Précis of Results on Linearity-Generating Processes

This paper uses the Linearity-Generating (LG) processes defined and analyzed in Gabaix (2009); this note offers a brief summary of parts of that paper.1 LG processes are given by $\mathcal{E}_t\left[\frac{M_{t+1}D_{t+1}}{M_tD_t}\right] = \alpha + \delta'X_t$ (1) and $\mathcal{E}_t\left[\frac{M_{t+1}D_{t+1}}{M_tD_t}X_{t+1}\right] = \gamma + \Gamma X_t$. (2)

Higher moments need not be specified. For instance, the distribution of the noise does not matter, which makes LG processes parsimonious. As a shorthand, $M_tD_t (1, X_t)$ is an LG process with generator $\Omega = \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix}$.

Stock and bond prices obtain in closed form. The price of a stock $P_t = \mathcal{E}_t\left[\sum_{s \geq t} M_sD_s\right] / M_t$ is, with $I_n$, the identity matrix of dimension $n$:

$$P_t = D_t \frac{\Gamma (I_n - \Gamma)^{-1} X_t}{1 - \alpha - \delta' (I_n - \Gamma)^{-1} \gamma} \quad (3)$$

The price-dividend ratio of a “bond,” or $Z_t(T) = \mathcal{E}_t [M_{t+T}D_{t+T}] / (M_tD_t)$, is:

$$Z_t(T) = \begin{pmatrix} 1 & 0_n \end{pmatrix} Q^T \begin{pmatrix} 1 \\ X_t \end{pmatrix} \quad (4)$$

$$= \alpha^T + \delta' \frac{\alpha^T I_n - \Gamma^T}{\alpha I_n - \Gamma} X_t \text{ when } \gamma = 0 \quad (5)$$

1 It is in particular the way it is used in Gabaix (forth.)
Hence, the “recipe” to solve a model using LG processes is very simple: First, calculate the LG moments (1)-(2), to obtain the values of $\alpha$, $\delta$, $\gamma$, and $\Gamma$. Second, use (3) and (4)-(5) to solve for stock and bond prices.

Conversely, the “recipe” to construct a model using LG processes is to force the model’s primitives (e.g., twists in the AR(1) processes) to satisfy (1)-(2). Then, the model is very easy to solve by the above procedure.

To ensure that the process is well-behaved (and, hence, will prevent prices from being negative), the volatility of the process has to go to zero near some boundary. Gabaix (2009) details these conditions.

Reference
