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STRING THEORY

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Abstract: Part III Course Notes. 24 Lectures.

Contents

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1. Why study String Theory?

- Because it's a consistent theory of perturbative quantum gravity, i.e. of gravitons. But the perturbation theory has zero radius of convergence, and so doesn't actually define a "theory". You need M-theory for that, but that's not yet a "theory" either.

- As a possible UV completion of the standard model. But the "landscape problem" makes it unclear whether String Theory will have any predictive power.

- Because it has applications to everything else (GR, QCD, fluid dynamics, condensed matter, . . .) via AdS/CFT correspondence. But maybe you don't really *need* String Theory.

- Because it lies on the borderline between the trivial and the insoluble. It's the "harmonic oscillator of the 21st century".

- None of the above.

2. The relativistic point particle

What is an elementary particle?

- (Maths) "A unitary irrep of the Poincaré group". These are classified by mass and spin.

- (Physics) "A particle without structure". The classical action for such a particle should depend only on the geometry of its worldline (plus possible variables describing its spin).

Let's pursue the physicist's answer, in the context of a D-dimensional Minkowski space-time. For zero spin the simplest geometrical action for a particle of mass m is

$$I = -mc^2 \int_A^B d\tau = -mc \int_A^B \sqrt{-ds^2} = -mc \int_{t_A}^{t_B} \sqrt{-\dot{x}^2}\, dt \,, \qquad \dot{x} = \frac{dx}{dt} \,, \tag{2.1}$$

where t is an arbitrary worldline parameter. In words, the action is the elapsed proper time between an initial point A and a final point B on the particle's worldline.

We could include terms involving the extrinsic curvature K of the worldline, which is essentially the D-acceleration, or yet higher derivative terms, i.e.

$$I = -mc \int dt \sqrt{-\dot{x}^2} \left[1 + \left(\frac{\ell K}{c^2} \right)^2 + \dots \right] \,, \tag{2.2}$$

where ℓ is a new length scale, which must be characteristic of some internal structure. In the long-wavelength approximation $c^2 K^{-1} \gg \ell$ this structure is invisible and we can neglect any extrinsic curvature corrections. Or perhaps the particle is truly elementary, and $\ell = 0$. In either case, quantization should yield a Hilbert space carrying a unitary irrep of the Poincaré group. For zero spin this means that the particle's wavefunction Ψ should satisfy the Klein-Gordan equation $(\Box - m^2)\,\Psi = 0$. There are many ways to see that this is true.

2.1 Gauge invariance

Think of the particle action

$$I[x] = -mc \int dt \sqrt{-\dot{x}^2} \tag{2.3}$$

as a "1-dim. field theory" for D "scalar fields" $x^m(t)$ $(m = 0, 1, \cdots D-1)$. For a different parameterization, with parameter t', we will have "scalar fields" $x'(t')$, s.t. $x'(t') = x(t)$. If $t' = t - \xi(t)$ for infinitesimal function ξ, then

$$x(t) = x'(t - \xi) = x(t - \xi) + \delta_\xi x(t) = x(t) - \xi\dot{x}(t) + \delta_\xi x(t)\,, \tag{2.4}$$

and hence

$$\delta_\xi x(t) = \xi(t)\dot{x}(t)\,. \tag{2.5}$$

This is a gauge transformation with parameter $\xi(t)$. Check:

$$\begin{aligned}\delta_\xi\sqrt{-\dot{x}^2} &= -\frac{1}{\sqrt{-\dot{x}^2}}\,\dot{x}\cdot\frac{d(\delta_\xi x)}{dt} = -\frac{1}{\sqrt{-\dot{x}^2}}\left(\dot{\xi}\dot{x}^2 + \xi\dot{x}\cdot\ddot{x}\right)\\ &= \dot{\xi}\sqrt{-\dot{x}^2} + \xi\frac{d\sqrt{-\dot{x}^2}}{dt} = \frac{d}{dt}\left(\xi\sqrt{-\dot{x}^2}\right)\,,\end{aligned} \tag{2.6}$$

so the action is invariant for any $\xi(t)$ subject to the b.c.s $\xi(t_A) = \xi(t_B) = 0$. The algebra of these gauge transformations is that of Diff_1, i.e. 1-dim. diffeomorphisms (maths) or 1-dim. general coordinate transformations (phys).

Gauge invariance is **not** a symmetry. Instead it implies a redundancy in the description. We can remove the redundancy by imposing a gauge-fixing condition. For example, in Minkowski space coords. $(x^0, \vec{x})$ we may choose the "temporal gauge"

$$x^0(t) = c\,t\,. \tag{2.7}$$

Since $\delta_\xi x^0 = c\,\xi$ when $x^0 = c\,t$, insisting on this gauge choice implies $\xi = 0$; i.e. no gauge transformation is compatible with the gauge choice, so the gauge is fixed. In this gauge

$$I = -mc^2\int dt\sqrt{1 - v^2/c^2} = \int dt\left\{-mc^2 + \frac{1}{2}mv^2\left[1 + \mathcal{O}\left(v^2/c^2\right)\right]\right\}\,, \tag{2.8}$$

where $v = |\dot{\vec{x}}|$. The potential energy is therefore the rest mass energy mc^2, which we can subtract because it is constant. We can then take the $c \to \infty$ limit to get the non-relativistic particle action

$$I_{NR} = \frac{1}{2}m\int dt\,|d\vec{x}/dt|^2\,. \tag{2.9}$$

From now on we set $c = 1$.

2.2 Hamiltonian formulation

If we start from the gauge-invariant action with $L = -m\sqrt{-\dot{x}^2}$, then

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{m\dot{x}}{\sqrt{-\dot{x}^2}} \quad\Rightarrow\quad p^2 + m^2 \equiv 0\,. \tag{2.10}$$

So not all components of p are independent, which means that *we cannot solve for* $\dot{x}$ *in terms of* p. Another problem is that

$$H = \dot{x} \cdot p - L = \frac{m\dot{x}^2}{\sqrt{-\dot{x}^2}} + m\sqrt{-\dot{x}^2} \equiv 0\,, \tag{2.11}$$

so *the canonical Hamiltonian is zero.*

What do we do? Around 1950 Dirac developed methods to deal with such cases. We call the mass-shell condition $p^2 + m^2 = 0$ a "primary" constraint because it is a direct consequence of the definition of conjugate momenta. Sometimes there are "secondary" constraints but we will never encounter them. According to Dirac we should take the Hamiltonian to be the mass-shell constraint times a Lagrange multiplier, so that

$$I = \int dt \left\{ \dot{x} \cdot p - \frac{1}{2} e \left(p^2 + m^2\right) \right\} , \tag{2.12}$$

where $e(t)$ is the Lagrange multiplier. We do not need to develop the ideas that lead to this conclusion because we can easily check the result by eliminating the variables p and e:

- Use the p equation of motion $p = e^{-1}\dot{x}$ to get the new action

  $$I[x;e] = \frac{1}{2} \int dt \left\{ e^{-1}\dot{x}^2 - em^2 \right\} . \tag{2.13}$$

  At this point it looks as though we have '1-dim. scalar fields" coupled to 1-dim. "gravity", with "cosmological constant" m^2; in this interpretation e is the square root of the 1-dim. metric, i.e. the "einbein".

- Now eliminate e from (2.13) using the e equation of motion $me = \sqrt{-\dot{x}^2}$, to get the standard point particle action $I = -m \int dt \sqrt{-\dot{x}^2}$.

<u>Elimination lemma</u>. When is it legitimate to solve an equation of motion and substitute the result back into the action to get a new action? Let the action $I[\psi, \phi]$ depend on two sets of variables ψ and ϕ, such that the equation $\delta I/\delta\phi = 0$ can be solved algebraically for the variables ϕ as functions of the variables ψ, i.e. $\phi = \phi(\psi)$. In this case

$$\left.\frac{\delta I}{\delta \phi}\right|_{\phi=\phi(\psi)} \equiv 0\,. \tag{2.14}$$

The remaining equations of motion for ψ are then equivalent to those obtained by variation of the new action $\hat{I}[\psi] = I[\psi, \phi(\psi)]$, i.e. that obtained by back-substitution. This follows from the chain rule and (2.14):

$$\frac{\delta \hat{I}}{\delta \psi} = \left.\frac{\delta I}{\delta \psi}\right|_{\phi=\phi(\psi)} + \frac{\delta\phi(\psi)}{\delta\psi} \left.\frac{\delta I}{\delta \phi}\right|_{\phi=\phi(\psi)} = \left.\frac{\delta I}{\delta \psi}\right|_{\phi=\phi(\psi)} . \tag{2.15}$$

Moral: If you use the field equations to eliminate a set of variables then you can substitute the result into the action, to get a new action for the remaining variables, only if the equations you used are those found by varying the original action with respect to the set of variables you eliminate. You can't back-substitute into the action if you use the equations of motion of A to solve for B (although you can still substitute into the remaining equations of motion).

The action (2.12) is still Diff_1 invariant. The gauge transformations are now

$$\delta_\xi x = \xi \dot{x}\,, \qquad \delta_\xi p = \xi \dot{p}\,, \qquad \delta_\xi e = \frac{d}{dt}(e\xi)\ . \tag{2.16}$$

However, the action is also invariant under the much simpler gauge transformations

$$\delta_\alpha x = \alpha(t) p\,, \qquad \delta_\alpha p = 0\,, \qquad \delta_\alpha e = \dot{\alpha}\,. \tag{2.17}$$

Let's call this the "canonical" gauge transformation (for reasons that will become clear). In fact,

$$\delta_\alpha I = \frac{1}{2}\left[\alpha\left(p^2 - m^2\right)\right]_{t_A}^{t_B} \tag{2.18}$$

which is zero if $\alpha(t_A) = \alpha(t_B) = 0$. The Diff_1 and canonical gauge transformations are equivalent because they differ by a "trivial" gauge transformation.

Trivial gauge invariances. Consider $I[\psi, \phi]$ again and transformations

$$\delta_f \psi = f \frac{\delta I}{\delta \phi}\,, \qquad \delta_f \phi = -f \frac{\delta I}{\delta \psi}\,, \tag{2.19}$$

for arbitrary function f. This gives $\delta_f I = 0$, so the action is gauge invariant. As the gauge transformations are zero "on-shell" (i.e. using equations of motion) they have no physical effect. *Any two sets of gauge transformations that differ by a trivial gauge transformation have equivalent physical implications.*

If we fix the gauge invariance by choosing the temporal gauge $x^0(t) = t$ we have

$$\dot{x}^m p_m = \dot{\vec{x}} \cdot \vec{p} - p^0\,, \tag{2.20}$$

so in this gauge the canonical Hamiltonian is

$$H = p^0 = \pm\sqrt{|\vec{p}|^2 + m^2}\,, \tag{2.21}$$

where we have used the constraint to solve for p^0. The sign ambiguity is typical for a relativistic particle.

The canonical Hamiltonian *depends on the choice of gauge.* Another possible gauge choice is *light-cone gauge.* Choose phase-space coordinates

$$\begin{aligned} x^\pm &= \frac{1}{\sqrt{2}}\left(x^1 \pm x^0\right)\,, \qquad \mathbf{x} = \left(x^2, \dots, x^{D-1}\right) \\ p_\pm &= \frac{1}{\sqrt{2}}\left(p_1 \pm p_0\right)\,, \qquad \mathbf{p} = \left(p_2, \dots, p_{D-1}\right)\ . \end{aligned} \tag{2.22}$$

Then

$$\dot{x}^m p_m = \dot{x}^+ p_+ + \dot{x}^- p_- + \dot{\mathbf{x}}\cdot\mathbf{p}\,, \qquad p^2 = 2p_+p_- + |\mathbf{p}|^2\,. \tag{2.23}$$

Notice too that

$$p^+ = p_-\,, \qquad p^- = p_+\,. \tag{2.24}$$

The light-cone gauge is

$$x^+(t) = t\,. \tag{2.25}$$

Since $\delta_\alpha x^+ = \alpha p^+ = \alpha p_-$ the gauge is fixed provided that $p_- \neq 0$. In this gauge

$$\dot{x}^m p_m = \dot{\mathbf{x}}\cdot\mathbf{p} + \dot{x}^- p_- + p_+\,, \tag{2.26}$$

so the canonical Hamiltonian is now

$$H = -p_+ = \frac{|\mathbf{p}|^2 + m^2}{2p_-}\,, \tag{2.27}$$

where we have used the mass-shell constraint to solve for p_+.

- *Poisson brackets.* For mechanical model with action

$$I[q,p] = \int dt\,\left[\dot{q}^I p_I - H(q,p)\right] \tag{2.28}$$

the Poisson bracket of any two functions (f,g) on phase space is

$$\{f,g\}_{PB} = \frac{\partial f}{\partial q^I}\frac{\partial g}{\partial p_I} - \frac{\partial f}{\partial p_I}\frac{\partial g}{\partial q^I}\,. \tag{2.29}$$

In particular,

$$\left\{q^I, p_J\right\}_{PB} = \delta^I_J\,. \tag{2.30}$$

- *More generally*, we start from a symplectic manifold, a phase-space with coordinates z^A and a symplectic (closed, invertible) 2-form $\Omega = \frac{1}{2}\Omega_{AB}\,dz^A\wedge dz^B$. Locally, since $d\Omega = 0$,

$$\Omega = d\omega\,, \qquad \omega = dz^A f_A(z)\,, \tag{2.31}$$

and the action in local coordinates is

$$I = \int dt\,\left[\dot{z}^A f_A(z) - H(z)\right]\,. \tag{2.32}$$

The PB of functions (f,g) is defined as

$$\{f,g\}_{PB} = \Omega^{AB}\frac{\partial f}{\partial z^A}\frac{\partial g}{\partial z^B}\,, \tag{2.33}$$

where Ω^{AB} is the inverse of Ω_{AB}. The PB is an antisymmetric bilinear product, from its definition. Also, for any three functions (f,g,h),

$$d\Omega = 0 \quad \Leftrightarrow \quad \{\{f,g\}_{PB}, h\}_{PB} + \quad \text{cyclic permutations} \equiv 0\,. \tag{2.34}$$

In other words, the PB satisfies the Jacobi identity, and is therefore a Lie bracket, as a consequence of the closure of the symplectic 2-form.

- *Darboux theorem.* This states that there exist local coordinates such that

$$\Omega = dp_I \wedge dq^I \quad \Rightarrow \quad \omega = p_I dq^I + d()\,. \tag{2.35}$$

This leads to the definition (2.29) of the PB.

2.2.1 Gauge invariance and first-class constraints

Consider the action

$$I = \int dt \left\{\dot{q}^I p_I - \lambda^i \varphi_i(q,p)\right\}\,, \qquad I = 1,\ldots,N \quad ; \; i = 1,\ldots,n < N\,. \tag{2.36}$$

The Lagrange multipliers λ^i impose the phase-space constraints $\varphi_i = 0$. Let us suppose that

$$\{\varphi_i, \varphi_j\}_{PB} = f_{ij}{}^k \varphi_k \tag{2.37}$$

for some phase-space *structure functions* $f_{ij}{}^k = -f_{ji}{}^k$. In this case we say that the constraints are "first-class" (Dirac's terminology. There may be "second-class" constraints, but we don't need to know about that now).

<u>Lemma</u>. For any phase space function, call it Q, we can define an infinitesimal transformation of any phase-space function f, with infinitesimal parameter $\epsilon(t)$, by $\delta_\epsilon f = \epsilon \{f, Q\}_{PB}$. In particular,

$$\delta_\epsilon x^I = \epsilon \frac{\partial Q}{\partial p_I}\,, \quad \delta_\epsilon p_I = -\epsilon \frac{\partial Q}{\partial x^I}\,. \tag{2.38}$$

This transformation is such that

$$\delta_\epsilon \left(\dot{x}^I p_I\right) = \dot{\epsilon} Q + \frac{d}{dt}\left[\epsilon \left(p_I \frac{\partial Q}{\partial p_I} - Q\right)\right]\,. \tag{2.39}$$

The special feature of first-class constraints is that *they generate gauge invariances*. From the lemma we see that the linear combination $\epsilon^i \varphi_i$ generates a transformation such that

$$\delta_\epsilon \left(\dot{x}^I p_I\right) = \dot{\epsilon}^i \varphi_i + \frac{d}{dt}\,() \tag{2.40}$$

and we also have

$$\begin{aligned} \delta_\epsilon \left(\lambda^i \varphi_i\right) &= \delta_\epsilon \lambda^i \varphi_i + \lambda^i \epsilon^j \{\varphi_i, \varphi_j\}_{PB} \\ &= \left(\delta_\epsilon \lambda^k + \lambda^i \epsilon^j f_{ij}{}^k\right) \varphi_k \end{aligned} \tag{2.41}$$

where we use (2.37) to get to the second line. Putting these result together, we have

$$\delta_\epsilon I = \int dt \left\{\left(\dot{\epsilon}^k - \delta_\epsilon \lambda^k - \lambda^i \epsilon^j f_{ij}{}^k\right) \varphi_k + \frac{d}{dt}\,()\right\}\,. \tag{2.42}$$

As the Lagrange multipliers are not functions of canonical variables, their transformations can be chosen independently. If we choose

$$\delta_\epsilon \lambda^k = \dot{\epsilon}^k + \epsilon^i \lambda^j f_{ij}{}^k \tag{2.43}$$

then $\delta_\epsilon I$ is a surface term, which is zero if we impose the b.c.s $\epsilon^i(t_A) = \epsilon^i(t_B) = 0$.

The point particle provides a very simple (abelian) example. The one constraint is

$$\varphi = \frac{1}{2}\left(p^2 + m^2\right) , \tag{2.44}$$

and it is trivially first-class. It generates the canonical gauge transformations:

$$\begin{aligned} \delta_\alpha x &= \frac{1}{2}\alpha \left\{x, p^2 + m^2\right\}_{PB} = \alpha p \,, \\ \delta_\alpha p &= \frac{1}{2}\alpha \left\{p, p^2 + m^2\right\}_{PB} = 0 \,, \end{aligned} \tag{2.45}$$

and if we apply the formula (2.43) to get the gauge transformation of the einbein, we find that $\delta_\alpha e = \dot{\alpha}$.

The general model (2.36) also includes the string, as we shall see later. This is still a rather simple case because the structure functions are constants, which means that the constraint functions φ_i span a (non-abelian) Lie algebra. In such cases the transformation (2.43) is a Yang-Mills gauge transformation for a 1-dim. YM gauge potential.

2.2.2 Gauge fixing

We can fix the gauge generated by a set of n first-class constraints by imposing n gauge-fixing conditions

$$\chi^i(q,p) = 0 \qquad i = 1, \ldots, n \,. \tag{2.46}$$

The gauge transformation of these constraints is

$$\delta_\epsilon \chi^i = \left\{\chi^i, \varphi_j\right\}_{PB} \epsilon^j \,, \tag{2.47}$$

so if we want $\delta_\epsilon \chi^i = 0$ to imply $\epsilon^j = 0$ for all j (which is exactly what we do want in order to fix the gauge completely) then we must choose the functions χ^i such that

$$\det \left\{\chi^i, \varphi_j\right\}_{PB} \neq 0 \,. \tag{2.48}$$

This is a useful test for any proposed gauge fixing condition.

In addition to requiring that the gauge-fixing conditions $\chi^i = 0$ actually do fix the gauge, it should also be possible to make a gauge transformation to ensure that $\chi^i = 0$ if this is not already the case. In particular, if $\chi^i = f^i$ for infinitesimal functions f^i , and $\hat{\chi}^i = \chi^i + \delta_\epsilon \chi^i$, then $\hat{\chi}^i = f^i + \delta_\epsilon \chi^i$ and we should be able to find parameters ϵ^i such that $\hat{\chi}^i = 0$. This requires us to solve the equation

$$\left\{\chi^i, \varphi_j\right\}_{PB} \epsilon^j = -f^i \tag{2.49}$$

for ϵ^i, but a solution exists for arbitrary f^i iff the matrix $\{\chi^i, \varphi_j\}_{PB}$ has non-zero determinant.

Corollary. Whenever $\{\chi^i, \varphi_j\}_{PB}$ has zero determinant, two problems arise. One is that the gauge fixing conditions don't completely fix the gauge, and the other is that you can't always arrange for the gauge fixing conditions to be satisfied by making a gauge transformation. This is a *very general point*. Consider the Lorenz gauge $\partial \cdot A = 0$ in electrodynamics (yes, that's Ludwig Lorenz, not Henrik Lorentz of the Lorentz transformation). A gauge transformation $A \to A + d\alpha$ of the gauge condition gives $\Box\alpha = 0$, which does *not* imply that $\alpha = 0$; the gauge has not been fixed completely. It is also true, and for the same reason, that you can't always make a gauge transformation to get to the Lorenz gauge if $\partial \cdot A$ is not zero, even if it is arbitrarily close to zero: the reason is that the operator $\Box$ is not invertible because there are non-zero solutions of the wave equation that cannot be eliminated by imposing the b.c.s permissible for hyperbolic partial differential operators. The Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ does not have this problem because ∇^2 is invertible for appropriate b.c.s (but it breaks manifest Lorentz invariance).

The same problem will arise if we try to fix the gauge invariance of the action (2.36) by imposing conditions on the Lagrange multipliers. More on this later.

2.2.3 Continuous symmetries and Noether's theorem

In addition to its gauge invariance, the point particle action is invariant (in a Minkowski background) under the Poincaré transformations

$$\delta_\Lambda X^m = A^m + \Lambda^m{}_n X^n \,, \qquad \delta_\Lambda P_m = \Lambda_m{}^n P_n \,, \tag{2.50}$$

where A^m and $\Lambda_{mn} = -\Lambda_{nm}$ are constant parameters of a spacetime translation and Lorentz transformation, respectively. Noether's theorem implies that there are associated constants of the motion, i.e. conserved charges. These can be found easily using the following version of Noether's theorem:

- **Noether's Theorem**. Let $I[\phi]$ be an action functional invariant under an infinitesimal transformation $\delta_\epsilon \phi$ for constant parameter ϵ. Then its variation when ϵ is an *arbitrary* function of t must be of the form

$$\delta_\epsilon I = \int dt \, \dot\epsilon \, Q \,. \tag{2.51}$$

The quantity Q is a constant of motion. To see this, choose $\epsilon(t)$ to be zero at the endpoints of integration. In this case, integration by parts gives us

$$\delta_\epsilon I = -\int dt \, \epsilon \, \dot Q \,. \tag{2.52}$$

But the left hand side is zero if we use the field equations because these extremize the action for *any* variation of ϕ, whereas the right-hand side is zero for any $\epsilon(t)$ (with the specified endpoint conditions) only if $\dot{Q}(t) = 0$ for any time t (within the integration limits).

This proves Noether's theorem: a continuous symmetry implies a conserved charge (i.e. constant of the motion); it has to be continuous for us to be able to consider its infinitesimal form. The proof is constructive in that it also gives us the corresponding Noether charge: it is Q. Also, given Q we can recover the symmetry transformation from the formula $\delta_\epsilon \phi = \{\phi, \epsilon Q\}_{PB}$. There may be conserved charges for which the RHS of this formula is zero. These are "topological charges", which do not generate symmetries; they are not Noether charges.

To apply this proof of Noether's theorem to Poincaré invariance of the point particle action, we allow the parameters A and $\Lambda^m{}_n$ of (2.50) to be time-dependent. A calculation then shows that

$$\delta I = \int dt \left\{ \dot{A}^m \mathcal{P}_m + \frac{1}{2} \dot{\Lambda}^n{}_m \mathcal{J}^m{}_n \right\} , \tag{2.53}$$

where

$$\mathcal{P}_m = P_m , \qquad \mathcal{J}^m{}_n = X^m P_n - X_n P^m , \tag{2.54}$$

which are therefore the Poincaré charges. Notice that they are gauge-invariant; this is obvious for $\mathcal{P}_m$, and for $\mathcal{J}^m{}_n$ we have

$$\delta_\alpha \mathcal{J}^m{}_n = \alpha \left(P^m P_n - P_n P^m \right) = 0 . \tag{2.55}$$

Gauge-fixing and symmetries. If we have fixed a gauge invariance by imposing gauge-fixing conditions $\chi^i = 0$, then what happens if our gauge choice does not respect a symmetry with Noether charge Q, i.e. what happens if $\{Q, \chi^i\}_{PB}$ is non-zero.

The answer is that the symmetry is *not* broken. The reason is that there is an intrinsic ambiguity in the symmetry transformation generated by Q whenever there are gauge invariances. We may take the symmetry transformation to be

$$\delta_\epsilon f = \{f, Q\}_{PB}\, \epsilon + \{f, \varphi_j\}_{PB}\, \alpha^j . \tag{2.56}$$

That is, a symmetry transformation with parameter ϵ combined with a gauge transformtion for which the parameters α^i are fixed, in a way to be determined, in terms of ϵ. Because gauge transformations have no physical effect, such a transformation is as good as the one generated by Q alone. The parameters $\alpha^i(\epsilon)$ are determined by requiring that the modified symmetry transformation respect the gauge conditions $\chi^i = 0$, i.e.

$$0 = \left\{\chi^i, Q\right\}_{PB} \epsilon + \left\{\chi^i, \varphi_j\right\}_{PB} \alpha^j(\epsilon)\,. \tag{2.57}$$

As long as $\{\chi^i, \varphi_j\}_{PB}$ has non-zero determinant, we can solve this equation for all α^i in terms of ϵ.

Moral: gauge-fixing never breaks symmetries, because it just removes redundancies. If a symmetry *is* broken by some gauge choice then there is something wrong with the gauge choice!

2.2.4 Quantization: canonical and Dirac's method

We will use the prescription

$$\left\{q^I, p_J\right\}_{PB} \to -\frac{i}{\hbar}\left[\hat{q}^I, \hat{p}_J\right]\,, \tag{2.58}$$

which gives us the canonical commutation relations for the operators $\hat{q}^I$ and $\hat{p}_I$ that replace the classical phase-space coordinates:

$$\left[\hat{q}^I, \hat{p}_J\right] = i\hbar\delta^I_J\,. \tag{2.59}$$

Let's apply this to the point particle in temporal gauge. In this case the canonical commutation relations are precisely (2.59) where $I = 1, \ldots, D-1$. We can realise this om eigenfunctions of $\hat{x}^I$, with e-value x^I, by setting $\hat{p}_I = -i\hbar\partial_I$. The Schroedinger equation is

$$H\Psi = i\hbar\frac{\partial\Psi}{\partial t}\,, \qquad H = \pm\sqrt{-\hbar^2\nabla^2 + m^2} \tag{2.60}$$

Iterating we deduce that

$$\left[-\nabla^2 + \partial_t^2 + (m/\hbar)^2\right]\Psi(x) = 0 \tag{2.61}$$

Since $t = x^0$, this is the Klein-Gordon equation for a scalar field Ψ and mass parameter $m/\hbar$ (the mass parameter of the field equation is the particle mass divided by $\hbar$). The final result is Lorentz invariant even though this was not evident at each step.

An alternative procedure is provided by Dirac's method for quantization of systems with first-class constraints. We'll use the point particle to illustrate the idea.

- Step 1. We start from the manifestly Lorentz invariant, but also gauge invariant, action, and we quantise as if there were no constraint. This means that we have the canonical commutation relations

  $$\left[\hat{q}^m, \hat{p}_n\right] = i\hbar\delta^m_n\,. \tag{2.62}$$

  We can realise this on eigenfunctions $\Psi(x)$ of $\hat{x}^m$ by setting $\hat{p}_m = -i\hbar\partial_m$.

- Step 2. Because of the gauge invariance there are unphysical states in the Hilbert space. We need to remove these with a constraint. The mass-shell constraint encodes the full dynamics of the particle, so we now impose this in the quantum theory as the *physical state condition*

$$\left(\hat{p}^2 + m^2\right)|\Psi\rangle = 0\,. \tag{2.63}$$

  This is equivalent to the Klein-Gordon equation

$$\left[\Box_D - (m/\hbar)^2\right]\Psi(x) = 0\,, \qquad \Psi(x) = \langle x|\Psi\rangle\,. \tag{2.64}$$

  where $\Box_D = \eta^{mn}\partial_m\partial_n$ is the wave operator in D-dimensions.

More generally, for the general model with first-class constraints, we impose the physical state conditions

$$\hat{\varphi}_i|\Psi\rangle = 0\,, \quad i = 1, \ldots, n\,. \tag{2.65}$$

The consistency of these conditions requires that

$$\left[\hat{\varphi}_i, \hat{\varphi}_j\right]|\Psi\rangle = 0 \quad \forall\, i, j\,. \tag{2.66}$$

This would be guaranteed if we could apply the PB-to-commutator prescription of (2.58) to arbitrary phase-space functions, because this would give

$$\left[\hat{\varphi}_i, \hat{\varphi}_j\right] = i\hbar f_{ij}{}^k\hat{\varphi}_k\,, \qquad (?) \tag{2.67}$$

and the RHS annihilates physical states. However, because of operator ordering ambiguities there is no guarantee that (2.67) will be true when the functions φ_i are non-linear. We can use some of the ambiguity to redefine what we mean by $\hat{\varphi}_i$, but this may not be sufficient. *There could be a quantum anomaly.* The string will provide an example of this.
From now on we set $\hbar = 1$.

3. The Nambu-Goto string

The string analog of a particle's worldline is its "worldsheet": the 2-dimensional surface in spacetime that the string sweeps out in the course of its time evolution. Strings can be *open*, with two ends, or *closed*, with no ends. We shall start by considering a closed string. This means that the parameter σ specifying position on the string is subject to a periodic identification. The choice of period has no physical significance; we will choose it to be 2π; i.e. ($\sim$ means "is identified with")

$$\sigma \sim \sigma + 2\pi\,. \tag{3.1}$$

The worldsheet of a closed string is topologically a cylinder, parametrised by σ and some arbitrary time parameter t. We can consider these together as σ^μ ($\mu = 0, 1$), i.e.

$$\sigma^\mu = (t, \sigma). \tag{3.2}$$

The map from the worldsheet to Minkowski space-time is specified by worldsheet fields $X^m(t, \sigma)$. Using this map we can pull back the Minkowski metric on space-time to the worldsheet to get the induced worldsheet metric

$$g_{\mu\nu} = \partial_\mu X^m \partial_\nu X^n \eta_{mn} \,. \tag{3.3}$$

The natural string analog of the point particle action proportional to the proper length of the worldline (i.e. the elapsed proper time) is the Nambu-Goto action, which is proportional to the area of the worldsheet in the induced metric, i.e.

$$I_{NG} = -T \int dt \oint d\sigma \sqrt{-\det g} \,, \tag{3.4}$$

where the constant T is the string tension. Varying with respect to X we get the NG equation of motion

$$\partial_\mu \left(\sqrt{-\det g} \, g^{\mu\nu} \partial_\nu X \right) = 0 \,. \tag{3.5}$$

This is just the 2-dimensional massless wave equation for a set of scalar fields $\{X^m\}$ (scalars with respect to the 2D local Lorentz group) propagating on a 2-dimensional spacetime, but with a metric g that depends on the scalar fields.

Denoting derivatives with respect to t by an overdot and derivatives with respect to σ by a prime, we have

$$g_{\mu\nu} = \begin{pmatrix} \dot{X}^2 & \dot{X} \cdot X' \\ \dot{X} \cdot X' & X'^2 \end{pmatrix} , \tag{3.6}$$

and hence the following alternative form of the NG action

$$I_{NG} = -T \int dt \oint d\sigma \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2} \,, \tag{3.7}$$

where

$$X(t, \sigma + 2\pi) = X(t, \sigma) \,. \tag{3.8}$$

This action is Diff_2 invariant; i.e. invariant under arbitrary local reparametrization of the worldsheet coordinates. From an active point of view (transform fields rather than the coordinates) a Diff_2 transformation of X is

$$\delta_\xi X^m = \xi^\mu \partial_\mu X^m \,, \tag{3.9}$$

where $\xi(t, \sigma)$ is an infinitesimal worldsheet vector field. This implies that

$$\delta_\xi \left(\sqrt{-\det g} \right) = \partial_\mu \left(\xi^\mu \sqrt{-\det g} \right) \,, \tag{3.10}$$

and hence that the action is invariant if ξ is zero at the initial and final times.

3.1 Hamiltonian formulation

The worldsheet momentum density $P_m(t,\sigma)$ canonically conjugate to the wordsheet fields $X^m(t,\sigma)$ is

$$P_m = \frac{\delta L}{\delta \dot{X}^m}\,, \qquad L = -T\oint d\sigma \sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}\,, \tag{3.11}$$

which gives

$$P_m = \frac{T}{\sqrt{-\det g}}\left[\dot{X}_m X'^2 - X'_m\left(\dot{X}\cdot X'\right)\right]. \tag{3.12}$$

This implies the following identities

$$P^2 + \left(TX'\right)^2 \equiv 0\,, \qquad X'^m P_m \equiv 0\,. \tag{3.13}$$

In addition, the canonical Hamiltonian is

$$H = \oint d\sigma \dot{X}^m P_m - L \equiv 0\,. \tag{3.14}$$

As for the particle, we should take the Hamiltonian to be a sum of Lagrange multipliers times the constraints, so we should expect the phase-space form of the action to be

$$I = \int dt \oint d\sigma \left\{\dot{X}^m P_m - \frac{1}{2}e\left[P^2 + \left(TX'\right)^2\right] - u\,X'^m P_m\right\} \tag{3.15}$$

where $e(t,\sigma)$ and $u(t,\sigma)$ are Lagrange multipliers (analogous to the the "lapse" and "shift" functions appearing in the Hamiltonian formulation of GR). To check this, we may eliminate P by using its equation of motion:

$$P = e^{-1}D_t X\,, \qquad D_t X \equiv \dot{X} - uX'\,. \tag{3.16}$$

We are assuming here that e is nowhere zero (but we pass over this point). Back substitution takes us to the action

$$I = \frac{1}{2}\int dt \oint d\sigma \left\{e^{-1}\left(D_t X\right)^2 - e\left(TX'\right)^2\right\}. \tag{3.17}$$

Varying u in this new action we find that

$$u = \frac{\dot{X}\cdot X'}{X'^2} \quad \Rightarrow \quad D_t X^2 = \frac{\det g}{X'^2}\,. \tag{3.18}$$

Here we assume that X'^2 is non-zero (but we pass over this point too). Eliminating u we arrive at the action

$$I = \frac{1}{2}\int dt \oint d\sigma \left\{e^{-1}\frac{\det g}{X'^2} - e\left(TX'\right)^2\right\}. \tag{3.19}$$

Varying this action with respect to e we find that

$$Te = \sqrt{-\det g}/X'^2\,, \tag{3.20}$$

and back-substitution returns us to the Nambu-Goto action in its original form.

3.1.1 Alternative form of phase-space action

Notice that the phase-space constraints are equivalent to

$$\mathcal{H}_\pm = 0\,, \qquad \mathcal{H}_\pm \equiv \frac{1}{4T}\left(P \pm TX'\right)^2\,, \tag{3.21}$$

so we may rewrite the action as

$$I = \int dt \oint d\sigma \left\{\dot{X}^m P_m - \lambda^- \mathcal{H}_- - \lambda^+ \mathcal{H}_+\right\}\,, \qquad \lambda^\pm = Te \pm u\,. \tag{3.22}$$

3.1.2 Gauge invariances

From the Hamiltonian form of the NG string action (3.15) we read off the canonical Poisson bracket relations[1]

$$\{X^m(\sigma), P_n(\sigma')\}_{PB} = \delta^m_n\, \delta(\sigma - \sigma')\,. \tag{3.23}$$

Using this one may now compute the PBs of the constraint functions. One finds that

$$\begin{aligned}
\{\mathcal{H}_+(\sigma), \mathcal{H}_+(\sigma')\}_{PB} &= \left[\mathcal{H}_+(\sigma) + \mathcal{H}_+(\sigma')\right] \delta'(\sigma - \sigma')\,,\\
\{\mathcal{H}_-(\sigma), \mathcal{H}_-(\sigma')\}_{PB} &= -\left[\mathcal{H}_-(\sigma) + \mathcal{H}_-(\sigma')\right] \delta'(\sigma - \sigma')\,,\\
\{\mathcal{H}_+(\sigma), \mathcal{H}_-(\sigma')\}_{PB} &= 0\,.
\end{aligned} \tag{3.24}$$

This shows that

- The constraints are "first-class", with constant structure functions, which are therefore the structure constants of a Lie algebra
- This Lie algebra is a direct sum of two isomorphic algebras ($-\mathcal{H}_-$ obeys the same algebra as $\mathcal{H}_+$). In fact, it is the algebra

  $$\text{Diff}_1 \oplus \text{Diff}_1\,. \tag{3.25}$$

  We will verify this later. Notice that this is a proper subalgebra of Diff_2. Only the $\text{Diff}_1 \oplus \text{Diff}_1$ subalgebra has physical significance because all other gauge transformations of Diff_2 are "trivial" in the sense explained earlier for the particle.

The gauge transformation of any function F on phase space is

$$\delta_\xi F = \left\{F, \oint d\sigma \left(\xi^- \mathcal{H}_- - \xi^+ \mathcal{H}_+\right)\right\}_{PB}\,. \tag{3.26}$$

[1] We can put the action into the form (2.36) by expressing the worldsheet fields as Fourier series; we will do this later. Then we can read off the PBs of the Fourier components, and use them to get the PBs of the worldsheet fields. The result is as given.

where $\xi^\pm$ are arbitrary parameters[2]. This gives

$$\begin{aligned}\delta X &= \frac{1}{2T}\xi^-\left(P - TX'\right) + \frac{1}{2T}\xi^+\left(P + TX'\right) ,\\ \delta P &= -\frac{1}{2}\Big[\xi^-\left(P - TX'\right)\Big]' + \frac{1}{2}\left[\xi^+\left(P + TX'\right)\right]' .\end{aligned} \tag{3.27}$$

Notice that

$$\delta_{\xi^-}\left(P + TX'\right) = 0 , \qquad \delta_{\xi^+}\left(P - TX'\right) = 0 , \tag{3.28}$$

and hence $\delta_{\xi^\mp}\mathcal{H}_\pm = 0$, as expected from the fact that the algebra is a direct sum ($\mathcal{H}_+$ has zero PB with $\mathcal{H}_-$).

To get invariance of the action we have to transform the Lagrange multipliers too. One finds that

$$\delta\lambda^- = \dot{\xi}^- + \lambda^-\left(\xi^-\right)' , \qquad \delta\lambda^+ = \dot{\xi}^+ - \lambda^+\left(\xi^+\right)' . \tag{3.29}$$

We see that $\lambda^\pm$ is a gauge potential for the $\xi^\pm$-transformation, with each being inert under the gauge transformation associated with the other, as expected from the direct sum structure of the gauge algebra.

3.1.3 Symmetries of NG action

The closed NG action has manifest Poincaré invariance, with Noether charges

$$\mathcal{P}_m = \oint d\sigma P_m , \qquad \mathcal{J}_{mn} = 2\oint d\sigma X_{[m}P_{n]} . \tag{3.30}$$

These are constants of the motion. [Exercise: verify that the NG equations of motion imply that $\dot{\mathcal{P}}_m = 0$ and $\dot{\mathcal{J}}_{mn} = 0$.]

N.B. We use the following notation

$$T_{[mn]} = \frac{1}{2}\left(T_{mn} - T_{nm}\right) , \qquad T_{(mn)} = \frac{1}{2}\left(T_{mn} + T_{nm}\right) . \tag{3.31}$$

In other words, we use square brackets for antisymmetrisation and round brackets for symmetrisation, in both cases with "unit strength" (which means, for tensors of any rank, that $A_{[m_1\dots m_n]} = A_{m_1\dots m_n}$ if A is totally antisymmetric, and $S_{(m_1\dots m_n)} = S_{m_1\dots m_n}$ if S is totally symmetric).

The closed NG string is also invariant under **worldsheet parity**: $\sigma \to -\sigma$ (mod 2π). The worldsheet fields (X, P) are parity even, which means that X' is parity odd and hence $(P + TX')$ and $(P - TX')$ are exchanged by parity. This implies that $\mathcal{H}_\pm$ are exchanged by parity.

[2] The relative sign cancels effects due to the sign difference in the algebras of $\mathcal{H}_+$ and $\mathcal{H}_-$.

3.2 Monge gauge

A natural analogue of the temporal gauge for the particle is a gauge in which we set not only $X^0 = t$, to fix the time-reparametrization invariance, but also (say) $X^1 = \sigma$, to fix the reparametrization invariance of the string[3]. This is often called the "static gauge" but this is not a good name because there is no restriction to static configurations. A better name is "Monge gauge", after the 18th century French geometer who used it in the study of surfaces. So, the Monge gauge for the NG string is

$$X^0(t,\sigma) = t \qquad X^1(t,\sigma) = \sigma\,. \tag{3.32}$$

In this gauge the action (3.15) becomes

$$\begin{aligned} I = \int dt \oint d\sigma \Big\{ \dot{X}^I P_I + P_0 - u\left(P_1 + X'^I P_I\right) \\ -\frac{1}{2} e \left[-P_0^2 + P_1^2 + |\mathbf{P}|^2 + T^2\left(1 + |\mathbf{X}'|^2\right)\right]\Big\}\,, \end{aligned} \tag{3.33}$$

where $I = 1, \ldots, D-2$, and $\mathbf{X}$ is the $(D-2)$-vector with components X^I (and similarly for $\mathbf{P}$). We may solve the constraints for P_1, and P_0^2. Choosing the sign of P_0 corresponding to positive energy, we arrive at the action

$$I = \int dt \oint d\sigma \left\{ \dot{X}^I P_I - T\sqrt{1 + |\mathbf{X}'|^2 + T^{-2}\left[|\mathbf{P}|^2 + (\mathbf{X}' \cdot \mathbf{P})^2\right]}\right\}\,. \tag{3.34}$$

The expression for the Hamiltonian in Monge gauge simplifies if $\mathbf{P}$ is momentarily zero; we then have

$$H = T \oint d\sigma \sqrt{1 + |\mathbf{X}'|^2} \qquad (\mathbf{P} = \mathbf{0})\,. \tag{3.35}$$

The integral equals the proper length L of the string. To see this, we observe that the induced worldsheet metric in Monge gauge is

$$\begin{aligned} ds^2\big|_{\rm ind} &= -dt^2 + d\sigma^2 + \left|\dot{\mathbf{X}}dt + \mathbf{X}'d\sigma\right|^2 \\ &= -\left(1 - |\dot{\mathbf{X}}|^2\right)dt^2 + 2\dot{\mathbf{X}}\cdot\mathbf{X}'d\sigma dt + \left(1 + |\mathbf{X}'|^2\right)d\sigma^2\,, \end{aligned} \tag{3.36}$$

and hence

$$L = \oint d\sigma \sqrt{ds^2|_{\rm ind}(t = \text{const.})} = \oint d\sigma \sqrt{1 + |\mathbf{X}'|^2}\,. \tag{3.37}$$

Also, when $\mathbf{P} = \mathbf{0}$ the equations of motion in Monge gauge imply that $\dot{\mathbf{X}} = \mathbf{0}$, so the string is momentarily at rest. The energy of such a string is $H = TL$, and hence the (potential) energy per unit length, or energy density, of the string is

$$\mathcal{E} = T\,. \tag{3.38}$$

[3]We could choose any linear combination of the space components of X to equal σ but locally we can always orient the axes such that this combination equals X^1.

This is the defining feature of an ultra-relativistic string. For small amplitude vibrations of a uniform string of energy density $\mathcal{E}$, the velocity of transverse waves on the string is

$$v = \sqrt{T/\mathcal{E}}\; c\,, \tag{3.39}$$

so $v \leq c$ requires

$$T \leq \mathcal{E}\,. \tag{3.40}$$

For non-relativistic strings $T \ll \mathcal{E}$ (note that $\mathcal{E}$ includes any rest-mass energy, so a string made of any available material will be non-relativistic). The Schwarzshild solution of GR viewed as a string solution of 5D GR has $T = \mathcal{E}/2$, so it is a relativistic string, but not ultra-relativistic. Ultra-relativistic strings can occur as defects in relativistic scalar fields; these are the relativistic analogs of the vortices that can appear in superfluids. In the context of cosmology, such string are called "cosmic strings"; they are well-described by the NG action at lengths scales large compared to the size of the string core, but the NG action is then just an "effective action", which will have worldsheet curvature corrections, similar to those mentioned earlier for the particle. In String Theory, we take the NG action at face value, as the action for an "elementary" string.

Because the NG string is ultra-relativistic, it cannot support tangential momentum. This is what the constraint $X'^m P_m = 0$ tells us; given that $X^0 = t$, we have

$$\vec{X}' \cdot \vec{P} = 0\,, \tag{3.41}$$

which states that the (space) momentum is orthogonal to the tangent to the string. This has various consequences, One is that there can be no longitudinal waves on the string (i.e. sound waves). Only transverse fluctuations are physical.

It also means that a plane circular loop of NG string cannot be supported against collapse by rotation in the plane (which can be done if $T < \mathcal{E}$). This does not mean that a plane circular loop of string cannot be supported against collapse by rotation in other planes; we'll see an example later.

3.3 Polyakov action

Consider the "Polyakov" action[4]

$$I[X,\gamma] = -\frac{T}{2}\int d^2\sigma \sqrt{-\det\gamma}\,\gamma^{\mu\nu} g_{\mu\nu}\,, \tag{3.42}$$

where γ is a new *independent* worldsheet metric. As before, $g_{\mu\nu} = \partial_\mu X \cdot \partial_\nu X$ is the induced metric. The action depends on the metric γ only through its conformal class; in other words, given an everywhere non-zero function Ω, a rescaling

$$\gamma_{\mu\nu} \to \Omega^2 \gamma_{\mu\nu} \tag{3.43}$$

[4]The Polyakov action was actually introduced by Brink, DiVecchia and Howe, and by Deser and Zumino. Polyakov used it in the context of a path-integral quantization of the NG string, which we will consider later.

has no effect; the action is "Weyl invariant". This is easily seen since

$$\gamma^{\mu\nu} \to \Omega^{-2}\gamma^{\mu\nu}\,, \qquad \sqrt{-\det\gamma} \to \Omega^2\sqrt{-\det\gamma}\,, \tag{3.44}$$

so the factors of Ω cancel from $\sqrt{-\det\gamma}\,\gamma^{\mu\nu}$, which is how γ appears in the Polyakov action.

Varying the Polyakov action with respect to $\gamma^{\mu\nu}$ we get the equation

$$g_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\left(\gamma^{\rho\sigma}g_{\rho\sigma}\right) = 0\,, \tag{3.45}$$

which we can rewrite as

$$\gamma_{\mu\nu} = \Omega^2 g_{\mu\nu} \qquad \left(\Omega^{-2} = \frac{1}{2}\gamma^{\mu\nu}g_{\mu\nu}\right). \tag{3.46}$$

In other words, the equation of motion for $\gamma_{\mu\nu}$ sets it equal to the induced metric $g_{\mu\nu}$ up to an irrelevant conformal factor. Back substitution gives us

$$I \to -\frac{T}{2}\int d^2\sigma\sqrt{-\det g}\,g^{\mu\nu}g_{\mu\nu} = -T\int d^2\sigma\sqrt{-\det g}\,, \tag{3.47}$$

so the Polyakov action is equivalent to the NG action.

3.3.1 Relation to phase-space action

Recall that elimination of P, using $P = e^{-1}D_tX$, gives

$$\begin{aligned} I &= \frac{1}{2}\int d^2\sigma\left\{e^{-1}\left(D_tX\right)^2 - e\left(TX'\right)^2\right\} \qquad \left(D_t = \dot{X} - uX'\right) \\ &= \frac{1}{2}\int d^2\sigma\left\{e^{-1}\dot{X}^2 - 2e^{-1}u\dot{X}\cdot X' + e^{-1}\left(u^2 - T^2e^2\right)\left(X'\right)^2\right\}. \end{aligned} \tag{3.48}$$

This has the Polyakov form

$$I = -\frac{T}{2}\int d^2\sigma\sqrt{-\det\gamma}\;\gamma^{\mu\nu}\partial_\mu X\cdot\partial_\nu X\,, \tag{3.49}$$

with

$$\sqrt{-\det\gamma}\,\gamma^{\mu\nu} = \frac{1}{Te}\begin{pmatrix} -1 & u \\ u & T^2e^2 - u^2 \end{pmatrix}. \tag{3.50}$$

This tells us that

$$\gamma_{\mu\nu} = \Omega^2\begin{pmatrix} u^2 - T^2e^2 & u \\ u & 1 \end{pmatrix}, \tag{3.51}$$

for some irrelevant conformal factor Ω. Equivalently

$$\begin{aligned} ds^2(\gamma) &= \Omega^2\left[\left(u^2 - T^2e^2\right)dt^2 + 2udtd\sigma + d\sigma^2\right] \\ &= \Omega^2\left(d\sigma + \lambda^+dt\right)\left(d\sigma - \lambda^-dt\right)\,, \end{aligned} \tag{3.52}$$

where

$$\lambda^\pm = Te \pm u\,. \tag{3.53}$$

You should recognise these as the Lagrange multipliers of the phase-space action in the form (3.22).

3.4 Conformal gauge

In 2D there exist local coordinates σ^μ, and a conformal factor Ω, such that

$$\gamma_{\mu\nu} = \Omega^2 \eta_{\mu\nu} \,, \tag{3.54}$$

i.e. such that the metric is conformally flat (in the same conformal class as the flat Minkowski metric). This mathematical fact is plausible because the metric has three independent functions, two of which can be eliminated by a coordinate transformation[5]. In these coordinate the Polyakov action becomes

$$I_{Poly} \to -\frac{T}{2}\int d^2\sigma \, \eta^{\mu\nu}\partial_\mu X \cdot \partial_\nu X \,, \tag{3.55}$$

i.e. a Minkowski space 2D field theory for D scalar fields $X^m(t,\sigma)$. By introducing worldsheet light cone coordinates

$$\sigma^\pm = \frac{1}{\sqrt{2}}\left(\sigma \pm t\right) \tag{3.56}$$

we can rewrite the action (3.55) as

$$I = -T\int d^2\sigma \, \partial_+ X \cdot \partial_- X \tag{3.57}$$

where $\partial_\pm = \partial/\partial\sigma^\pm$.

In the conformal gauge the NG equation of motion is the 2D wave equation

$$\Box X^m = 0 \,, \qquad \Box \equiv -\partial_t^2 + \partial_\sigma^2 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu \equiv 2\partial_+\partial_- \,. \tag{3.58}$$

This can be seen either by using the conformal gauge condition (3.54) in the NG equation (3.5), or by varying X in the conformal gauge action (3.55). The general solution is

$$X = X_L(\sigma^+) + X_R(\sigma^-) \,, \tag{3.59}$$

where X_L depends *only* on σ^+ and X_R depends *only* on σ^-. In other words, we have some wave profile X_L that moves to the left at the speed of light, superposed on another wave profile X_R that moves to the right at the speed of light. However, not all components of X_L and X_R are independent because the equation of motion for the independent metric, i.e. eq. (3.46), tells us that it is in the same conformal class as the induced metric, and in the conformal gauge this condition is $g_{\mu\nu} = \Omega^2\eta_{\mu\nu}$, or

$$\begin{pmatrix} \dot{X}^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & (X')^2 \end{pmatrix} = \begin{pmatrix} -\Omega^2 & 0 \\ 0 & \Omega^2 \end{pmatrix} \,, \tag{3.60}$$

[5] The proof starts from the observation that the curvature of a 2D space is entirely determined by its Ricci scalar R, which becomes a function of Ω and its derivatives for a conformally-flat metric; so given R we get a differential equation for Ω with R as a "source" term. Locally, this can be solved for Ω in terms of R, so for any R there exists an Ω for which the metric is locally conformally flat.

This equation determines the irrelevant conformal factor Ω; more importantly, it also imposes the constraints

$$\dot{X}^2 + (X')^2 = 0 \quad \& \quad \dot{X} \cdot X' = 0 \quad \Leftrightarrow \quad \left(\dot{X} \pm X'\right)^2 = 0\,. \tag{3.61}$$

These constraints are equivalent to the Hamiltonian constraints $\mathcal{H}_\pm = 0$. To see this recall that $P = T\dot{X}$ in conformal gauge, so that

$$\mathcal{H}_\pm \equiv \frac{1}{4T}\left(P \pm TX'\right)^2 \;\to\; \frac{T}{4}\left(\dot{X} + X'\right)^2\,. \tag{3.62}$$

Notice that we can write the constraints using the worldsheet light cone coordinates $\sigma^\pm$ as

$$\left(\partial_- X\right)^2 = 0 \quad \& \quad \left(\partial_+ X\right)^2 = 0\,. \tag{3.63}$$

<u>A puzzle</u>: We found the constraints, starting from the Polyakov action, from the conformal gauge condition that we imposed to reduce this action to the simple form (3.55) of a massless 2D scalar field theory. If we were given only the conformal gauge action, and not told where it came from, how would we deduce the constraints? It looks as though the action no longer "knows" about the constraints once we have gone to conformal gauge. We shall address this puzzle in the context of the Hamiltonian formulation.

From (3.52) we see that the conformal gauge condition (3.54) is equivalent to the conditions

$$\lambda^+ = \lambda^- = 1 \qquad \left(\Leftrightarrow \quad e = \frac{1}{T} \quad \& \quad u = 0\right). \tag{3.64}$$

In other words, it corresponds to imposing particular conditions on the Lagrange multipliers that appear in the Hamiltonian form of the action.

<u>Check</u>. Setting $e = 1/T$ and $u = 0$ we get the action

$$I = \int d^2\sigma \left\{ \dot{X}^m P_m - \frac{1}{2T}P^2 - \frac{T}{2}(X')^2 \right\} \tag{3.65}$$

Eliminating P by its equation of motion $P = T\dot{X}$, we find the same action $I[X]$ for massless 2D fields as we found from the Polyakov action. Variation of $I[X]$ with respect to X yields the 2D wave equation $\Box X = 0$, but that's it. We can't vary $\lambda^\pm$ to deduce the constraints because we have set them both to unity. You might think "yes, but when you set a field to a constant you should expect to lose the equation that you found before by varying it, and that just means that you should add this equation 'by hand' to those you get by varying the gauge-fixed action."

However, **it is not possible to "lose" equations by fixing a gauge, as long as the gauge fixing is legitimate**. This is because we remove redundancy, and only redundancy, when we fix a gauge. It follows that the conformal gauge is not legitimate, strictly speaking, because it must remove more than redundancy. To see what is going on, it is helpful to return to the particle.

3.4.1 "Conformal gauge" for the particle

The particle analog of the conformal gauge is

$$e = 1/m\,. \tag{3.66}$$

Substituting into the action we find that

$$I \;\rightarrow\; \int dt \left\{ \dot{x}^m p_m - \frac{p^2}{2m} - \frac{m}{2} \right\}. \tag{3.67}$$

Eliminating p by its equation of motion $p = m\dot{x}$ we arrive at the action

$$I[x] = \frac{m}{2} \int dt \left\{ \dot{x}^2 - 1 \right\}. \tag{3.68}$$

This is the particle analog of the conformal gauge action. The equation of motion is

$$\ddot{x} = 0\,, \tag{3.69}$$

which is the "1D wave equation". That's all we get from $I[x]$ but if we recall that the particle equations of motion, prior to gauge fixing, imply that $me = \sqrt{-\dot{x}^2}$, which becomes $1 = \sqrt{-\dot{x}^2}$ when $e = 1/m$, we see that the $e = 1/m$ choice of gauge is equivalent, using the equations of motion, to the constraint

$$\dot{x}^2 = -1\,. \tag{3.70}$$

We appear to have lost this constraint from the gauge-fixed action. But this is impossible, so there must be something wrong with the gauge choice.

To see why we cannot use the gauge invariance of the particle action to set $e = 1/m$, recall that the gauge variation of e is $\delta e = \dot{\alpha}$, so

$$\delta_\alpha \left[\int_{t_A}^{t_B} dt\, e \right] = [\alpha]_{t_A}^{t_B} = 0 \qquad \text{by b.c.s on } \alpha(\mathrm{t}). \tag{3.71}$$

The integral is the lapsed proper time divided by the mass. This quantity is gauge invariant and *cannot be changed by a gauge transformation*, so setting e to any particular constant fixes a gauge-invariant quantity; this is more than just fixing the gauge. The best that we can do is to set

$$e = s\,, \tag{3.72}$$

for variable constant s, on which the gauge-fixed action still depends. Varying the gauge-fixed phase-space action with respect to s yields the *integrated constraint*

$$\int_{t_A}^{t_B} dt\, \left(p^2 + m^2\right) = 0\,. \tag{3.73}$$

However, since p^2+m^2 is a constant of the motion, this is equivalent, when combined with the equation of motion $\dot{p}=0$ to the unintegrated constraint $p^2+m^2=0$, which we can think of as a *initial condition*. This interpretation would be obvious if, instead of choosing $e=s$, we choose $e=1/m$ *almost everywhere*, leaving it free in a neighbourhood of the initial time.

There is another, related, problem with the gauge-fixing condition $e=1/m$, which is that *it doesn't completely fix the gauge*. Setting to zero the gauge variation of the gauge-fixing condition we find that

$$0=\delta_\alpha\left(e-1/m\right)=\dot{\alpha} \tag{3.74}$$

which does **not** imply that $\alpha=0$. Instead it implies that $\alpha=\bar{\alpha}$ for some constant $\bar{\alpha}$. So the gauge choice leaves a *residual* invariance, a rather simple one for the particle. This residual invariance is a symmetry of the action (3.68) but it is really a gauge invariance once we take into account that the Noether charge is precisely what the apparently lost constraint sets to zero.

3.4.2 Residual invariance of the string in conformal gauge

The analog of the conformal gauge for the particle was both *too strong* and *too weak*

- Too strong because it sets to a gauge invariant variable, precisely the variable needed to derive the constraint as an initial condition by varying the action.
- Too weak because it doesn't completely fix the gauge; it leaves a residual invariance, which is a symmetry of the gauge fixed action, but it's really a gauge invariance because the Noether charge is precisely what is set to zero by the constraint.

As mentioned earlier, the same happens for Lorenz gauge in electrodynamics, and the same is true of the conformal gauge of the string. The conformal gauge is too strong; we cannot choose the conformal gauge everywhere on the string worldsheet. A careful analysis would lead to the conclusion that the conformal gauge action should depend on additional variables, and variation with respect to them leads to the constraints $(\partial_\pm X)^2=0$ being imposed, e.g. as initial conditions.

The conformal gauge is also too weak. It leaves a residual invariance, which turns out to play a major role in String Theory. We can see what this invariance is by returning to the gauge transformations of the Lagrange multipliers $\lambda^\pm$, given in (3.29). Setting $\lambda^\pm=1$, and requiring that this be maintained by a gauge transformation, leads to

$$\begin{aligned} 0&=\delta\left(\lambda^- -1\right)|_{\lambda^-=1}=\dot{\xi}^- +(\xi^-)'=\sqrt{2}\partial_+\xi^-\,, \\ 0&=\delta\left(\lambda^+ -1\right)|_{\lambda^+=1}=\dot{\xi}^- -(\xi^+)'=\sqrt{2}\partial_-\xi^+\,, \end{aligned} \tag{3.75}$$

so we have a residual invariance with parameters $\xi^\pm$ satisfying

$$\partial_- \xi^+ = \partial_+ \xi^- = 0\,. \tag{3.76}$$

In other words,

$$\xi^- = \xi^-(\sigma^-)\,, \qquad \xi^+ = \xi^+\left(\sigma^+\right)\,. \tag{3.77}$$

Let's check this. Recalling the gauge transformation of X given in (3.27), and using the conformal gauge relation $P = T\dot{X}$, we see that

$$\delta_\xi X = \frac{1}{\sqrt{2}}\left[\xi^- \partial_- X + \xi^+ \partial_+ X\right]\,. \tag{3.78}$$

Using this in the conformal gauge action $I[X] = T\int d^2\sigma\, \partial_+ X \cdot \partial_- X$, we find that the ξ^+ variation of $I[X]$ is

$$\delta_{\xi^+} I = \frac{T}{\sqrt{2}} \int d^2\sigma \left\{\partial_+\left(\xi^+ \partial_+ X\right) \cdot \partial_- X + \partial_+ X \cdot \partial_- \left(\xi^+ \partial_+ X\right)\right\}\,. \tag{3.79}$$

Now we integrate by parts in the first term (with respect to ∂_+); in doing so we pick up a total time derivative, which we can omit. We are left with a term that almost cancels the second term, leaving

$$\delta_{\xi^+} I = \frac{T}{\sqrt{2}} \int d^2\sigma \left\{\partial_- \xi^+ \left(\partial_+ X\right)^2\right\}\,. \tag{3.80}$$

A similar calculation yields

$$\delta_{\xi^-} I = \frac{T}{\sqrt{2}} \int d^2\sigma \left\{\partial_+ \xi^- \left(\partial_- X\right)^2\right\}\,. \tag{3.81}$$

We thus confirm that the action is invariant if the parameters $\xi^\pm$ are restricted by (3.76).

The residual invariance is a *symmetry* of the conformal gauge action. Using the above calculation, we can compute the Noether charges by allowing the parameters to have an additional t-dependence; i.e.

$$\xi^- = \xi^-(\sigma^-, t)\,, \qquad \xi^+ = \xi^+(\sigma^+, t)\,, \tag{3.82}$$

then we find that

$$\delta_\xi I = \frac{T}{2} \int d^2\sigma \left\{\partial_t \xi^+ \left(\partial_+ X\right)^2 + \partial_t \xi^- \left(\partial_- X\right)^2\right\}\,, \tag{3.83}$$

so the Noether charges are

$$\Theta_{++} = T(\partial_+ X)^2 \quad \& \quad \Theta_{--} = T(\partial_- X)^2\,. \tag{3.84}$$

These are the non-zero components of the stress tensor in light-cone coordinates. Viewing the conformal gauge action as a massless 2D scalar field theory, its stress tensor is

$$\Theta_{\mu\nu} = T\left[\partial_\mu X \cdot \partial_\nu X - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\left(\partial_\rho X \cdot \partial_\sigma X\right)\right]. \tag{3.85}$$

Because this tensor is traceless its Θ_{+-} component is zero (*Exercise*: check this). The only non-zero components are Θ_{++} and Θ_{--}.

Puzzle: How can the residual invariance be a *symmetry* when it is a residual *gauge* invariance? The answer is that it is a symmetry of the conformal gauge action, rather than a gauge invariance, precisely because the Hamiltonian constraints are "missing" from this action. Notice that **the Noether charges of the residual symmetry are precisely what the missing constraints set to zero**. So if we properly take into account the origin of the conformal gauge action, and include the constraints as initial conditions, then we have a residual *gauge* invariance. But if we view the conformal gauge action as just a 2D scalar field theory, it is a symmetry.

3.4.3 Conformal invariance

What is the residual symmetry of the string in conformal gauge?

The conformal group of any space-time, with metric g, is the group of conformal isometries, which is the subgroup of the group of diffeomorphisms, generated by vector fields, that leave invariant the conformal class of g. This subgroup is generated by the conformal Killing vector fields, for which[6]

$$\left(\mathcal{L}_\xi g\right)_{\mu\nu} = \chi g_{\mu\nu} \tag{3.86}$$

for any function χ. The symbol $\mathcal{L}_\xi$ means "Lie derivative with respect to vector field ξ", and its definition is such that

$$\left(\mathcal{L}_\xi g\right)_{\mu\nu} = \xi^\rho\partial_\rho g_{\mu\nu} + 2\partial_{(\mu}\xi^\rho g_{\nu)\rho}\,. \tag{3.87}$$

The infinitesimal action of this group on a scalar X is $X \to X + \delta_\xi X$, where

$$\delta_\xi X = \mathcal{L}_\xi X = \xi^\mu\partial_\mu X\,. \tag{3.88}$$

For Minkowski space-time with metric η, which is constant in cartesian coordinates, a vector field ξ is a conformal Killing vector field if $2\partial_{(\mu}\xi_{\nu)} = \chi\eta_{\mu\nu}$. The number of linearly independent conformal Killing vector fields is finite for $n > 2$. Let's

[6] If $\chi = 0$ the vector field ξ generates an isometry, which is a special case of a conformal isometry. The isometry group of Minkowski space-time is the Poincaré group, which is a proper subgroup of the conformal group.

now consider the $n = 2$ case; in light-cone coordinates the condition (3.87) can be expressed as

$$\begin{pmatrix} 2\partial_+\xi^- & \partial_+\xi^+ + \partial_-\xi^- \\ \partial_+\xi^+ + \partial_-\xi^- & 2\partial_-\xi^+ \end{pmatrix} = \begin{pmatrix} 0 & \chi \\ \chi & 0 \end{pmatrix} . \tag{3.89}$$

This equation determines the irrelevant factor χ but it also restricts $\xi^\pm$ to satisfy $\partial_\mp\xi^\pm = 0$, which is precisely (3.76). We conclude that the residual symmetry of the NG action in conformal gauge is 2D conformal invariance.

What is the algebra of the residual symmetry transformations?

A symmetry of the action is also a symmetry of the equations of motion[7]. For any solution of the equations of motion we have

$$X = X_L(\sigma^+) + X_R(\sigma^-) \tag{3.90}$$

subject to the constraints

$$(\partial_- X_R)^2 = 0 \quad \& \quad (\partial_+ X_L)^2 = 0\,, \tag{3.91}$$

which follow from (3.63). The residual symmetry transformation (3.78) can now be written as

$$\left[\delta_{\xi^+} X_L(\sigma^+) - \frac{1}{\sqrt{2}}\xi^+(\sigma_+)\partial_+ X_L(\sigma^+)\right] + \left[\delta_{\xi^-} X_R(\sigma^-) - \frac{1}{\sqrt{2}}\xi^-(\sigma_-)\partial_- X_R(\sigma^-)\right] = 0\,. \tag{3.92}$$

As the terms in the first (second) bracket are all functions of σ^+ (σ^-) only, both sets of bracketed terms are separately zero[8], so

$$\delta_{\xi^+} X_L(\sigma^+) = \frac{1}{\sqrt{2}}\xi^+(\sigma_+)\partial_+ X_L(\sigma^+)\,, \qquad \delta_{\xi^-} X_R(\sigma^-) = \frac{1}{\sqrt{2}}\xi^-(\sigma_-)\partial_- X_R(\sigma^-)\,. \tag{3.93}$$

These are just the transformations due to separate 1D coordinate transformations; the algebra of these transformations is

$$\mathrm{Diff}_1 \oplus \mathrm{Diff}_1\,. \tag{3.94}$$

Whereas this was originally an algebra of *gauge transformations* of the phase-space form of the NG action, with unrestricted parameters, it is now the algebra of a symmetry of the conformal gauge action.

N.B. A basis for Diff_1 is

$$\{z^n\partial_z\ ;\ n = 0, 1, 2, \ldots\} \tag{3.95}$$

[7]But the converse is not true; there can be symmetries of the equations of motion that are not symmetries of the action.

[8]Actually, they are both equal to constants, which have to sum to zero, but we shall ignore shifts of X_L and X_R by constants.

where z can be either σ^+ or σ^-. An arbitrary Diff_1 *symmetry* transformation, with *constant* parameters $\{\xi_n; n = 0, 1, 2\ldots\}$ has a parameter

$$\xi(z) = \xi_0 + \xi_1 z + \frac{1}{2}\xi_2 z^2 + \ldots \tag{3.96}$$

In other words, the linear combination of parameters ξ depend only on z, not on t. This accords with the fact that the parameters $\xi^\pm$ of the residual invariance in conformal gauge are functions only of $\sigma^\pm$. If we allow them to also have an independent dependence on t, then both $\xi^\pm$ become arbitrary functions of both σ and t, which is what they were before we chose the conformal gauge.

Moral: The NG action is a gauge theory of the 2D conformal group, for which the algebra is $\text{Diff}_1 \oplus \text{Diff}_1$. In conformal gauge there is a residual conformal symmetry, with the same algebra but the parameters are now constants.

The algebra Diff_1 has a finite-dimensional subalgebra, for which a basis of vector fields is

$$\left\{ J_- = \partial_z\,, \quad J_3 = z\partial_z\,, \quad J_+ = \frac{1}{2}z^2\partial_z \right\}. \tag{3.97}$$

The commutation relations of these vector fields are [Exercise: check this]

$$[J_3, J_\pm] = \pm J_\pm\,, \qquad [J_-, J_+] = J_3\,. \tag{3.98}$$

This is the algebra of $Sl(2;\mathbb{R})$, so the finite dimensional conformal algebra is $Sl(2;\mathbb{R}) \oplus Sl(2;\mathbb{R}) \cong SO(2,2)$. In n dimensions the conformal algebra is $SO(2,n)$; for example, for $D = 4$ it is $SO(2,4)$.

3.5 Solving the NG equations in conformal gauge

Locally, the NG solutions reduce to the 2D wave equation in conformal gauge, which is easily solved. In particular, we can solve the 2D wave-equation for X^0 by setting $X^0(t,\sigma) = t$. In fact, we can use the residual conformal symmetry in conformal gauge to set $X^0 = t$ without loss of generality. Let's check whether this does fix the residual conformal symmetry. Recall that the residual conformal transformation of X is $\delta_\xi X = \xi^+(\sigma^+)\partial_+ X + \xi^-(\sigma^-)\partial_- X$, so

$$0 = \sqrt{2}\,\delta_\xi \left(X^0 - t\right)\Big|_{X^0=t} = \xi^+(\sigma^+) - \xi^-(\sigma^-)\,, \tag{3.99}$$

which implies that $\xi^+(\sigma^+) = \xi^-(\sigma^-)$ and hence that $\xi^+ = \xi^- = \bar{\xi}$, a constant. The only surviving part of the conformal transformation is therefore $\delta X \propto \bar{\xi} X'$, which corresponds to a constant shift of σ (a change of where we choose $\sigma = 0$ on the string).

We can easily write down the general solution for $\vec{X}$ that satisfies the 2D wave equation but to have a global solution we must also solve the conformal gauge constraints, which are now

$$\left|\sqrt{2}\partial_\pm \vec{X}\right|^2 = 1\,. \tag{3.100}$$

Given a solution of the 2D wave equation for $\vec{X}$ we may check directly to see whether the constraints are satisfied. Alternative, we may compute the induced metric to see if it is conformally flat; if it is then the conformal gauge constraints will be satisfied because they are precisely the conditions for conformal flatness of the induced metric.

Let's apply these ideas to the closed string configuration in a 5-dimensional space-time with $X^0 = t$ and

$$Z \equiv X^1 + iX^2 = \frac{1}{2n}e^{in(\sigma - t)}\,, \qquad W \equiv X^3 + iX^4 = \frac{1}{2m}e^{im(\sigma+t)}\,. \tag{3.101}$$

This configuration clearly solves the 2D wave equation. If the induced metric is conformally flat then it will also solve the full NG equations, including the constraints. A calculation gives

$$\begin{aligned} ds^2\big|_{\text{ind}} &= -\left(dX^0\right)^2 + |dZ|^2 + |dW|^2 = -dt^2 + \frac{1}{2}\left(dt^2 + d\sigma^2\right) \\ &= \frac{1}{2}\left(-dt^2 + d\sigma^2\right)\,. \end{aligned} \tag{3.102}$$

In other words,

$$g_{\mu\nu} = \frac{1}{2}\eta_{\mu\nu}\,. \tag{3.103}$$

This is flat, and hence conformally flat, so the given configuration is a solution of the NG equations.

This solution has the special property of being stationary; the string is motionless in a particular rotating frame. To see this, we first compute the proper length L of the string. Setting $t = t_0$ in the induced worldsheet metric (for some constant t_0) we see that $d\ell^2 = \frac{1}{2}d\sigma^2$, so

$$L = \frac{1}{2}\oint d\sigma = \sqrt{2}\,\pi\,. \tag{3.104}$$

It is rather surprising that this should be constant, i.e. independent of t_0; it means that the motion of the string is supporting it against collapse due to its tension. To check this, we may compute the total energy, which is

$$H = \oint d\sigma P^0 = T\oint d\sigma \dot{X}^0 = 2\pi T\,. \tag{3.105}$$

We see that

$$H = \sqrt{2}\,TL = TL + \left(\sqrt{2} - 1\right)TL \tag{3.106}$$

The first term is the potential energy of the string. The second term is therefore kinetic energy. The string is supported against collapse by rotation in the Z and W planes. The string is circular for $n = m$, and planar, so a circular planar loop of string can be supported against collapse by rotation in two orthogonal planes provided that neither of them coincides with the plane of the string loop.

3.6 Open string boundary conditions

An open string has two ends. We shall choose the parameter length to be π, so the action in Hamiltonian form is

$$I = \int dt \int_0^\pi d\sigma \left\{ \dot{X}^m P_m - \frac{1}{2} e \left[P^2 + (TX')^2 \right] - u X' \cdot P \right\} . \tag{3.107}$$

What are the possible b.c.s at the ends of the string?

Principle: the action should be stationary when the equations of motion are satisfied. In other words, when we vary the action to get the equations of motion, the boundary terms arising from integration by parts must be zero; otherwise the functional derivative of the action is not defined.

Applying this principle to the above action, we see that boundary terms can arise only when we vary X' and integrate by parts to get the derivative with respect to σ off the δX variation (we can ignore any boundary terms in time). These boundary terms are

$$\delta I\big|_{\text{on-shell}} = -\int dt \left[\left(T^2 e X' + uP \right) \cdot \delta X \right]_{\sigma=0}^{\sigma=\pi} . \tag{3.108}$$

Here, "on-shell" is shorthand for "using the equations of motion". [Exercise: check this].

It would make no physical sense to fix X^0 at the endpoints, and if X^0 is free then so is $\dot{X}^0$ and hence P^0 when we use the equations of motion, so the boundary term with the factor of δX^0 will be zero only if we impose the conditions

$$u|_{ends} = 0 , \qquad \left(X^0\right)'\Big|_{ends} = 0 . \tag{3.109}$$

Given that $e \neq 0$ (we'll pass over this point) we conclude that

$$\vec{X}' \cdot \delta \vec{X}\Big|_{ends} = 0 . \tag{3.110}$$

What this means is that *at each end* and *for each component*, call it X_*, of $\vec{X}$, we have

$$\begin{aligned} &\textbf{either} \quad X_*' = 0 \qquad \text{(Neumann b.c.s)} , \\ &\textbf{or} \quad \delta X_* = 0 \quad \Rightarrow \quad X_* = \bar{X}_* \text{ (constant)} \qquad \text{(Dirichlet b.c.s)} \end{aligned} \tag{3.111}$$

There are many possibilities. The simplest is *free-end* b.c.s for which $\vec{X}' = 0$ at both ends. In this case

$$X'|_{ends} = 0 \,. \tag{3.112}$$

This implies that $(X')^2$ is zero at the ends of the string. The open string mass-shell constraint then implies that P^2 is zero at the endpoints, and since

$$P|_{ends} = e^{-1}\left(\dot{X} - uX'\right)\Big|_{ends} = e^{-1}\dot{X}\Big|_{ends} = 0\,, \tag{3.113}$$

we deduce that $\dot{X}^2$ is zero at the ends of the string; i.e. *the string endpoints move at the speed of light*.

3.7 Fourier expansion: closed string

The worldsheet fields of the closed string are periodic functions of σ with (by convention) period 2π, so we can express them as Fourier series. It is convenient to express (X, P) as Fourier series by starting with the combinations $P \pm TX'$ (because the gauge transformations act separately on $P + TX'$ and $P - TX'$), so we write

$$\begin{aligned} P - TX' &= \sqrt{\frac{T}{\pi}} \sum_{k\in\mathbb{Z}} e^{ik\sigma}\alpha_k \qquad (\alpha_{-k}(t) = \alpha_k^*) \\ P + TX' &= \sqrt{\frac{T}{\pi}} \sum_{k\in\mathbb{Z}} e^{-ik\sigma}\tilde{\alpha}_k \qquad (\tilde{\alpha}_{-k}(t) = \tilde{\alpha}_k^*) \end{aligned} \tag{3.114}$$

Recall that worldsheet parity $\sigma \to -\sigma$ exchanges $P + TX'$ with $P - TX'$. Because of the relative minus sign in the exponent of the Fourier series, this means that worldsheet parity exchanges α_k with $\tilde{\alpha}_k$:

$$\alpha_k \quad \leftrightarrow \quad \tilde{\alpha}_k \,. \tag{3.115}$$

We can integrate either of the above equations to determine the total D-momentum in terms of Fourier modes, since X' integrates to zero for a closed string; this gives us

$$p = \oint P d\sigma = \begin{cases} \sqrt{4\pi T}\alpha_0 \\ \sqrt{4\pi T}\tilde{\alpha}_0 \end{cases} \quad \Rightarrow \quad \alpha_0 = \tilde{\alpha}_0 = \frac{p}{\sqrt{4\pi T}}\,. \tag{3.116}$$

By adding the Fourier series expressions for $P \pm TX'$ we now get

$$P(t,\sigma) = \frac{p(t)}{2\pi} + \sqrt{\frac{T}{4\pi}} \sum_{k\neq 0} e^{ik\sigma}\left[\alpha_k(t) + \tilde{\alpha}_{-k}(t)\right]\,, \tag{3.117}$$

By subtracting we get

$$X' = -\frac{1}{\sqrt{4\pi T}} \sum_{k\neq 0} e^{ik\sigma}\left(\alpha_k - \tilde{\alpha}_{-k}\right)\,, \tag{3.118}$$

which we may integrate to get the Fourier series expansion for X:

$$X(t,\sigma) = x(t) + \frac{1}{\sqrt{4\pi T}} \sum_{k\neq 0} \frac{i}{k} e^{ik\sigma} \left[\alpha_k(t) - \tilde\alpha_{-k}(t)\right] . \tag{3.119}$$

The integration constant (actually a function of t) can be interpreted as the position of the centre of mass of the string; we should expect it to behave like a free particle.

Using the Fourier series expansions of (X,P) we now find that

$$\oint d\sigma\, \dot X^m P_m = \dot x^m p_m + \sum_{k=1} \frac{i}{k}\left(\dot\alpha_k\cdot\alpha_{-k} + \dot{\tilde\alpha}_k\cdot\tilde\alpha_{-k}\right) + \frac{d}{dt}\,() \tag{3.120}$$

<u>Exercise</u>: check this [*Hint.* Cross terms that mix α with $\tilde\alpha$ are all in the total time derivative term, and the $k<0$ terms in the resulting sum double the $k>0$ terms].

Next we Fourier expand the constraint functions $\mathcal{H}_\pm$:

$$\mathcal{H}_- = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{in\sigma} L_n\,, \qquad \mathcal{H}_+ = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{-in\sigma}\tilde L_n\,. \tag{3.121}$$

Inverting to get the Fourier coefficients in terms of the functions $\mathcal{H}_\pm$, we get

$$\begin{aligned} L_n &= \oint d\sigma\, e^{-in\sigma}\mathcal{H}_- = \frac{1}{4T}\oint d\sigma\, e^{-in\sigma}\left(P - TX'\right)^2 \\ \tilde L_n &= \oint d\sigma\, e^{in\sigma}\mathcal{H}_+ = \frac{1}{4T}\oint d\sigma\, e^{in\sigma}\left(P + TX'\right)^2 . \end{aligned} \tag{3.122}$$

Inserting the Fourier expansions (3.114) we find that (<u>Exercise</u>: check this)

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}}\alpha_k\cdot\alpha_{n-k}\,, \qquad \tilde L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}}\tilde\alpha_k\cdot\tilde\alpha_{n-k}\,. \tag{3.123}$$

We may similarly expand the Lagrange multipliers as Fourier series but it should be clear in advance that there will be one Fourier mode of λ^- for each L_n (let's call this λ_{-n}) and one Fourier mode of λ^+ for each $\tilde L_n$ (let's call this $\tilde\lambda_{-n}$). We may now write down the closed string action in terms of Fourier modes. It is

$$I = \int dt \left\{ \dot x^m p_m + \sum_{k=1}\frac{i}{k}\left(\dot\alpha_k\cdot\alpha_{-k} + \dot{\tilde\alpha}_k\cdot\tilde\alpha_{-k}\right) - \sum_{n\in\mathbb{Z}}\left(\lambda_{-n}L_n + \tilde\lambda_{-n}\tilde L_n\right)\right\} . \tag{3.124}$$

This action is manifestly Poincaré invariant. The Noether charges are

$$\begin{aligned} \mathcal{P}_m &= \oint d\sigma P_m = p_m\,, \\ \mathcal{J}^{mn} &= 2\oint d\sigma X^{[m}P^{n]} = 2x^{[m}p^{n]} + S^{mn} \end{aligned} \tag{3.125}$$

where the spin part of the Lorentz charge is (<u>Exercise</u>: check this)

$$S^{mn} = -2\sum_{k=1}^{\infty}\frac{i}{k}\left(\alpha_{-k}^{[m}\alpha_k^{n]} + \tilde\alpha_{-k}^{[m}\tilde\alpha_k^{n]}\right) . \tag{3.126}$$

- **Lemma**. For a Lagrangian of the form

$$L = \frac{i}{c}\,\dot{\alpha}\alpha^* - H(\alpha,\alpha^*) \tag{3.127}$$

for constant c, the PB of the canonical variables takes the form

$$\{\alpha,\alpha^*\}_{PB} = -ic\,. \tag{3.128}$$

To see this set $\alpha = \sqrt{c/2}[q + i\,\mathrm{sign}(c)p]$ to get $L = \dot{q}p - H$, for which we know that $\{q,p\}_{PB} = 1$. This implies the above PB for α and α^*.

Using this lemma we may read off from the action that the non-zero Poisson brackets of canonical variables are $\{x^m, p_n\}_{PB} = \delta^m_n$ and

$$\left\{\alpha^m_k, \alpha^n_{-k}\right\}_{PB} = -ik\eta^{mn}\,, \quad \left\{\tilde{\alpha}^m_k, \tilde{\alpha}^n_{-k}\right\}_{PB} = -ik\eta^{mn}\,. \tag{3.129}$$

Using these PBs, and the Fourier series expressions for (X,P), we may compute the PB of $X(\sigma)$ with $P(\sigma')$. [Exercise: check that the result agrees with (3.23).]

We may also use the PBs (3.129) to compute the PBs of the constraint functions $(L_n, \tilde{L}_n)$. The non-zero PBs are (Exercise: check this)

$$\{L_k, L_j\}_{PB} = -i\,(k-j)\,L_{k+j}\,, \qquad \left\{\tilde{L}_k, \tilde{L}_j\right\}_{PB} = -i\,(k-j)\,\tilde{L}_{k+j}\,. \tag{3.130}$$

We may draw a number of conclusions from this result:

- The constraints are first class, so the L_n and $\tilde{L}_n$ generate gauge transformations, for each $n \in \mathbb{Z}$.

- The structure functions of the algebra of first-class constraints are *constants*. This means that the $(L_n, \tilde{L}_n)$ span an infinite dimensional *Lie algebra*.

- The Lie algebra of the gauge group is a direct sum of two copies of the same algebra, sometimes called the Witt algebra.

The Witt algebra is also the algebra of diffeomorphisms of the circle. Suppose we have a circle parameterized by $\theta \sim \theta + 2\pi$ (we could take θ to be σ^+ or σ^-). The algebra Diff_1 of diffeomorphisms is spanned by the vector fields on the circle, and since these are periodic we may take as a basis set the vector fields $\{V_n; n \in \mathbb{Z}\}$, where

$$V_n = e^{in\theta}\frac{d}{d\theta}\,. \tag{3.131}$$

The commutator of two basis vector fields is

$$[V_k, V_j] = -i\,(k-j)\,V_{k+j}\,. \tag{3.132}$$

Corollary: the algebra of the gauge group is $\mathrm{Diff}_1 \oplus \mathrm{Diff}_1$, as claimed previously.

3.8 Open string

The open string has two ends. We will choose the ends to be at $\sigma = 0$ and $\sigma = \pi$, so the parameter length of the string is π (this is just a convention). We shall first consider the case of free-end (Neumann) boundary conditions. Then we shall go on to see how the results change when the string ends are not free to move in certain directions (mixed Neumann/Dirichlet b.c.s).

3.8.1 Fourier expansion: Free-ends

If the ends of the string are free, we must require X' to be zero at the ends, i.e. at $\sigma = 0$ and $\sigma = \pi$. We shall proceed in a way that will allow us to take over results from the closed string; we shall use a "doubling trick".

- First we extend the definition of (X, P) from the interval $[0, \pi]$ to the interval $[0, 2\pi]$ in such a way that (X, P) are periodic on this doubled interval. This will allow us to use the closed string Fourier series expressions.
- Next, we impose a condition that relates (X, P) in the interval $[\pi, 2\pi]$ to (X, P) in the interval $[0, \pi]$; this will ensure that any additional degrees of freedom that we have introduced by doubling the interval are removed; everything will depend only on what the piece of string in the interval $[0, \pi]$ is doing. The condition we impose should be consistent with periodicity in the doubled interval, but it should also imply the free-end b.c.s at $\sigma = 0, \pi$.
- Since we have periodicity in σ with period 2π, if $\sigma \in [0, \pi]$ then $-\sigma \sim -\sigma + 2\pi \in [\pi, 2\pi]$, so we need to relate the worldsheet fields at σ to their values at $-\sigma$.

The solution to these requirements is to impose the condition

$$(P + TX')(\sigma) = (P - TX')(-\sigma)\,. \tag{3.133}$$

This is consistent with periodicity, and setting $\sigma = 0$ it implies that $X'(0) = 0$. It also implies that $X'(\pi) = 0$ because $-\pi \sim \pi$ by periodicity. In terms of Fourier modes, the condition (3.133) becomes

$$\tilde{\alpha}_k = \alpha_k \qquad (k \in \mathbb{Z}). \tag{3.134}$$

Using this in (3.114) we have

$$P \pm TX' = \sqrt{\frac{T}{\pi}} \sum_{k \in \mathbb{Z}} e^{\mp ik\sigma} \alpha_k \qquad (\alpha_{-k} = \alpha_k^*)\,. \tag{3.135}$$

Equivalently,

$$P = \sqrt{\frac{T}{\pi}} \sum_{k \in \mathbb{Z}} \cos(k\sigma)\, \alpha_k\,, \qquad X' = -\frac{i}{\sqrt{\pi T}} \sum_{k \in \mathbb{Z}} \sin(k\sigma)\, \alpha_k \tag{3.136}$$

Integrating to get X, and defining $p(t)$ by

$$\alpha_0 = \frac{p}{\sqrt{\pi T}}\,, \tag{3.137}$$

we have

$$\begin{aligned} X(t,\sigma) &= x(t) + \frac{1}{\sqrt{\pi T}} \sum_{k\neq 0} \frac{i}{k} \cos(k\sigma)\,\alpha_k\,, \\ P(t,\sigma) &= \frac{p(t)}{\pi} + \sqrt{\frac{T}{\pi}} \sum_{k\neq 0} \cos(k\sigma)\,\alpha_k\,. \end{aligned} \tag{3.138}$$

Notice that p is again the total momentum since

$$p = \int_0^\pi P(t,\sigma) d\sigma\,, \tag{3.139}$$

but its relation to α_0 differs from that of the closed string.

Using the Fourier series expansions for (X,P) we find that

$$\int_0^\pi \dot{X}^m P_m\, d\sigma = \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \dot{\alpha}_k \cdot \alpha_{-k} + \frac{d}{dt}()\ . \tag{3.140}$$

Because of (3.133) we also have $\mathcal{H}_+(\sigma) = \mathcal{H}_-(-\sigma)$, so we should impose a similar relation on the Lagrange multipliers

$$\lambda^+(\sigma) = \lambda^-(-\sigma)\,. \tag{3.141}$$

Then

$$\begin{aligned} \int_0^\pi d\sigma \left(\lambda^-\mathcal{H}_- + \lambda^+\mathcal{H}_+\right) &= \int_0^\pi d\sigma\, \lambda^-(\sigma)\mathcal{H}_-(\sigma) + \int_0^\pi d\sigma\, \lambda^-(-\sigma)\mathcal{H}_-(-\sigma) \\ &= \int_0^\pi d\sigma\, \lambda^-(\sigma)\mathcal{H}_-(\sigma) + \int_\pi^{2\pi} d\sigma\, \lambda^-(\sigma)\mathcal{H}_-(\sigma) \\ &= \oint d\sigma\, \lambda^-\mathcal{H}_-\,, \end{aligned} \tag{3.142}$$

where we can now use the Fourier series expansions of the closed string.

The final result for the open string action in Fourier modes is

$$I = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1} \frac{i}{k} \dot{\alpha}_k \cdot \alpha_{-k} - \sum_{n\in\mathbb{Z}} \lambda_{-n} L_n \right\}\,. \tag{3.143}$$

The difference with the closed string is that we have one set of oscillator variables instead of two. We can now read off the non-zero PB relations

$$\{x^m, p_n\}_{PB} = \delta^m_n\,, \qquad \left\{\alpha^m_k, \alpha^n_{-k}\right\}_{PB} = -ik\eta^{mn}\,, \tag{3.144}$$

and we can use this to show that

$$\{L_n, \alpha_k^m\}_{PB} = ik\alpha_{n+k}\,. \tag{3.145}$$

This means that the gauge variation of α_k is

$$\delta_\xi \alpha_k^m = \sum_{n\in\mathbb{Z}} \xi_{-n}\, \{\alpha_k^m, L_n\}_{PB} = -ik \sum_{n\in\mathbb{Z}} \xi_{-n}\alpha_{n+k}^m\,, \tag{3.146}$$

where ξ_n are parameters. To compute the gauge transformation of (x,p) we use the fact that

$$L_0 = \frac{1}{2}\alpha_0^2 + \dots\,, \qquad L_n = \alpha_0 \cdot \alpha_n + \dots \tag{3.147}$$

where the dots indicate terms that do not involve α_0, and the relation $p = \sqrt{\pi T}\,\alpha_0$ to compute

$$\delta_\xi x^m = \frac{1}{\sqrt{\pi T}} \sum_{n\in\mathbb{Z}} \xi_{-n}\alpha_n\,, \qquad \delta p_m = 0\,. \tag{3.148}$$

Finally, one may verify that the action is invariant if

$$\delta_\xi \lambda_n = \dot{\xi}_n + i \sum_{k\in\mathbb{Z}} (2k-n)\, \xi_k \lambda_{n-k}\,. \tag{3.149}$$

3.8.2 Mixed free-end/fixed-end boundary conditions

Let's now suppose that we have mixed Neumann and Dirichlet boundary conditions. We will divide the cartesian coordinates X^m into two sets

$$\left\{X^{\hat m}\,; \hat m = 0,1,\dots,p\right\}\,, \qquad \left\{X^{\check m}\,; \check m = p+1,\dots,D-1\right\}\,. \tag{3.150}$$

We will suppose that $X^{\hat m}(t,\sigma)$ are subject to Neumann b.c.s. and that $X^{\check m}(t,\sigma)$ are subject to Dirichlet b.c.s, so the string is stretched between a p-plane at the origin and a parallel p-plane situated at $X^{\check m} = L^{\check m}$. The boundary conditions corresponding to this situation are

$$\left(X^{\hat m}\right)'\Big|_{ends} = 0 \qquad X^{\check m}\Big|_{\sigma=0} = 0 \quad \& \quad X^{\check m}\Big|_{\sigma=\pi} = L^{\check m}\,. \tag{3.151}$$

Notice that these boundary conditions break invariance under the $SO(1,D-1)$ Lorentz group to invariance under the subgroup $SO(1,p) \times SO(D-p-1)$. In particular the D-dimensional Lorentz invariance is broken to a $(p+1)$-dimensional Lorentz invariance.

To get the Fourier series expansions of $P \pm TX'$ for these b.c.s. we may again use the doubling trick, but the constraint relating the components of $(P \pm TX')$ at σ to the components at $-\sigma$ now depends on whether it is a $\hat m$ component or a $\check m$ component. For the $\hat m$ components we choose the (3.133) condition, which implies that $(X^{\hat m})'$ is zero at the endpoints. For the $\check m$ components we impose the condition

$$\left(P + TX'\right)^{\check m}(\sigma) = -\left(P + TX'\right)^{\check m}(-\sigma)\,, \tag{3.152}$$

which implies that $P_{\check{m}}$ is zero at the endpoints, and hence (since the equations of motion imply that $P = e^{-1}\dot{X}$ at the endpoints) that

$$\dot{X}^{\check{m}}\Big|_{\text{ends}} = 0\,. \tag{3.153}$$

In terms of the oscillator variables, the condition (3.152) becomes

$$\tilde{\alpha}_k^{\check{m}} = -\alpha_k^{\check{m}}\,, \tag{3.154}$$

which gives

$$(P \pm TX')^{\check{m}} = \mp\sqrt{\frac{T}{\pi}}\sum_{k\in\mathbb{Z}} e^{\mp ik\sigma}\alpha_k^{\check{m}}\,. \tag{3.155}$$

Taking sums and differences, we find that

$$P^{\check{m}} = i\sqrt{\frac{T}{\pi}}\sum_{k\in\mathbb{Z}} \sin(k\sigma)\alpha_k^{\check{m}}\,, \tag{3.156}$$

which is indeed zero at $\sigma = 0, \pi$, and

$$\left(X^{\check{m}}\right)' = \frac{\alpha_0^{\check{m}}}{\sqrt{\pi T}} + \frac{1}{\sqrt{\pi T}}\sum_{k\neq 0}\cos(k\sigma)\alpha_k^{\check{m}}\,, \tag{3.157}$$

which we may integrate to get $X^{\check{m}}$. For the b.c.s of (3.150), we get

$$X^{\check{m}} = \frac{L^{\check{m}}\sigma}{\pi} - \frac{1}{\sqrt{\pi T}}\sum_{k\neq 0}\frac{1}{k}\sin(k\sigma)\alpha_k^{\check{m}}\,, \qquad L^{\check{m}} = \sqrt{\frac{\pi}{T}}\alpha_0^{\check{m}}\,. \tag{3.158}$$

Using the Fourier series expressions for $X^{\check{m}}$ and $P_{\check{m}}$, we find that

$$\int_0^\pi \dot{X}^{\check{m}}P_{\check{m}} = \sum_{k=1}^\infty \frac{i}{k}\sum_{\check{m}}\alpha_k^{\check{m}}\alpha_{-k}^{\check{m}} \quad + \quad \frac{d}{dt}() \tag{3.159}$$

The only effect of the Dirichlet b.c.s (relative to Neumann b.c.s) is the absence of any zero mode term.

Irrespective of whether the b.c.s are Neumann or Dirichlet, we still have $\mathcal{H}_+(\sigma) = \mathcal{H}_-(-\sigma)$, so the Fourier series for the constraint functions are formally independent of the b.c.s. So, adding the contributions from the Neumann and Dirichlet directions, we arrive at the action

$$I = \int dt\left\{\dot{x}^{\hat{m}}p_{\hat{m}} + \sum_{k=1}^\infty \frac{i}{k}\dot{\alpha}_k\cdot\alpha_{-k} - \sum_{n\in\mathbb{Z}}\lambda_{-n}L_n\right\}\,, \tag{3.160}$$

where, as before, $L_n = \frac{1}{2}\sum_k \alpha_k\cdot\alpha_{n-k}$. Notice the *absence* of a $\dot{x}^{\check{m}}p_{\check{m}}$ term. Apart from this, the only difference to the free-end case is in the zero-mode contribution to the L_n. Whereas we had $\alpha_0 = p/\sqrt{\pi T}$, we now have

$$\alpha_0^{\hat{m}} = \frac{p^{\hat{m}}}{\sqrt{\pi T}}\,, \qquad \alpha_0^{\check{m}} = \sqrt{\frac{T}{\pi}}\,L^{\check{m}}\,. \tag{3.161}$$

3.9 The NG string in light-cone gauge

We shall start with the open string (with free-end b.c.s). We shall impose the gauge conditions

$$X^+(t,\sigma) = x^+(t)\,, \qquad P_-(t,\sigma) = p_-(t)\,. \tag{3.162}$$

It customary to also set $x^+(t) = t$, as for the particle, but it is simpler not to do this. This means that we will not be fixing the gauge completely since we will still be free to make σ-independent reparametrizations of the worldsheet time t.

The above gauge-fixing conditions are equivalent to

$$(P \pm TX')^+ = p_-(t) \quad \Leftrightarrow \quad \alpha_k^+ = 0 \quad \forall k \neq 0\,. \tag{3.163}$$

In other words, we impose a light-cone gauge condition only on the oscillator variables of the string, not on the zero modes (centre of mass variables). Let's check that the gauge has been otherwise fixed. We can investigate this using the criterion summarised by the formula (2.48); we compute

$$\begin{aligned} \left\{L_n, \alpha_{-k}^+\right\} &= -ik\alpha_{n-k}^+ \\ &= -ik\alpha_0^+ \quad \text{(using gauge condition)} \\ &= -i\frac{kp_-}{\sqrt{\pi T}}\delta_{nk} \end{aligned} \tag{3.164}$$

This is invertible if we exclude $n = 0$ and $k = 0$, so we have fixed all but the gauge transformation generated by L_0. Now we have, since $\alpha_k^+ = 0$,

$$\sum_{k=1}^{\infty} \frac{i}{k}\dot{\alpha}_k \cdot \alpha_{-k} = \sum_{k=1}^{\infty} \frac{i}{k}\dot{\boldsymbol{\alpha}}_k \cdot \boldsymbol{\alpha}_{-k}\,, \tag{3.165}$$

where the $(D-2)$-vectors $\boldsymbol{\alpha}_k$ are the *transverse* oscillator variables.

We also have,

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{k=1}^{\infty} \boldsymbol{\alpha}_{-k} \cdot \boldsymbol{\alpha}_k = \frac{1}{2\pi T}\left(p^2 + 2\pi TN\right)\,, \tag{3.166}$$

where N is the *level number*:

$$N = \sum_{k=1}^{\infty} \boldsymbol{\alpha}_{-k} \cdot \boldsymbol{\alpha}_k\,. \tag{3.167}$$

For $n \neq 0$,

$$\begin{aligned} L_n &= \frac{1}{2}\sum_{k\in\mathbb{Z}} \left(\alpha_k^+\alpha_{n-k}^- + \alpha_k^-\alpha_{n-k}^+\right) + \frac{1}{2}\sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{n-k} \\ &= \alpha_0^+\alpha_n^- + \frac{1}{2}\sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{n-k} \qquad \text{(using gauge condition).} \end{aligned} \tag{3.168}$$

We can solve this for α_n^-; using $p = \sqrt{\pi T}\,\alpha_0$, we get

$$\alpha_n^- = -\frac{\sqrt{\pi T}}{2p_-}\sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{n-k} \qquad (n \neq 0). \tag{3.169}$$

As we have solved the constraints $L_n = 0$ for $n \neq 0$, only the $L_0 = 0$ constraint will be imposed by a Lagrange multiplier in the gauge-fixed action, which is

$$I = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \dot{\boldsymbol{\alpha}}_k \cdot \boldsymbol{\alpha}_{-k} - \frac{1}{2} e_0 \left(p^2 + M^2\right) \right\} \tag{3.170}$$

where $e_0 = \lambda_0/(\pi T)$ and

$$M^2 = 2\pi T N = N/\alpha' \qquad \left(\alpha' \equiv \frac{1}{2\pi T}\right). \tag{3.171}$$

Notice that the action does not involve α_n^- (for $n \neq 0$) but the Lorentz charges do. Recall that the spin part of the Lorentz charge $\mathcal{J}^{mn}$ is $S^{mn} = -2\sum_{k=1}^{\infty} \frac{i}{k}\alpha_{-k}^{[m}\alpha_k^{n]}$. Its non-zero components of S^{mn} in light-cone gauge are $(I, J, = 1, \ldots D-2)$

$$\begin{aligned} S^{IJ} &= -2\sum_{k=1}^{\infty} \frac{i}{k} \alpha_{-k}^{[I}\alpha_k^{J]}\,, \\ S^{-I} &= -\sum_{k=1}^{\infty} \frac{i}{k}\left(\alpha_{-k}^- \alpha_k^I - \alpha_{-k}^I \alpha_k^-\right) . \end{aligned} \tag{3.172}$$

The canonical PB relations that we read off from the action (3.170) are

$$\{x^m, p_n\}_{PB} = \delta_n^m\,, \qquad \left\{\alpha_k^I, \alpha_{-k}^J\right\}_{PB} = -ik\,\delta^{IJ}\,. \tag{3.173}$$

These may be used to compute the PBs of the Lorentz generators; since $\mathcal{J} = L + S$ where $\{L, S\}_{PB} = 0$, the PB relations among the components of S alone must be the same as those of $\mathcal{J}$. The PBs of S^{IJ} are those of the Lie algebra of the transverse rotation group $SO(D-2)$, and their PBs with S^{-K} are those expected from the fact that S^{-K} is a $(D-2)$ vector. Finally, Lorentz invariance requires that

$$\left\{S^{-I}, S^{-J}\right\}_{PB} = 0\,. \tag{3.174}$$

This can be confirmed by making use of the PB relations

$$\left\{\alpha_k^-, \alpha_\ell^-\right\}_{PB} = i\frac{\sqrt{\pi T}}{p_-}(k-\ell)\,\alpha_{k+j}^-\,, \qquad \left\{\alpha_k^-, \alpha_\ell^I\right\}_{PB} = -i\frac{\sqrt{\pi T}}{p_-}\ell\,\alpha_{k+\ell}^I\,. \tag{3.175}$$

This has to work because gauge fixing cannot break symmetries; it can only obscure them.

3.9.1 Light-cone gauge for mixed Neumann/Dirichlet b.c.s

What changes if we change the boundary conditions to the mixed Neumann/Dirichlet case? Assuming that Dirichlet b.c.s apply only in some, or all, of the $(D-2)$ transverse directions[9] we can still impose the gauge-fixing condition[10]

$$\alpha_k^+ = 0 \quad \forall k\,, \tag{3.176}$$

and then proceed as before. We can again solve the constraints $L_n = 0$ $(n \neq 0)$ for α_k^- $(k \neq 0)$. The $L_0 = 0$ constraint, which still has to be imposed via a Lagrange multiplier, is

$$0 = \frac{1}{2}\alpha_0^2 + N = \frac{1}{2\pi T}\left(\hat{p}^2 + (TL)^2 + 2\pi T N\right) \tag{3.177}$$

where

$$\hat{p}^2 = p^{\hat{m}} p_{\hat{m}}\,, \qquad L^2 = \left|L^{\check{m}}\right|\,. \tag{3.178}$$

In other words, the boundary conditions affect only the zero modes. The action is

$$I = \int dt \left\{ \dot{x}^{\hat{m}} p_{\hat{m}} + \sum_{k=1}^{\infty} \frac{i}{k} \dot{\boldsymbol{\alpha}}_k \cdot \boldsymbol{\alpha}_{-k} - \frac{1}{2} e_0 \left(\hat{p}^2 + M^2\right) \right\}, \tag{3.179}$$

where

$$M^2 = (TL)^2 + N/\alpha'\,. \tag{3.180}$$

Classically, $N \geq 0$ and the minimum energy configuration has $N = 0$. In this case, $M = TL$, which can be interpreted as the statement that the minimal energy string is a straight string stretched orthogonally between the two p-planes; since they are separated by a distance L the potential energy in the string is TL.

3.9.2 Closed string in light-cone gauge

Now we fix the gauge invariances associated with L_n and $\tilde{L}_n$ for $n \neq 0$ by setting

$$\alpha_k^+ = 0 \quad \& \quad \tilde{\alpha}_k^+ = 0 \qquad \forall k \neq 0\,. \tag{3.181}$$

This leaves unfixed the gauge invariances generated by L_0 and $\tilde{L}_0$, which are now

$$\begin{aligned} L_0 &= \frac{1}{2}\alpha_0^2 + \sum_{k=1}^{\infty} \boldsymbol{\alpha}_{-k} \cdot \boldsymbol{\alpha}_k = \frac{p^2}{8\pi T} + N\,, \\ \tilde{L}_0 &= \frac{1}{2}\tilde{\alpha}_0^2 + \sum_{k=1}^{\infty} \tilde{\boldsymbol{\alpha}}_{-k} \cdot \tilde{\boldsymbol{\alpha}}_k = \frac{p^2}{8\pi T} + \tilde{N}\,. \end{aligned} \tag{3.182}$$

[9] This excludes only the case in which one end is free and other end is completely fixed, i.e. tethered to a fixed point.

[10] This is no longer looks so simple in terms of X and P.

Here we have used the closed string relation (3.116) between p and $\alpha_0 = \tilde{\alpha}_0$. By adding and subtracting the two constraints $L_0 = 0$ and $\tilde{L}_0 = 0$ we get the two equivalent constraints

$$p^2 + 4\pi T\left(N + \tilde{N}\right) = 0 \quad \& \quad \tilde{N} - N = 0\,, \tag{3.183}$$

which will be imposed by Lagrange multipliers e_0 and u_0 in the gauge-fixed action. The remaining constraints $L_n = 0$ and $\tilde{L}_n = 0$ for $n \neq 0$ we solve for α_k^- and $\tilde{\alpha}_k^-$, as for the open string. The closed string action in light-cone gauge is therefore

$$I = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k}\left(\dot{\boldsymbol{\alpha}}_k \cdot \boldsymbol{\alpha}_{-k} + \dot{\tilde{\boldsymbol{\alpha}}}_k \cdot \tilde{\boldsymbol{\alpha}}_{-k}\right) - \frac{1}{2}e_0\left(p^2 + M^2\right) - u_0\left(N - \tilde{N}\right)\right\}\,, \tag{3.184}$$

where

$$\begin{aligned} M^2 &= 4\pi T\left(N + \tilde{N}\right) \\ &= 8\pi T N \qquad \left(\text{using } \tilde{N} = N \text{ constraint}\right). \end{aligned} \tag{3.185}$$

4. Interlude: Light-cone gauge in field theory

We shall consider Maxwell's equation, for the vector potential A_m, and the linearised Einstein equations for a symmetric tensor potential h_{mn}, which may be interpreted as the perturbation of the space-time metric about a Minkowski vacuum metric.

We choose light-cone coordinates (x^+, x^-, x^I) $(I = 1, \dots, D-2)$. Recall that for the particle we assumed that $p_- \neq 0$, which is equivalent to the assumption that the differential operator ∂_- is invertible. We shall make the same assumption in the application to field theory.

4.0.3 Maxwell in light-cone gauge

Maxwell's equations are

$$\Box_D A_m - \partial_m\left(\partial \cdot A\right) = 0\,, \qquad m = 0, 1, \dots, D-1\,. \tag{4.1}$$

They are invariant under the gauge transformation $A_m \to A_m + \partial_m \alpha$. The light-cone gauge is

$$A_- = 0\,. \tag{4.2}$$

To see that the gauge is fixed we set to zero a gauge variation of the gauge-fixing condition: $0 = \delta_\alpha A_- = \partial_- \alpha$. This implies that $\alpha = 0$ if the differential operator ∂_- is invertible.

In this gauge the $m = -$ Maxwell equation is $\partial_-\left(\partial \cdot A\right) = 0$, which implies that $\partial \cdot A = 0$ since we assume invertibility of ∂_-. And since $0 = \partial \cdot A = \partial_- A_+ + \partial_I A_I$, we can solve for A_+:

$$A_+ = -\partial_-^{-1}\left(\partial_I A_I\right)\,. \tag{4.3}$$

This leaves A_I as the only independent variables. The $m = +$ equation is $\Box_D A_+ = 0$, but this is a consequence of the $m = I$ equation, which is

$$\Box_D A_I = 0\,, \qquad I = 1, \ldots, D-2 \tag{4.4}$$

So this is what Maxwell's equations look like in light-cone gauge: wave equations for $D-2$ independent polarisations.

4.0.4 Linearized Einstein in light-cone gauge

The linearised Einstein equations are

$$\Box_D h_{mn} - 2\partial_{(m} h_{n)} + \partial_m \partial_n h = 0\,, \qquad h_m \equiv \partial^n h_{nm}\,, \quad h = \eta^{mn} h_{mn}\,. \tag{4.5}$$

They are invariant under the gauge transformation (Exercise: verify this)

$$h_{mn} \to h_{mn} + 2\partial_{(m}\xi_{n)}\,. \tag{4.6}$$

The light-cone gauge choice is

$$h_{-n} = 0 \qquad (n = -, +, I) \qquad \Rightarrow \quad h_- = 0 \qquad \& \quad h = h_{JJ}\,. \tag{4.7}$$

In the light-cone gauge the "$m = -$" equation is $-\partial_- h_n + \partial_- \partial_n h_{JJ} = 0$, which can be solved for h_n:

$$h_n = \partial_n h_{JJ}\,. \tag{4.8}$$

But since we already know that $h_- = 0$, this tells us that $h_{JJ} = 0$, and hence that $h_n = 0$ and $h = 0$. At this point we see that the equations reduce to $\Box_D h_{mn} = 0$, but

$$\begin{aligned} h_+ = 0 \quad &\Rightarrow \quad h_{++} = -\partial_-^{-1}\left(\partial_I h_{I+}\right)\,, \\ h_I = 0 \quad &\Rightarrow \quad h_{+I} = -\partial_-^{-1}\left(\partial_J h_{JI}\right)\,, \end{aligned} \tag{4.9}$$

so the only independent components of h_{mn} are h_{IJ}, and this has zero trace. We conclude that the linearised Einstein equations in light-cone gauge are

$$\Box_D h_{IJ} = 0 \quad \& \quad h_{II} = 0\,. \tag{4.10}$$

The number of polarisation states of the graviton in D dimensions is therefore

$$\frac{1}{2}(D-2)(D-1) - 1 = \frac{1}{2}D(D-3)\,. \tag{4.11}$$

For example, for $D = 4$ there are two polarisation states, and the graviton is a massless particle of spin-2.

5. Quantum NG string

Now we pass to the quantum theory. We start by fixing the gauge invariances by a variant of the light-cone gauge, which has the advantage of eliminating, prior to quantization, all unphysical components of the oscillator variables. Then we consider how the same results could be found by Lorentz-covariant quantization; this is the "old covariant" approach.

5.1 Light-cone gauge quantization: open string

The canonical PB relations of the open string in Fourier modes are (3.173). These become the canonical commutation relations

$$[\hat{x}^m, \hat{p}_n] = i\delta^m_n\,, \qquad \left[\hat{\alpha}^I_k, \hat{\alpha}^J_{-k}\right] = k\delta^{IJ}\,, \tag{5.1}$$

where the hats now indicate operateors, and the hermiticity of the operators $(\hat{X}, \hat{P})$ requires that

$$\hat{\boldsymbol{\alpha}}_{-k} = \hat{\boldsymbol{\alpha}}^\dagger_k\,. \tag{5.2}$$

A state of the string of definite momentum is the tensor product of a momentum eigenstate $|p\rangle$ with a state in the oscillator Fock space, built upon the Fock vacuum state $|0\rangle$ annihilated by all annihilation operators:

$$\hat{\boldsymbol{\alpha}}_k|0\rangle = \mathbf{0} \quad \forall\, k \in \mathbb{Z}^+\,. \tag{5.3}$$

We get other states in the Fock space by acting on the oscillator vacuum with the creation operators $\hat{\boldsymbol{\alpha}}_{-k}$ any number of times, and for any $k > 0$. This gives us a basis for the entire infinite-dimensional space.

Next, we need to replace the level number N by a level number operator $\hat{N}$, but there is an operator ordering ambiguity; different orderings lead to operators $\hat{N}$ that differ by a constant. We shall choose to call $\hat{N}$ the particular operator that annihilates the oscillator vacuum; i.e.

$$\hat{N} = \sum_{k=1}^{\infty} \hat{\boldsymbol{\alpha}}_{-k}\cdot\hat{\boldsymbol{\alpha}}_k \qquad \Rightarrow \qquad \hat{N}|0\rangle = 0\,. \tag{5.4}$$

So the oscillator vacuum has level number zero. Notice now that

$$\left[\hat{N}, \hat{\boldsymbol{\alpha}}_{-k}\right] = k\,\hat{\boldsymbol{\alpha}}_{-k}\,. \tag{5.5}$$

This tell us that acting on a state with any component of $\hat{\boldsymbol{\alpha}}_{-k}$ raises the level number by k, and this tells that $\hat{N}$ is diagonal in the Fock state basis constructed in the way described above, and that the possible level numbers (eigenvalues of $\hat{N}$) are $N = 0, 1, 2, \ldots, \infty$. We can therefore organise the states according to their level number. There is only one state in the Fock space with $N = 0$, the oscillator vacuum. At $N = 1$ we have the $(D-2)$ states $\hat{\alpha}^I_{-1}|0\rangle$. At $N = 2$ we have the states $\hat{\alpha}^I_{-2}|0\rangle$ and $\hat{\alpha}^I_{-1}\hat{\alpha}^J_{-1}|0\rangle$. At $N = 3$ we have the states

$$\hat{\alpha}^I_{-3}|0\rangle\,, \qquad \hat{\alpha}^I_{-2}\hat{\alpha}^J_{-1}|0\rangle\,, \qquad \hat{\alpha}^I_{-1}\hat{\alpha}^J_{-1}\hat{\alpha}^K_{-1}|0\rangle\,, \tag{5.6}$$

and so on.

A generic state of the string at level N in a momentum eigenstate takes the form

$$|p\rangle \otimes |\Psi_N\rangle\,, \tag{5.7}$$

where p is the D-momentum and Ψ_N some state in the oscillator Fock space with level number N. The mass-shell constraint for such a state implies that $p^2 = -M^2$, where

$$M^2 = 2\pi T\,(N-a)\ . \tag{5.8}$$

The constant a is introduced to take care of the operator ordering ambiguity in passing from the classical to the quantum theory. We chose to define $\hat{N}$ in a particular way, such that its eigenvalues are non-negative integers, but there is nothing to tell us that this is what we must use in the quantum mass-shell constraint (imposed as a physical state condition, as for the particle). As mentioned, different definitions of N would differ by a constant, so we introduce the constant a to allow for this.

In fact, if we had defined $\hat{N}$ using the conventional Weyl ordering that leads to the usual zero-point energy for a harmonic oscillator, we would find that its eigenvalues are

$$N + \frac{(D-2)}{2}\sum_{k=1}^{\infty} k\,, \tag{5.9}$$

This is because the $(D-2)$ oscillators associated to the pair $(\boldsymbol{\alpha}_k, \boldsymbol{\alpha}_k^\dagger)$ have angular frequency $|k|$, and we have to sum over all oscillators. This would lead us to make the identification

$$-\,a = \frac{(D-2)}{2}\sum_{k=1}^{\infty} k \tag{5.10}$$

The sum on the RHS is infinite, it would seem. In fact, it is ill-defined. One way to define it is as the $s \to -1$ limit of

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}\,. \tag{5.11}$$

When this sum converges, it defines an analytic function in the complex s-plane: the Riemann zeta function. Remarkably, $\zeta(s)$ can be analytically continued to $s = -1$, where it is finite; in fact

$$\zeta(-1) = -\frac{1}{12}\,. \tag{5.12}$$

Using this in (5.10) we find that

$$a = \frac{(D-2)}{24}\,. \tag{5.13}$$

This looks rather dubious, so let's leave it aside for the moment and proceed to analyse the string spectrum for arbitrary a, and level by level. We shall use the standard notation

$$2\pi T = 1/\alpha'\,. \tag{5.14}$$

- $N = 0$. There is one state, and hence a scalar, with $M^2 = -a/\alpha'$. For $a > 0$ (as (5.13) suggests) this scalar is a tachyon.
- $N = 1$. There are now $(D-2)$ states, $\hat{\boldsymbol{\alpha}}_{-1}|0\rangle$ with $M^2 = (1-a)/\alpha'$. The only way that these states could be part of a Lorentz-invariant theory is if they describe the polarization states of a *massless* vector (a massive vector has $(D-1)$ polarisation states), so Lorentz invariance requires

$$a = 1\,. \tag{5.15}$$

- $N = 2$. Since $a = 1$ the $N = 2$ states are massive, with $M^2 = 1/\alpha'$. The states are those of (5.6), which are in the symmetric 2nd-rank tensor plus vector representation of $SO(D-2)$. These states form a symmetric traceless tensor of $SO(D-1)$ and hence describe a massive spin-2 particle[11].

We now know that the string ground state is a scalar tachyon, and its first excited state is a massless vector, a "photon". All higher level states are massive, and so should be in $SO(D-2)$ representations that can be combined to form $SO(D-1)$ representations (i.e. representations of the rotation group). We have seen that this is true for $N = 2$ and it can be shown to be true for all $N \geq 2$. The $N = 1$ states are exceptional in this respect.

Notice that if $a = 1$ is used in (5.13) we find that $D = 26$. Remarkably, it is indeed true that Lorentz invariance requires $D = 26$, as we shall now see.

5.1.1 Critical dimension

Because Lorentz invariance is not *manifest* in light-cone gauge it might be broken when we pass to the quantum theory. To check Lorentz invariance we have to compute the commutators of the quantum Lorentz charges $\hat{\mathcal{J}}^{mn}$. In fact, $\hat{\mathcal{J}} = \hat{L} + \hat{S}$ and it is easy to see that $[\hat{L}, \hat{S}] = 0$ so we can focus on the spin $\hat{S}^{mn}$; its components should obey the same algebra as those of $\hat{\mathcal{J}}$, and this requires that

$$\left[\hat{S}^{-I}, \hat{S}^{-J}\right] = 0\,. \tag{5.16}$$

[11] A symmetric traceless tensor field of rank n is usually said to describe a particle of "spin n" even though "spin" is not sufficient to label states in space times of dimension $D > 4$.

If the $\{,\}_{PB} \to -i[,]$ rule were to apply to these charges then Lorentz invariance of the quantum string would be guaranteed because the classical theory is Lorentz invariant, even in the light-cone gauge. But it does not apply because the S^{-I} are cubic in the transverse oscillators; a product of two of them is therefore 6th-order in transverse oscillators, but the commutator reduces this to 4th order. The classical PB computation gives zero for this 4th order term, but to achieve this in the quantum theory we might have to change the order of operators, which would produce a term quadratic in oscillators. So, potentially, the RHS of (5.16) might end up being an expression quadratic in transverse oscillators.

Because of this possibility, we need to check (5.16); there is no guarantee that it will be true. We can do this calculation once we have the quantum analogs of the PB relations (3.175). The commutator $[\hat{\alpha}^-_k, \hat{\alpha}^-_\ell]$ is the one we have to examine carefully. There is no ordering ambiguity in the quantum version of the expression (3.169) for α^-_n as long as $n \neq 0$, so we are taking the commutator of well-defined operators as long as $k\ell \neq 0$. Looking first at the $k+\ell = 0$ case, we find that[12]

$$\left[\hat{\alpha}^-_k, \hat{\alpha}^-_{-k}\right] = 2k\frac{\pi T}{p_-^2}\left(\frac{|\mathbf{p}|^2}{2\pi T} + \hat{N}\right) + \frac{2\pi T}{p_-^2}\left(\frac{D-2}{24}\right)\left(k^3 - k\right) . \tag{5.17}$$

Using the mass-shell condition in the operator form

$$\left[\frac{|\mathbf{p}|^2}{2\pi T} + \hat{N}\right] = \left[a - 2\frac{p_+p_-}{2\pi T}\right] , \tag{5.18}$$

which is valid when the operators act on any physical state[13], and using the the fact that $p_+ = \sqrt{\pi T}\,\alpha^-_0$, we can rewrite (5.17) as

$$\left[\hat{\alpha}^-_k, \hat{\alpha}^-_{-k}\right] = -2k\frac{\sqrt{\pi T}}{p_-}\alpha^-_0 + \frac{2\pi T}{p_-^2}\left(k\left[a - \frac{(D-2)}{24}\right] + \frac{(D-2))}{24}k^3\right) \tag{5.19}$$

More generally, one finds that

$$\left[\hat{\alpha}^-_k, \hat{\alpha}^-_\ell\right] = -\frac{\sqrt{\pi T}}{p_-}\left(k-\ell\right)\hat{\alpha}^-_{k+\ell} + \frac{2\pi T}{p_-^2}\left(k\left[a - \frac{(D-2)}{24}\right] + \frac{(D-2)}{24}k^3\right)\delta_{k+\ell} \tag{5.20}$$

where

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 . \end{cases} \tag{5.21}$$

Compare this result with the analogous PB relation of (3.175); he second term in (5.20) has no classical counterpart. Using this result leads to the further result that

$$\left[\hat{S}^{-I}, \hat{S}^{-J}\right] = \frac{4\pi T}{p_-^2}\sum_{k=1}^{\infty}\left(\left[\frac{(D-2)}{12} - 2\right]k + \frac{1}{k}\left[2a - \frac{(D-2)}{12}\right]\right)\hat{\alpha}^{[I}_{-k}\hat{\alpha}^{J]}_k , \tag{5.22}$$

[12]A very similar calculation will be explained in more detail later.

[13]In the light-cone gauge used here the mass-shell constraint is the only physical-state condition.

which is zero for $D > 3$ iff

$$a = 1 \quad \& \quad D = 26\,. \tag{5.23}$$

We therefore confirm that Lorentz invariance requires $a = 1$, but we now see that it also requires $D = 26$; this is the *critical dimension* of the NG string.

From now on we drop the hats on operators.

5.1.2 Quantum string with mixed N/D b.c.s

We shall consider only the case for which both ends of an open string are constrained to move in the same p-plane. In this case the light-cone gauge action is

$$I = \int dt \left\{ \dot{x}^{\hat{m}} p_{\hat{m}} + \sum_{k=1}^{\infty} \frac{i}{k} \dot{\boldsymbol{\alpha}}_k \cdot \boldsymbol{\alpha}_{-k} - \frac{1}{2} e_0 \left(\hat{p}^2 + 2\pi T N\right) \right\}, \tag{5.24}$$

where the level number N is exactly the same as it was for the open string with free ends. The hats here are **not** "operator hats"; they indicate directions in the $(p+1)$-dimensional subspace of D-dimensional Minkowski space-time in which the string ends (and centre-or-mass) move: $\hat{m} = 0, 1, \ldots, p$.

Quantization proceeds exactly as for the string with free ends, *except that the mass-shell condition at given level N is now a wave-equation in the $(p+1)$-dimensional Minkowski space-time.* In particular, the mass-squared at level N is again $2\pi T(N - a)$, and the $N = 1$ excited states are $|\hat{p}\rangle \otimes \boldsymbol{\alpha}_{-1}|0\rangle$, where $\boldsymbol{\alpha}_{-1}$ is a transverse $(D-2)$-vector. However, the boundary conditions preserve only an $SO(D-p-1)$ subgroup of the transverse rotation group $SO(D-2)$ (we'll assume that $p \geq 2$, so that $D - p - 1 < D - 2$). This means that the generic level-1 state of $(p+1)$-momentum $\hat{p}$ is

$$|\hat{p}\rangle \otimes \left[A_{\hat{I}}(\hat{p}) \alpha_{-1}^{\hat{I}} + A_{\check{I}}(\hat{p}) \alpha_{-1}^{\check{I}} \right] |0\rangle\,. \tag{5.25}$$

We can identify $A_{\hat{I}}$ as the $(p-2)$ physical components of a $(p+1)$-vector potential, and $A_{\check{I}}$ as $(D-p-1)$ scalars, all propagating in the Mink_{p+1} subspace of Mink_D. Because a massive photon would have $(p-1)$ physical components, it must be massless[14]., and this again tells us that $a = 1$.

To check that the $(p+1)$-dimensional Lorentz invariance is preserved in the quantum theory we need to check that

$$\left[S^{-\hat{I}}, S^{-\hat{J}} \right] = 0\,, \tag{5.26}$$

and this again turns out to be true only if $a = 1$ and $D = 26$.

Because the b.c.s break translation invariance in the directions orthogonal to the fixed p-plane on which the string ends, the string can lose or gain momentum in

[14] We can't use one of the $(D-p-1)$ scalars as the extra component of the massive photon field because this would break the transverse rotation invariance

these directions. This was considered unphysical for many years, but there is now a physical interpretation. The fixed p-plane is now a p-brane, of a special kind called a D-brane (or Dp-brane) where the D here is for "Dirichlet". What makes this interpretation possible is that small fluctuations of a p-brane in D dimensions are described by $D-p-1$ massless scalar fileds "on the brane", and this is precisely the number of such fields that we have found at level-one; all higher levels give massive fields. The ground state is still a tachyon, which indicates that NG D-branes are unstable.

5.1.3 Quantum closed string

Finally, we consider light-cone gauge quantization of the closed string. There are now two sets of oscillator operators, with commutation relations

$$\left[\alpha_k^I, \alpha_{-k}^J\right] = k\delta^{IJ} = \left[\tilde{\alpha}_k^I, \tilde{\alpha}_{-k}^J\right] \tag{5.27}$$

The oscillator vacuum is now

$$|0\rangle = |0\rangle_R \otimes |0\rangle_L \,, \qquad \boldsymbol{\alpha}_k|0\rangle_R = 0 \quad \& \quad \tilde{\boldsymbol{\alpha}}_k|0\rangle_L = 0 \qquad \forall k > 0\,. \tag{5.28}$$

We define the level number operators $\hat{N}$ and $\hat{\tilde{N}}$ such that they annihilate the oscillator vacuum. Again, we can choose a basis for the Fock space built on the oscillator vacuum for which these operators are diagonal, with eigenvalues N and $\tilde{N}$. In the space of states of definite p and definite $(N, \tilde{N})$ the mass-shell and level matching constraints are

$$p^2 + 4\pi T\left[(N - a_R) + \left(\tilde{N} - a_L\right)\right] = 0 \quad \& \quad (N - a_R) = \left(\tilde{N} - a_L\right)\,. \tag{5.29}$$

Since we want $|0\rangle$ to be a physical state we must choose $a_L = a_R = a$, and then we have

$$p^2 + 8\pi T\,(N - a) = 0 \quad \& \quad \tilde{N} = N\,. \tag{5.30}$$

This means that we can organise the states according to the level N, with $M^2 = 8\pi T(N-a)$. We must do this respecting the level-matching condition $\tilde{N} = N$. Let's look at the first few levels

- $N = 0$. There is one state, and hence a scalar, with $M^2 = -4a/\alpha'$.
- $N = 1$. There are now $(D-2)\times(D-2)$ states

$$\alpha_{-1}^I|0\rangle_R \otimes \alpha_{-1}^J|0\rangle_L \tag{5.31}$$

We can split these into irreducible representations by taking the combinations

$$\left[h_{IJ}(p) + \delta_{IJ}\phi(p) + b_{IJ}(p)\right]\alpha_{-1}^I|0\rangle_R \otimes \alpha_{-1}^J|0\rangle_L\,, \tag{5.32}$$

where h_{IJ} is symmetric traceless tensor, b_{IJ} an antisymmetric tensor and ϕ a scalar. The only way that these could be part of a Lorentz-invariant theory i s if h_{IJ} are the physical components of a *massless* spin-2 field because massive spin-2 would require a symmetric traceless tensor of the full rotation group $SO(D-1)$. Then b_{IJ} must be the physical components of a massless antisymmetric tensor field, and ϕ a massless scalar (the dilaton).

Since we require $M^2 = 0$ we must choose $a = 1$ again[15]. This means that the ground state is a tachyon, indicating an instability of the Minkowski vacuum.

- $N = 2$. Since $a = 1$ the $N = 2$ states are massive, with $M^2 = 8\pi T$. Recalling that the open string states at level-2 combined into a symmetric traceless tensor of $SO(D-1)$, we see that the level-2 states of the closed string will combine into those $SO(D-1)$ representations found in the product of two symmetric traceless $SO(D-1)$ tensors. This includes a 4-order symmetric traceless tensor describing a particle of spin-4; there will be several lower spins too.

The most remarkable fact about these results is that the closed string spectrum contains a massless spin-2 particle, suggesting that a closed string theory will be a theory of quantum gravity. As for the open string, one finds that Lorentz invariance is preserved only if $D = 26$ (the calculation needed to prove this is a repeat of the open string calculation because the spin operator is a sum of a contribution from "left" oscillators and a contribution from "right" oscillators). The ground state is a tachyon, but the tachyon is absent in superstring theory, for which the critical dimension is $D = 10$, and there are various ways to compactify dimensions so as to arrive at more-or-less realistic models of gravity coupled to matter.

5.2 "Old covariant" quantization

Dirac's method of dealing with first-class constraints would appear to allow us to quantise the string in a way that preserves manifest Lorentz invariance. Let's consider the open string with free-end b.c.s. Recall that the action in terms of Fourier modes is

$$I = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \dot{\alpha}_k \cdot \alpha_{-k} - \sum_{n\in\mathbb{Z}} \lambda_{-n} L_n \right\} , \qquad L_n = \frac{1}{2} \sum_{k\in\mathbb{Z}} \alpha_k \cdot \alpha_{n-k} \, , \tag{5.33}$$

and that $\{L_k, L_\ell\}_{PB} = -i\,(k-\ell)\,L_{k+\ell}$. Applying the $\{,\} \to -i[,]$ rule to the PBs of the canonical variables, we get the canonical commutation relations

$$[x^m, p_n] = i\delta_n^m \, , \qquad \left[\alpha_k^m, \alpha_{-k}^n\right] = k\eta^{mn} \, . \tag{5.34}$$

[15]This is usually given as $a = 2$ but that's due to a different definition of a for the closed string.

Now we define the oscillator vacuum $|0\rangle$ by

$$\alpha_k^m|0\rangle = 0 \quad \forall k > 0 \qquad (m = 0, 1, \dots, D-1). \tag{5.35}$$

The Fock space is built on $|0\rangle$ by the action of the creation operators α_{-k}^m, but this gives a space with many unphysical states since we now have D creation operators for each k, whereas we know (from light-cone gauge quantization) that $D-2$ suffice to construct the physical states.

Can we remove unphysical states by imposing the physical state conditions

$$L_n|\text{phys}\rangle = 0 \quad \forall n\ ? \tag{5.36}$$

Notice that we do not encounter an operator ordering ambiguity when passing from the classical phase-space function L_n to the corresponding operator L_n except when $n = 0$, so the operator L_n is unambiguous for $n \neq 0$ and it is easy to see that

$$L_n|0\rangle = 0\,, \qquad n > 0\,. \tag{5.37}$$

However, it is also easy to see [exercise: check these statements] that

$$\begin{aligned} L_{-1}|0\rangle &\equiv \frac{1}{2}\sum_k \alpha_k \cdot \alpha_{-1-k}|0\rangle = \alpha_0 \cdot \alpha_{-1}|0\rangle \\ L_{-2}|0\rangle &\equiv \frac{1}{2}\sum_k \alpha_k \cdot \alpha_{-2-k}|0\rangle = \left(\alpha_0 \cdot \alpha_{-2} + \frac{1}{2}\alpha_{-1}^2\right)|0\rangle\,, \end{aligned} \tag{5.38}$$

so it looks as though not even $|0\rangle$ is physical. In fact, there are no states in the Fock space satisfying (5.36) because the algebra of the operators L_n has a quantum anomaly, which is such that the set of operators $\{L_n; n \in \mathbb{Z}\}$ is not "first-class". That is what we shall now prove.

Since the L_n are quadratic in oscillator variables, the product of two of them is quartic but the commutator $[L_m, L_n]$ is again quadratic. That is what we expect from the PB, which is proportional to L_{m+n}, but to get the operator L_{m+n} from the expression that results from computing the commutator $[L_m, L_n]$, we may need to re-order operators, and that would produce a constant term. So, we must find that

$$[L_m, L_n] = (m-n)\,L_{m+n} + A_{mn} \tag{5.39}$$

for some constants A_{mn}. We can compute the commutator using the fact that

$$[L_m, \alpha_k] = -k\,\alpha_{k+m}\,. \tag{5.40}$$

This can be verified directly but it also follows from the corresponding PB result because no ordering ambiguity is possible either on the LHS or the RHS. Using this, we find that

$$
\begin{aligned}
[L_m, L_n] &= \sum_k \left([L_m, \alpha_k] \cdot \alpha_{n-k} + \alpha_k \cdot [L_m, \alpha_{n-k}]\right) \\
&= -\frac{1}{2}\sum_k k\, \alpha_{k+m} \cdot \alpha_{n-k} - \frac{1}{2}\sum_k (n-k)\, \alpha_k \cdot \alpha_{n+m-k} \,. \qquad (5.41)
\end{aligned}
$$

As long as $n + m \neq 0$ this expression is not affected by any change in the order of operators, so it must equal what one gets from an application of the $\{,\}_{PB} \to -i[,]$ rule. In other words, $A_{mn} = 0$ unless $m + n = 0$. We can check this by using the fact that $\alpha_{-k} = \alpha_k^\dagger$, so that

$$
\alpha_k|0\rangle = 0 \quad \Leftrightarrow \quad \langle 0|\alpha_{-k} = 0\,. \qquad (5.42)
$$

From this we see that for $m + n \neq 0$

$$
\langle 0|\alpha_{k+m} \cdot \alpha_{n-k}|0\rangle = 0 = \langle 0|\alpha_k \cdot \alpha_{n+m-k}|0\rangle \qquad (m + n \neq 0) \qquad (5.43)
$$

and hence that $\langle 0|[L_m, L_n]|0\rangle = 0$ unless $m + n = 0$. This tells us that $A_{mn} = A(m)\delta_{m+n}$.

We now focus on the $m + n = 0$ case, for which

$$
[L_m, L_{-m}] = 2mL_0 + A(m) \quad \Rightarrow \quad A(-m) = -A(m)\,. \qquad (5.44)
$$

Because of an operator ordering ambiguity, the operator L_0 is only defined up to the addition of a constant, so what we find for $A(m)$ will obviously depend on how we define the operator L_0. We shall define it as

$$
L_0 = \frac{1}{2}\alpha_0^2 + \sum_{k=1}^{\infty} \alpha_{-k} \cdot \alpha_k\,, \qquad (5.45)
$$

but we should keep in mind that it is possible to redefine L_0 by adding a constant to it. We could now return to (5.41), set $n = -m$, and then complete the computation to find $A(m)$. This can be done, but it has to be done with great care to avoid illegitimate manipulations of infinite sums. For that reason we here take an indirect route.

First we use the Jacobi identity

$$
[L_k, [L_m, L_n]] + \text{cyclic permutations} \ \equiv 0\,, \qquad (5.46)
$$

to deduce that [Exercise]

$$
[(m-n)A(k) + (n-k)A(m) + (k-m)A(n)]\, \delta_{m+n+k} = 0\,. \qquad (5.47)
$$

Now set $k = 1$ and $m = -n - 1$ (so that $m + n + k = 0$) to deduce that

$$
A(n+1) = \frac{(n+2)A(n) - (2n+1)A(1)}{n-1} \quad n \geq 2\,. \qquad (5.48)
$$

This is a recursion relation that determines $A(n)$ for $n \geq 3$ in terms of $A(1)$ and $A(2)$, so there are two independent solutions of the recursion relation. You may verify that $A(m) = m$ and $A(m) = m^3$ are solutions, so now we have

$$[L_m, L_{-m}] = 2mL_0 + c_1 m + c_2 m^3\,, \tag{5.49}$$

for some constants c_1 and c_2. Observing that

$$\langle 0|\,[L_m, L_{-m}]\,|0\rangle = \langle 0|L_m L_{-m}|0\rangle = ||L_{-m}|0\rangle||^2\,, \tag{5.50}$$

and that

$$\langle 0|L_0|0\rangle = \frac{1}{2}\alpha_0^2 = \frac{p^2}{2\pi T}\,, \tag{5.51}$$

we deduce that

$$||L_{-m}|0\rangle||^2 - \left(\frac{p^2}{\pi T}\right) m = c_1 m + c_2 m^3\,. \tag{5.52}$$

We can now get two equations for the two unknown constants (c_1, c_2) by evaluating $||L_{-m}|0\rangle||^2$ for $m = 1$ and $m = 2$. Using (5.38) we find that

$$||L_{-1}|0\rangle||^2 = \frac{1}{\pi T}\langle 0|p\cdot\alpha_{-1} p\cdot\alpha_{-1}|0\rangle = \frac{p_m p_n}{\pi T}\langle 0|\alpha_1^m \alpha_{-1}^n|0\rangle = \frac{p^2}{\pi T}\,, \tag{5.53}$$

and that

$$\begin{aligned} ||L_{-2}|0\rangle||^2 &= \langle 0|\left(\alpha_0\cdot\alpha_2 + \frac{1}{2}\alpha_1^2\right)\left(\alpha_0\cdot\alpha_{-2} + \frac{1}{2}\alpha_{-1}^2\right)|0\rangle \\ &= \frac{1}{\pi T}\langle 0|p\cdot\alpha_2\, p\cdot\alpha_{-2}|0\rangle + \frac{1}{4}\langle 0|\alpha_1^2\alpha_{-1}^2|0\rangle \\ &= \frac{2p^2}{\pi T} + \frac{D}{2}\,, \end{aligned} \tag{5.54}$$

from which we see that

$$||L_{-1}|0\rangle||^2 - \frac{p^2}{\pi T} = 0\,, \qquad ||L_{-2}|0\rangle||^2 - \frac{2p^2}{\pi T} = \frac{D}{2}\,, \tag{5.55}$$

and hence that

$$c_1 + c_2 = 0\,, \qquad c_1 + 4c_2 = \frac{D}{4} \qquad \Rightarrow \qquad c_2 = -c_1 = \frac{D}{12}\,. \tag{5.56}$$

Inserting this result into (5.49), we have

$$[L_m, L_{-m}] = 2mL_0 + \frac{D}{12}\left(m^3 - m\right)\,, \tag{5.57}$$

and hence that

$$[L_m, L_n] = (m-n)\,L_{m+n} + \frac{D}{12}\left(m^3 - m\right)\delta_{m+n}\,. \tag{5.58}$$

This is an example of the *Virasoro algebra*. In general, the Virasoro algebra takes the form

$$[L_m, L_n] = (m-n)\,L_{m+n} + \frac{c}{12}\left(m^3 - m\right)\delta_{m+n}\,, \tag{5.59}$$

where c is the *central charge*. In the current context we get this algebra with $c = D$.

5.2.1 The Virasoro constraints

We have just seen that without breaking manifest Lorentz covariance, it is not possible to impose the physical state conditions that we would need to impose to eliminate all unphysical degrees of freedom. In view of our light-cone gauge results, this should not be too much of a surprise. We saw that the level-one states are the polarisation states of a massless spin-1 particle. The covariant description of a massless spin-1 particle in terms of a vector field (which is what is needed to allow interactions) *necessarily* involves unphysical degrees of freedom.

This argument suggests that we should aim to impose weaker conditions that leave unphysical states associated to gauge invariances, but that remove all other unphysical states. For correspondence with the classical NG string in a semi-classical limit, these weaker conditions should have the property that

$$\langle\Psi'|\left(L_n - a\delta_n\right)|\Psi\rangle = 0 \quad \forall n\,, \tag{5.60}$$

for any two allowed states $|\Psi\rangle$ and $|\Psi'\rangle$. Because of the operator ordering ambiguity in L_0, we should allow for the possibility that the L_0 operator of relevance here is shifted by some constant relative to how we defined it in (5.45), hence the constant a (which will turn out to be the same constant a that we introduced in the light-cone gauge quantization). We can achieve (5.60) without encountering inconsistencies by imposing the *Virasoro constraints*

$$L_n|\Psi\rangle = 0 \quad \forall n > 0 \quad \& \quad \left(L_0 - a\right)|\Psi\rangle = 0\,. \tag{5.61}$$

Let's call states $|\Psi\rangle$ satisfying the Virasoro constraints "Virasoro-allowed".

As for the light-cone gauge quantization, we can define the level number operator

$$N = \sum_{k=1}^{\infty} \alpha_{-k}\cdot\alpha_k\,. \tag{5.62}$$

This differs from the level-number operator of the light-cone gauge in that it includes all D components of the oscillator annihilation and creation operators, not just the $D-2$ transverse components. Otherwise it plays a similar role. We can choose a basis such that the operators p and N are diagonal, in which case p and N will mean their respective eigenvalues. For given p and N, the Virasoro mass-shell condition is

$$p^2 + M^2 = 0\,, \qquad M^2 = 2\pi T\left(N - a\right)\,. \tag{5.63}$$

The mass-squared is a linearly increasing function of the level number, as in the light-cone case. Let's consider the first few levels. The only $N = 0$ state in the oscillator Fock space is $|0\rangle$ (which gives the string states $|p\rangle\otimes|0\rangle$).

The general $N = 1$ state is

$$A_m(p)\alpha_{-1}^m|0\rangle\,. \tag{5.64}$$

The norm-squared of this state is

$$||A \cdot \alpha_{-1}|0\rangle||^2 = A_m A_n \langle 0|\alpha_1^m \alpha_{-1}^n|0\rangle = \eta^{mn} A_m A_n\,. \tag{5.65}$$

This could be negative but we still have to impose the other Virasoro conditions $L_k(A \cdot \alpha_{-1})|0\rangle = 0$ for $k > 0$. For $k \geq 2$ these conditions are trivially satisfied, but for $k = 1$ we find that

$$0 = L_1(A \cdot \alpha_{-1})|0\rangle = A \cdot \alpha_0|0\rangle \quad \Rightarrow \quad p \cdot A = 0\,. \tag{5.66}$$

Let's now consider the implications of this for the constant a:

- $a > 1$. Then $M^2 < 0$ so p is spacelike. In a frame where $p = (0; p, \mathbf{0})$ the constraint $p \cdot A = 0$ is equivalent to $A_1 = 0$ and then

  $$||A \cdot \alpha_{-1}|0\rangle||^2 = -A_0^2 + |\mathbf{A}|^2 \tag{5.67}$$

  which allows "ghosts", i.e. states of negative norm (e.g. those with $\mathbf{A} = \mathbf{0}$ but $A_0 \neq 0$). This implies a violation of unitarity (non-conservation of probability) so we should not allow $a > 1$.

- $a = 1$. Then $M^2 = 0$, which agrees with the light-cone gauge result. In this case p is null. In a frame where $p = (1; 1, \mathbf{0})$ the constraint $p \cdot A = 0$ implies that $A_0 = A_1$, and hence that

  $$||A \cdot \alpha_{-1}|0\rangle||^2 = |\mathbf{A}|^2 \geq 0\,. \tag{5.68}$$

  There are no ghosts but the state $|\chi\rangle = (\alpha_{-1}^0 + \alpha_{-1}^1)|0\rangle$ is one with $p \cdot A = 0$ but has zero-norm. This state is also orthogonal to all other $N = 1$ states satisfying $P \cdot A = 0$, so given one such state $|\psi\rangle$ we have

  $$|||\psi\rangle + c|\chi\rangle||^2 = |||\psi\rangle||^2 \tag{5.69}$$

  for any complex constant c. This means that we may identify the physical polarisation with equivalence classes of states, where any two states that differ by a null state are considered equivalent. The space of such equivalence classes has dimension $D - 2$.

  What we are finding here is essentially the Gupta-Bleuler quantization of electrodynamics (in D space-time dimensions).

- $a < 1$. Then $M^2 > 0$, so p is timelike. In a frame where $p = (p, \vec{0})$, the constraint $p \cdot A = 0$ implies that $A_0 = 0$, so

  $$||A \cdot \alpha_{-1}|0\rangle||^2 = |\vec{A}|^2 \geq 0\,, \tag{5.70}$$

  and now there are no non-zero zero-norm states satisfying $p \cdot A = 0$.

  There is nothing obviously unphysical about $a < 1$ but it involves the inclusion of states that are absent in the light-cone gauge. The quantization procedure has introduced new degrees of freedom that were absent classically.

Let's now look at the level-2 states. The general level-2 state is

$$|\Psi_2\rangle = \left(A_{mn}\alpha^m_{-1}\alpha^n_{-1} + B_m\alpha^m_{-2}\right)|0\rangle \tag{5.71}$$

This is trivially annihilated by L_k for $k > 2$. However, $L_1|\Psi_2\rangle = 0$ imposes the condition

$$B_n = -\alpha_0^m A_{mn}\,, \tag{5.72}$$

and $L_2|\Psi_2\rangle = 0$ imposes the condition

$$\eta^{mn}A_{mn} = -2\alpha_0 \cdot B \tag{5.73}$$

This means that only the traceless part of A_{mn} is algebraically independent, so the dimension of the Virasoro-allowed level-2 space is

$$\frac{1}{2}D(D+1) - 1 = \left[\frac{1}{2}D(D-1) - 1\right] + D \tag{5.74}$$

The dimension is D larger than the physical level-2 space that we found from light-cone gauge quantization (that was spanned by the polarisation states of a massive spin-2 particle, so a symmetric traceless tensor of the $SO(D-1)$ rotation group). However, equivalence with the light-cone gauge results is still possible if there are sufficient null states, and no ghosts.

To analyse this we need to consider the norm-squared of $|\Psi_2\rangle$, which is

$$|||\Psi_2\rangle||^2 = 2A^{mn}A_{mn} + 2B^2\,. \tag{5.75}$$

Then we need to consider the implications for this norm of (5.72) and (5.73). We will not pursue this further, but the final result is that there are no ghosts only if $D \leq 26$ and then there are sufficient null states for equivalence with the light-cone gauge results iff $D = 26$.

It is simple to see that there are ghosts if $D > 26$. Consider the special case of (5.71) for which

$$A^{mn} = \eta^{mn} + k_1\alpha_0^m\alpha_0^n\,, \qquad B^m = k_2\alpha_0^m\,. \tag{5.76}$$

The conditions (5.72) and (5.73) determine the constants (k_1, k_2) to be

$$k_1 = \frac{D+4}{10}\,, \qquad k_2 = \frac{D-1}{5}\,, \tag{5.77}$$

and then one finds that the norm-squared is

$$-\frac{2}{25}(D-1)(D-26)\,. \tag{5.78}$$

This is negative for $D > 26$, so the state being considered is a ghost. It is null for $D = 26$. For $D < 26$ the state has positive norm but it now counts as a physical state that increases the dimension of the space of level-2 states. For $D < 26$ there are more states than those that can be found from quantization of the physical degrees of freedom of the classical NG string, so it is unclear to what extent any "sub-critical string theory" would be related to strings.

6. Interlude: Path integrals and the point particle

Let $A(X)$ be the quantum-mechanical amplitude for a particle to go from the origin of Minkowski coordinates to some other point in Minkowski space-time with cartesian coordinates X. As shown by Feynman, $A(X)$ has a path-integral representation. In the case of a relativistic particle of mass m, with phase-space action $I[x,p;e]$ we have

$$A(x) = \int [de] \int [dxdp]\, e^{iI}\,, \qquad x(0) = 0\,, \quad x(1) = X\,. \tag{6.1}$$

Here we are parametrising the path such that it takes *unit parameter time* to get from the space-time origin to the space-time point with coordinates X. The integrals have still to be defined, but we proceed formally for the moment.

We now allow t to be complex and we "Wick rotate": first set $t = -i\tilde{t}$ to get

$$I = \int_0^i d\tilde{t} \left\{ \dot{x}^m p_m + \frac{i}{2} e \left(p^2 + m^2\right) \right\} \qquad \left(\dot{x} = dx/d\tilde{t}\right). \tag{6.2}$$

As it stands, $\tilde{t}$ is pure imaginary, but we can rotate the contour in the complex $\tilde{t}$-plane back to the real axis; if we choose to call this real integration variable t then this procedure takes

$$-iI \to \int_0^1 dt \left\{ -i\dot{x}^m p_m + \frac{1}{2} e \left(p^2 + m^2\right) \right\} = I_E \tag{6.3}$$

where I_E is the "Euclidian action". Notice that on successive elimination of p and e,

$$I_E[x,p;e] \to \frac{1}{2} \int_0^1 dt \left\{ e^{-1}\dot{x}^2 + m^2 e \right\} \to m \int_0^1 dt \sqrt{\dot{x}^2}\,, \tag{6.4}$$

which is positive. The amplitude A is now given by the "Euclidean path integral"

$$A = \int [de] \int [dxdp] e^{-I_E}\,. \tag{6.5}$$

We will fix the gauge invariance by setting $e = s$ for constant s. As discussed previously, the variable s is gauge-invariant, so it is not possible to use gauge invariance to bring it to a particular value, so we have to integrate over s, which (being proportional to the elapsed proper time) could be any number from zero to infinity. In other words, we can use gauge invariance to reduce the functional integral over $e(t)$ to an ordinary integral over s. We now have

$$A(x) = \int_0^\infty ds \int [dxdp] e^{\int_0^1 \left\{ i\dot{x}^m p_m - \frac{s}{2}\left(p^2+m^2\right)\right\}}\,. \tag{6.6}$$

This is not quite right, for reasons to be explained soon, but it will suffice for the moment.

To define the $\int[dx\,dp]$ integral we first approximate the path in some way. We could do this by n straight-line segments. We would then have n D-momentum integral to do (one for each segment) and $(n-1)$ integrals over the D-vector positions of the joins. This illustrates the general point that in any multiple-integral approximation to the phase-space path integral there will be some number of phase-space pairs of integrals plus one extra D-momentum integral, which is the average of $p(t)$. Consider the $n=1$ case, for which

$$x(t) = Xt \quad \& \quad p(t) = P \quad \Rightarrow \quad -I_E = iX^m P_m - \frac{s}{2}\left(P^2 + m^2\right) . \tag{6.7}$$

The only free variable on which the Euclidean action depends is the particle's D-momentum P, so $\int[dxdp]$ is approximated by the momentum-space integral $\int d^D P$, and we find that

$$A_1(X) = \int d^D P\, e^{iX\cdot P} \int_0^\infty ds\, e^{-\frac{s}{2}\left(P^2+m^2\right)} \propto \int d^D P \frac{e^{iX\cdot P}}{P^2+m^2} , \tag{6.8}$$

which is the Fourier transform of the standard momentum-space Feynman propagator for a particle of mass m and zero spin.

That was just the $n=1$ approximation! A simpler alternative to approximation by segments (which differs from it only for $n>1$) is approximation by polynomials, nth order for $x(t)$ and $(n-1)$th order for $p(t)$. Consider the $n=2$ case:

$$x(t) = (X - x_1)\, t + x_1 t^2 \quad \& \quad p(t) = P + 2q\left(t - \frac{1}{2}\right) . \tag{6.9}$$

Notice that $x(t)$ satisfies the b.c.s and P is the integral of $p(t)$ (i.e. the average D-momentum). We have to integrate over the pair (x_1, q) and P. Using these expressions, we find that

$$-I_E = iX\cdot P - \frac{s}{2}\left(P^2+m^2\right) - \frac{s}{6}p_1^2 - \frac{1}{6s}x_1^2 , \qquad (p_1 = q - ix_1/s). \tag{6.10}$$

This gives

$$A_2(X) = \int d^D P e^{iX\cdot P} \int_0^\infty ds\, e^{-\frac{s}{2}\left(P^2+m^2\right)} \left[\int d^D x_1 e^{-\frac{x_1^2}{6s}} \int d^D p_1 e^{-\frac{sp_1^2}{6}}\right] . \tag{6.11}$$

The bracketed pair of Gaussian integrals is an s-independent constant, so $A_2(X) \propto A_1(X)$. One finds, similarly, that $A_n(X) = A_1(X)$. Taking the $n\to\infty$ limit we then have $A(X) \propto A_1(X)$, so $A(X)$ is the Feynman propagator in configuration space.

6.1 Faddeev-Popov determinant

Now we return to the problem of gauge-fixing. The problem with the formula (6.1) is that, because of gauge-invariance, we are integrating over too many functions.

Implicitly, we are integrating over functions $\alpha(t)$ that are maps from the (one-dimensional) gauge group to the worldline. If this integral were explicit we could just omit the integral, but it is only implicit, so it is not immediately obvious how we should proceed.

Since we can choose a gauge for which $e(t) = s$, for variable constant s, it must be possible to write an arbitrary function $e(t)$ as a gauge transform of $e = s$:

$$e(t) = s + \dot{\alpha}(t) = e_s\left[\alpha(t)\right] . \tag{6.12}$$

We have now expressed the general $e(t)$ in terms of the gauge group parameter $\alpha(t)$ and the constant s. As a corollary, we have

$$\int [de] = \int_0^\infty ds \int [d\alpha]\, \Delta_{FP} \tag{6.13}$$

where Δ_{FP} is the Jacobian for the change of variables from $e(t)$ to $\{s, \alpha(t)\}$:

$$\Delta_{FP} = \det\left[\frac{\delta e_s[\alpha(t)]}{\delta\alpha(t')}\right] . \tag{6.14}$$

This is the *Faddeev-Popov determinant*. Using this result in the formula

$$1 = \int [de]\delta[e(t) - s] , \tag{6.15}$$

which definees what we mean by the delta functional, we deduce that

$$1 = \int_0^\infty ds \int [d\alpha]\, \Delta_{FP}\, \delta[e(t) - s] \tag{6.16}$$

We now return to the initial path-integral expression for $A(X)$ and "insert 1" into the integrand; i.e. we insert the RHS of (6.16) to get

$$A(X) = \int [de] \left[\int_0^\infty ds \int [d\alpha]\, \Delta_{FP}\, \delta[e(t) - s]\right] \int [dxdp] e^{iI} . \tag{6.17}$$

Re-ordering the integrals and using the delta functional to do the $[de]$ integral (this sets $e = s$ elsewhere in the integrand) we get

$$A(X) = \int [d\alpha] \int_0^\infty ds\, \Delta_{FP} \int [dxdp] e^{iI_{e=s}} . \tag{6.18}$$

By these manipulations we have made *explicit* the integral over maps from the gauge group to the worldline, so we can remedy the problem of too many integrals by simply omitting the $[d\alpha]$ integral. This gives us

$$A(X) = \int_0^\infty ds\, \Delta_{FP} \int [dxdp] e^{iI_{e=s}} . \tag{6.19}$$

The Δ_{FP} factor was missing from (6.6), which is why that formula was "not quite right", but the Δ_{FP} factor only effects the normalisation of $A(X)$, which anyway depends on the detailed definitions of the path integral measures.

Although the FP determinant is not very relevant to the computation of $A(X)$ it is relevant to other computations, and it is very important to the path-integral quantization of the NG string, which we will get to soon.

6.2 Fadeev-Popov ghosts

Let (b_i, c^i) $(i = 1, \ldots, n)$ be n pairs of *anticommuting* variables. This means that

$$\{b_i, b_j\} = 0\,, \quad \left\{b_i, c^j\right\} = 0\,, \quad \left\{c^i, c^j\right\} = 0 \quad \forall\, i,j = 1, \ldots, n\,, \tag{6.20}$$

where $\{,\}$ means anticommutator: $\{A, B\} = AB + BA$. Any function of anticommuting variables has a terminating Taylor expansion because no one anticommuting variable can appear twice. Consider the $n = 1$ case

$$f(b,c) = f_0 + bf_1 + cf_{-1} + bc\tilde{f}_0\,, \tag{6.21}$$

where $(f_0, f_{\pm 1}, \tilde{f}_0)$ are independent of both b and c. Then

$$\frac{\partial}{\partial b} f = f_1 + c\tilde{f}_0 \quad \Rightarrow \quad \frac{\partial}{\partial c}\frac{\partial}{\partial b} f = \tilde{f}_0 \tag{6.22}$$

Essentially, a derivative with respect to b removes the part of f that is independent of b and then strips b off what is left. However, we should move b to the left of anything else before stripping it off; this is equivalent to the definition of the derivative as a "left derivative". Using this definition we have

$$\frac{\partial}{\partial c} f = f_{-1} - b\tilde{f}_0 \quad \Rightarrow \quad \frac{\partial}{\partial b}\frac{\partial}{\partial c} f = -\tilde{f}_0\,. \tag{6.23}$$

There is minus sign relative to (6.22) because we had to move c to the left of b. This result shows that

$$\left\{\frac{\partial}{\partial b}, \frac{\partial}{\partial c}\right\} = 0\,. \tag{6.24}$$

That is, *partial derivatives with respect to anticommuting variables anti-commute.*

We can also integrate over anticommuting variables. The (Berezin) integral over an anticommuting variable is defined to the same as the partial derivative with respect to it. Consider, for example, the Gaussian integral

$$\int d^n c\, d^n b\; e^{-b_i M^i{}_j c_j} = \left[\frac{\partial}{\partial b_n} \cdots \frac{\partial}{\partial b_1}\right]\left[\frac{\partial}{\partial c^n} \cdots \frac{\partial}{\partial c^1}\right] e^{-b_i M^i{}_j c_j}\,. \tag{6.25}$$

If we expand the integrand in powers of bMc the expansion terminates at the nth term, which is also the only term that contributes to the integral because it is the only one to contain all b_i and all c^i. Because of the anti-commutativity of the partial derivatives, we then find that

$$\int d^n c\, d^n b\ e^{-b_i M^i{}_j c_j} \ \propto\ \frac{1}{n!}\varepsilon_{i_1\cdots i_n} M^{i_1}{}_{j_1}\cdots M^{i_n}{}_{j_n}\, \varepsilon^{j_1\cdots j_n} = \det M \tag{6.26}$$

We can use a functional variant of this result to write the FP determinant as a Gaussian integral over anticommuting "worldline fields" $b(t)$ and $c(t)$:

$$\begin{aligned}\det\left[\delta(t-t')\partial_t\right] &= \int [dbdc] \exp\left[-\int_0^1 dt \int_0^1 dt'\, b(t')\left[\delta(t-t')\partial_t\right] c(t)\right] \\ &= \int [dbdc] \exp\left[-\int_0^1 dt\, b\dot c\right] .\end{aligned} \tag{6.27}$$

The anticommuting worldline fields are known collectively as the FP ghosts, although it is useful to distinguish between them by calling c the ghost and b the anti-ghost.

N.B. There is no relation between the FP ghosts and the ghosts that appear in the NG string spectrum for $D > 26$. The same word is being used for two entirely different things!

Using (6.27) in the expression (6.19) we arrive at the result

$$A(X) = \int_0^\infty ds \int [dxdp] \int [dcdb]\, e^{iI_{qu}} , \tag{6.28}$$

where the "quantum" action is[16]

$$I_{qu} = \int dt \left\{\dot x^m p_m + ib\dot c - H_{qu}\right\} , \qquad H_{qu} = \frac{s}{2}\left(p^2 + m^2\right) . \tag{6.29}$$

We now have a mechanical system with an extended phase-space with additional, anticommuting, coordinates (b, c). The factor of i multiplying the $b\dot c$ term stems from the convention (and it is only a convention) that a product of two "real" anti-commuting variables is "imaginary", so an i is needed for reality[17].

There is a simple extension of these ideas to the general mechanical system with n first class constraints φ_i imposed by Lagrange multipliers λ^i. The FP operator is found by varying the gauge-fixing condition with respect to the gauge parameter, so in the gauge $\lambda^i = \bar\lambda^i$, for constants $\bar\lambda^i$, we find that

$$\Delta_{FP} = \det\left(\left.\frac{\delta_\epsilon \lambda^i(t')}{\delta\epsilon^j(t)}\right|_{\lambda^i=\bar\lambda^i}\right) \tag{6.30}$$

[16]We can see now why the FP determinant does not affect our previous computation of $A(X)$; it changes only the overall normalisation factor, which we did not compute.

[17]Of course, an anticommuting number cannot really be real; it is "real" if we declare it to be unchanged by complex conjugation. Given two such anti-commuting numbers μ and ν we may construct the complex anti-commuting number $\mu + i\nu$, which will then have complex conjugate $\mu - i\nu$. According to the convention, the product $i\mu\nu$ is real.

where $\delta_\epsilon \lambda^i$ is the gauge variation of λ^i with parameters ϵ^i, given in (2.43). This gives

$$\Delta_{FP} = \det\left[\delta(t-t')\left(\delta^i_j \partial_t + \bar\lambda^k f_{jk}{}^i\right)\right] \quad \propto \quad \int [dbdc]\, e^{iI_{FP}[b,c]}\,, \tag{6.31}$$

where the FP action is

$$I_{FP}[b,c] = \int dt\left\{ib_i\left[\dot c + \bar\lambda^k c^j f_{jk}{}^i\right]\right\}\,. \tag{6.32}$$

This must be added to the original action to get the "quantum action"

$$I_{qu} = \int dt\left\{\dot q^I p_i + ib_i\dot c^i - H_{qu}\right\}\,, \qquad H_{qu} = \bar\lambda^k\left(\varphi_k + ic^j f_{jk}{}^i b_i\right)\,. \tag{6.33}$$

We now have an action for a mechanical system with an extended phase space (actually a *superspace*) for which some coordinates are anticommuting. On this space we have the following closed non-degenerate (i.e. invertible) 2-form

$$\Omega = dp_m \wedge dx^m + db_i \wedge dc^i\,. \tag{6.34}$$

This is "orthosymplectic" rather than "symplectic" because the anticommutativity of b and c means that[18]

$$db_i \wedge dc^i = dc^i \wedge db_i\,. \tag{6.35}$$

This leads to a canonical Poisson bracket for b and c that is *symmetric* rather than antisymmetric

$$\left\{b_i, c^j\right\}_{PB} = \left\{c^j, b_i\right\}_{PB} = -i\delta^j_i\,. \tag{6.36}$$

The PB defined by the inverse of an orthosymplectic 2-form is no longer a Lie bracket because it is not always antisymmetric, but it is a super-Lie bracket satisfying a super-Jacobi identity. Let C, C', C'' be commuting functions on phase space, and let A, A', A'' be anticommuting functions on phase space. Then the super-Jacobi identity states that

$$\begin{aligned}
\{C,\{C',C''\}_{PB}\}_{PB} + \text{ cyclic perms.} &\equiv 0 \\
\{A,\{C,C'\}_{PB}\}_{PB} + \text{ cyclic perms.} &\equiv 0 \\
\{A,\{A',A''\}_{PB}\}_{PB} + \text{ cyclic perms.} &\equiv 0\,,
\end{aligned} \tag{6.37}$$

but

$$\{C,\{A,A'\}_{PB}\}_{PB} - \{A',\{C,A\}_{PB}\}_{PB} + \{A,\{A',C\}_{PB}\}_{PB} \equiv 0\,. \tag{6.38}$$

We get a minus sign in the second term because we had to cycle A'' past A', and this sign is changed back to plus in the next term for the same reason.

[18]The usual minus sign coming from the antisymmetry of the wedge product of 1-forms is cancelled by the minus sign coming from changing the order of b and c.

In the quantum theory, these identities become

$$\left[\hat{C},\left[\hat{C}',\hat{C}''\right]\right] + \text{ cyclic perms.} \equiv 0$$
$$\left[\hat{A},\left[\hat{C},\hat{C}'\right]\right] + \text{ cyclic perms.} \equiv 0 \qquad (6.39)$$

and

$$[A,\{A',A''\}] + \text{ cyclic perms.} \equiv 0 \qquad (6.40)$$

but

$$[C\{A,A'\}] - \{A'[C,A]\} + \{A[A',C]\} \equiv 0\,. \qquad (6.41)$$

All of these identities hold for *arbitrary* operators $\hat{A},\hat{A}',\hat{A}''$ and $\hat{C},\hat{C}',\hat{C}''$; e.g. for arbitrary square matrices [**Exercise: check this**].

6.3 BRST invariance

The "quantum" point particle action (6.29) is invariant under the following transformations with constant anticommuting parameter Λ:

$$\delta_\Lambda x = i\Lambda cp\,,\quad \delta_\Lambda p = 0\,,\qquad \delta_\Lambda b = -\frac{1}{2}\left(p^2+m^2\right)\Lambda\,,\quad \delta_\Lambda c = 0\,. \qquad (6.42)$$

Notice that the transformations of (x,p) are gauge transformations with parameter $\alpha(t) = -i\Lambda c(t)$; the factor of i is included because Λ is assumed to anticommute with c and we use the convention that complex conjugation changes the order of anticommuting numbers, so we need an i for "reality". Allowing Λ to be t-dependent we find [Exercise: check this]

$$\delta_\Lambda I_{qu} = \frac{i}{2}\int dt\,\dot{\Lambda}\left(p^2+m^2\right)c\,. \qquad (6.43)$$

This confirms the invariance for constant Λ and tells us that the (anti-commuting) Noether charge is

$$Q_{BRST} = \frac{1}{2}c\left(p^2+m^2\right)\,. \qquad (6.44)$$

This is the BRST charge[19] . It generates the BRST transformations (6.42) via Poisson brackets defined for functions on the extended phase space. Let's check this:

$$\delta_\Lambda x = \{x, i\Lambda Q_{BRST}\}_{PB} = i\Lambda c\left\{x,\frac{1}{2}p^2\right\}_{PB} = i\Lambda cp$$
$$\delta_\Lambda b = \{b, i\Lambda Q_{BRST}\}_{PB} = -\frac{i}{2}\Lambda\{b,c\}_{PB}\left(p^2+m^2\right) = -\frac{1}{2}\left(p^2+m^2\right)\Lambda\,. \quad (6.45)$$

BRST symmetry seems rather mysterious, but it's not just a special feature of the point particle. Consider the quantum action (6.33) of the general mechanical

[19]BRST stands for Becchi, Rouet, Stora and Tyutin, who discovered BRST symmetry in the context of YM theory.

model with first class constraints (which includes, as we have seen, the NG string). If we assume that the constraint functions φ_i span a *Lie algebra* then the structure functions f_{ij}^k will be constants satisfying (as a consequence of the Jacobi identity)

$$f_{[ij}{}^{\ell} f_{k]\ell}{}^{m} \equiv 0 \,. \tag{6.46}$$

In this case[20] the action is invariant under the transformations generated by

$$Q_{BRST} = c^i \varphi_i + \frac{i}{2} c^i c^k f_{ki}{}^{j} b_j \,, \tag{6.47}$$

which satisfies

$$\{Q_{BRST}, Q_{BRST}\}_{PB} = 0 \,. \tag{6.48}$$

This is not a trivial property of Q_{BRST} because the PB is *symmetric* under exchange of its two arguments if these are both anticommuting.

We still need to check the invariance. Because the variation of $(\dot{q}^I p_I + ib\dot{c})$ is guaranteed to be a total time derivative by an extension to the extended phase-space of the lemma summarised by (2.39), we need only check that

$$0 = \{Q_{BRST}, H_{qu}\}_{PB} \,, \qquad H_{qu} = \bar{\lambda}^k \left(\varphi_k + ic^j f_{jk}{}^{i} b_i\right) \,. \tag{6.49}$$

This can be checked directly, but an alternative is to first check that

$$H_{qu} = i \left\{\bar{\lambda}^i b_i, Q_{BRST}\right\}_{PB} \,, \tag{6.50}$$

and then use the super-Jacobi identity to show that

$$\left\{Q_{BRST}, \left\{\bar{\lambda}^i b_i, Q_{BRST}\right\}_{PB}\right\}_{PB} = \frac{1}{2} \left\{\{Q_{BRST}, Q_{BRST}\}_{PB} \,, \bar{\lambda}^i b_i\right\}_{PB} \,, \tag{6.51}$$

which is zero by (6.48).

6.4 BRST Quantization

Consider first general model with "quantum" action (6.33). The canonical commutation relations for the extended phase-space coordinates are

$$[\hat{x}^m, \hat{p}_n] = i\delta_n^m \,, \qquad \{\hat{b}_i, \hat{c}^j\} = \delta_i^j \,. \tag{6.52}$$

In addition,

$$\{\hat{b}_i, \hat{b}_j\} = 0 \qquad \left\{\hat{c}^i, \hat{c}^j\right\} = 0 \,. \tag{6.53}$$

The Hamiltonian now becomes the operator

$$\hat{H} = \bar{\lambda}^k \left(\hat{\varphi}_i + i\hat{c}^j f_{jk}{}^{i} \hat{b}_i\right) \,. \tag{6.54}$$

[20]There is still a BRST charge if the structure functions are *not* constants, but it is more complicated.

Notice that $\hat{H}$ commutes with the *ghost number* operator

$$n_{gh} = \hat{c}^i \hat{b}_i \,. \tag{6.55}$$

This operator also has the property that

$$\left[n_{gh}, \hat{c}^i\right] = c^i \qquad \left[n_{gh}, \hat{b}_i\right] = -\hat{b}_i \,, \tag{6.56}$$

so the c-ghosts have ghost number 1 and the b-ghosts have ghost number -1.

The BRST charge becomes the following operator of unit ghost number:

$$\hat{Q}_{BRST} = \hat{c}^i \hat{\varphi}_i + \frac{i}{2} \hat{c}^i \hat{c}^k f_{ki}{}^j \hat{b}_j \,. \tag{6.57}$$

Assuming that the PB to (anti)commutator rule applies, we learn from (6.48) that

$$\hat{Q}^2_{BRST} = 0 \,. \tag{6.58}$$

This is the fundamental property of the BRST charge. As a consequence of this property, it is consistent to impose the physical state condition

$$\hat{Q}_{BRST} |\text{phys}\rangle = 0 \,. \tag{6.59}$$

This condition has the following motivation. The Hamiltonian operator $\hat{H}$ is gauge-dependent[21] so its matrix elements cannot be physical. However, the quantum version of (6.50) is

$$\hat{H} = \left[\bar{\lambda}^i \hat{b}_i, \hat{Q}_{BRST}\right] \,, \tag{6.60}$$

so it follows from (6.59) that, for any two physical states $|\text{phys}\rangle$ and $|\text{phys}'\rangle$,

$$\langle \text{phys}' | \hat{H} | \text{phys} \rangle = 0 \,. \tag{6.61}$$

In other words, the BRST physical state condition ensure that all physical matrix elements are gauge-independent.

The "physical state" condition (6.59) does not actually remove all unphysical states because for any state $|\chi\rangle$ the state $\hat{Q}_{BRST}|\chi\rangle$ will be "physical", by the definition (6.59), as a consequence of the nilpotency of Q_{BRST}, but it will also be null if we assume an inner product for which $\hat{Q}_{BRST}$ is hermitian:

$$\begin{aligned} \left|\left| \hat{Q}_{BRST} |\chi\rangle \right|\right|^2 &= \langle \chi | \hat{Q}^\dagger_{BRST} \hat{Q}_{BRST} | \chi \rangle \\ &= \langle \chi | \hat{Q}^2_{BRST} | \chi \rangle \qquad \left(\text{if } \hat{Q}^\dagger_{BRST} = \hat{Q}_{BRST}\right) \\ &= 0 \end{aligned} \tag{6.62}$$

[21]Although the constants $\bar{\lambda}^i$ may be gauge invariant if the gauge condition $\lambda^i = \bar{\lambda}^i$ is assumed to hold for all t, these constants could still be changed locally. The main point is that different gauge-fixing conditions lead to a different $\hat{H}$.

So we should really define physical states as equivalence classes (cohomology classes of $\hat{Q}_{BRST}$), where the equivalence relation is

$$|\Psi\rangle \sim |\Psi\rangle + \hat{Q}_{BRST}|\chi\rangle \tag{6.63}$$

for any state $|\chi\rangle$. This is consistent because $\hat{Q}_{BRST}|\chi\rangle$ is orthogonal to *all* states that are physical by the definition (6.59).

Let's now see how these ideas apply to the point particle. In a basis for which $(\hat{x}, \hat{c})$ are diagonal, with eigenvalues (x, c), the the canonical (anti)commutation relations are realised by the operators

$$\hat{p}_m = -i\partial_m \,, \qquad \hat{b} = \frac{\partial}{\partial c} \,, \tag{6.64}$$

acting on wavefunctions $\Psi(x, c)$, which we can expand as

$$\Psi(x,c) = \psi_0(x) + c\psi_1(x) \,. \tag{6.65}$$

The BRST charge is now

$$\hat{Q}_{BRST} = -\frac{1}{2}c\left(\Box - m^2\right) \,, \tag{6.66}$$

so the physical state condition is $c(\Box - m^2)\psi_0(x) = 0$, which implies that ψ_0 is a solution to the Klein-Gordon equation. We learn nothing about $\psi_1(x)$, but the equivalence relation (6.63) tells us that

$$\psi_1 \sim \psi_1 + \left(\Box - m^2\right)\chi_0 \,, \tag{6.67}$$

for any function $\chi_0(x)$, which implies that ψ_1 is equivalent to zero unless it too is a solution of the KG equation[22]. So we actually get a doubling of the expected physical states (solutions of the KG equation). For this reason, we have to impose the additional condition

$$b\,|\text{phys}\rangle = 0 \qquad \Rightarrow \qquad \psi_1 = 0 \,. \tag{6.68}$$

All this depends on a choice of inner product for which $\hat{Q}_{BRST}$ is hermitian, despite being nilpotent. This is achieved in the point particle case by the choice

$$\langle\Psi|\Psi'\rangle = \int d^Dx \int dc\, \Psi^*\Psi' = \int d^Dx\, \frac{\partial}{\partial c}\left[\Psi^*\Psi'\right] \,. \tag{6.69}$$

With respect to this inner product, the operators $\hat{b}$ and $\hat{c}$ are hermitian, and hence $\hat{Q}_{BRST}$ is hermitian. Using this inner product we can construct a *field theory* action

[22]Expand both sides in terms of eigenfunctions of the KG operator, and compare coefficients; all coefficients in the expansion of $(\Box - m^2)\chi_0$ are arbitrary except the coefficients of zero modes of the KG operator, which are zero.

from which the BRST physical state condition emerges as a field squation. This action is

$$
\begin{aligned}
S[\Psi(x,c)] &= \langle\Psi|\hat{Q}_{BRST}|\Psi\rangle \\
&= \frac{1}{2}\int d^Dx \int \frac{\partial}{\partial c}\left[\psi_0 c\left(\Box - m^2\right)\psi_0\right] \\
&= \frac{1}{2}\int d^Dx\, \psi_0\left(\Box - m^2\right)\psi_0\,,
\end{aligned}
\tag{6.70}
$$

which is the Klein-Gordon action.

7. BRST for the NG string

We now aim to use the conformal gauge in a path-integral approach to quantisation of the closed NG string. Recall that the conformal gauge for the NG string is $\lambda^\pm = 1$, and that in this gauge the gauge variation of $\lambda^\pm$ is

$$
\delta\lambda^\pm = \mp\sqrt{2}\,\partial_\mp\xi^\pm\,. \tag{7.1}
$$

The FP contribution to the action is therefore

$$
I_{FP} = \sqrt{2}\int dt \oint d\sigma\left\{ib\partial_+ c + \tilde{b}\partial_-\tilde{c}\right\}\,. \tag{7.2}
$$

Notice that it is precisely because the conformal gauge fails to completely fix the gauge invariance that we get a non-trivial FP action. Whenever the gauge is fixed completely, the FP action is one for which the FP ghosts can be trivially eliminated; that's not the case here because we can't invert $\partial_\pm$, and that is also the reason that there is a residual conformal symmetry. From this fact, one can see that the FP ghosts are, in a sense made precise by the BRST formalism, subtracting out the residual unphysical degrees of freedom that survive the gauge-fixing condition.

Adding the FP action we get the "quantum" action for the closed NG string in conformal gauge

$$
\begin{aligned}
I_{qu} &= \int dt \oint d\sigma\left\{\dot{X}P_m + ib\dot{c} + i\tilde{b}\dot{\tilde{c}} - \mathcal{H}_{qu}\right\}\,, \\
\mathcal{H}_{qu} &= \frac{P^2}{2T} + T(X')^2 - i\left(bc' - \tilde{b}\tilde{c}'\right)\,.
\end{aligned}
\tag{7.3}
$$

From this action we can read off the PB relations; in particular

$$
\{b(\sigma), c(\sigma')\}_{PB} = -i\delta(\sigma-\sigma') = \{\tilde{b}(\sigma), \tilde{c}(\sigma')\}_{PB}\,. \tag{7.4}
$$

The BRST charge can be written as

$$
Q_{BRST} = Q_- + Q_+\,, \tag{7.5}
$$

and it follows from the general formula, using the fact that the algebra of constraint functions is $\mathrm{Diff}_1 \oplus \mathrm{Diff}_1$, that

$$Q_- = \oint d\sigma \left\{ c\mathcal{H}_- + icc'b \right\} , \qquad Q_+ = \oint d\sigma \left\{ \tilde{c}\mathcal{H}_+ - i\tilde{c}\tilde{c}'\tilde{b} \right\} . \tag{7.6}$$

Using the PB relations obeyed by $\mathcal{H}_\pm$, given in (3.24), it is not difficult to verify that

$$\left.\begin{array}{l} \{Q_\pm, Q_\pm\}_{PB} = 0 \\ \{Q_+, Q_-\}_{PB} = 0 \end{array}\right\} \quad \Rightarrow \quad \{Q_{BRST}, Q_{BRST}\}_{PB} = 0 \,. \tag{7.7}$$

We now pass to the Fourier mode formulation. In addition to the Fourier series for $P \pm TX'$, we will need the Fourier series expansions

$$\begin{aligned} c &= \sum_{k\in\mathbb{Z}} e^{ik\sigma} c_k \,, & b &= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} e^{ik\sigma} b_k \,, \\ \tilde{c} &= \sum_{k\in\mathbb{Z}} e^{-ik\sigma} \tilde{c}_k \,, & \tilde{b} &= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} e^{-ik\sigma} \tilde{b}_k \,. \end{aligned} \tag{7.8}$$

The "quantum" action in terms of Fourier modes is

$$\begin{aligned} I_{qu} &= \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \left(\dot{\alpha}_k \cdot \alpha_{-k} + \dot{\tilde{\alpha}}_k \cdot \tilde{\alpha}_{-k} \right) + \sum_{n\in\mathbb{Z}} i \left(b_{-n}\dot{c}_n + \tilde{b}_{-n}\dot{\tilde{c}}_n \right) - H_{qu} \right\} , \\ H_{qu} &= \left[L_0 + N_{(gh)} \right] + \left[\tilde{L}_0 + \tilde{N}_{(gh)} \right] , \end{aligned} \tag{7.9}$$

where the ghost level numbers are

$$N_{(gh)} = \sum_{k=1}^{\infty} k \left(b_{-k}c_k + c_{-k}b_k \right) , \qquad \tilde{N}_{(gh)} = \sum_{k=1}^{\infty} k (\tilde{b}_{-k}\tilde{c}_k + \tilde{c}_{-k}\tilde{b}_k) \,. \tag{7.10}$$

It will be convenient to define

$$\left.\begin{array}{l} \mathcal{L}_0 = L_0 + N_{(gh)} = \frac{1}{2}\alpha_0^2 + N + N_{(gh)} \\ \tilde{\mathcal{L}}_0 = \tilde{L}_0 + \tilde{N}_{(gh)} = \frac{1}{2}\alpha_0^2 + \tilde{N} + \tilde{N}_{(gh)} \end{array}\right\} \quad \Rightarrow \quad H_{qu} = \mathcal{L}_0 + \tilde{\mathcal{L}}_0 \,. \tag{7.11}$$

We can now read off the PBs of the Fourier modes. For the new, anticommuting, variables we have

$$\{c_n, b_{-n}\}_{PB} = -i \,, \qquad \left\{ \tilde{c}_n, \tilde{b}_{-n} \right\}_{PB} = -i \,, \qquad (n \in \mathbb{Z}). \tag{7.12}$$

Notice that $n = 0$ is included, although the (anti)ghost zero modes (b_0, c_0) and $(\tilde{b}_0, \tilde{c}_0)$ do not appear in the Hamiltonian. These anticommutation relations are equivalent to (7.4).

A Fourier decomposition of $Q_\pm$ yields the result

$$
\begin{aligned}
Q_- &= \sum_{n\in\mathbb{Z}} c_{-n} L_n - \frac{1}{2}\sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{Z}} (p-q)\, c_{-p} c_{-q} b_{p+q}\,, \\
Q_+ &= \sum_{n\in\mathbb{Z}} \tilde{c}_{-n} \tilde{L}_n - \frac{1}{2}\sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{Z}} (p-q)\, \tilde{c}_{-p} \tilde{c}_{-q} \tilde{b}_{p+q}\,.
\end{aligned}
\tag{7.13}
$$

Let us rewrite these expressions as

$$
Q_- = \sum_{m\in\mathbb{Z}} \left[L_m + \frac{1}{2} L_m^{(gh)} \right] c_{-m}\,, \qquad Q_+ = \sum_{m\in\mathbb{Z}} \left[\tilde{L}_m + \frac{1}{2} \tilde{L}_m^{(gh)} \right] c_{-m}\,, \tag{7.14}
$$

where

$$
L_m^{(gh)} = \sum_n (m-n)\, b_{m+n} c_{-n}\,, \qquad \tilde{L}_m^{(gh)} = \sum_n (m-n)\, \tilde{b}_{m+n} \tilde{c}_{-n}\,. \tag{7.15}
$$

Notice that

$$
L_0^{(gh)} \equiv N_{(gh)}\,, \qquad \tilde{L}_0^{(gh)} \equiv \tilde{N}_{(gh)}\,. \tag{7.16}
$$

A check on the expressions for $Q_\pm$ is to verify that

$$
\left.\begin{aligned} \mathcal{L}_0 &= i\,\{b_0, Q_-\}_{PB} \\ \tilde{\mathcal{L}}_0 &= i\{\tilde{b}_0, Q_+\}_{PB} \end{aligned}\right\} \quad \Rightarrow \quad H_{qu} = i \left\{ \left(b_0 + \tilde{b}_0 \right), Q_{BRST} \right\}_{PB}\,, \tag{7.17}
$$

which is the formula (6.50) for our case. In fact, if we define

$$
\mathcal{L}_m = L_m + L_m^{(gh)}\,, \qquad \tilde{\mathcal{L}}_m = \tilde{L}_m + \tilde{L}_m^{(gh)}\,, \tag{7.18}
$$

then we have the generalisation

$$
\mathcal{L}_m = i\,\{b_m, Q_-\}_{PB}\,, \qquad \tilde{\mathcal{L}}_m = i\{\tilde{b}_m, Q_+\}_{PB}\,. \tag{7.19}
$$

A calculation using the PBs (7.12) shows that

$$
\left\{ L_m^{(gh)}, L_n^{(gh)} \right\}_{PB} = -i\,(m-n)\, L_{m+n}^{(gh)}\,, \tag{7.20}
$$

and similarly for $\tilde{L}_m^{(gh)}$, from which it follows that

$$
\begin{aligned}
\{\mathcal{L}_m, \mathcal{L}_n\}_{PB} &= -i\,(m-n)\,\mathcal{L}_{m+n} \\
\{\mathcal{L}_n, \tilde{\mathcal{L}}_m\}_{PB} &= 0 \\
\{\tilde{\mathcal{L}}_m, \tilde{\mathcal{L}}_n\}_{PB} &= -i\,(m-n)\,\tilde{\mathcal{L}}_{m+n}\,.
\end{aligned}
\tag{7.21}
$$

This was to be expected; the conformal gauge preserves a residual conformal invariance, which is preserved by the FP ghost action. The FP ghosts therefore contribute to the associated Noether charges, which obey the same conformal algebra as before.

In fact, the $\mathcal{L}_n$ are the Fourier modes of the non-zero components of the energy-momentum stress tensor of the quantum action (7.3). The PBs of the $\mathcal{L}_m$ with the Fourier modes of the various fields determine the transformations of these fields under the residual conformal invariance of the conformal gauge. For example

$$\begin{aligned} \{\mathcal{L}_m, \alpha_n\}_{PB} &= in\alpha_{n+m} \,, \\ \{\mathcal{L}_m, c_n\}_{PB} &= i\,(2m+n)\,c_{n+m} \,, \\ \{\mathcal{L}_m, b_n\}_{PB} &= -i\,(m-n)\,b_{n+m} \end{aligned} \tag{7.22}$$

In general, for any phase-space function $\mathcal{O}^{(J)}$ of conformal dimension J,

$$\left\{\mathcal{L}_m, \mathcal{O}_n^{(J)}\right\}_{PB} = -i\left[m\,(J-1) - n\right]\mathcal{O}_{n+m}^{(J)} \,, \tag{7.23}$$

so we can see that $P \pm TX'$ have conformal dimension 1 while b has conformal dimension 2 and c has conformal dimension -1. This is all classical, but it carries over to the quantum theory; in particular the conformal dimensions of $P \pm TX'$ and (b,c) are as above. However, whereas the product $\mathcal{O}^{(J)}\mathcal{O}^{(J')}$ of phase-space functions has conformal dimension $J + J'$, this will **not** generally be true in the quantum theory; products of operators can have *anomalous* conformal dimensions.

We should also consider the transformations generated by $\tilde{\mathcal{L}}_m$; these are entirely analogous, so the worldsheet fields and (b,c) actually have two equal conformal dimensions: $P \pm TX'$ has conformal dimensions $(1,1)$, the ghost c has conformal dimensions $(-1,-1)$ and the antighost b has conformal dimensions $(2,2)$. For any closed string theory preserving worldsheet parity, the left/right conformal dimensions will be equal.

7.1 Quantum BRST

Passing to the quantum theory, the non-zero anticommutators of the (anti)ghost modes are

$$\{c_n, b_{-n}\} = 1\,, \qquad \{\tilde{c}_n, \tilde{b}_{-n}\} = 1\,. \tag{7.24}$$

We define the (anti)ghost oscillator vacuum as the state

$$|0\rangle_{gh} = |0\rangle_R^{gh} \otimes |0\rangle_L^{gh}\,, \tag{7.25}$$

such that

$$c_n|0\rangle_R^{gh} = 0\,, \quad n > 0 \qquad \& \qquad b_n|0\rangle_R^{gh} = 0\,, \quad n \geq 0\,. \tag{7.26}$$

and similarly for the tilde operators acting on $|0\rangle_L^{gh}$. We can act on this with the (anti)ghost creation operators c_0 and (c_{-n}, b_{-n}) for $n > 0$ to get states in an (anti)ghost Fock space. The full oscillator vacuum is now the tensor product state

$$|0) = |0\rangle \otimes |0\rangle_{gh}\,. \tag{7.27}$$

We should first deal with some operator ordering ambiguities in expressions involving (anti)ghost operators. There is no ambiguity in the expression for $L_m^{(gh)}$ as long as $m \neq 0$; for $m = 0$ we choose the ordering given in (7.10), which ensures that

$$L_0^{(gh)} \equiv N_{(gh)} = \sum_{k=1}^{\infty} k \left(b_{-k} c_k + c_{-k} b_k\right) \quad \Rightarrow \quad N_{(gh)}|0\rangle = 0 \,. \tag{7.28}$$

We then have

$$L_m^{(gh)}|0\rangle_{gh} = 0 \,, \quad m \geq 0 \,. \tag{7.29}$$

Notice that

$$\left[N_{(gh)}, c_{-k}\right] = k c_{-k} \,, \qquad \left[N_{(gh)}, b_{-k}\right] = k b_{-k} \,, \tag{7.30}$$

so that acting with either the ghost creation operator c_{-k} or the anti-ghost creation operator b_{-k} increases the ghost level number by k. It follows that the eigenvalues of $N_{(gh)}$ and $\tilde{N}_{(gh)}$ are the non-negative integers.

There is also an ordering ambiguity in Q_- that allows us to add to it any multiple of $\hat{c}_0$; we choose the order such that

$$Q_- = \left(L_0 + \frac{1}{2} N_{(gh)} - a\right) \hat{c}_0 + \sum_{m=1}^{\infty} \left[\left(L_{-m} + \frac{1}{2} L_{-m}^{(gh)}\right) c_m + c_{-m} \left(L_m + \frac{1}{2} L_m^{(gh)}\right)\right] , \tag{7.31}$$

for some constant a. This definition is such that

$$Q_-|0\rangle = 0 \quad \Rightarrow \quad \left(L_0 - a\right)|0\rangle = 0 \,. \tag{7.32}$$

It is also such that [Exercise: check this]

$$\{b_m, Q_-\} = \mathcal{L}_m - a\delta_m \,, \tag{7.33}$$

which is the quantum version of (7.19). Let's also record here that

$$[\mathcal{L}_m, b_n] = (m-n)\, b_{n+m} \,, \tag{7.34}$$

which is the statement that the operator b has conformal dimension 2.

Now we show how the Virasoro anomaly in the algebra of the $\mathcal{L}_n$ is related to a BRST anomaly. Using (7.33) we find that

$$\begin{aligned} [\mathcal{L}_m, \mathcal{L}_n] &= [\{b_m, Q_-\}, \mathcal{L}_n] \\ &= -\{[\mathcal{L}_n, b_m], Q_-\} + \{[Q_-, \mathcal{L}_n], b_m\} \\ &= (m-n)\{b_{m+n}, Q_-\} + \{[Q_-, \mathcal{L}_n], b_m\} \,, \end{aligned} \tag{7.35}$$

where the second line follows from the super-Jacobi identity, and the last line uses (7.34). Now we use (7.33) again, and again the super-Jacobi identity, to show that

$$[Q_-, \mathcal{L}_n] = [Q_-, \{b_n, Q_-\}] = -\left[Q_-^2, b_n\right] \,. \tag{7.36}$$

Using this in (7.35) we deduce that

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\left(\mathcal{L}_{m+n} - a\delta_{m+n}\right) + \left\{\left[Q_-^2, b_n\right], b_m\right\} . \tag{7.37}$$

This shows that $Q_-^2 = 0$ implies no Virasoro anomaly (i.e. zero central charge c). If Q_-^2 is non-zero it will be some expression quadratic in oscillator operators (the classical result ensures that the quartic term cancels) and it must have ghost number 2, so

$$Q_-^2 = \frac{1}{2}\sum_{k\in\mathbb{Z}} c_k c_{-k} A(k) \tag{7.38}$$

for some function $A(k)$. We then find that

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\left(\mathcal{L}_{m+n} - a\delta_{m+n}\right) + A(m)\delta_{m+n} \tag{7.39}$$

This shows that no Virasoro anomaly implies $Q_-^2 = 0$. The same argument applies to Q_+, so we now see that

$$Q_{BRST}^2 = 0 \quad \Leftrightarrow \quad c = 0 . \tag{7.40}$$

For there to be no BRST anomaly we require that the $\mathcal{L}_n - a\delta_n$ satisfy the Witt algebra for some constant a; i.e. we require that

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\left(\mathcal{L}_{m+n} - a\delta_{m+n}\right) . \tag{7.41}$$

7.1.1 Critical dimension again

We already know that the algebra satisfied by the L_m is the Virasoro algebra with central charge D, i.e.

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{D}{12}\left(m^3 - m\right)\delta_{m+n} . \tag{7.42}$$

Now we need to determine the algebra satisfied by the $L_m^{(gh)}$, which must take the form

$$\left[L_m^{(gh)}, L_n^{(gh)}\right] = (m-n)\left(L_{m+n}^{(gh)} - a\delta_{m+n}\right) + \frac{c_{gh}}{12}\left(m^3 - m\right)\delta_{m+n} . \tag{7.43}$$

for some constants a and c_{gh}. Setting $n = -m$ and taking the oscillator vacuum expectation value of both sides, we deduce (using the fact that $L_m^{(gh)}|0\rangle = 0$ for $m \geq 0$)

$$\left|\left|L_{-m}^{(gh)}|0\rangle\right|\right|^2 = -2ma + \frac{c_{gh}}{12}\left(m^3 - m\right) . \tag{7.44}$$

In particular

$$\left|\left|L_{-1}^{(gh)}|0\rangle\right|\right|^2 = -2a , \qquad \left|\left|L_{-2}^{(gh)}|0\rangle\right|\right|^2 = -4a + \frac{c_{gh}}{2} . \tag{7.45}$$

But we may also compute the LHSs directly using

$$\begin{aligned} L^{(gh)}_{-1}|0\rangle &= -\left(b_{-1}c_0 + 2b_0c_{-1}\right)|0\rangle \\ L^{(gh)}_{-2}|0\rangle &= -\left(b_{-2}c_0 + 3b_{-1}c_{-1} + 4b_0c_{-2}\right)|0\rangle \end{aligned} \tag{7.46}$$

For example[23]

$$\begin{aligned} \left|\left|L^{(gh)}_{-1}|0\rangle\right|\right|^2 &= \langle 0|\left(c_0b_1 + 2c_1b_0\right)\left(b_{-1}c_0 + 2b_0c_{-1}\right)|0\rangle \\ &= -2\langle 0|\left(c_0b_0b_1c_{-1} + b_0c_0c_1b_{-1}\right)|0\rangle \quad (\text{using } b_0^2 = c_0^2 = 0) \\ &= -2\langle 0|\left(c_0b_0\left\{b_1, c_{-1}\right\} + \left\{c_1, b_{-1}\right\}b_0c_0\right)|0\rangle \quad (\text{using } b_1|0\rangle = c_1|0\rangle = 0) \\ &= -2\langle 0|\left\{c_0, b_0\right\}|0\rangle = -2\,, \end{aligned} \tag{7.47}$$

from which we conclude that $a = 1$. Similarly,

$$\begin{aligned} \left|\left|L^{(gh)}_{-2}|0\rangle\right|\right|^2 &= \langle 0|\left(2c_0b_2 + 3c_1b_1 + 4c_2b_0\right)\left(2b_{-2}c_0 + 3b_{-1}c_{-1} + 4b_0c_{-2}\right)|0\rangle \\ &= -8\langle 0|\left(c_0b_0b_2c_{-2} + b_0c_0c_2b_{-2}\right)|0\rangle - 9\langle 0|c_1b_{-1}b_1c_{-1}|0\rangle \\ &= -8\langle 0|\left(c_0b_0\left\{b_2, c_{-2}\right\} + b_0c_0\left\{c_2, b_{-2}\right\}\right)|0\rangle - 9\langle 0|\left\{c_1, b_{-1}\right\}\left\{b_1, c_{-1}\right\}|0\rangle \\ &= -8\langle 0|(\left\{c_0, b_0\right\}|0\rangle - 9 = -17\,, \end{aligned} \tag{7.48}$$

from which we conclude that

$$-4 + \frac{c_{gh}}{2} = -17 \quad \Rightarrow \quad c_{gh} = -26\,. \tag{7.49}$$

Using these values for a and c_{gh} in (7.43) we find that

$$\left[L^{(gh)}_m, L^{(gh)}_n\right] = (m-n)\left(L^{(gh)}_{m+n} - \delta_{m+n}\right) - \frac{26}{12}\left(m^3 - m\right)\delta_{m+n}\,, \tag{7.50}$$

and combining this with (7.42), we deduce that

$$\left[\mathcal{L}_m, \mathcal{L}_n\right] = (m-n)\left(\mathcal{L}_{m+n} - \delta_{m+n}\right) + \frac{D-26}{12}\left(m^3 - m\right)\,. \tag{7.51}$$

This is a Virasoro algebra with central charge $D - 26$, which is zero (as required for $Q_-^2 = 0$) iff $D = 26$.

In conclusion, we find agreement with the light-cone gauge result that $a = 1$ and $D = 26$. In this case it is consistent to impose the BRST physical-state condition $Q_{BRST}|\text{phys}\rangle = 0$. The physical states are then cohomology classes of Q_{BRST}. Consider the state

$$|\psi\rangle_R = |\Psi\rangle_R \times |0\rangle_R^{gh} \tag{7.52}$$

[23] Subtleties of the inner product for the (anti)ghost zero modes can be passed over here because all that is needed is their anti-commutation relation.

where $|\psi\rangle_R$ is a state in the α-oscillator Fock space. Then

$$Q_-|\psi)_R = (L_0 - 1)\,|\psi\rangle_R \otimes |0\rangle_R^{gh} + \sum_{m=1}^{\infty} L_m|\psi\rangle_R \otimes c_{-m}|0\rangle_R^{gh}\,, \tag{7.53}$$

which is zero only if ψ satisfies

$$(L_0 - 1)\,|\psi\rangle_R = 0 \quad \& \quad L_m|\psi\rangle_R = 0 \;\forall m > 0\,. \tag{7.54}$$

These are the Virasoro conditions of the "old covariant" method of quantization.

There is an entirely analogous result for the states built on the $|0)_L$ vacuum, and it can be shown that the physical states of the light-cone gauge are in one-to-one correspondence with the cohomology classes of Q_{BRST} at ghost number $-1/2$. This is known as the "no-ghost" theorem.

8. Interactions

So far we have seen that each excited state of a string can be viewed as a particle with a particular mass and spin. Now we are going to see that the stringy origin of these particles leads naturally to interactions between them. We shall explore this in the context of a path integral quantization of a closed string, where we sum over world sheets weighted by the NG string action. We shall assume that the conformal gauge has been chosen, and that we have done the Gaussian integral over P, which effectively sets $P = T\dot{X}$, so the conformal gauge action is

$$I_{c.g.} = -T \int dt \oint d\sigma\; \partial_+ X \cdot \partial_- X\,, \qquad \partial_\pm = \frac{\partial}{\partial \sigma^\pm}\,. \tag{8.1}$$

We now "Wick rotate": set $t = -i\tau$ and then take τ to be real rather than imaginary:

$$\sigma^- \to \frac{1}{\sqrt{2}}\,(\sigma + i\tau) = z\,, \qquad \sigma^+ \to \bar{z}\,. \tag{8.2}$$

Then

$$-iI_{c.g} \to I_E = T \int d^2z\; \partial X \cdot \bar{\partial} X\,, \qquad \partial = \frac{\partial}{\partial z}\,, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}}\,. \tag{8.3}$$

The functional $I_E[X]$ is the Euclidean NG action in conformal gauge; the integral is over the cylindrical worldsheet. The Euclidean FP ghost action is

$$I_E^{(FP)} = \int d^2z \left\{ ib\bar{\partial}c + i\tilde{b}\partial\tilde{c} \right\}\,. \tag{8.4}$$

For path integral purposes, the conformal gauge action is the "quantum" Euclidean action

$$I_E^{(qu)} = I_E[X] + I_E^{(FP)}[b, c; \tilde{b}, \tilde{c}]\,. \tag{8.5}$$

Now suppose that the cylindrical worldsheet is infinite in both directions, corresponding to a string in the far past propagating into the far future. It can be formally mapped to the complex plane with the complex coordinate $w = e^{-iz}$. Notice that

$$\sigma \to \sigma + 2\pi \quad \Leftrightarrow \quad w \to e^{2\pi i} w\,, \tag{8.6}$$

so the constant τ slices of the cylinder are mapped to circles centred on the origin in the complex w-plane. Also

$$\tau \to -\infty \quad \Leftrightarrow \quad w \to 0\,, \qquad \tau \to \infty \quad \Leftrightarrow \quad w \to \infty \tag{8.7}$$

Here we view the complex w-plane as the Riemann sphere, with two punctures: one at the south pole ($w = 0$) associated to the incoming string, and one at the north pole ($w = \infty$) associated to the outgoing string. Conformal invariance is essential to the consistency of this idea, as is the cancellation of the conformal anomaly; for example

$$L_0|_{cylinder} = L_0|_{R.sphere} + \frac{c}{24}\,. \tag{8.8}$$

The central charge represents a Casimir energy due to the periodic identification of the string coordinate σ on the cylinder. As we have seen $c = 0$ when $D = 26$ due to the FP ghost contribution to it.

8.1 Ghost zero modes and $Sl(2;\mathbb{C})$

When we go from the cylinder to the R. sphere we must take account of how the various fields behave under conformal transformations. Integrals such as $I_E^{(FP)}$ will take the the same form in terms of the transformed fields, now functions of w, only if the integrand has conformal dimensions $(1,1)$ (i.e. 1 with respect to both "left" and "right" factors of the conformal group). This is the case for $I_E^{(FP)}$; for example, $b\bar\partial c$ has conformal dimension $(1,1)$ because b has conformal dimension $(2,0)$ and c has conformal dimension $(-1,0)$, while $\bar\partial$ has conformal dimension $(0,1)$.

We shall now suppose that all fields are defined on the R. sphere, and we relabel the coordinate $w \to z$, so the Euclidean FP action is again given precisely by (8.4). Path integrals representing amplitudes should include a functional integral over the FP ghosts. For the amplitudes to be discussed here, this is just a product of gaussian integrals that only contribute to an overall normalisation, which we are ignoring, except that it would not be a gaussian for modes of b or c on the R. sphere for which

$$either \quad \bar\partial c = 0 \qquad or \quad \bar\partial b = 0\,. \tag{8.9}$$

In either case the FP action would be zero so Berezin integration over these modes would give us a zero amplitude. We should make the replacement

$$\int [dbdc] \quad \to \quad \int [dbdc]' \tag{8.10}$$

where the prime indicates an absence of integration over (anti)ghost "zero modes".

What are these (anti)ghost zero modes. From (8.9) we see that they are analytic, but they are not analytic functions because both b and c have a non-zero conformal dimension. The fact that b has conformal dimension 2 means that it is a quadratic differential. By a variant of the argument to follow for c, it can be shown that there are no analytic quadratic differentials on the R. sphere. The fact that c has conformal dimension -1 means that it is a vector (a one-form, or co-vector, has conformal dimension 1). This is as expected because c must be of the same tensorial type as the gauge parameter ξ^-, which is the one component of the vector $\xi^-\partial_-$ which becomes $\xi(z)\partial$ in Euclidean space. To see how this behaves at infinity, set $z = -1/\zeta$ to get

$$\xi(z)\partial_z = \xi\left(z(\zeta)\right)\zeta^2\partial_\zeta \qquad (\zeta = -1/z). \tag{8.11}$$

This is finite at $\zeta = 0$ as long as $\xi\zeta^2$ is finite as $\zeta \to 0$, which is equivalent to the requirement that $\xi(z)$ grow no faster that z^2 as $z \to \infty$. So the analytic vector fields that are well defined on the R. sphere have the form

$$\xi(z) = \alpha_1 + \alpha_2 z + \alpha_3 z^2\,, \qquad \alpha_k \in \mathbb{C} \quad k = 1, 2, 3. \tag{8.12}$$

These vector fields are the conformal Killing vector fields of the R. sphere. They span the Lie algebra of $Sl(2;\mathbb{C})$. Recall that the infinite-dimensional 2D conformal algebra $\mathrm{Diff}_1 \oplus \mathrm{Diff}_1$ has $sl(2;\mathbb{R}) \oplus sl(2;\mathbb{R})$ as a finite-dimensional sub algebra. This 6-dimensional algebra becomes the 3-complex dimensional algebra $sl(2;\mathbb{C})$ in Euclidean signature.

The defining representation of the group $Sl(2;\mathbb{C})$ is in terms of the following matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{C}\,, \qquad\qquad ad - bc = 1\,. \tag{8.13}$$

The group $Sl(2;C)$ acts on the coordinate z of the R. sphere via the fractional linear transformation

$$z \to z' = \frac{az + b}{cz + d}\,. \tag{8.14}$$

Near the identity we can write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \alpha_1 & 2\alpha_2 \\ 2\alpha_3 & -\alpha_1 \end{pmatrix} + \dots \tag{8.15}$$

where $(\alpha_1, \alpha_2, \alpha_3)$ are three complex parameters and the omitted terms are at least quadratic in these parameters; to first order in them, one finds that

$$z' = z + \xi(z)\,, \qquad \xi(z) = \alpha_1 + \alpha_2 z + \alpha_3 z^2\,. \tag{8.16}$$

8.2 Virasoro-Shapiro amplitude from the path integral

If an incoming or outgoing string can be mapped to the poles of the R. sphere, what is to stop us considering more strings, incoming or outgoing, mapped to other points? For example, a four-punctured sphere could represent the scattering of two strings into two other strings. Actually, each string can be in any of its excited states, so we should have an option at each puncture of an insertion into the path-integral corresponding to each excited state of the string. We shall explore this idea by answering the following question. What is the amplitude $A(X_1,\ldots,X_N)$ for the string to pass through N points in Minkowski space-time with coordinates $(X_1,\ldots,X_N)$? We could represent this formally as the path integral

$$A(X_1,\ldots,X_N) = \int [dX]\; e^{-I_E} \prod_{i=1}^{N} \int dz_i^2\, \delta^D\left(X(z_i) - X_i\right) , \qquad (8.17)$$

where the functional integral is over maps $X(z,\bar{z})$ from the Riemann sphere to Minkowski space-time (we are supposing here that we have done the gaussian integrals over P and over the (anti)ghosts), and the $\int dz_i$ integrals are over the R. sphere. In the path integral computation of the propagator for the point particle, we had to sum over the gauge-invariant proper time of the particle worldline (proportional to the "modular" parameter s). If there were some analogous parameters of the R. sphere then we would have to integrate over them, and we would have to consider how the P and the (anti)ghosts path integrals dependence on them, but the R. sphere has no modular parameters[24]; as Riemann proved, the conformally flat metric on the R. sphere is unique.

It is simpler to consider the Fourier transform[25]

$$\begin{aligned} A(p_1,\ldots,p_N) &= \prod_{i=1}^{N} \int d^D X_i\; e^{ip_i\cdot X_i} A(X_1,\ldots,X_N) \\ &= \int [dX]\, e^{-I_E} \prod_{i=1}^{N} \int d^2 z_i\; e^{ip_i\cdot X_i} \\ &= \int [dX] \prod_{i=1}^{N} \int d^2 z_i \exp\left(-I_E + i\sum_{j=1}^{N} p_j\cdot X_j\right) \end{aligned} \qquad (8.18)$$

Now we observe that

$$-I_E + i\sum_{j=1}^{N} p_j\cdot X_j = -T\int d^2 z \left\{ \partial X\cdot\bar{\partial} X - \frac{i}{T}\left[\sum_{j=1}^{N} \delta^2(z-z_j) p_j\right]\cdot X \right\}. \qquad (8.19)$$

[24] This is actually equivalent to the earlier claim that there are no solutions of $\bar{\partial} b = 0$ on the R. sphere since any analytic quadratic differential on a Riemann surface corresponds to the possibility of deforming a conformally flat metric metric to a conformally inequivalent one.

[25] Although we should set $D = 26$ to avoid the Virasoro anomaly, this really becomes an unavoidable issue only when we consider string loops, so we keep the space-time dimension D arbitrary here.

Integrating by parts to write $\partial X \cdot \bar{\partial} X = -X \cdot \nabla^2 X + \partial()$, where $\nabla^2 = \partial\bar{\partial}$ is the Laplacian on the R. sphere, we have[26]

$$-I_E + i\sum_{j=1}^N p_j \cdot X_j = T\int d^2z \left\{ X \cdot \left[\nabla^2 X + \frac{i}{T}\sum_{j=1}^N \delta^2(z-z_j)p_j \right] \right\} . \tag{8.20}$$

The idea now is to complete the square in $X(z)$ but to do this we need to invert ∇^2 and there is a problem with this because ∇^2 has a zero eigenvalue on the sphere. The eigenfunction is the constant function, i.e. $X(z) = X_0$, so we should write

$$\int [dX] = \int d^D X_0 \int [dX]' , \tag{8.21}$$

where $[dX]'$ is an integral over all functions *except* the constant function. Isolating the X_0-dependence we now have

$$\begin{aligned} A(p_1, \ldots, p_N) &= \left[\int d^D X_0 \, e^{i(\sum_j p_j)\cdot X_0} \right] \hat{A}(p_1, \ldots, p_N) \\ &\propto \; \delta\left(\sum_{j=1}^N p_j \right) \hat{A}(p_1, \ldots, p_N) , \end{aligned} \tag{8.22}$$

where the path integral for $\hat{A}$ excludes the integration over the constant function. The delta-function prefactor imposes conservation of the total D-momentum.

We can now invert ∇^2; the inverse is the Green function $G(z, z_i)$ on the R. sphere:

$$\nabla^2 G(z, z_i) = \delta^2(z - z_i) \quad \Rightarrow \quad G(z, z_i) = \frac{1}{2\pi}\ln|z - z_i|^2 . \tag{8.23}$$

Setting

$$X(z) = Y(z) - \frac{i}{2T}\sum_{i=1}^N G(z, z_i)p_i , \tag{8.24}$$

we have[27] $[dX]' = [dY]'$, and [Exercise]

$$-I_E + \sum_{j=1}^N p_j \cdot X_j = T\int d^2z \; Y \cdot \nabla^2 Y \; + \; \frac{1}{8\pi T}\sum_i \sum_j p_i \cdot p_j \ln|z_i - z_j|^2 . \tag{8.25}$$

The terms in the double sum are infinite when $i = j$, so we omit these terms (this amounts to a renormalisation to remove "self-energies"). We can now do the gaussian $[dY]'$ path integral. This contributes only to the overall normalisation, and we are left with

$$\hat{A}(p_1, \ldots, p_N) \; \propto \; \prod_{i=1}^N \int d^2 z_i \prod_{j<k} |z_j - z_k|^{\alpha_{jk}} , \qquad \alpha_{ij} = \frac{p_i \cdot p_j}{2\pi T} . \tag{8.26}$$

[26]There is no surface term because the sphere has no boundary.

[27]A shift in the integration variable has no effect because we integrate over all values of the (non-constant) functions X.

As the derivation of this formula assumed conformal invariance, this result should be invariant under the $Sl(2;\mathbb{C})$ conformal isometry group of the R. sphere. Let's check this; using (8.14) one finds that

$$z_i' - z_j' = \frac{z_i - z_j}{(cz_i + d)(cz_j + d)}, \qquad d^2 z' = \frac{dz^2}{(cz+d)^4}, \tag{8.27}$$

and hence that

$$\prod_{i=1}^{N} d^2 z_i' \prod_{j<k} |z_j' - z_k'|^{\alpha_{jk}} = \left[\prod_{i=1}^{N} d^2 z_i \prod_{j<k} |z_j - z_k|^{\alpha_{jk}}\right] \left[\prod_{i=1}^{N} |cz_i + d|^{-4-\sum_j' \alpha_{ij}}\right], \tag{8.28}$$

where

$$\begin{aligned} \sum_j{}' \alpha_{ij} &= \sum_{j=1}^{N} \alpha_{ij} - \alpha_{ii} \qquad (i = 1, \ldots, N) \\ &= \frac{1}{2\pi T} p_i \cdot \left(\sum_{j=1}^{N} p_j\right) - \frac{p_i^2}{2\pi T} \\ &= -\frac{p_i^2}{2\pi T} \qquad \text{(by momentum conservation).} \end{aligned} \tag{8.29}$$

We see from this that the amplitude is $Sl(2;\mathbb{C})$ invariant only if

$$-4 + \frac{p_i^2}{2\pi T} = 0 \quad \Leftrightarrow \quad p_i^2 = 8\pi T\,. \tag{8.30}$$

This is the mass-shell condition for the tachyonic ground state of the string! We learn a few things from this computation

- We might have supposed that the N momenta $p_1, \ldots, p_N$ could be chosen arbitrarily, consistent with overall momentum conservation, but we have now seen that these must be the momenta of strings in their ground state, i.e. we have a tachyon scattering amplitude.

- The tachyon scattering amplitude that we have constructed is "on-shell", i.e. the external particles have momenta satisfying their mass-shell condition. This can be contrasted with the amplitude that we found earlier for propagation of a particle from one point to another in Minkowski space time; this was the Fourier transform of $1/(p^2 + m^2)$, which is infinite on the mass shell. This should lead us to expect that the string scattering amplitude of (8.26) makes sense only for $N > 2$.

Recall that the FP ghosts in the "quantum" action used in the path integral are the result of a gauge-fixing that is needed to eliminate unnecessary integration over redundant variables; it amounts to a trick that allows us to divide by the gauge group volume. If this volume were finite it would only affect the normalisation but it is usually infinite. Recall also, that we had to restrict the functional integral over the (anti)ghosts; we excluded integration over the ghost "zero modes", which are conformal Killing vectors of the R. sphere. As a result, the amplitude (8.26) is still infinite because of its $Sl(2;\mathbb{C})$ invariance. We can deal with this by gauge fixing.

We choose the gauge fixing conditions

$$f_i \equiv z_i - u_i = 0\,, \qquad i = 1, 2, 3. \tag{8.31}$$

That is, we chose three of the N points on the sphere to be at $z_i = u_i$ for arbitrarily chosen complex numbers u_i. This will involve an insertion of delta functions into the integrals defining the amplitude, but this must be accompanied by an FP determinant. Since the infinitesimal $Sl(2;\mathbb{C})$ variation of z is $\delta z = \alpha_1 + \alpha_2 z + \alpha_3 z^2$, we have

$$\left|\left|\frac{\partial f_i}{\partial \alpha_j}\right|\right| = \left|\left|\begin{matrix} 1 & z_1 & z_1^2 \\ 1 & z_2 & z_2^2 \\ 1 & z_3 & z_3^2 \end{matrix}\right|\right| = (z_1 - z_2)(z_2 - z_3)(z_3 - z_1)\,. \tag{8.32}$$

The FP determinant is the modulus squared of this, so

$$\Delta_{FP} = |z_1 - z_2|^2 |z_2 - z_3|^2 |z_3 - z_1|^2\,. \tag{8.33}$$

Following the earlier argument for gauge fixing the particle action, the insertion of the delta functions with the FP determinant allows us to factor out the (infinite) volume Ω of $Sl(2;\mathbb{C})$; dividing by this volume we then get

$$\Omega^{-1}\hat{A}(p_1, \ldots, p_N) \;\propto\; \prod_{i=1}^{N} \int d^2 z_i\, \delta^2(f_1)\delta^2(f_2)\delta^2(f_3)\Delta_{FP} \prod_{j<k} |z_j - z_k|^{\alpha_{jk}}\,. \tag{8.34}$$

This can be checked as follows. Multiply both sides by $|(u_1 - u_2)(u_2 - u_3)(u_3 - u_1)|^{-2}$ and integrate over (u_1, u_2, u_3). On the RHS the u integrals can be done using the delta functions, the Δ_{FP} is then cancelled and we recover the expression (8.26). On the LHS the integral cancels the factor of Ω^{-1} because, formally,

$$\Omega = \int \frac{d^2 u_1 d^2 u_2 d^2 u_3}{|(u_1 - u_2)(u_2 - u_3)^2 (u_3 - u_1)|^2}\,. \tag{8.35}$$

This integral is infinite but the integrand is the $Sl(2;\mathbb{C})$ invariant measure on the $Sl(2;\mathbb{C})$ group manifold, parametrised by three complex coordinates on which $Sl(2;\mathbb{C})$ acts by the fractional linear transformation (8.14).

We may now do the (z_1, z_2, z_3) integrals of (8.34) to get

$$\begin{aligned} \Omega^{-1}\hat{A}(p_1, \ldots, p_N) \;\propto\; & |u_1 - u_2|^{2+\alpha_{12}} |u_2 - u_3|^{2+\alpha_{23}} |u_3 - u_1|^{2+\alpha_{13}} \times \\ & \times \prod_{i=4}^{N} \int d^2 z_i \prod_{i=4}^{N} |u_1 - z_i|^{\alpha_{1i}} |u_2 - z_i|^{\alpha_{2i}} |u_3 - z_i|^{\alpha_{3i}} \prod_{4 \le j < k} |z_i - z_j|^{\alpha_{jk}}\,. \end{aligned} \tag{8.36}$$

This can be simplified enormously by the choice

$$u_3 = 1\,, \qquad u_2 = 0\,, \qquad u_1 \to \infty\,. \tag{8.37}$$

In this limit we get a factor of

$$|u_1|^{4-\alpha_{11}^2+\sum_i \alpha_{1i}} = 1\,, \tag{8.38}$$

where the equality follows upon using both the mass-shell condition and momentum conservation. The remaining terms give the Virasoro-Shapiro amplitude

$$\hat{A}_{VS}(p_1,\ldots,p_N) = \prod_{i=4}^{N} \int d^2 z_i \prod_{i=4}^{N} |z_i|^{\alpha_{2i}} |z_i - 1|^{\alpha_{3i}} \prod_{4\le j<k} |z_i - z_j|^{\alpha_{jk}}\,. \tag{8.39}$$

The result for $N = 3$ is a constant, which can be interpreted as a coupling constant. The $N = 4$ case was found by Virasoro, and generalised to arbitrary N by Shapiro.

8.2.1 The Virasoro amplitude and its properties

The $N = 4$ amplitude is

$$\hat{A}(p_1,p_2,p_3,p_4) = \int d^2 z\, |z|^{\alpha_{24}} |z-1|^{\alpha_{34}}\,. \tag{8.40}$$

If we view this as the amplitude for two identical particles of momenta (p_1, p_2) to scatter into two other identical particles of the same kind with momenta $(-p_3, -p_4)$ then we can define the associated (rescaled) Mandelstam invariants as

$$\begin{aligned} s &= -\frac{1}{8\pi T}\,(p_1+p_2)^2 = -2 - \frac{1}{2}\alpha_{12} \\ t &= = -\frac{1}{8\pi T}\,(p_1+p_3)^2 = -2 - \frac{1}{2}\alpha_{13} \\ u &= -\frac{1}{8\pi T}\,(p_1+p_4)^2 = -2 - \frac{1}{2}\alpha_{14}\,. \end{aligned} \tag{8.41}$$

Recall that, for the amplitude of interest here,

$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = 8\pi T\,, \tag{8.42}$$

and that momentum conservation requires[28]

$$p_1 + p_2 = -p_3 - p_4\,. \tag{8.43}$$

In the centre of mass frame we have

$$p_1 = (E,\vec{p})\,, \quad p_2 = (E,-\vec{p}) \quad \& \quad p_3 = (-E,\vec{p}\,')\,, \quad p_4 = (-E,-\vec{p}\,')\,, \tag{8.44}$$

[28]The Virasoro amplitude was derived assuming, for simplicity, that all particles were either incoming. In this convention, outgoing particles will have negative energy.

where

$$|\vec{p}|^2 = |\vec{p}\,'|^2 = E^2 + 8\pi T\,. \tag{8.45}$$

From this we see that

$$s = \frac{E^2}{2\pi T}\,, \quad t = -2\left(1 + \frac{s}{4}\right)(1 - \cos\theta_s)\,, \tag{8.46}$$

where θ_s is the scattering angle. Of course, it doesn't make much sense to talk about scattering of tachyons, but the amplitude provides an illustration of some properties that are typical of scattering amplitudes in string theory, and superstring theory (which has no tachyons).

From (8.41) and momentum conservation it follows that (Exercise)

$$\begin{aligned} \alpha_{34} &= \alpha_{12} = -4 - 2s\,, \\ \alpha_{24} &= \alpha_{13} = -4 - 2t\,, \\ \alpha_{23} &= \alpha_{14} = -4 - 2u\,, \end{aligned} \tag{8.47}$$

and

$$s + t + u = -4\,. \tag{8.48}$$

We will now use the following identity

$$\int d^2z\, |z|^{2\alpha}|z-1|^{2\beta} \equiv \pi\, \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(-\alpha-\beta-1)}{\Gamma(\alpha+\beta+2)\Gamma(-\alpha)\Gamma(-\beta)}\,. \tag{8.49}$$

We can use this to do the integrals of (8.40) by setting

$$\alpha = -2 - t\,, \qquad \beta = -2 - s\,. \tag{8.50}$$

This gives the Virasoro amplitude

$$A(s,t) \;\propto\; \frac{\Gamma(-1-t)\Gamma(-1-s)\Gamma(-1-u)}{\Gamma(u+2)\Gamma(s+2)\Gamma(t+2)} \qquad (u = -4 - s - t). \tag{8.51}$$

For fixed t the Virasoro amplitude A becomes a function of s, which has poles when[29]

$$s = -1, 0, 1, 2, \ldots \tag{8.52}$$

These poles correspond to resonances, i.e. to other particles in the spectrum (stable particles, in fact, because the poles are on the real axis in the complex s-plane). The position of the pole on the real axis gives the mass-squared of the particle in units of $-m^2 = 8\pi T$. The pole at $s = -1$ is therefore the tachyon. The pole at $s = 0$ implies

[29]The poles come from the $\Gamma(-1-s)$ factor. Recall that $\Gamma(z)$ has no zeros in the complex z-plane but has simple poles on the real axis at non-negative integers.

the existence of massless particles in the spectrum. The residue of the pole at $s = 0$ is

$$-\frac{\Gamma(-1-t)\Gamma(3+t)}{\Gamma(-2+t)\Gamma(t+2)} = t^2 - 4\,. \tag{8.53}$$

This is a quadratic function of t and hence of $\cos\theta_s$, which implies that there must be a massless particle of spin 2 (but none of higher spin). The residue of the pole at $s = n$ is a polynomial in t of order $2(n+1)$, so that $2(n+1)$ is the maximum spin of particles in the spectrum with mass-squared $n \times (8\pi T)$. In a plot of J_{max} against s, such particles appear at integer values of J_{max} on a straight line with slope $\alpha'/2$ and intercept 2 (value of J_{max} at $s = 0$). This is the leading Regge trajectory. All other particles in the spectrum appear on parallel "daughter" trajectories in the $J - s$ plane (e.g. the massless spin-zero particle in the spectrum is the first one on the trajectory with zero intercept.

Another feature of the Virasoro amplitude is its $s \leftrightarrow t$ symmetry. Poles in A as a function of s at fixed t therefore reappear as poles in A as a function of t at fixed s. These correspond to the exchange of a particle. In particular, a massless spin-2 particle is exchanged, and general arguments imply that such a particle must be the quantum associated to the gravitational force, so a theory of interacting closed strings is a theory of quantum gravity.

There is an analog of the Virasoro amplitude for open strings, which was proposed earlier; this is the Veneziano amplitude[30]

$$A = \frac{\Gamma(-1-s)\Gamma(-1-t)}{\Gamma(-2-s-t)}\,, \tag{8.54}$$

where now

$$s = -\frac{1}{2\pi T}\left(p_1 + p_2\right)^2\,, \qquad t = -\frac{1}{2\pi T}\left(p_1 + p_3\right)^2\,. \tag{8.55}$$

Thie Veneziano amplitude also has poles at $s = -1, 0, 1, 2, \ldots$, but the maximum spin for $s = n$ is now $J_{\max} = n+1$, and the leading Regge trajectory has slope α' and intercept 1 (this is the constant a that equals the zero point energy in the light-cone gauge quantization of the open string).

8.3 Other amplitudes and vertex operators

Consider the following modification of the Euclidean conformal gauge action to include a potential function $U(X)$:

$$I_E \to I = I_E + \int d^2z\, U(X)\,. \tag{8.56}$$

The term involving U is not conformal invariant. However, if we write the action as

$$I = \frac{1}{2\pi\alpha'}\int d^2z\, \left\{\partial X \cdot \bar\partial X + 2\pi\alpha'\, U(X)\right\} \tag{8.57}$$

[30] Veneziano did not do a computation to get this amplitude; he just proposed it on the basis of its properties, but it turns out to be the amplitude for elastic scattering of two open-string tachyons.

then we see that α' plays the role $\hbar$ so that a term in the integrand with a factor of α', such as the $U(X)$ term, must be considered along with first-order quantum corrections to the $2D$ classical field theory defined by the action (8.56). Under renormalisation, all operators of a given classical conformal dimension mix and those of definite quantum dimension are eigenfunctions of an anomalous dimension matrix. In this case, the matrix is $-\Box_D/(8\pi T)$ and the functions $U(X)$ satisfying

$$-\frac{\Box_D}{8\pi T}U = \lambda U \tag{8.58}$$

have quantum conformal dimension (λ, λ). Its integral over the worldsheet will be conformal invariant if $\lambda = 1$, and this is the case if $p^2 = 8\pi T$. This agrees with the calculation that we have already done; if we treat $U(X)$ as a perturbation and write

$$e^{-I} = e^{-I_E} \sum_{N=0}^{\infty} \prod_{i=1}^{N} \int d^2 z_i U\left(X(z_i)\right) , \tag{8.59}$$

and take $U(X)$ to be some linear combination of the functions $U_p(X) = e^{ip\cdot X}$, then the corresponding expansion of the vacuum to vacuum amplitude includes scattering amplitudes of the type already considered, which we found to be conformal invariant iff $p^2 = 8\pi T$.

In an operator version of this calculation we would have to compute vacuum expectation values of products of integrals of "vertex operators" of the form

$$V_0 =: e^{ip\cdot X} : \tag{8.60}$$

where the colons indicate "normal ordering" (all creation operators appear to the left of all annihilation operators, so that V_0 annihilates the oscillator vacuum). Each particle in the string spectrum has its own vertex operator of conformal dimension $(1, 1)$. For example, at level 1 we have the vertex "operator"

$$V_1 = \epsilon_{mn} : \partial X^m \bar{\partial} X^n \, e^{ip\cdot X} : , \tag{8.61}$$

where ϵ_{mn} is a constant polarisation tensor. Since $\partial X^m \bar{\partial} X^n$ already has dimension $(1, 1)$ we need the $e^{ip\cdot X}$ factor to have conformal dimension zero, which it does if $p^2 = 0$. It turns out that one also needs to impose the conditions $p^m \epsilon_{mn} = p^n \epsilon_{mn} = 0$. So V_1 is a vertex operator for massless particles in the string spectrum. Notice that if $\epsilon_{mn} = \xi_m p_n$ then

$$\int d^2 z \, V_1 = -i\xi_m \int d^2 z \partial X^m \bar{\partial}\left(e^{ip\cdot X}\right) , \tag{8.62}$$

so integrating by parts we have

$$\int d^2 z V_1 = i\xi_m \int d^2 z \nabla^2 X^m e^{ip\cdot X} , \tag{8.63}$$

which is zero if we use the X field equation[31]. This shows that there is a gauge invariance for zero mass. Notice that the polarisation tensor has the decomposition

$$\epsilon_{mn} = \epsilon_{(mn)} + \epsilon_{[mn]} \,. \tag{8.64}$$

The symmetric part is the polarisation tensor for the massless graviton and the antisymmetric part is the polarisation tensor for the massless antisymmetric tensor.

Now consider the level-2 vertex operator

$$V_2^{(4)} = \epsilon_{mnpq} : \partial X^m \partial X^n \bar{\partial} X^p \bar{\partial} X^q \, e^{ip\cdot X} : \tag{8.65}$$

Since $\partial X^m \partial X^n \bar{\partial} X^p \bar{\partial} X^q$ has dimension $(2,2)$ we need $: e^{ip\cdot X} :$ to have conformal dimension $(-1,-1)$, which will be the case if $p^2 = -8\pi T$ and the polarisation tensor ϵ_{mnpq} is orthogonal to p on all four indices. Another level-2 vertex operator is

$$V_2^{(3)} = \epsilon_{mnp} : \partial^2 X^m \bar{\partial} X^n \bar{\partial} X^p e^{ip\cdot X} : \tag{8.66}$$

Notice that level-matching now becomes the requirement that both "left" and "right" conformal dimensions equal 1.

As for the tachyon field, we can also introduce the massless level-1 fields as background fields[32]. For example, we may modify the Euclidean conformal gauge action to

$$I = \frac{1}{2\pi\alpha'} \int d^2z \left\{ \partial X^m \bar{\partial} X^n \left[\eta_{mn} + h_{mn}(X) + b_{mn}(X) \right] \right\} \,, \tag{8.67}$$

where $h_{mn}(X)$ is a symmetric tensor field and $b_{mn}(X)$ is an antisymmetric tensor field. Notice that we can define $\eta_{mn} + h_{mn}(X)$ to be the space-time metric $g_{mn}(X)$. The conditions for conformal invariance to first-order in h and b are, respectively, the linear Einstein equations and Maxwell-type equations.

However, we are missing one of the level-1 states: the scalar "dilaton". The dilaton field $\Phi(X)$ couples to the string, through its derivative $\partial_m \Phi$, via the interaction

$$I_{dil} \propto \frac{1}{2\pi\alpha'} \int d^2z \left[cb\bar{\partial} X^m + \tilde{c}\tilde{b}\partial X^m \right] \partial_m \Phi(X) \,. \tag{8.68}$$

However, this is equivalent, because of a *ghost number anomaly* to a much simpler term that we will now discuss.

8.3.1 The dilaton and the string-loop expansion

The way that the dilaton field $\Phi(X)$ couples to the string in its Polyakov formulation is through the scalar curvature of the independent worldsheet metric $\gamma_{\mu\nu}$. In two

[31] Such terms can be removed by a field redefinition.

[32] This is also true, but less useful, for higher-level fields.

dimensions the Riemann curvature tensor is entirely determined by its double trace, the Ricci scalar $R(\gamma)$, but this allows us to add to the Euclidean NG action the term

$$I_\Phi = \frac{1}{4\pi}\int d^2z\, \Phi(X)\sqrt{\gamma}\, R(\gamma)\,. \tag{8.69}$$

Here are some features of this term:

- If $\Phi = \phi_0$, a constant then

  $$I_\Phi = \phi_0 \chi\,, \qquad \chi = \frac{1}{4\pi}\int d^2z\, \sqrt{\gamma}\, R(\gamma)\,. \tag{8.70}$$

  The integral χ is a topological invariant of the worldsheet, called the Euler number. For a compact Riemann surface with boundary (which we'll abbreviate to "Riemann surface" in what follows) the Euler number is related to the genus g (the number of doughnut-type "holes") by the formula

  $$\chi = 2\,(1-g)\ . \tag{8.71}$$

- In conformal gauge, we can write the line element for the (Euclidean signature) metric γ as $ds^2(\gamma) = 2e^\sigma dz d\bar{z}$, i.e. a conformal factor e^σ (an arbitrary function of z and $\bar{z}$) times the Euclidean metric. We then find that

  $$\sqrt{\gamma}\, R(\gamma) = 2\nabla^2\sigma \tag{8.72}$$

  and hence, after integrating by parts,

  $$I_\Phi = \frac{1}{2\pi}\int d^2z\ \sigma\, \partial X^m \bar{\partial} X^n \partial_n \partial_m \Phi\,. \tag{8.73}$$

  This shows that I_Φ is *not* conformal invariant because it depends on the conformal factor (through its σ-dependence). This is allowed because (like the tachyon potential term) it comes with an additional factor of α' relative to the other terms; we have to consider the lack of conformal invariance of I_Φ at the same time that we consider possible conformal anomalies of action (8.67).

- When we consider conformal invariance to second-order in α' we find the condition $\Box_D \Phi = 0$ to leading order in Φ; this confirms that $\Phi(X)$ is a massless field.

So far we have considered amplitudes for which the Euclidean worldsheet is a Riemann sphere with some number of punctures (corresponding to some vertex "operator" insertion into the path integral). However, in QM we have to sum over all possibilities, and hence over Euclidean worldsheets of any genus g. So far we have considered only the $g = 0$ case, i.e. the Riemann sphere. How should we weight this sum over different topologies? To answer this question, at least partially, suppose that

$$\Phi = \phi_0 + \phi(X)\,, \tag{8.74}$$

where $\phi(X)$ is zero in the vacuum; i.e. the constant ϕ_0 is the "vacuum expectation value" of $\Phi(X)$. Then there will appear a factor in the path integral of the form

$$e^{-\phi_0 \chi} = \left(g_s^2\right)^{g-1}\,, \qquad g_s \equiv e^{\phi_0}\,. \tag{8.75}$$

For $g = 0$ this tells us that the R. sphere contribution to scattering amplitudes is weighted by a factor of $1/g_s^2$. If we use these amplitudes to construct an effective field theory action S from which we could read off the amplitudes directly (by looking at the various interaction terms) then this action will come with a factor of g_s (we can then absorb all other dimensionless factors into a redefinition of g_s, i.e. of ϕ_0). If we focus on the amplitudes for scattering of level-1 particles then we find that

$$S[h,g,\phi] = \frac{1}{g_s^2}\frac{1}{\ell_s^{(D-2)}}\int d^Dx\,\sqrt{-\det g}\,e^{-2\phi}\left[2\Lambda + R(g) - \frac{1}{3}H^2 + 4\left(\partial\phi\right)^2 + \mathcal{O}(\alpha')\right]\,, \tag{8.76}$$

where the cosmological constant is

$$\Lambda = \frac{(D-26)}{3\alpha'}\,. \tag{8.77}$$

and ℓ_s is a length scale, required by dimensional analysis. This scale must be set by the string tension, for the following reason. The NG string action in a background metric g_{mn} is unchanged if

$$\alpha' \to \lambda\alpha' \quad \& \quad g_{\mu\nu} \to \lambda g_{\mu\nu}\,, \tag{8.78}$$

and this is also true of the spacetime effective action if $\ell_s \propto \sqrt{\alpha'}$. The constant of proportionality can be absorbed into the dimensionless constant g_s, so we may choose, without loss of generality, to set

$$\ell_s = \sqrt{\alpha'}\,. \tag{8.79}$$

Some other features of the effective space-time action are

- The exact result for S will involve a series of all order in α' since the coupling of the background fields to the string introduces interactions into the 2D QFT defined by the string worldsheet action.

- The leading term is the cosmological constant Λ. Unless we know the entire infinite series in α', we must set $\Lambda = 0$; i.e. we must choose $D = 26$. It is then consistent to consider the string as perturbation about the Minkowski vacuum, which is what we implicitly assumed when we earlier derived the condition $D = 26$.

- It is consistent to exclude the coupling to the string of the fields associated to massive modes of the string because without them the worldsheet action defines a renormalizable 2D QFT. Coupling to the fields associated to the massive particles in the string spectrum leads to a non-renormalizable 2D QFT for which it is necessary to consider all possible terms of all dimensions. But if all fields of level $N > 1$ are all zero initially then they stay zero.

- The integrand involves a factor of $e^{-2\phi}$. This is because the action must be such that $\phi_0 \equiv \ln g_s$ and $\phi(X)$ must appear only through the combination $\Phi = \phi_0 + \phi(X)$.

In the effective spacetime action, g_s^2 plays the role of $\hbar$. This suggests that we have been considering so far only the leading term in a semi-classical expansion. This is true because we have still to consider R. surfaces with genus $g > 0$, and a string amplitude at genus g is weighted, according to to (8.75), by a factor of $(g_s^2)^{g-1}$, i.e. a factor of $(g_s^2)^g$ relative to the zero-loop amplitude. This confirms that the string-loop expansion is a semi-classical expansion in powers of g_s^2. Taking into account all string loops gives us a *double expansion* of the effective field theory[33]

$$S = \frac{1}{g_s^2 \ell_s^{(D-2)}} \int d^D x \sqrt{-\det g} \sum_{g=0}^{\infty} g_s^{2g} \, e^{2(g-1)\phi} \, L_g \, , \qquad L_g = \sum_{l=0}^{\infty} \ell_s^{2l} L_g^{(l)} \, . \tag{8.80}$$

In effect, the expansion in powers of ℓ_s comes from first-quantisation of the string, and the expansion in powers of g_s comes from second-quantisation. How can we quantise twice? Is there not a single $\hbar$? The situation is actually not so different from that of the point particle. When we first-quantise we get a Klein-Gordon equaltion but with a mass $m/\hbar$; we then relabel this as m so that it becomes the mass parameter of the classical field equation, and then we quantise again. For the string, first quantization would have led to $\alpha'\hbar$ as the expansion parameter if we had not set $\hbar = 1$; if we relabel this as α' then $\hbar$ appears only in the combination $g_s^2\hbar$.

To lowest order in α' we have what looks like GR coupled to an antisymmetric tensor and a scalar. The D-dimensional Newton gravitational constant G_D is

$$G_D \;\propto\; g_s^2 \ell_s^{(D-2)} \, . \tag{8.81}$$

Consistency of the string-loop expansion (in powers of g_s^2) relies on this formula. Particles in the string spectrum have masses proportional to $1/\ell_s$, independent of g_s,

[33]There is a lot of freedom in the form of the Lagrangians $L_g^{(l)}$ beyond leading order. Recall that the construction of S involves a prior determination of scattering amplitudes of the level-1 fields, which we then arrange to replicate from a local spacetime Lagrangian. Since the amplitudes are all "on-shell" they actually determine only the field equations for the background fields, and then only up to field redefinitions. Even with all this freedom it is not obvious why it should be possible to replicate the string theory scattering amplitudes in this way, although this has been checked to low orders in the expansion and there are general arguments that purport to prove it.

so their contribution to the gravitational potential in D dimensions is proportional to g_s^2, and hence zero at zero string coupling. This means that the strings of free ($g_s = 0$) string theory do not back-react on the space-time metric; the metric is changed by the presence of strings only within perturbation theory. If this had not been the case it would not have been consistent to start (as we did) by considering a string in Minkowski spacetime.

Why is g_s called the string coupling constant? Consider a g string-loop vacuum to vacuum diagram with the appearance of a chain of g tori connected by long "throats", and think of it as "fattened" Feymann diagram in which a chain of g loops connected by lines; where each line meets a loop we have a 3-point vertex. As there are $(g-1)$ lines that link the loops, and each of the two ends of each line ends at a vertex, we have a total of $2(g-1)$ vertices. If we associate a coupling constant to each vertex, call it g_s, we see that this particular diagram comes with a factor of $(g_s^2)^{g-1}$, which agrees with our earlier result.

Is there a g-loop R. surface with the appearance just postulated. Yes, there is. For $g > 0$ there is no longer a unique flat metric; for example, there is a one complex parameter f smily of conformally inequivalent flat metrics. For $g \geq 2$ there is a $3(g-1)$-parameter family of conformally inequivalent flat metrics; these parameters are called "moduli". This number can be understood intuitively from the "chain of tori" diagram if we associate one parameter with each propagator. For g loops we have, in addition to the $(g-1)$ links, $(g-2)$ "interior" loops with 2-propagators each, and two "end of chain" loops with one propagator each. The total number of propagators is therefore

$$(g-1) + 2(g-2) + 2 = 3(g-1)\,. \tag{8.82}$$

This is also, and not coincidentally, the dimension of the space of quadratic differentials on a Riemann surface of genus $g \geq 2$.

8.4 String theory at 1-loop: taming UV divergences

We will now take a brief look at what happens at one string-loop. In this case amplitudes are found from the path integral by considering vertex operator insertions at points on a conformally flat complex torus. In contrast to the R. sphere, for which all conformally flat metrics are conformally equivalent, there is a one complex parameter family of conformally flat tori. We can define a flat torus by a doubly periodic identification in the complex z-plane. Without loss of generality we may choose one identification to be under a unit shift along the real axis. The other identification must then be by some complex number τ (as it is traditionally called) that has a non-zero imaginary part, so

$$z \sim z + 1 \quad \& \quad z \sim z + \tau\,, \qquad \Im\tau > 0\,. \tag{8.83}$$

We may restrict τ to take values in the upper-half τ-plane without loss of generality since otherwise we could take the complex conjugate and consider the $\bar{z}$-plane. Different values of the "modulus" τ correspond to "shapes" that are conformally inequivalent, which we should sum over in the path integral. However not all points on the upper-half τ-plane correspond to different shapes. It is not difficult to see that the unit translation

$$\tau \to \tau + 1 \tag{8.84}$$

gives the same torus; the $z = \tau + 1$ in the z-plane is equivalent to the point $z = \tau$ because $z \sim z + 1$. It is also true that the inversion

$$\tau \to -\frac{1}{\tau} \tag{8.85}$$

gives the same torus (after a rescaling). Consider the special case for which τ is pure imaginary, say $\tau = 2i$; we then have a rectangular torus with sides in the ratio $2 : 1$. After taking $\tau \to -1/\tau$, which takes $2i$ to $i/2$ we get a rectangular torus with sides in the ratio $1 : 2$, but this is the same as $2 : 1$ if we don't care which side we consider first. More generally, the lengths of the two sides are in equivalent ratios and the angle between them is preserved by any analytic map, so we get an equivalent torus. The operations of unit translation and inversion do not commute; they close to the discrete fractional linear transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1. \tag{8.86}$$

This is a realisation of the group

$$PSl(2;\mathbb{Z}) \cong Sl(2;\mathbb{Z})/\{\pm 1\}\ . \tag{8.87}$$

We have to identify $Sl(2;\mathbb{Z})$ matrices that differ by a sign since this sign cancels from (8.86). This is called the "modular group" of the torus. Because tori with modular parameters related by the action of the modular group have the same shape, we should integrate τ in the path integral only over one fundamental domain of the modular group.

It is convenient to choose this fundamental domain to be the one, let's call it F, in which we may take $\Im\tau \to \infty$. This is because in this limit the torus becomes long and thin and it starts to look like a one-loop Feynmann diagram (with vertices at various points if we had vertex operators at points on the torus). Specifically, we can think of $\Im\tau$ as the modular parameter s of a particle worldline in the $s \to \infty$ limit, which is an IR limit. We should expect string theory to reduce to a field theory in this limit because the massive particles in the string spectrum should be unimportant and we should be able to replace the string theory by its effective field theory.

In the particle case we have to integrate s from $s = 0$ to $s = \infty$, and $s = 0$ is the UV limit in which we find the UV divergences of QFT. The situation is quite different for string theory because the fundamental domain F is the region of the upper-half τ-plane defined by

$$|\tau| \geq 1 \quad \& \quad -\frac{1}{2} \leq \Re\tau < \frac{1}{2} \,. \tag{8.88}$$

This does not include the imaginary axis below $\tau = i$, so there is no UV limit! A better way to say this is that an approach to theUV limit is equivalent, by a modular transfermation, to an approach to the IR limit. So UV divergences will be absent as long as the IR divergences cancel.

8.5 Beyond String Theory

So far we have considered only the NG string, or "bosonic string" as it is often called to distinguish it from string theories that incorporate fermions in some way. There are not many consistent ways to do this. One, called the "spinning string" involves a worldline supersymmetric extension of the NG string. The "spinning string" has critical dimension $D = 10$ but still has a tachyon. However, it turns out that it is consistent to simply "throw away" the tachyon, along with a lot of other states, leaving a spectrum that exhibits *spacetime* supersymmetry. This is called the "superstring", and another formulation of it was later found for which the spacetime supersymmetry is manifest (as is then the absence of a tachyon since tachyonic unitary irreps of the super-Poincaré group do not exist). Actually, there are several superstring theories, which are roughly in correspondence with the possible low-energy $D = 10$ supergravity theories (after excluding the anomalous ones).

Superstring theory provides a way of consistently computing perturbative corrections to a $D = 10$ supergravity theory. However the string-loop expansions an asymptotic one with zero radius of divergence; it does not define a theory and is useful only if $g_s \ll 1$. Because of (8.81) this means that

$$l_s \gg \ell_{Planck} \qquad \left(\ell_{Planck} = G_D^{1/(D-2)}\right) \,, \tag{8.89}$$

so we are not going to learn much from perturbative string theory about the real problems of quantum gravity, which occur at the Planck length. For that a non-perturbative extension of string theory is required. Fortunately, supersymmetry in $D = 10$ is a strong constraint from which a lot has been learned about this non-perturbative theory. What we have have learned is

- The non-perturbative theory has $D = 11$, not $D = 10$. From this perspective all the different $D = 10$ perturbative superstring theories are unified into one theory, called "M-theory" (best not to ask why).

- Strings are just one of many kinds of "branes" and M-theory is an Owellian democracy in which all branes are equal but some are more equal than others.