Predicate logic

**Discussion:** (i) Does (1a) contradict (1b)? [Two sentences are contradictory iff they cannot be simultaneously true.] (ii) If it does, can you show this contradiction by propositional logic?

(1) a. Mary is wearing a blue skirt.
   b. Nobody is wearing a blue skirt.

**Discussion:** How did we translate the following quantificational sentences in set-theoretic notations?

(2) a. Kitty is a cat.
   b. Some cat meows.
   c. Every cat meows.

1 Vocabulary and syntax of predicate Logic

**Vocabulary of predicate logic**

(3) a. Individual constants: $j, m, \ldots$
   b. Individual variables: $x, y, z, \ldots$
      (Individual variables and constants together are called *terms*.)
   c. Predicates: $P, Q, R, \ldots$, each with a fixed finite number of argument places.
   d. Connectives: $\neg, \lor, \land, \rightarrow, \leftrightarrow$
   e. Quantifiers:
      i. existential quantifier $\exists$ (means ‘some’ in the sense of ‘at least one, possibly more’),
      ii. universal quantifier $\forall$ (means ‘all, each, every’)
   f. Constituency labels: ( ), [ ], commas

**Well-formed formulas (wffs) of predicate logic**

(4) a. If $P$ is an $n$-place predicate and $t_1, t_2, \ldots, t_n$ are terms, then $P(t_1, t_2, \ldots, t_n)$ is a wff.
   b. If $\alpha$ is a wff and $x$ is an individual variable, then $\exists x \alpha$ and $\forall x \alpha$ are wffs.
   c. If $\alpha$ and $\beta$ are wffs, then $\neg \alpha$, $(\alpha \land \beta)$, $(\alpha \lor \beta)$, $(\alpha \rightarrow \beta)$, and $(\alpha \leftrightarrow \beta)$ are wffs.
   d. Nothing else is a wff of predicate logic.

We can also abbreviate some of the brackets like what we did in propositional logic.

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1This definition allows vacuous quantification (e.g., $\exists x P(j)$) and is more permissive than the one in Allwood et al. (1977).
Exercise: Following the recursive rules above, identify whether each of the following strings is a wff of predicate logic. (H is a one-place predicate, L is a two-place predicate, jd are individual constants, xy are individual variables.)

(5) a. H
b. j
c. H(j)
d. L(j)
e. L(j, x)
f. ∀xH(y)
g. ∀x∃y[H(x) → L(d, x)]

• Scope, bound/free variables, closed/open formula
  – In ∃xα and ∀xα, α is the scope of ∃x and ∀x.
    (6) a. ∀x∃y[H(x) → L(d, x)] (scope of ∀x)
       b. ∃x[Q(x) ∧ ∀y[P(y) → ∃zS(x, y, z)]] (scope of ∃x)
       c. ∀x∃y[H(x) → L(d, x)] (scope of ∀y)
       d. ∃x[Q(x) ∧ ∀y[P(y) → ∃zS(x, y, z)]] (scope of ∃z)
  – Bound/free variables, closed/open formula:
    * An occurrence of a variable x is bound if it occurs in the scope of ∃x/∀x, otherwise it is free.
      (7) a. ∃x[P(x), ∀x[P(y) → Q(x)]
         b. P(x), ∀yP(x)
    * Every occurrence of a variable x can only be bound at most once. Example: In (8), the occurrence of x in Q(x) has been bound by its closest eligible binder ∃x, and thus is not bound by ∀x.
      (8) ∀x[P(x) → ∃xQ(x)] is equivalent to ∀x[P(x) → ∃yQ(y)]

    * A wff is closed iff it does not contain any free variables, otherwise it is open.

Exercise: Identify (i) the scope of each quantifier and (ii) the binder of each individual variable.

(9) a. ∃xP(x) ∧ ∀xQ(x)
b. ∃x[P(x) ∧ ∀xQ(x)]
c. ∃y[P(y) ∧ ∀xQ(x)]
d. ∀x∃y[P(x) ∧ Q(y)]
e. ∀x[P(x) ∧ ∃yQ(y)]
2 Semantics of predicate logic

- The truth value of any statement in predicate logic depends on the domain of discourse and the choice of semantic values for the constants and predicates.

Thus, we interpret expressions of predicate logic in models. A model $M$ is a pair $(D, I)$.

- $D$ is the domain of discourse (i.e., the set of considered individuals).
- $I$ is an interpretation function that assigns a semantic value to each basic constant expression.

For an expression $\alpha$, $[\alpha]^M$ is called the interpretation/denotation of $\alpha$ relative to $M$.

Example: a toy language $L$

<table>
<thead>
<tr>
<th>Category</th>
<th>Basic expressions</th>
<th>NL counterpart</th>
</tr>
</thead>
<tbody>
<tr>
<td>Names</td>
<td>$a, b, c, d$</td>
<td>Andy, Billy, Cindy, Danny</td>
</tr>
<tr>
<td>1-place predicates</td>
<td>$H, C$</td>
<td>Happy, cried</td>
</tr>
<tr>
<td>2-place predicates</td>
<td>$L, K$</td>
<td>dislike, know</td>
</tr>
</tbody>
</table>

(10) $M_1 = \langle D_1, I_1 \rangle$, where

a. $D_1 = \{\text{Andy, Billy, Cindy, Danny}\}$

b. $I_1(a) = \text{Andy}, I_1(b) = \text{Billy}, I_1(c) = \text{Cindy}, I_1(d) = \text{Danny}$

c. $I_1(H) = \{\text{Andy, Billy}\}$

d. $I_1(C) = \{\text{Andy, Billy, Cindy}\}$

e. $I_1(L) = \langle \langle \text{Andy, Danny}\rangle, \langle \text{Billy, Cindy}\rangle \rangle$

f. $I_1(K) = \{\langle \text{Andy, Danny}\rangle, \langle \text{Billy, Cindy}\rangle, \langle \text{Cindy, Andy}\rangle \}$

Semantics of formulas of predicate logic


b. $[P(a_1, a_2, ..., a_n)]^M = 1$ iff $[a_1]^M, [a_2]^M, ..., [a_n]^M \in [P]^M$.

c. $[-\phi]^M = 1$ iff $[\phi]^M = 0$.


e. $[\phi \lor \psi]^M = 1$ iff $[\phi]^M = 1$ or $[\psi]^M = 1$.


h. $[\forall \phi]^M = 1$ iff $[\phi_{(d/x)}]^M = 1$ for every constant $d \in D$.

i. $[\exists \phi]^M = 1$ iff $[\phi_{(d/x)}]^M = 1$ for some constant $d \in D$.

[$\phi_{(d/x)}$ means replacing the occurrence(s) of $x$ in $\phi$ with $d$.]

Exercise: Determine the truth values of the following sentences relative to $M_1$.

(12) a. $\exists x H(x)$

b. $\exists x \neg H(x)$

c. $\forall x C(x)$

d. $\forall x \exists y K(x, y)$

e. $\forall x [H(x) \rightarrow C(x)]$
3 Translations between predicate logic and English

• Predicates

\[ P(t) \] 1-place predicate  \( \text{run, happy, teacher, meet John, who arrived} \)
\[ P(t_1, t_2) \] 2-place predicate  \( \text{meet, taller than, come from} \)
\[ P(t_1, t_2, t_3) \] 3-place predicate  \( \text{give, show} \)
\[ P(t_1, t_2, ..., t_n) \] \( n \)-place predicate

- Translation of an atomic formula:

(13) a. Kitty meows
   Key: \( M(x): x \) meows; \( k: \) Kitty
   Translation: \( M(k) \)

b. John introduced Andy to Billy.
   Key: \( I(x, y, z): x \) introduce \( y \) to \( z; j: \) John; \( a: \) Andy; \( b: \) Billy
   Translation: \( I(j, a, b) \)

• Quantifiers

- Existential quantification:

(14) John admires some teacher.
   a. \( T(c) \land A(j, c) \) Cindy is a teacher and John admires Cindy.
   b. \( \exists x[T(x) \land A(j, x)] \) There is some \( x \) such that \( x \) is a teacher and John admires \( x \).

- Universal quantification:

(15) Every teacher is friendly.
   a. \( T(p) \rightarrow F(p) \) If Peter is a teacher, then he is friendly.
   b. \( T(b) \rightarrow F(b) \) If Bill is a teacher, then he is friendly.
   c. \( \forall x[T(x) \rightarrow F(x)] \) For every \( x \), if \( x \) is a teacher, then \( x \) is friendly.

Discussion: Why is that the following wffs are not appropriate translations for (14) and (15)?

(16) a. \( \exists x[T(x) \rightarrow A(j, x)] \)
    b. \( \forall x[T(x) \land F(x)] \)

Exercise: Translate the following formulas into good English sentences. \( (H: \text{be human}, L: \text{like}, m: \text{Mary}) \)

(17) a. \( \exists x[H(x) \land L(x, m)] \)
    b. \( \forall x[H(x) \rightarrow L(x, m)] \)
    c. \( \exists x[H(x) \land \forall y[H(y) \rightarrow L(x, y)]] \)
    d. \( \forall x[H(x) \land L(x, m) \rightarrow \exists y[H(y) \land L(x, y)]] \)
**Exercise:** Translate the following sentences into predicate logic. (Ignore the details of tense and aspect.)

(18) a. Charles is nice, but Elsa isn’t.

b. If Peter didn’t hear the news from Charles, he heard it from Elsa.

c. No one is going to visit John.

d. Every student enrolled in LING 106 is interested in some linguistic topic.

**More about translations of quantificational expressions**

- Sometimes, the quantificational force of a sentence might not be explicitly expressed:

(19) a. A whale is a mammal. (Generic sentence)

\[ \forall x [W(x) \rightarrow M(x)] \]

b. Students who are late are to be punished. (Bare plurals)

\[ \forall x [S(x) \land L(x) \rightarrow P(x)] \]

- Scope ambiguity of quantifiers

(20) Every boy invited some girl.

a. Every boy is such that he invited a girl.

\[ \forall x [B(x) \rightarrow \exists y [G(x) \land I(x, y)]] \]

b. There is a girl such that she is invited by every boy.

\[ \exists y [G(y) \land \forall x [B(x) \rightarrow I(x, y)]] \]

- Interactions between negation and existential quantifiers

(21) a. John saw someone.

\[ \exists x [H(x) \land S(j, x)] \]

b. John didn’t see someone.

c. John didn’t see anyone.

d. John didn’t see a person.