb'(T)}{b(T)} - \phi_J T, \tag{46}$$

so that $\beta = -1 - \frac{2\psi_j - \psi_I}{3}T + O(T^2)$ when $T \to 0$, and $\beta = -\psi_J T + o(T)$ when $T \to \infty$. The reasoning in the text of the article comes from the fact that for small T, $\frac{\mathbb{E}_t[dy_t(T)]}{dt} = -\frac{\phi_J T}{2}\pi_t$, $-\frac{\partial y_t(T)}{\partial T} = \left(-\frac{1}{2} + O(T)\right)\pi_t$, so $\frac{y_{t+\Delta t}(T-\Delta t) - y_t(T)}{\Delta t} \simeq -\frac{\partial y_t(T)}{\partial T}$.

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SUPPLEMENTARY MATERIAL

An Online Appendix for this article can be found at QJE online (qje.oxfordjournals.org).

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