Predicate Logic

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1 Review

1.1 Set theory

1.2 Propositional Logic

- Connectives
- Syntax of propositional logic:
  - A recursive definition of well-formed formulas
  - Abbreviation rules
- Semantics of propositional logic:
  - Truth tables
  - Logical equivalence
  - Tautologies, contradictions, contingencies
  - Indirect reasoning
  - Relations between propositions: equivalence, contradiction, entailment
- **Question**: (i) Does (a) contradicts (b) (viz. two sentences are contradictory iff they cannot be simultaneously true)? (ii) If it does, can you show this contradiction by propositional logic?

  (1) a. Mary is wearing a blue skirt.
      b. Nobody is wearing a blue skirt.

2 The syntax of predicate logic

2.1 The vocabulary of predicate Logic

- Vocabulary

(2) a. Individual constants: $j, m, ...$
b. Individual variables: $x, y, z, ...$
The individual variables and constants are the *terms*.
c. Predicates: $P, Q, R, ...$
Each predicate has a fixed and finite number of arguments called its *arity*.
d. Connectives: $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$
e. Quantifiers: $\forall$ (the universal quantifier, '*all, each, every*'), $\exists$ (the existential quantifier, '*some*' in the sense of "at least one, possibly more")
f. Constituency labels: parentheses, square brackets and commas.

- Existential quantifier: $\exists$

(3) a. $A(j, x)$ — John admires x.
b. $\exists x A(j, x)$ — John admires something.

- Universal quantifier: $\forall$

(4) a. Every teacher is friendly.
b. $T(p) \rightarrow F(p)$ — If Peter is a teacher, then he is friendly.
c. $T(b) \rightarrow F(b)$ — If Bill is a teacher, then he is friendly.
d. $\forall x(T(x) \rightarrow F(x))$ — For every x, If x is a teacher, then x is friendly.

Discussion: Why is that the representations of the universal quantification and the existential quantification have to be an implication and a conjunction, respectively? What is the meaning of $\forall x(T(x) \wedge F(x))$?

Exercise 1: Translate the following formulas into 'good' English sentences. (H: *be human*, L: *like*, m: *Mary*)

(5) a. $\exists x L(m, x)$
b. $\exists x(H(x) \wedge (x, m))$
c. $\forall x(H(x) \rightarrow L(x, m))$
d. $\exists x(H(x) \wedge \forall y(H(y) \rightarrow L(x, y))$

- Some quantifying expressions and their translations

(6)
a. Everything is subject to decay.
b. John gave something to Peter.
c. John gave Peter nothing.
d. John gave Peter a book.
e. A whale is a mammal.
f. Boys who are late are to be punished.

Exercise 2: Translate the following sentences into predicate logic. (Details about tense and aspect can be ignored)

(7)
a. Charles is nice, but Elsa isn't.
b. If peter didn't hear the news from Charles, he heard it from Elsa.
c. No one is going to visit John.
d. Every student enrolled in Ling97r is interested in some linguistic topic.

- Predicate logic vs. set theory

  We can use set theoretical notions (or venn diagrams) to represent the meanings of quantifiers.

2.2 Well-formed formulas

- Syntactic rules to specify the well-formed formulas of predicate logic:

(8)
a. If $t_1, t_2, ..., t_n$ are individual terms and P is an n-place predicate, then $P(t_1, t_2, ..., t_n)$ is a wff.
b. If x is an individual variable and α is a wff, then $\exists x\alpha$ and $\forall x\alpha$ are wffs.
c. If α and β are wffs, then $\neg\alpha$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$, and $(\alpha \leftrightarrow \beta)$ are wffs.
d. Only the formulas generated in accordance with these rules are wffs.

This syntax allows "vacuous" quantification (e.g. $\forall xP(j)$) and is weaker than the one in Allwood et al. (1977).

Exercise 3: According to the simpler syntactic rules above, for each of the following expressions, determine whether it is a wff or not. (H is a one-place predicate, and L is a two-place predicate.)

(9)
a. H
b. $H(j)$
c. $L(j)$
d. $L(j, x)$
e. $\forall xH(y)$
f. $\forall x\exists y(H(x) \rightarrow L(d, x))$

2.3 Scope, bound and free, closed and open

- If x is a variable and α a formula to which a quantifier is attached, than α is the **scope** of the quantifier.

- Bound vs. free:

(10) a. An occurrence of a variable x is **bound** if it occurs in the scope of $\exists x$ or $\forall x$, (existentially bound and universally bound, respectively).

b. A variable is **free** if it is not bound.
E.g. $P(x)$, $\forall y P(x)$

c. Every variable is either free or bound. If bound, it is bound exactly once.
E.g. $\forall x(P(x) \wedge \exists x Q(x))$

- A more precise definition (from Gamut 1):

(11) a. An occurrence of a variable x in the formula ϕ (which is not a quantifier) is said to be *free* if this occurrence of x doesn't fall within the scope of a quantifier $\forall x$ or a quantifier $\exists x$ appearing in ϕ.

b. If $\forall x \psi$ or $\exists x \psi$ is a subformula of ϕ and x is free in ψ, then this occurrence of x is said to be *bound* by the quantifier $\forall x$ or $\exists x$.

- Closed vs. open:

(12) a. A **closed** formula (also called *sentence/proposition*) is one that does not contain any free variables.

b. Any well-formed formula of predicate logic which contains at least one free individual variable is an **open** formula (also called *sentential/ propositional function*).
E.g. $P(x)$, $\forall y P(x)$, $P(x) \wedge \exists x Q(x)$

Exercise 4: Identify the scope of $\exists x$, $\forall x$, $\forall y$ and $\exists z$.

(13) a. $\exists x \forall y(R(x, y) \rightarrow K(x, x))$

b. $\exists x[Q(x) \wedge \forall y(P(y) \rightarrow \exists z S(x, y, z))]$

c. $\exists x P(x) \wedge \forall x Q(x)$

Exercise 5: For each of the following formulas, indicate its free variables and whether it is a sentence.

(14) a. $\exists x(P(x, y) \wedge Q(x))$

b. $\forall x \neg \exists y P(x, y)$

c. $\forall x \forall y P(x, y) \rightarrow Q(x)$

3 The semantics of predicate logic

3.1 Interpretation functions and modals

- Models

  Expressions are interpreted in models. A model M is a pair $\langle D, I \rangle$, where D is the domain, a (nonempty) set of individuals, and I is an interpretation function: an assignment of semantic values to every basic expression (constant) in the language.

  Models are distinguished both by the objects in their domains and by the values assigned to the expressions of the language by I by the particular way that the words of the language are "linked" to the things in the world.

- Domain

  In order to judge the truth value of the following sentence, it is necessary to know what we are talking about, viz. what the *domain of discourse* is.

  (15) Everyone is friendly.

- Interpretation functions

  An interpretation relates L to the world (or a possible world) by giving the extensions/values of the expressions of the language, i.e. the objects of the world that are designated by the expressions of L.

  The interpretation of an arbitrary expression α relative to M: $[\![\alpha]\!]^M$

  $[\![c]\!]^M$ is called the *interpretation* of a constant c, or its *reference/denotation*, and if e is the entity in D s.t. $[\![c]\!]^M =$ e, then c is said to be one of e's *names* (e may have several different names.)

- Example: a toy language L

  The toy language L only has three categories of expressions: names, one-place predicates, and two-place predicates.

Category	Basic expressions	NL counterpart
Names	s, a, t, m	Sharon, Anna, Tiphanie, Martin
One place (unary) predicates	H, C	Happy, cries
Two place (binary) predicates	D, K	dislike, know

(16) $M_1 = \langle D_1, I_1 \rangle$, where

a. $D_1 =$ {Sharon, Anna, Tiphanie, Martin}

b. I_1 determines the following mapping mapping between names and predicate terms in L and objects in D_1

Name	Value		Predicate	Value
s	Sharon		H	{Sharon, Anna}
a	Anna		C	{Sharon, Anna, Tiphanie}
t	Tiphanie		D	{⟨ Sharon, Martin ⟩, ⟨ Anna, Tiphanie⟩}
m	Martin		K	{⟨ Sharon, Martin ⟩, ⟨ Anna, Tiphanie⟩, ⟨ Tiphanie, Sharon⟩}

Composition rules of L (part 1)

(17) a. If P is a one place predicate and α is a name, then $[\![P(\alpha)]\!]^M = 1$ iff $[\![\alpha]\!]^M \in [\![P]\!]^M$.
b. If Q is a two place predicate and α and β are names, then $[\![P(\alpha,\beta)]\!]^M = 1$ iff $\langle [\![\alpha]\!]^M, [\![\beta]\!]^M \rangle \in [\![Q]\!]^M$.
c. If ϕ is a formula, then $[\![\neg\phi]\!]^M = 1$ i $[\![\phi]\!]^M = 0$.
d. If ϕ and ψ are formulas then $[\![\phi \wedge \psi]\!]^M = 1$ iff both $[\![\phi]\!]^M$ and $[\![\psi]\!]^M = 1$.
e. If ϕ and ψ are formulas then $[\![\phi \vee \psi]\!]^M = 1$ iff
f. If ϕ and ψ are formulas then $[\![\phi \rightarrow \psi]\!]^M = 1$ iff
g. If ϕ and ψ are formulas then $[\![\phi \leftrightarrow \psi]\!]^M = 1$ iff

Exercise 6:

Give one sentence with a negation which is true in M_1.

Give one sentence with an implication which is true in M_1.

3.2 *Assignment function

- So far we have assumed that interpretations of basic expressions are given by I, which assigns values in D to names and predicates. The interpretation of individual variables requires a further semantic component, called an **assignment function**, notated g. The assignment function assigns individuals in D to individual variables in formulas.

	Variable	Value		Variable	Value
g_1	x	Anna	g_2	x	Anna
	y	Sharon		y	Sharon
	z	Martin		z	Tiphanie

- Composition rules of L (Part 2)

(18) a. If ϕ is a formula, then $[\![\forall x\phi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g[d/x]} = 1$ for all $d \in D$.
b. If ϕ is a formula, then $[\![\exists x\phi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g[d/x]} = 1$ for some $d \in D$.

- **Discussion:** Assume that none of the individuals in M_1 is a teacher, what is the value of the following sentence under M_1?

(19) Every teacher is friendly.

3.3 Properties of relations

- Reflexivity

  If $\forall x R(x, x)$ holds in M, then R is reflexive in M.

- Symmetry

  If $\forall x \forall y (R(x, y) \rightarrow R(y, x))$ holds in M, then R is symmetric in M.

- Transitivity

  If $\forall x \forall y \forall z (R(x, y) \wedge R(y, z) \rightarrow R(x, z))$ holds in M, then R is transitive in M.

- Converse

  A relation R is said to be the converse of another relation S, if $R(x, y)$ is true whenever $S(y, x)$ is true.

  E.g. 'parent of' vs. 'children of'; 'is seen by' vs. 'see'.