Core percolation on complex networks: Supplemental Material

Yang-Yu Liu, Endre Csőka, Haijun Zhou, and Márton Pósfai
(Dated: October 2, 2012)

Contents

I. Locality assumption 2

II. Proof of $\alpha$ is the smallest fixpoint of $A(A(x))$ 2

III. Greedy leaf removal 2

IV. Normalized number of edges in the core 3

V. Core percolation on random graphs with specific degree distributions 3
   A. Poisson-distributed graphs 3
   B. Exponentially distributed graphs 4
   C. Purely power-law distributed graphs 4
   D. Power-law distribution with exponential cutoff 5
   E. Static model 6

VI. Percolation threshold and discontinuity 7

VII. Critical exponents 9
   A. non-degenerate case: $P^+(k) \neq P^-(k)$ 9
   B. degenerate case: $P^+(k) = P^-(k) = P(k)$ 15

VIII. Numerical Simulations 19

IX. Real networks 21

References 24
I. LOCALITY ASSUMPTION

By locality we mean that $\forall \epsilon > 0, \exists k \in \mathbb{N}$, if we take a random node $v$ of a random graph, then there is probability at most $\epsilon$ that $v$ is removable in a particular way, but not removable this way in $k$ steps.

II. PROOF OF $\alpha$ IS THE SMALLEST FIXPOINT OF $A(A(x))$

Proof. Let $\alpha_k$ denote the probability that a random neighbor of a random node $v$ of a network $G$ is $\alpha$-removable in $G \setminus v$ in at most $k$ steps. Then $\alpha_{k+1} = A(A(\alpha_k))$. Therefore, $\alpha = \lim_{k \to \infty} \alpha_k = \lim_{k \to \infty} A^{(2k)}(0)$, which converges to the smallest fixpoint of $A(A(x))$. \qed

III. GREEDY LEAF REMOVAL

The core of a undirected network is defined as a spanned subgraph which remains in the network after the following greedy leaf removal (GLR) procedure [1]: As long as the network has leaves, i.e. nodes of degree one, choose an arbitrary leaf $v_1$, and its neighbor $v_2$, and remove them together with all the edges incident with $v_2$. Finally, we remove all isolated nodes. The resulting graph is independent of the order of removals [2]. One can easily tell whether the core exists in two very special cases: (1) If a network has no cycles, i.e. a tree or a forest (a disjoint union of trees), then eventually all nodes will be removed, hence no core. For example, the Barabási-Albert (BA) model with parameter $m = 1$ yields a tree network, hence no core exists. (2) If a network has no leaf nodes, e.g. regular graphs with all nodes having the same degree $k > 1$ or the networks generated by the BA model with $m > 1$, then the GLR procedure will not even be initiated, hence all the nodes belong to the core.

In the main text, GLR was extended for directed networks and the core of a directed network $G$ was defined as the core of its corresponding bipartite graph $B$ obtained by applying GLR to $B$ as if $B$ is a unipartite undirected network. In a previous work [3], the GLR on directed networks has been defined in a slightly different way. Given a directed network $G$, as long as there are in-leaves ($k_{in} = 1$) or out-leaves ($k_{out} = 1$), we choose an arbitrary in-leaf (or out-leaf), prune its neighbor’s all outgoing (or incoming) edges, respectively. Finally, we remove all isolated nodes. The core is then defined to be all the non-single nodes (and edges between them) which remain after the iterative GLR.

The two different GLR procedures will yield different core structures in directed networks.
Up to our knowledge, the core percolation problem associated with the latter GLR has not been analytically solved.

IV. NORMALIZED NUMBER OF EDGES IN THE CORE

The normalized number of edges in the core \((l_{\text{core}} \equiv L_{\text{core}}/N)\) can be calculated as follows. Consider a uniform random edge, which remains in the core if and only if both of its endpoints are non-removable without removing the edge. The probability of one endpoint being non-removable without removing the edge is \(1 - \alpha - \beta\), and for the two endpoints the probabilities are independent. Therefore, the expected normalized number of edges in the core is

\[
l_{\text{core}} = \frac{c}{2} (1 - \alpha - \beta)^2 \tag{S1}\]

with \(c/2 = L/N\) the normalized number of edges in the network. Similar argument yields the result for directed networks

\[
l_{\text{core}} = c (1 - \alpha^+ - \beta^+)(1 - \alpha^- - \beta^-) \tag{S2}\]

where \(c = L/N\) is the mean in-degree (or out-degree) of the directed network.

V. CORE PERCOLATION ON RANDOM GRAPHS WITH SPECIFIC DEGREE DISTRIBUTIONS

In the following, we apply the analytical approach described in the main text onto some examples of specific random undirected graphs [4].

A. Poisson-distributed graphs

For Erdős-Rényi random graphs, in the thermodynamic limit, \(P(k)\) follows the ordinary Poisson distribution, i.e.

\[
P(k) = e^{-c} \frac{c^k}{k!} \tag{S3}\]

with \(c = \langle k \rangle = \sum_{k=0}^{\infty} kP(k)\) the mean degree. The generating functions are given by

\[
G(x, c) = e^{-c(1-x)} \tag{S4}
\]

\[
A(x, c) = e^{-cx}. \tag{S5}\]
where we have explicitly considered the $c$-dependence.

At the critical point $(c^*, \alpha^*)$, $f(\alpha^*, c^*) = f'(\alpha^*, c^*) = 0$ where $f(x, c) \equiv A(A(x, c), c) - x$. For $c \leq c^*$, the root of $f(x, c)$ is given by $\alpha = A(\alpha, c)$. Hence, $f'(x, c) = A'(A(x, c), c)A'(x, c) = \left[A'(x, c)\right]^2$. By solving the two equations $A(\alpha^*) = e^{-c^*\alpha^*} = \alpha^*$ and $A'(\alpha^*) = -c^*e^{-c^*\alpha^*} = -1$, we have $c^* = e$ and $\alpha^* = 1/e$ for Poisson-distributed graphs.

### B. Exponentially distributed graphs

For exponentially distributed graphs,

$$P(k) = (1 - e^{-1/\kappa})e^{-k/\kappa}. \quad (S6)$$

The mean degree is given by $c = \frac{e^{-1/\kappa}}{1 - e^{-1/\kappa}}$ and the generating functions are

$$G(x) = \frac{1 - e^{-1/\kappa}}{1 - xe^{-1/\kappa}}. \quad (S7)$$

$$A(x) = \left(\frac{1 - e^{-1/\kappa}}{1 - (1 - x)e^{-1/\kappa}}\right)^2. \quad (S8)$$

Solving the two equations $A(\alpha^*) = \alpha^*$ and $A'(\alpha^*) = -1$ yields $c^* = 4$ and $\alpha^* = 1/4$. We also calculate $n_{core}, l_{core}, \alpha$ and $\beta$ at different mean degrees (see Fig. S1).

### C. Purely power-law distributed graphs

For purely power-law distributed graph,

$$P(k) = \frac{k^{-\gamma}}{\zeta(\gamma)} \quad \text{for} \quad k \geq 1. \quad (S9)$$
Figure S3: Core percolation on random graphs with power-law degree distribution and exponential cutoff.

with \( \zeta(\gamma) = \sum_{k=1}^{\infty} k^{-\gamma} \) the Riemann Zeta function. \( P(k) \) is normalizable for \( \gamma > 2 \). The generating functions are

\[
G(x) = \frac{\text{Li}_\gamma(x)}{\zeta(\gamma)} \quad \text{(S10)}
\]

\[
A(x) = \frac{\text{Li}_{\gamma-1}(1 - x)}{(1 - x)\zeta(\gamma - 1)} \quad \text{(S11)}
\]

with \( \text{Li}_n(x) \) the \( n \)th polylogarithm of \( x \).

We calculate \( n_{\text{core}}, l_{\text{core}}, \alpha \) and \( \beta \) at different \( \gamma \) values (see Fig.S2). Interestingly, we find that the core does not exist in purely power-law distributed graphs for all \( \gamma > 2 \).

D. Power-law distribution with exponential cutoff

Consider a purely power-law distribution with exponent \( \gamma \) and exponential cutoff

\[
P(k) = \frac{k^{-\gamma} e^{-k/\kappa}}{\text{Li}_\gamma(e^{-1/\kappa})} \quad \text{for} \quad k \geq 1. \quad \text{(S12)}
\]
Note that with the exponential cutoff $e^{-k/\kappa}$, the distribution is normalizable for any $\gamma$. The generating functions are

$$G(x) = \frac{\text{Li}_\gamma(xe^{-1/\kappa})}{\text{Li}_\gamma(e^{-1/\kappa})} \quad (S13)$$

$$A(x) = \frac{\text{Li}_{\gamma-1}(1-x)e^{-1/\kappa})}{(1-x)\text{Li}_{\gamma-1}(e^{-1/\kappa})} \quad (S14)$$

We calculate $n_{\text{core}}$, $l_{\text{core}}$, $\alpha$ and $\beta$ at different $\gamma$ and $\kappa$ values (see Fig.S3). We find that core percolation occurs for $\gamma < \gamma_c(\kappa)$, where the threshold value $\gamma_c(\kappa)$ increases and approaches 1 as $\kappa$ increases. Clearly, no core exists in such graphs for all $\gamma > 1$.

E. Static model

Static model is often used to generate asymptotically scale-free undirected networks with $\gamma > 2$ [5]. Starting from $N$ disconnected nodes indexed by integer number $i$ ($i = 1, \ldots, N$), we assign a weight or expected degree $w_i \sim i^{-\xi}$ to each node, with $\xi$ a real number in the range $[0, 1)$ and $\sum_{i=1}^N w_i = 2E$. We randomly select two different nodes $i$ and $j$ from the set of $N$ vertices with probability proportional to $w_i$ and $w_j$, respectively. If $i$ and $j$ have not been connected, then connect them. Otherwise we randomly choose another pair of nodes. This process is repeated until $E = mN$ links are created, resulting in a network of mean degree $c = 2m$. Note that in case $\xi = 0$, this model is equivalent to the classical Erdős-Rényi random graph model.

In the thermodynamic limit the degree distribution of the static model can be analytically derived:

$$P(k) = \frac{(m(1-\xi))^{1/\xi} \Gamma(k-1/\xi, m(1-\xi))}{\Gamma(k+1)} \quad (S15)$$

with $\Gamma(s)$ the gamma function and $\Gamma(s, x)$ the upper incomplete gamma function [6, 7]. In the large $k$ limit, $P(k) \sim k^{-(1+\frac{1}{\xi})} = k^{-\gamma}$ where $\gamma = 1 + \frac{1}{\xi}$. Hence, by tuning $\xi$ we can construct asymptotically scale-free random networks with different degree exponent $\gamma$.

The generating functions are given by

$$G(x) = \frac{1}{\xi} E_{1+\frac{1}{\xi}}[(1-x)m(1-\xi)] \quad (S16)$$

$$A(x) = \frac{1}{\xi} E_{\frac{1}{\xi}}[x m(1-\xi)] \quad (S17)$$

with $E_n(x) \equiv \int_1^\infty dy e^{-xy} y^{-n}$ the exponential integral.

The static model can also be easily generalized to construct directed SF random networks with different degree exponents $\gamma_{\text{in}}$ and $\gamma_{\text{out}}$ by assigning two weights (or expected in- and out-degree)
Figure S4: Core percolation on random graphs generated by the static model.

$w_i^{in} \sim i^{-\xi_i}$ and $w_i^{out} \sim i^{-\xi_i}$ to each node and chose a source node $i$ and a target node $j$ with probability proportional to $w_i^{out}$ and $w_j^{in}$, respectively, to construct an directed edge ($i \to j$).

For undirected SF networks generated by the static model, we find that $c^{\ast} \to e$ as $\gamma \to \infty$. (This is obvious because in case $\xi \to 0$ and $\gamma \to \infty$, this model is equivalent to the classical Erdős-Rényi random graph model, which has $c^{\ast} = e$.) As $\xi \to 1$ and $\gamma \to 2$, we find that $c^{\ast} \to \infty$. In other words, there is no core percolation for such asymptotically SF networks as $\gamma \to \gamma_c = 2$.

We calculate $n_{core}$, $l_{core}$, $\alpha$ and $\beta$ at different $\gamma$ and $c$ values (see Fig.S4). Note that for small $k$, $P(k)$ deviates significantly from the power-law distribution [6] and there are much fewer small-degree nodes than the purely scale-free networks, which results in a drastically different core percolation behavior.

VI. PERCOLATION THRESHOLD AND DISCONTINUITY

At the critical point $c^{\ast}$, $f^\pm(x)$ touches the $x$-axis at its new root (see main text Fig.3c,d), hence we have either $f^+(\alpha^+) = (f^+)'(\alpha^+) = 0$ (or $f^-(1 - \beta^-) = (f^-)'(1 - \beta^-) = 0$), which enable us to calculate the core percolation threshold $c^{\ast}$. In the degenerate case, if $c \leq c^{\ast}$ then $f(\alpha) = f'(\alpha) = 0$.
Figure S5: Threshold and discontinuity of core percolation. a, Analytical solution of the core percolation threshold $c^*$ calculated by solving $f^\pm(x) = f^{\mp'}(x) = 0$ for model networks. For ER networks, $c^* = e$. For undirected asymptotically SF networks generated by the static model, $c^* \rightarrow \infty$ as $\gamma \rightarrow 2$, and and $c^* \rightarrow e$ as $\gamma \rightarrow \infty$. b, The discontinuity $\Delta_n$ in $n_{\text{core}}$ at $c = c^*$ for model networks. For undirected or directed networks with $P^+(k) = P^-(k)$, $\Delta_n = 0$. For directed network, $\Delta_n$ increases as the difference between the in- and out-degree distributions (quantified by the difference between the degree exponents $\gamma_{\text{in}}$ and $\gamma_{\text{out}}$) increases.

can be further simplified as $A(\alpha) = \alpha$ and $[A'(\alpha)]^2 = 1$. The results of $c^*$ for ER and SF networks generated by the static model are shown in Fig.S5a.

The discontinuity in $n_{\text{core}}$ and $l_{\text{core}}$ at $c^*$, denoted by $\Delta_n$ and $\Delta_l$ respectively, can be calculated as follows:

$$
\Delta_n = \frac{1}{2} (\Delta_n^+ + \Delta_n^-) \quad \text{(S18)}
$$

$$
\Delta_l = c^* (1 - \beta^+ - \alpha^-) (1 - \beta^+ - \alpha^-) \quad \text{(S19)}
$$

with $\Delta_n^\pm = G^\pm (1 - \alpha^\pm) - G^\pm (\beta^\pm) - c^* (1 - \beta^+ - \alpha^-) \alpha^\pm$. The results of $\Delta_n$ for ER and SF networks generated by the static model are shown in Fig.S5b.

We find that $\Delta_n \rightarrow 0$ as $\gamma_{\text{in}} \rightarrow \gamma_{\text{out}}$, consistent with the result that core percolation is continuous for undirected networks or directed networks with $P^+(k) = P^-(k)$. We also find that $\Delta_n$ increases as the differences between $\gamma_{\text{in}}$ and $\gamma_{\text{out}}$ increases.
VII. CRITICAL EXponents

A. non-degenerate case: $P^+(k) \neq P^-(k)$

We consider the explicit $c$-dependence in $A^\pm(x)$ for a general random directed network ensemble with mean degree $c$ as a continuously tunable parameter and $P^+(k) \neq P^-(k)$. We define

$$f^\pm(x,c) \equiv A^\pm(A^\pm(x,c),c) - x.$$  \hspace{1cm} (S20)

$\alpha^\pm$ is the smallest root of $f^\pm(x,c)$. Denote $\alpha^{\pm,*}$ and $\beta^{\pm,*}$ as the $\alpha^\pm$- and $\beta^\pm$-value, respectively, at the percolation threshold $c^*$. We can perform the Taylor expansion of $f^\pm(x,c)$ around $(x^*,c^*)$ and yield

$$f^\pm(x,c) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\partial^p \partial_q^q f^\pm(p,q)}{p! q!} f^\pm(p,q).$$  \hspace{1cm} (S21)

where $\delta \equiv x - x^*$, $\epsilon \equiv c - c^*$, and $f^\pm(p,q) \equiv \frac{\partial^p \partial_q^q f^\pm}{\partial x^p \partial c^q}(x^*,c^*)$.

At the critical point,

$$f^\pm(\alpha^{\pm,*},c^*) = 0.$$  \hspace{1cm} (S22)

Moreover, we either have

$$\text{case-1: } f^{+,(1,0)}(\alpha^{+,*},c^*) = f^{-,(1,0)}(1 - \beta^{-,*},c^*) = 0$$  \hspace{1cm} (S23)

or

$$\text{case-2: } f^{-,(1,0)}(\alpha^{-,*},c^*) = f^{+,(1,0)}(1 - \beta^{+,*},c^*) = 0.$$  \hspace{1cm} (S24)

Eqs.S22 and S23 (or S24) enable us to calculate the critical point $(\alpha^{+,*},c^*)$ or $(\alpha^{-,*},c^*)$. Note that case-1 and case-2 can be mapped onto each other by switching $P^+(k)$ with $P^-(k)$. Hence, without loss of generality, we focus on case-1 hereafter.

Consider the expansion of $f^+(\alpha^+,c)$ around the critical point $(\alpha^{+,*},c^*)$ and define

$$\delta_{\alpha^+} \equiv \alpha^+ - \alpha^{+,*}. $$  \hspace{1cm} (S25)

Since $f^{+,(0,0)}_s = f^{+,(1,0)}_s = 0$ (due to Eq.22 and 23), the leading-order terms of $\epsilon$, $\delta_{\alpha^+}$, and $\epsilon \delta_{\alpha^+}$ are $f^{+,(0,1)}_s$, $\frac{1}{2} f^{+,(2,0)}_s \delta_{\alpha^+}^2$, and $f^{+,(1,1)}_s \epsilon \delta_{\alpha^+}$, respectively. (Note that in general $f^{+,(0,1)}_s$, $f^{+,(2,0)}_s$, and $f^{+,(1,1)}_s$ are nonzero.) To make sure that the leading-order terms are canceled with each other, the self-consistent scaling relation between $\delta_{\alpha^+}$ and $\epsilon$ has to be

$$\delta_{\alpha^+} = a \epsilon^\frac{1}{2} + b \epsilon + \mathcal{O}(\epsilon^\frac{3}{2}),$$  \hspace{1cm} (S26)
where $a$ and $b$ are constants determined by the following equations:

\[
\begin{align*}
    f^+_{s, (0,1)} + \frac{1}{2} f^+_{s, (2,0)} a^2 &= 0 \\
    f^+_{s, (1,1)} + f^+_{s, (2,0)} ab + \frac{1}{6} f^+_{s, (3,0)} a^3 &= 0.
\end{align*}
\]  

(S27)

Similarly, consider the expansion of $f^-(\alpha^-, c)$ around $(\alpha^{-*}, c^*)$ and define

\[
\delta_{\alpha^-} \equiv \alpha^- - \alpha^{-*}.
\]  

(S28)

Since in case-1 $f^+_{s, (0,0)} = 0$ but $f^+_{s, (1,0)} \neq 0$ in general, the leading-order terms of $\epsilon$, $\delta_{\alpha^-}$, and $\epsilon \delta_{\alpha^-}$ are $f^+_{s, (0,1)}$, $f^+_{s, (1,0)} \delta_{\alpha^-}$, and $f^+_{s, (1,1)} \epsilon \delta_{\alpha^-}$, respectively. To make sure that the leading-order terms are canceled with each other, the self-consistent scaling relation between $\delta_{\alpha^-}$ and $\epsilon$ has to be

\[
\delta_{\alpha^-} = d \epsilon + O(\epsilon^2)
\]  

(S29)

where $d$ is a constant determined by the following equation:

\[
\begin{align*}
    f^+_{s, (0,1)} + f^+_{s, (1,0)} d &= 0.
\end{align*}
\]  

(S30)

Expand $A^- (\alpha^+, c)$ around the critical point $(\alpha^{+*}, c^*)$

\[
A^- (\alpha^+, c) = 1 - \beta^-
\]

\[
= A^-_{s, (0,0)} + A^-_{s, (1,0)} \delta_{\alpha^+} + \frac{1}{2} A^-_{s, (2,0)} \delta_{\alpha^+}^2 + O(\delta_{\alpha^+}^3)
\]

\[
+ \left( A^-_{s, (0,1)} + A^-_{s, (1,1)} \delta_{\alpha^+} + \frac{1}{2} A^-_{s, (2,1)} \delta_{\alpha^+}^2 + O(\delta_{\alpha^+}^3) \right) \epsilon + O(\epsilon^2)
\]

\[
= 1 - \beta^{-*} + \left( A^-_{s, (1,0)} a \right) \epsilon^2 + \left( \frac{1}{2} A^-_{s, (2,0)} a^2 + A^-_{s, (1,1)} \right) \epsilon + O(\epsilon^3)
\]  

(S31)

where $\frac{\partial^{p+q} A^- (x, c)}{\partial x^p \partial c^q} \bigg|_{(\alpha^{+*}, c^*)}$, $A^-_{s, (0,0)} \equiv A^- (\alpha^{+*}, c^*) = 1 - \beta^{-*}$, and we have implicitly use the result of Eq.S26 to track the correct order of $\delta_{\alpha^+}$ and $\epsilon$. Consequently, we can define

\[
\delta_{\beta^-} \equiv \beta^- - \beta^{-*} = - \left( A^-_{s, (1,0)} a \right) \epsilon^2 - \left( \frac{1}{2} A^-_{s, (2,0)} a^2 + A^-_{s, (1,1)} \right) \epsilon + O(\epsilon^3).
\]  

(S32)

Similarly, we expand $A^+ (\alpha^-, c)$ around $(\alpha^{-*}, c^*)$

\[
A^+ (\alpha^-, c) = 1 - \beta^+
\]

\[
= A^+_{s, (0,0)} + A^+_{s, (1,0)} \delta_{\alpha^-} + \frac{1}{2} A^+_{s, (2,0)} \delta_{\alpha^-}^2 + O(\delta_{\alpha^-}^3)
\]

\[
+ \left( A^+_{s, (0,1)} + A^+_{s, (1,1)} \delta_{\alpha^-} + \frac{1}{2} A^+_{s, (2,1)} \delta_{\alpha^-}^2 + O(\delta_{\alpha^-}^3) \right) \epsilon + O(\epsilon^2)
\]

\[
= 1 - \beta^{+*} + \left( A^+_{s, (1,0)} d + A^+_{s, (0,1)} \right) \epsilon + O(\epsilon^2)
\]  

(S33)
where \( A^+_{\pm}(p,q) \equiv \left. \frac{\partial^{p+q} G(x,c)}{\partial x^p \partial c^q} \right|_{(\alpha^+,\epsilon')} \), \( A^+_{\pm}(0,0) \equiv A^+(\alpha^-; \epsilon') = 1 - \beta^+, \) and we have implicitly use the result of Eq.(S29) to track the correct order of \( \delta\alpha^+ \) and \( \epsilon \). Hence, we can define

\[
\delta \beta^+ \equiv \beta^+ - \beta^+ = - \left( A^+_{\pm}(1,0) d + A^+_{\pm}(0,1) \right) \epsilon + \mathcal{O}(\epsilon^2). \tag{S34}
\]

Now we consider the Taylor expansion of

\[
\n_{\text{core}}^\pm = G^\pm(1 - \alpha^+) - G^\pm(\beta^+) - c(1 - \beta^+ - \alpha^+) \alpha^\pm \tag{S35}
\]

around the critical point term by term.

First of all,

\[
1 - \beta^- - \alpha^- = 1 - \beta^- + A^-_{\pm}(1,0) a \epsilon^2 + \left( \frac{1}{2} A^-_{\pm}(2,0) a^2 + A^-_{\pm}(0,1) \right) \epsilon + \mathcal{O}(\epsilon^2) - (\alpha^- + c \epsilon + \mathcal{O}(\epsilon^2))
\]

\[
= \rho^- + \left( A^-_{\pm}(1,0) a \right) \epsilon^2 + \left( \frac{1}{2} A^-_{\pm}(2,0) a^2 + A^-_{\pm}(0,1) - d \right) \epsilon + \mathcal{O}(\epsilon^2) \tag{S36}
\]

and

\[
1 - \beta^+ - \alpha^+ = 1 - \beta^+ + A^+_{\pm}(1,0) d + A^+_{\pm}(0,1) \epsilon + \mathcal{O}(\epsilon^2) - (\alpha^+ + a \epsilon^2 + b \epsilon + \mathcal{O}(\epsilon^2))
\]

\[
= \rho^+ - a \epsilon^2 + \left( A^+_{\pm}(1,0) d + A^+_{\pm}(0,1) - b \right) \epsilon + \mathcal{O}(\epsilon^2) \tag{S37}
\]

where

\[
\rho^\mp \equiv (1 - \beta^\pm; \epsilon') - \alpha^\mp; \epsilon'
\]

denotes the difference between the largest and smallest root of \( f^\pm(x, c) \) at the critical threshold, which is the origin of the discontinuous core percolation.

Then

\[
c(1 - \beta^- - \alpha^-) \alpha^+
\]

\[
= (c^{+} + \epsilon) \left[ \rho^+ + A_{\pm}^{-(1,0)} a \epsilon^2 + \left( \frac{1}{2} A_{\pm}^{-(2,0)} a^2 + A_{\pm}^{-(0,1)} - d \right) \epsilon + \mathcal{O}(\epsilon^2) \right] (\alpha^+ + a \epsilon^2 + b \epsilon + \mathcal{O}(\epsilon^2))
\]

\[
= c^{+} \alpha^+ \rho^+ + (c^{+} A_{\pm}^{-(1,0)} \alpha^+ a + c^{+} A_{\pm}^{-1} a) \epsilon^2
\]

\[
+ \left[ c^{+} \alpha^+ + \left( \frac{1}{2} A_{\pm}^{-(2,0)} a^2 + A_{\pm}^{-(0,1)} - d \right) \right] + \alpha^+ \rho^+ + A_{\pm}^{-1} a^2 + c^{+} \rho^+ b \right] \epsilon + \mathcal{O}(\epsilon^2) \tag{S39}
\]
and
\[
c(1 - \beta^+ - \alpha^+)\alpha^-
= (c^* + c) \left[ \rho^+ - a e^2 + \left( A^+_{\alpha^+}(1,0) d + A^+_{\alpha^+}(0,1) - b \right) \right] \epsilon + O(\epsilon^2)\]
\[
= c^* \alpha^+ \rho^+ + (-c^* \alpha^- a) \epsilon^2 + \left[ c^* \alpha^+ (A^+_{\alpha^+}(1,0) d + A^+_{\alpha^+}(0,1) - b) + \alpha^- \rho^+ + c^* \rho^+ d \right] \epsilon + O(\epsilon^3).
\]

(S40)

Secondly,
\[
G^\pm(1 - \alpha^+, c)
= G^\pm(1 - \alpha^+, c^*) + G^\pm(1,0) \left|_{(1-\alpha^+, \alpha^+)} \right. (-\delta_{\alpha^+}) + \left. \frac{1}{2} G^\pm(2,0) \right|_{(1-\alpha^+, \alpha^+)} \delta_{\alpha^+}^2 + O(\delta_{\alpha^+}^3)
+ \left( G^\pm(0,1) \left|_{(1-\alpha^+, \alpha^+)} \right. + G^\pm(1,1) \left|_{(1-\alpha^+, \alpha^+)} \right. (-\delta_{\alpha^+}) + \left. \frac{1}{2} G^\pm(2,1) \right|_{(1-\alpha^+, \alpha^+)} \delta_{\alpha^+}^2 + O(\delta_{\alpha^+}^3) \right) \epsilon
+ O(\epsilon^2).
\]

(S41)

So
\[
G^+(1 - \alpha^-, c)
= G^+(1 - \alpha^-, c^*) + G^+(1,0) \left|_{(1-\alpha^-, \alpha^-)} \right. (-\delta_{\alpha^-}) + \left. \frac{1}{2} G^+(2,0) \right|_{(1-\alpha^-, \alpha^-)} \delta_{\alpha^-}^2 + O(\delta_{\alpha^-}^3)
+ \left( G^+(0,1) \left|_{(1-\alpha^-, \alpha^-)} \right. + G^+(1,1) \left|_{(1-\alpha^-, \alpha^-)} \right. (-\delta_{\alpha^-}) + \left. \frac{1}{2} G^+(2,1) \right|_{(1-\alpha^-, \alpha^-)} \delta_{\alpha^-}^2 + O(\delta_{\alpha^-}^3) \right) \epsilon
+ O(\epsilon^2).
\]

(S42)

and
\[
G^-(1 - \alpha^+, c)
= G^-(1 - \alpha^+, c^*) + G^-(1,0) \left|_{(1-\alpha^+, \alpha^-)} \right. (-\delta_{\alpha^+}) + \left. \frac{1}{2} G^-(2,0) \right|_{(1-\alpha^+, \alpha^-)} \delta_{\alpha^+}^2 + O(\delta_{\alpha^+}^3)
+ G^-(0,1) \left|_{(1-\alpha^+, \alpha^-)} \right. \epsilon + O(\epsilon^2)
+ \left[ -c^* (1 - \beta^+ \beta^-) b + \frac{1}{2} G^-(2,0) \left|_{(1-\alpha^+, \alpha^-)} \right. a^2 + G^-(0,1) \left|_{(1-\alpha^+, \alpha^-)} \right. \epsilon + O(\epsilon^2) \right)
\]

(S43)

where we have used the fact that \( G^\pm(x) = \sum_{k=0}^{\infty} P^\pm(k) k x^{k-1} = c \sum_{k=0}^{\infty} Q^\pm(k+1) x^k = cA^\pm(1 - x) \)

and hence \( G^\pm(1,0) \left|_{(1-\alpha^+, \alpha^-)} \right. = c^* A^\pm(\alpha^+, \alpha^-) = c^* (1 - \beta^+ \beta^-) \).

Similarly
\[
G^\pm(\beta^+, c)
= G^\pm(\beta^+, c^*) + G^\pm(1,0) \left|_{(\beta^+, \alpha^+)} \right. \delta_{\beta^+} + \left. \frac{1}{2} G^\pm(2,0) \right|_{(\beta^+, \alpha^+)} \delta_{\beta^+}^2 + O(\delta_{\beta^+}^3)
+ \left( G^\pm(0,1) \left|_{(\beta^+, \alpha^+)} \right. + G^\pm(1,1) \left|_{(\beta^+, \alpha^+)} \right. \delta_{\beta^+} + \left. \frac{1}{2} G^\pm(2,1) \right|_{(\beta^+, \alpha^+)} \delta_{\beta^+}^2 + O(\delta_{\beta^+}^3) \right) \epsilon
+ O(\epsilon^2).
\]

(S44)
So

\[ G^+(\beta^-, c) = G^+(\beta^-, c^*) + G^{+,(1,0)}|_{(\beta^-, \cdot, c^*)} \delta_{\beta^-} + \frac{1}{2} G^{+,(2,0)}|_{(\beta^-, \cdot, c^*)} \delta_{\beta^-} + G^{+,(0,1)}|_{(\beta^-, \cdot, c^*)} \epsilon + \mathcal{O}(\epsilon^2) \]

\[ = G^+(\beta^-, c^*) + \left[ c^* \alpha^+ (A_{\cdot}^{-,(1,0)} a) \right] \epsilon + \frac{1}{2} G^{+,(2,0)}|_{(\beta^-, \cdot, c^*)} \left( A_{\cdot}^{-,(1,0)} a \right)^2 + G^{+,(0,1)}|_{(\beta^-, \cdot, c^*)} \epsilon + \mathcal{O}(\epsilon^2) \]  

(S45)

and

\[ G^-(\beta^+, c) = G^- - (\beta^+, \cdot, c^*) + G^{-,(1,0)}|_{(\beta^+, \cdot, c^*)} \delta_{\beta^+} + G^{-,(0,1)}|_{(\beta^+, \cdot, c^*)} \epsilon + \mathcal{O}(\epsilon^2) \]

\[ = G^- - (\beta^+, c^*) + \left[ c^* \alpha^- (A_{\cdot}^+,(1,0) d) - A_{\cdot}^+,(1,0) \right] + G^{-,(0,1)}|_{(\beta^+, \cdot, c^*)} \epsilon + \mathcal{O}(\epsilon^2) \]  

(S46)

where we have used \( G^{+,(1,0)}|_{(\beta^+, \cdot, c^*)} = c^* A^\pm (1 - \beta^+ c^*), c^* = c^* \alpha^\pm \).

Now we have

\[ n^+_{\text{core}} = G^+(1 - \alpha^-, c) - G^+(\beta^-, c) - c(1 - \beta^- - \alpha^-) \alpha^+ \]

\[ = \left[ G^+(1 - \alpha^-, c^*) - G^+(\beta^-, c^*) - c^* \alpha^+ \rho^- \right] + (-c^* \rho^- a) \epsilon^2 \]

\[ + \left[ -c^* \rho^+ d + \left( G^{+,(0,1)}|_{(1-\alpha^-, \cdot, c^*)} - G^{+,(0,1)}|_{(\beta^-, \cdot, c^*)} \right) - \alpha^+ \rho^- - c^* \rho^- b \]

\[ + \frac{1}{2} G^{+,(2,0)}|_{(\beta^-, \cdot, c^*)} \left( A_{\cdot}^{-,(1,0)} a \right)^2 - c^* A_{\cdot}^{-,(1,0)} a \]  

\[ \epsilon + \mathcal{O}(\epsilon^2) \]

\[ = \Delta^+_n + B \epsilon^2 + C^+ \epsilon + \mathcal{O}(\epsilon^2) \]  

(S47)

and

\[ n^-_{\text{core}} = G^-(1 - \alpha^+, c) - G^-(\beta^+, c) - c(1 - \beta^+ - \alpha^+) \alpha^- \]

\[ = \left[ G^-(1 - \alpha^+, c^*) - G^- (\beta^+, c^*) - c^* \alpha^- \rho^+ \right] + (-c^* \rho^+ a) \epsilon^2 \]

\[ + \left[ -c^* \rho^+ b + \left( G^{-,(0,1)}|_{(1-\alpha^+, \cdot, c^*)} - G^{-,(0,1)}|_{(\beta^+, \cdot, c^*)} \right) - \alpha^- \rho^+ - c^* \rho^+ d \]

\[ + \frac{1}{2} G^{-,(2,0)}|_{(1-\alpha^+, \cdot, c^*)} a \]  

\[ \epsilon + \mathcal{O}(\epsilon^2) \]

\[ = \Delta^-_n + B \epsilon^2 + C^- \epsilon + \mathcal{O}(\epsilon^2) \]  

(S48)
\[ \Delta_n^\pm = G^\pm(1 - \alpha^\pm, c) - G^\pm(\beta^\pm, c) - c^* \alpha^\pm \rho^\pm \]  
\[ B = -c^* \rho^- a \]  
\[ C^+ = -c^* \rho^+ d + \left( G^+,(0,1)|_{(1 - \alpha^+, c^*)} - G^+,,(0,1)|_{(\beta^+, c^*)} \right) - \alpha^+, \rho^- - c^* \rho^- b \]  
\[ -\frac{1}{2} G^+,(2,0)\|(\beta^+, c^*) (A^-,(1,0) a^2 - c^* A^-(1,0) a^2 \]  
\[ C^- = -c^* \rho^- b + \left( G^-,(0,1)|_{(1 - \alpha^+, c^*)} - G^-,,(0,1)|_{(\beta^+, c^*)} \right) - \alpha^- \rho^+ - c^* \rho^+ d \]  
\[ + \frac{1}{2} G^-,,(2,0)\|(1 - \alpha^+, c^*) a^2. \]  

So
\[ n_{\text{core}} = \frac{1}{2} (n_{\text{core}}^+ + n_{\text{core}}^-) = \Delta_n + B \epsilon^2 + C \epsilon + O(\epsilon^3) \]  
\[ n_{\text{core}} = \Delta_n^+ + \Delta_n^- \]  

where \( \Delta_n \equiv \frac{1}{2} (\Delta_n^+ + \Delta_n^-) \) is the jump in \( n_{\text{core}} \) at the critical point and the coefficient \( C \equiv \frac{1}{2} (C^+ + C^-) \).

Similarly
\[ l_{\text{core}} = c(1 - \beta^+ - \alpha^+)(1 - \beta^- - \alpha^-) \]  
\[ = (c^* + \epsilon) \left( \rho^+ - a \epsilon^2 + \left( A^+,(1,0) d + A^+,(0,1) - b \right) \epsilon + O(\epsilon^3) \right) \]  
\[ - \left[ \rho^- + \left( A^-(1,0) a \right) \epsilon^2 + \left( \frac{1}{2} A^-,(2,0) a^2 + A^-,(0,1) - d \right) \epsilon + O(\epsilon^3) \right] \]  
\[ = c^* \rho^- \rho^+ + \left( c^* A^+,(1,0) a \rho^+ - c^* a \rho^- \right) \epsilon^2 \]  
\[ + \left[ -a^2 c^* A^+,(1,0) + c^* \left( -d + A^+,(0,1) + \frac{1}{2} a^2 A^+,(2,0) \right) \rho^+ \right. \]  
\[ \left. + \rho^- \left( -bc^* + c^* A^+,(0,1) + c^* d A^+,(1,0) + \rho^+ \right) \right] \epsilon + O(\epsilon^3) \]  
\[ = \Delta_1 + D \epsilon^2 + E \epsilon + O(\epsilon^3) \]  

where
\[ \Delta_1 \equiv c^* \rho^- \rho^+ \]  
\[ D \equiv c^* a \left( A^+,(1,0) \rho^+ - \rho^- \right) \]  
\[ E \equiv \left[ -a^2 c^* A^+,(1,0) + c^* \left( -d + A^+,(0,1) + \frac{1}{2} a^2 A^+,(2,0) \right) \rho^+ \right. \]  
\[ \left. + \rho^- \left( -bc^* + c^* A^+,(0,1) + c^* d A^+,(1,0) + \rho^+ \right) \right]. \]  
\[ (S54) \]
In sum, in the critical regime $\epsilon = c - c^* \to 0^+$

$$
\begin{align*}
    n_{\text{core}} - \Delta_n & \sim (c - c^*)^\eta \\
    l_{\text{core}} - \Delta_l & \sim (c - c^*)^\theta
\end{align*}
$$

(S58)

with the critical exponents

$$
\eta = \theta = \frac{1}{2}.
$$

(S59)

Note that $\eta$ and $\theta$ have to be equal. Otherwise, the mean degree of the core will either diverge or vanish at the critical point. Both cases are unrealistic.

We emphasize that the above calculations do not use any specific functional form of $A^\pm(x, c)$. Instead, we only assume that $A^\pm(x, c)$ is a continuous function of $c$. Therefore we conclude that $\eta = \theta = \frac{1}{2}$ are universal critical exponents for all random network ensembles with $P^+(k) \neq P^-(k)$ when both $P^+(k)$ and $P^-(k)$ are parametrized continuously in mean degree $c$.

Eq. S58 clearly demonstrates that in the general non-degenerate case core percolation is a hybrid phase transition, i.e. $n_{\text{core}}$ (or $l_{\text{core}}$) has a jump at the critical point as at a first-order phase transition but also has a critical singularity as at a continuous transition.

B. degenerate case: $P^+(k) = P^-(k) = P(k)$

In the degenerate case, i.e. either directed networks with the same out- and in-degree distributions or undirected networks, we define

$$
f(x, c) \equiv A(A(x)) - x.
$$

(S60)

$\alpha$ is the smallest root of $f(x, c) = A(A(x), c) - x$. Denote $\alpha^*$ as the $\alpha$-value at the percolation threshold $c^*$. We can perform the Taylor expansion of $f(x, c)$ around the critical point $(\alpha^*, c^*)$ and yield

$$
f(x, c) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \delta^p \epsilon^q \frac{1}{p!q!} f_*(p,q)
$$

(S61)

where $\delta \equiv x - x^*$, $\epsilon \equiv c - c^*$, and $f_*(p,q) \equiv \frac{\partial^{p+q} f}{\partial x^p \partial c^q} \bigg|_{(x^*,c^*)}$.

As discussed in main text, we have

$$
    f_*(0,0) = f_*(1,0) = 0,
$$

(S62)

which are exactly the equations we used to calculate the critical point $(\alpha^*, c^*)$. Here we prove that $f(x, c)$ satisfies two additional equations:

$$
    f_*(0,1) = f_*(2,0) = 0.
$$

(S63)
Proof. We first prove a very useful fact that
\[ A^{(1,0)}_s = \frac{\partial A(x, c)}{\partial x} \bigg|_{(\alpha^*, c^*)} = -1. \]  
(S64)

Note that
\[ f^{(1,0)}_s = \frac{\partial f(x, c)}{\partial x} \bigg|_{(\alpha^*, c^*)} = \frac{\partial A(x, c)}{\partial x} \bigg|_{(A(\alpha^*, c^*), c^*)} \cdot \frac{\partial A(x, c)}{\partial x} \bigg|_{(\alpha^*, c^*)} - 1 = \left( A^{(1,0)}_s \right)^2 - 1 = 0 \]  
(S65)
where we have used the fact \( \alpha^* = A(\alpha^*, c^*) = 1 - \beta^* \) (because \( \alpha = A(\alpha, c) \) is always a solution of the equation \( \alpha = A(A(\alpha, c), c) \), and for \( c \leq c^* \) there is only one solution, which is just \( \alpha = A(\alpha, c) \)).

Since \( A(x, c) \) is a monotonically decreasing function in the range \( x \in [0, 1] \), so we have \( \frac{\partial A(x, c)}{\partial x} < 0 \).

Consequently, Eq. S65 yields \( A^{(1,0)}_s = -1 \).

According to the chain rule of differentiation we have
\[ f^{(0,1)}_s = \frac{\partial f(x, c)}{\partial c} \bigg|_{(\alpha^*, c^*)} = \frac{\partial A(x, c)}{\partial c} \bigg|_{(A(\alpha^*, c^*), c^*)} + \frac{\partial A(x, c)}{\partial x} \bigg|_{(A(\alpha^*, c^*), c^*)} \cdot \frac{\partial A(x, c)}{\partial c} \bigg|_{(\alpha^*, c^*)} = A^{(0,1)}_s + A^{(1,0)}_s A^{(0,1)}_s = A^{(0,1)}_s - A^{(0,1)}_s = 0 \]  
(S66)

and
\[ f^{(2,0)}_s = \frac{\partial^2 f(x, c)}{\partial x^2} \bigg|_{(\alpha^*, c^*)} = \frac{\partial^2 A(x, c)}{\partial x^2} \bigg|_{(A(\alpha^*, c^*), c^*)} \cdot \left( \frac{\partial A(x, c)}{\partial x} \bigg|_{(\alpha^*, c^*)} \right)^2 + \frac{\partial A(x, c)}{\partial x} \bigg|_{(A(\alpha^*, c^*), c^*)} \cdot \frac{\partial^2 A(x, c)}{\partial x^2} \bigg|_{(\alpha^*, c^*)} = A^{(2,0)}_s \cdot A^{(1,0)}_s^2 + A^{(1,0)}_s \cdot A^{(2,0)}_s = A^{(2,0)}_s - A^{(2,0)}_s = 0. \]  
(S67)

Q.E.D.
Now we consider the Taylor expansion of \( f(x, c) \), e.g. Eq.S61. Since \( f_s^{(0,0)} = f_s^{(1,0)} = f_s^{(0,1)} = f_s^{(2,0)} = 0 \), the leading-order terms of \( \epsilon, \delta \), and \( \epsilon \delta \) are

\[
\frac{1}{2} f_s^{(0,2)} \epsilon^2; \quad \frac{1}{6} f_s^{(3,0)} \delta^3; \quad \text{and} \quad f_s^{(1,1)} \epsilon \delta
\]

respectively. (Note that in general \( f_s^{(0,2)} \), \( f_s^{(3,0)} \) and \( f_s^{(1,1)} \) are nonzero.) To make sure that the leading-order terms are canceled with each other, the only self-consistent scaling relation between \( \delta \) and \( \epsilon \) is given by

\[
\delta = a \epsilon^2 + b \epsilon + O(\epsilon^3)
\]

(S68)

where \( a \) and \( b \) are constants determined by the following equations:

\[
f_s^{(1,1)} a + \frac{1}{6} f_s^{(3,0)} a^3 = 0
\]

\[
f_s^{(1,1)} b + \frac{1}{2} f_s^{(3,0)} a^2 b + \frac{1}{2} f_s^{(2,0)} + \frac{1}{24} f_s^{(4,0)} a^4 = 0.
\]

(S69)

For example, for ER random graph we have \( a = -\frac{\sqrt{6}e}{c^3} = -\sqrt{6}e^{-\frac{3}{2}} \), and \( b = (\alpha^*)^2 = e^{-2} \).

Now we consider the Taylor expansion of \( n_{\text{core}} = G(1 - \alpha) - G(\beta) - c(1 - \beta - \alpha) \alpha \) around the critical point \((\alpha^*, c^*)\). First of all,

\[
1 - \beta = A(\alpha, c)
\]

\[
= A_s^{(0,0)} + A_s^{(1,0)} \delta + \frac{1}{2} A_s^{(2,0)} \delta^2 + O(\delta^3) + \left( A_s^{(0,1)} + A_s^{(1,1)} \delta + \frac{1}{2} A_s^{(2,1)} \delta^2 + O(\delta^3) \right) \epsilon + O(\epsilon^2)
\]

\[
= \alpha^* - \delta + \frac{1}{2} A_s^{(2,0)} \delta^2 + A_s^{(0,1)} \epsilon + O(\delta^3)
\]

\[
= 1 - \beta^* - \delta + \frac{1}{2} A_s^{(2,0)} \delta^2 + A_s^{(0,1)} \epsilon + O(\delta^3)
\]

(S70)

where we have implicitly use the fact that \( \alpha^* = 1 - \beta^* \) and the result of Eq.S68 to correctly track the orders of \( \delta \) and \( \epsilon \). So

\[
\delta_\beta \equiv \beta - \beta^* = \delta - \frac{1}{2} A_s^{(2,0)} \delta^2 - A_s^{(0,1)} \epsilon + O(\delta^3),
\]

(S71)

\[
1 - \beta - \alpha = \left( \alpha^* - \delta + \frac{1}{2} A_s^{(2,0)} \delta^2 + A_s^{(0,1)} \epsilon + O(\delta^3) \right) - (\alpha^* + \delta)
\]

\[
= -2\delta + \frac{1}{2} A_s^{(2,0)} \delta^2 + A_s^{(0,1)} \epsilon + O(\delta^3),
\]

(S72)

\[
c(1 - \beta - \alpha) = (c^* + \epsilon) \left( -2\delta + \frac{1}{2} A_s^{(2,0)} \delta^2 + A_s^{(0,1)} \epsilon + O(\delta^3) \right) (\alpha^* + \delta)
\]

\[
= -2c^* \alpha^* \delta + c^* \alpha^* \left( \frac{1}{2} A_s^{(2,0)} \delta^2 + A^{(0,1)} \epsilon \right) - 2c^* \delta^2 + O(\delta^3)
\]

(S73)
Secondly,

\[ G(1 - \alpha, c) = G(1 - \alpha^*, c^*) + G^{(1,0)}|(1-\alpha^*,c^*)(-\delta) + \frac{1}{2} G^{(2,0)}|(1-\alpha^*,c^*)\delta^2 + \mathcal{O}(\delta^3) \]
\[ + \left( G^{(0,1)}|(1-\alpha^*,c^*) + G^{(1,1)}|(1-\alpha^*,c^*)(-\delta) + \frac{1}{2} G^{(2,1)}|(1-\alpha^*,c^*)\delta^2 + \mathcal{O}(\delta^3) \right) \epsilon + \mathcal{O}(\epsilon^2) \]
\[ = G(1 - \alpha^*, c^*) - c^*\alpha^* \delta + \frac{1}{2} G^{(2,0)}|(1-\alpha^*,c^*)\delta^2 + G^{(0,1)}|(1-\alpha^*,c^*)\epsilon + \mathcal{O}(\delta^3) \]

(S74)

where we have used the fact that \( G'(x) = \sum_{k=0}^{\infty} P(k) k x^{k-1} = c \sum_{k=0}^{\infty} Q(k+1) x^k = c A(1 - x) \) and \( G^{(1,0)}|(1-\alpha^*,c^*) = c^* A(\alpha^*, c^*) = c^* \alpha^* \).

Similarly

\[ G(\beta, c) = G(\beta^*, c^*) + G^{(1,0)}|(\beta^*,c^*)\delta_\beta + \frac{1}{2} G^{(2,0)}|(\beta^*,c^*)\delta^2_\beta + \mathcal{O}(\delta^3_\beta) \]
\[ + \left( G^{(0,1)}|(\beta^*,c^*) + G^{(1,1)}|(\beta^*,c^*)\delta_\beta + \frac{1}{2} G^{(2,1)}|(\beta^*,c^*)\delta^2_\beta + \mathcal{O}(\delta^3_\beta) \right) \epsilon + \mathcal{O}(\epsilon^2) \]
\[ = G(1 - \alpha^*, c^*) + c^*\alpha^* \left( \delta - \frac{1}{2} A^2(2,0) \delta^2 - A^2(0,1) \epsilon \right) + \frac{1}{2} G^{(2,0)}|(1-\alpha^*,c^*)\delta^2 \\
+ G^{(0,1)}|(1-\alpha^*,c^*)\epsilon + \mathcal{O}(\delta^3) \].

(S75)

Finally, we have

\[ n_{\text{core}} = G(1 - \alpha, c) - G(\beta, c) - c(1 - \beta - \alpha)\alpha \]
\[ = 2c^* \delta^2 + \mathcal{O}(\delta^3) \]
\[ = 2c^* \left( a\epsilon^{\frac{3}{2}} + b\epsilon + \mathcal{O}(\epsilon^{\frac{3}{2}}) \right)^2 + \mathcal{O}(\delta^3) \]
\[ = 2c^* a^2 \epsilon + \mathcal{O}(\epsilon^{\frac{3}{2}}) \]

(S76)

and

\[ l_{\text{core}} = \frac{1}{2} c(1 - \beta - \alpha)^2 \]
\[ = \frac{1}{2} (c^* + \epsilon) \left( -2\delta + \frac{1}{2} A^2(2,0) \delta^2 + A^2(0,1) \epsilon + \mathcal{O}(\delta^3) \right)^2 \]
\[ = 2c^* \delta^2 + \mathcal{O}(\delta^3) \]
\[ = 2c^* a^2 \epsilon + \mathcal{O}(\epsilon^{\frac{3}{2}}) \].

(S77)

In sum, in the critical regime \( \epsilon = c - c^* \to 0^+ \)

\[ \begin{cases} 
  n_{\text{core}} \sim (c - c^*)^{\eta'} \\
  l_{\text{core}} \sim (c - c^*)^{\theta'} 
\end{cases} \]

with the critical exponents \( \eta' = \theta' = 1 \).
VIII. NUMERICAL SIMULATIONS

We check our analytical results with extensive numerical calculations by performing the GLR procedure on finite discrete networks generated by the static model [5, 6].

Fig.S6a and S6b show \( n_{\text{core}} \) and \( l_{\text{core}} \) (in symbols) for undirected ER networks and asymptotically SF networks with different degree exponents. For comparison, analytical results for infinite large networks are also shown (in lines). Clearly, core percolation is continuous in this case. This is fundamentally different from the \( k \geq 3 \)-core percolation, which becomes discontinuous for ER networks and SF networks with \( \gamma > 3 \) [8, 9].

Fig.S6c and S6d show the results of \( n_{\text{core}} \) and \( l_{\text{core}} \) for directed networks. For directed networks with the same in- and out-degree distributions, e.g. directed ER networks or directed SF networks with \( \gamma_{\text{in}} = \gamma_{\text{out}} \) generated by the static model, the core percolation is still continuous. But for directed networks with different in- and out-degree distributions, e.g. directed SF networks with \( \gamma_{\text{in}} \neq \gamma_{\text{out}} \) generated by the static model, the core percolation looks discontinuous. The discontinuity in \( n_{\text{core}} \) (or \( l_{\text{core}} \)) increases as the difference between \( \gamma_{\text{in}} \) and \( \gamma_{\text{out}} \) increases (see Fig.S6e,f).
Directed SF networks with fixed ER and asymptotically SF model networks. The core percolation is continuous if the out- and in-degree becomes discontinuous for ER networks and SF networks with core percolation is continuous, which is fundamentally different from the different values of Rényi (ER) and asymptotically scale-free (SF) networks imposing degree cutoff in constructing the SF networks [10].

Finite size effects are more discernable for $\gamma \to 2$, which can be eliminated by imposing degree cutoff in constructing the SF networks [10]. a-b, The normalized core size ($n_{core} = N_{core}/N$) and the normalized number of edges in the core ($l_{core} = L_{core}/N$) for undirected model networks: Erdős-Rényi (ER) and asymptotically scale-free (SF) with different values of $\gamma$. For both model networks, the core percolation is continuous, which is fundamentally different from the $k \geq 3$-core percolation, which becomes discontinuous for ER networks and SF networks with $\gamma > 3$ [8, 9]. c-d, $n_{core}$ and $l_{core}$ for directed ER and asymptotically SF model networks. The core percolation is continuous if the out- and in-degree distributions are the same ($P^+(k) = P^-(k)$) while it becomes discontinuous if $P^+(k) \neq P^-(k)$. e-f, For directed SF networks with fixed $\gamma_{out} = 3.0$, by tuning $\gamma_{in}$ we see that the discontinuity in both $n_{core}$ and $l_{core}$ become larger as the difference between $\gamma_{in}$ and $\gamma_{out}$ increases.
IX. REAL NETWORKS

We also apply our theory to real-world networks with known degree distributions. In Fig.S7 we demonstrate that in some cases our analytical results calculated with degree distribution as the only input predict with surprising accuracy the core size of real networks. Yet, in other cases there is a noticeable difference between theory and reality, which suggests the presence of extra structure in the real-world networks that is not captured by the degree distribution. In particular we find that almost all the directed real-world networks have larger core sizes than the theoretical predictions (see Fig.S7a,b). In other words, those networks are “overcored”. While if we treat those networks as undirected ones, their core sizes deviate from our theory in a more complicated manner. The effects of higher order correlations (e.g. degree correlations [11, 12], clustering [13], loop structure [14] and modularity [15]) may play very important roles to explain the discrepancy between theory and reality.

All the real-world networks analyzed in the paper are listed and briefly described in Table S1. For each network, we show its type, name and reference; number of nodes ($N$) and edges ($L$); and brief description.
Figure S7: Normalized core size for real networks, compared with analytical predictions. All the real networks considered here are directed. For data sources and references, see Table 1. 

**a-b** By applying the GLR procedure we yield $n_{\text{real}}^{\text{core}}$ and $l_{\text{real}}^{\text{core}}$. Using out- and in-degree distributions ($P^+(k)$ and $P^-(k)$) as the only inputs, we also analytically calculate $n_{\text{analytic}}^{\text{core}}$ and $l_{\text{analytic}}^{\text{core}}$. 

**c-d** By ignoring the direction of the edges, we can treat the original directed networks as undirected ones and apply the GLR procedure to get $n_{\text{real}}^{\text{core}}$ and $l_{\text{real}}^{\text{core}}$. Similarly, we can calculate $n_{\text{analytic}}^{\text{core}}$ and $l_{\text{analytic}}^{\text{core}}$ with the degree distribution $P(k)$ as the only input.
<table>
<thead>
<tr>
<th>Name</th>
<th>N</th>
<th>L</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Regulatory</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TRN-Yeast-1 [16]</td>
<td>4,441</td>
<td>12,873</td>
<td>Transcriptional regulatory network of <em>S. cerevisiae</em></td>
</tr>
<tr>
<td>TRN-Yeast-2 [17]</td>
<td>688</td>
<td>1,079</td>
<td>Same as above (compiled by different group).</td>
</tr>
<tr>
<td>TRN-EC-1 [18]</td>
<td>1,550</td>
<td>3,340</td>
<td>Transcriptional regulatory network of <em>E. coli</em></td>
</tr>
<tr>
<td>TRN-EC-2 [17]</td>
<td>418</td>
<td>519</td>
<td>Same as above (compiled by different group).</td>
</tr>
<tr>
<td><strong>Trust</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>college student [20, 21]</td>
<td>32</td>
<td>96</td>
<td>Social networks of positive sentiment (college students).</td>
</tr>
<tr>
<td>prison inmate [20, 21]</td>
<td>67</td>
<td>182</td>
<td>Same as above (prison inmates).</td>
</tr>
<tr>
<td>Slashdot [22]</td>
<td>82,168</td>
<td>948,464</td>
<td>Social network (friend/foe) of Slashdot users.</td>
</tr>
<tr>
<td><strong>Food Web</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Power Grid</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Metabolic</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><em>E. coli</em> [27]</td>
<td>2,275</td>
<td>5,763</td>
<td>Metabolic network of <em>E. coli</em>.</td>
</tr>
<tr>
<td><em>S. cerevisiae</em> [27]</td>
<td>1,511</td>
<td>3,833</td>
<td>Metabolic network of <em>S. cerevisiae</em>.</td>
</tr>
<tr>
<td><em>C. elegans</em> [27]</td>
<td>1,173</td>
<td>2,864</td>
<td>Metabolic network of <em>C. elegans</em>.</td>
</tr>
<tr>
<td><strong>Electronic</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>s838 [17]</td>
<td>512</td>
<td>819</td>
<td>Electronic sequential logic circuit.</td>
</tr>
<tr>
<td><strong>Circuits</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>s420 [17]</td>
<td>252</td>
<td>399</td>
<td>Same as above.</td>
</tr>
<tr>
<td>s208 [17]</td>
<td>122</td>
<td>189</td>
<td>Same as above.</td>
</tr>
<tr>
<td><strong>Neuronal</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ArXiv-HepTh [28]</td>
<td>27,770</td>
<td>352,807</td>
<td>Citation networks in HEP-TH category of Arxiv.</td>
</tr>
<tr>
<td>ArXiv-HepPh [28]</td>
<td>34,546</td>
<td>421,578</td>
<td>Citation networks in HEP-PH category of Arxiv.</td>
</tr>
<tr>
<td><strong>WWW</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>nd.edu [29]</td>
<td>325,729</td>
<td>1,497,134</td>
<td>WWW from nd.edu domain.</td>
</tr>
<tr>
<td>stanford.edu [22]</td>
<td>281,903</td>
<td>2,312,497</td>
<td>WWW from stanford.edu domain.</td>
</tr>
<tr>
<td><strong>Internet</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>p2p-2 [31]</td>
<td>8,846</td>
<td>31,839</td>
<td>Same as above (at different time).</td>
</tr>
<tr>
<td>p2p-3 [31]</td>
<td>8,717</td>
<td>31,525</td>
<td>Same as above (at different time).</td>
</tr>
<tr>
<td><strong>Social</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>UCIonline [32]</td>
<td>1,899</td>
<td>20,296</td>
<td>Online message network of students at UC, Irvine.</td>
</tr>
<tr>
<td><strong>Communication</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Email-epoch [33]</td>
<td>3,188</td>
<td>39,256</td>
<td>Email network in a university.</td>
</tr>
<tr>
<td>Cellphone [34]</td>
<td>36,595</td>
<td>91,826</td>
<td>Call network of cell phone users.</td>
</tr>
<tr>
<td><strong>Intra-organizational</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Freemans-2 [35]</td>
<td>34</td>
<td>830</td>
<td>Social network of network researchers.</td>
</tr>
<tr>
<td>Freemans-1 [35]</td>
<td>34</td>
<td>695</td>
<td>Same as above (at different time).</td>
</tr>
<tr>
<td>Manufacturing [36]</td>
<td>77</td>
<td>2,228</td>
<td>Social network from a manufacturing company.</td>
</tr>
<tr>
<td>Consulting [36]</td>
<td>46</td>
<td>879</td>
<td>Social network from a consulting company.</td>
</tr>
</tbody>
</table>


