Recursive Stochastic Choice*

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Abstract

This paper provides axiomatic characterizations of two sorts of recursive stochastic choice rules, where the agent makes his current decisions using a forward-looking value function that takes into account his future randomizations. Both of the choice rules we examine generalize logistic choice and are equivalent to it in static problems. The rules differ in how the agent views future choice sets and how he views his future randomizations. One rule is equivalent to the discounted logit used in applied work, and exhibits a "preference for flexibility;" the other is "error-averse" and penalizes the addition of undesirable choices to a menu.

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1 Introduction

Observed individual choice is typically stochastic. Most of the theoretical literature on stochastic choice has focused on static models, though dynamic random utility models are commonly used in estimation and much of modern economics emphasizes dynamic choice.

This paper provides the first characterization of stochastic choice in a dynamic setting, where choices made today can influence the possible choices available tomorrow. The models we consider are recursive, in the sense that the agent's choice in each period is made taking into account the continuation value of the future problem, where this continuation value incorporates the agent's awareness that he will choose randomly in the future. We focus on two sorts of choice rules that have many properties in common; in particular, they both coincide with the logit model in static problems. The extra information provided by the agent's choices between future menus lets us distinguish between the two rules; in particular the two choice rules correspond to different relationships between the agent's choice of a menu of future outcomes in some period t and the choice *from* that menu in period $t+1$.

Under one rule the agent has a "preference for flexibility" in the sense of preferring larger choice sets, as in Kreps (1979) and Dekel, Lipman, and Rustichini (2001); in particular the agent prefers to add duplicate items to a menu. One interpretation of this comes from a representation in which stochastic choice arises from privately observed payoff shocks; here adding items to a menu cannot hurt the agent as even an item that is ex ante unlikely to be optimal will only be chosen if it turns out to be the best choice; an alternative representation suggests that the agent simply enjoys randomization.

With the other, "error-averse," choice rule, the agent dislikes adding inferior items to a menu; this corresponds to a representation with consideration costs based on menu size, and also to a representation where the agent has to expend effort to avoid choosing the wrong item by accident. In this case the agent's choices satisfy a version of "set betweenness"as in Gul and Pesendorfer (2001), and the agent is "duplicate averting", which we interpret as aversion to potential errors.

To motivate and explain our representations and results it is useful to recall several explanations from the literature for stochastic choice in static problems:

1) Agents might maximize their expected utility given privately observed payoff shocks as in McFadden (1973) and Harsanyi (1973a), so that even choices that are typically unappealing could be optimal when the payoff shock is large. This is the starting point for the discounted logit model used in estimation (see, e.g., Miller, 1984; Rust, 1989; Hendel and Nevo, 2006; Kennan and Walker, 2011; Sweeting, 2011; Gowrisankaran and Rysman, 2012; and Eckstein and Wolpin, 1989; Rust, 1994; Aguirregabiria and Mira, 2010 for surveys) and corresponds to

what we call the discounted logit representation (Definition 1). This paper provides axiomatic foundations for this model, and also proposes and axiomatizes a closely related error-averse alternative, discounted logit with menu costs (Definition 4).

2) Agents might randomize as the result of error or inattention under a cost of attending to the decision, with no attention resulting in a prespecified error distribution, increasing costs for making sure the desired action is selected, and prohibitive costs for implementing a pure action. This explanation is explored in van Damme (1991) and van Damme and Weibull (2002). This corresponds to our discounted relative entropy representation (Definition 6), which is equivalent to discounted logit with menu costs.

3) Agents might maximize the sum of expected utility and a non-linear perturbation function that makes it optimal to assign positive probability to every action, as in Harsanyi (1973b) and Machina (1985). This corresponds to our discounted entropy representation (Definition 3). This sort of objective function is also analyzed in Fudenberg and Levine (1995), Hart and Mas-Colell (2001), Hofbauer and Hopkins (2005), Hofbauer and Sandholm (2002), Fudenberg and Takahashi (2011), all of which focus on the case of repeated stochastic choice in static games.[1] One motivation for this paper is to extend that work to allow for dynamic considerations, such as would arise in learning to play an extensive-form game.

4) Finally, observed choices might be the result of psychophysical "weighting functions, "as in Thurstone (1927) and Luce (1959), who characterized stochastic choice in static problems under the additional assumptions of positivity (all actions have positive probability) and Independence of Irrelevant Alternatives or "IIA." Under these assumptions, the observed choice distribution can be generated by assigning weights $w(z)$ to each action z , and then picking an action with probability equal to its share of the total weight. This corresponds to our discounted flexibility preferring Luce and discounted error-averse Luce representations (Definitions 9 and 10).

In the static setting, there are well-known equivalences between the above explanations of stochastic choice. The Luce choice rule from point 4) can equivalently be described by the choice rule that arises from a version of point 3) in which the agent maximizes the sum of expected utility u and a constant η times the entropy of the distribution.[2] This choice rule also corresponds to logit choice, meaning a Harsanyi model from point 1) where the payoff shocks are i.i.d. with extreme value type-1 distributions. Finally, it will be important in what follows that the same choice rule also arises from maximizing the sum of expected utility and a constant η times the *relative entropy* of the distribution of actions as this relative entropy is

[1] Fudenberg and Levine (1995) show that this generates a choice rule that is Hannan consistent (Hannan, 1957). This property might be of interest to non-Bayesian agents who are not completely sure that the outcomes (or the map from actions to outcomes) will be as specified and so want to randomize to guard against malevolent choices by Nature.

[2] The corresponding Luce weight on action z is $w(z) = \exp(u(z)/\eta)$.

just the difference between the entropy of the chosen action distribution and the entropy of a baseline distribution. Here, the baseline distribution corresponds to the action distribution if the agent does not pay attention, and the relative entropy corresponds to the cost of attending to decisions from point 2).

Our main goal is to better understand the issues involved in modeling an agent who makes random choices not only over actions with immediate consumption consequences but also over actions that can alter the choice sets that will be available in the future. An additional benefit of our approach is that the relationship between choices in various periods can help distinguish between interpretations and representations that are equivalent in the static setting. To make this first step in characterizing dynamic stochastic choice we maintain the IIA assumption throughout the paper.[3] Although this assumption is restrictive, and can make implausible predictions about the impact of adding duplicate choices, we maintain it here to focus on the new issues that arise when modeling stochastic choice in a dynamic setting.[4] Also, one of our motivations is to axiomatize the widely used discounted logit model and that model implies IIA.

Our axioms for the discounted logit model are (a) the IIA assumption (so that static choice is logit); (b) an axiom that implies that preferences over future decision problems are independent of the outcome in the current period, and conversely; (c) a separability axiom to arrive at the convenient discounted sum formulation; and finally (d) an "aggregate recursivity"axiom that says that a future choice problem A is more likely to be selected now than some other B if elements of A are more likely to be selected (in the aggregate) than elements of B when the union of these menus is presented as an immediate decision next period. The error-averse form uses the same axioms, except that aggregate recursivity is replaced by "average recursivity,"which says roughly that menus are judged by their average as opposed to aggregate future attractiveness.

Each set of axioms leads to "recursive" representations that express the agent's choice at time t in terms of the utility of time-t outcomes and a continuation value. We give two equivalent versions of each representation; one version generalizes the dynamic logit of empirical work,

[3]There is a theoretical literature that characterizes the static stochastic choices that can be generated by general Harsanyi random-utility models without the IIA assumption (Falmagne, 1978; Barberá and Pattanaik, 1986; Gul and Pesendorfer, 2006), and recent work by Gul, Natenzon, and Pesendorfer (2012) that characterizes generalizations of nested logit, but there is not an analogous characterization of the stochastic choices consistent with the perturbed utility functions described in point 3) above. (Hofbauer and Sandholm (2002) consider a related but different setting where whe size of the choice set is held fixed while the underlying utilities—which are known to the analyst—vary.)

[4]The issue with duplicates is not due to IIA but arises in any random utility model with independent shocks. A similar issue arises in nested-logit estimation, where adding similar alternatives makes some purchase more likely than the alternative "no purchase,"and in the limit of a very large set of goods almost everyone must purchase. Ackerberg and Rysman (2005) propose two alternative responses to this issue in a static model: either scale the variance of the extreme-value shocks with the number of goods in the menu, or add a term to the utility function that depends on various characteristics of the menu. This is similar in spirit to logit with menu costs representation, see Definition 4.

and the other is a more general version of the rule "maximize the discounted sum of expected utility and a constant η times the (relative) entropy of the selected distribution." To arrive at precisely the discounted-sum formulations, however, requires additional assumptions to ensure that future periods enter into the value function in an additively separable way, and a further assumption is needed to ensure a constant discount factor; the conditions we use here parallel those of Koopmans (1960) and Fishburn (1970).

A preference for flexibility—that is, for larger choice sets—arises in the discounted entropy representation from the fact that larger choice sets have a higher maximum entropy, and in the discounted logit specification from the fact that each new object added to a menu provides another chance for a good realization of the random shocks. In the error-averse discounted relative entropy representation, enlarging the menu does not change the maximum relative entropy, so adding duplicate items does not change maximum utility; in the associated discounted logit representation, adding a duplicate doesn't change maximum utility because the benefit of the additional random draw is exactly offset by a "menu" or "consideration" cost.

As should be clear by now, this paper relates to several strands of the axiomatic decision theory literature, to foundational literature in game theory, and to empirical work on dynamic choice. We discuss these relationships in the concluding section, after we have developed our representations and the associated axioms. The appendix gives a detailed outline of the proofs, as well as additional definition and results on recursive representations. The Online Appendix contains all the proofs.

2 Primitives

For any set S let $K(S)$ be the collection of nonempty finite subsets of S, to be interpreted as the collection of possible *choice problems*. For any set S let $\Delta(S)$ be the collection of probability measures on S with finite support. Let $\Delta^n := \Delta(\{1, \ldots, n\})$.

We assume that time is discrete, $t = 0, 1, \ldots, T$ with T finite. Let Z be the set of all one-period *outcomes*.[5] In any period t, an individual choice problem is called a *menu*; we denote period t menus by letters $A_t, B_t, C_t, \ldots$ and the space in which all menus live by $\mathcal{M}_t$. The elements of the menu are called *actions* and are denoted by $a_t, b_t, c_t, \ldots$; the space in which all actions live is denoted by $\mathcal{A}_t$. We construct the set of *dynamic choice problems* recursively. Let $\mathcal{A}_T := Z$ and $\mathcal{M}_T := K(\mathcal{A}_T)$; in period T actions are synonymous with one-period outcomes because in the terminal period there is no future, and period T menus are just finite collections of one-period outcomes. Now we define the possible menus and actions in earlier time periods

[5] Our richness axiom will imply that Z is infinite, but we do not assume any structure on this set; possible cases include: a subset of $\mathbb{R}$ (monetary payoffs), or $\mathbb{R}^n$ (consumption bundles or acts), and $\Delta(\mathbb{R}^n)$ (lotteries).

$t = 0, 1, \dots, T-1$ as follows:

$$\mathcal{A}_t := Z \times \mathcal{M}_{t+1} \quad \text{and} \quad \mathcal{M}_t := K(\mathcal{A}_t).$$

Thus, an action a_t at time t is a pair (z_t, A_{t+1}) of current outcome and time-$t+1$ menu, while a menu A_t at time t is a finite set of such actions. For notational convenience, we set $\mathcal{M}_{T+1} = \emptyset$ and use the convention that $Z \times \mathcal{M}_{T+1} = Z$.

It is important that the actions today can restrict future opportunities without having any impact on the current outcome; for example the agent might face the period $T-1$ menu $\{(z, A_T), (z, A'_T)\}$. Moreover, the agent might face the choice at time $T-3$ of whether to commit to her time-T outcome in period $T-2$ or in period $T-1$ As we will see, our flexibility-preferring and error-averse representations predict different choices here; this is one advantage of allowing a general finite horizon as opposed to restricting the model to have only two time periods.

A *dynamic stochastic choice rule* is a collection of mappings $\{\Phi_t\}_{t=0}^{T}$ such that $\Phi_t : \mathcal{M}_t \to \Delta(\mathcal{A}_t)$, with the property that for any $A_t \in \mathcal{M}_t$ the support of $\Phi_t(A_t)$ is a subset of A_t.[6] For any $A_t \in \mathcal{M}_t$, $\Phi_t(A_t)$ is the probability distribution on actions that represents the stochastic choice from A_t. For notational convenience, we write $\Phi_t[B_t|A_t]$, to denote the probability that the chosen action will belong to the set B_t when the choice set is A_t. For $(z, A_{t+1}) \in A_t$ we write $\Phi_t[(z, A_{t+1})|A_t]$ instead of $\Phi_t[\{(z, A_{t+1})\}|A_t]$; note that $\Phi_t[B_t|A_t] = \sum_{b_t \in B_t} \Phi_t[b_t|A_t]$.

3 Representations

We will provide two sets of additively separable representations, the first set for what we call "flexibility-preferring" preferences and the second for preferences that are "error averse." Paralleling past work on static stochastic choice, we will give several alternate representations for each sort of preference; some of these may be better suited for estimation while others may be more tractable in theoretical work.

3.1 Representations of Flexibility-Preferring Choice Rules

We will show that the following discounted-sum formulations are equivalent, and characterize their consequences for dynamic stochastic choice. Each of them has a "preference for flexibility" in the sense that even if a is preferred to b, the menu $\{a, b\}$ is preferred to {a}.

Definition 1 (Discounted Logit Representation). A dynamic stochastic choice rule has a *Discounted Logit Representation* iff there exist surjective felicity functions $v_t : Z \to \mathbb{R}$, *discount*

[6]Note that this implicitly assumes that choice at time t is independent of past history. We believe that our approach could be extended to allow for history dependence.

factor $\delta_t > 0$, and value functions $U_t : \mathcal{A}_t \to \mathbb{R}$ recursively defined by

$$U_t(z_t, A_{t+1}) = v_t(z_t) + \delta_t \mathbb{E}\left[\max_{a_{t+1} \in A_{t+1}} U_{t+1}(a_{t+1}) + \epsilon_{a_{t+1}}\right] \tag{1}$$

such that for all $t = 0, \ldots, T$, all A_t, and all $a_t \in A_t$

$$\Phi_t[a_t \mid A_t] = \text{Prob}\left(U_t(a_t) + \epsilon_{a_t} \geq \max_{b_t \in A_t} U_t(b_t) + \epsilon_{b_t}\right), \tag{2}$$

where ϵ_{a_t} are i.i.d. with the extreme value distribution $F(\theta) = \exp(-\exp(-\theta/\eta_t - \gamma))$, γ is Euler's constant, and $\eta_t > 0$ are noise level parameters.

The representation is stationary iff $v_0 = \cdots = v_{T-1} = U_T$, $\delta_0 = \cdots = \delta_{T-1}$, and $\eta_0 = \cdots = \eta_T$; a stationary representation is impatient iff $\delta < 1$.

In this representation, the ε terms correspond to payoff shocks that are observed by the decision maker but not by the analyst, as in static random utility models. Note well that these payoff shocks apply to every action in every menu, just as they do under the "Assumption AS" or equation (3.7) of Rust (1994). For example, if a consumer first decides how much canned tuna fish to buy and later decides how much to consume each day, payoff shocks apply to the purchase decision as well as to consumption.[7] Intuitively, the reason that these preferences are flexibility-preferring is that each new object added to the menu provides another chance for a good realization of the random shocks ϵ_{z_T}. Note also that this is the simplest sort of discounted logit representation, as it does not include a state variable and assumes stationarity. This is often relaxed in empirical work, but we maintain them here to focus on the issues related to recursive choice.

Definition 2. For any $q \in \Delta^n$, let $H(q) := -\sum_{i=1}^n q_i \log(q_i)$ be the *entropy* of q.

Definition 3 (Discounted Entropy Representation). A dynamic stochastic choice rule has a *Discounted Entropy Representation* if and only there exist $\eta_t > 0$, surjective felicity functions $v_t : Z \to \mathbb{R}$, *discount factor* $\delta_t > 0$, and value functions $U_t : \mathcal{A}_t \to \mathbb{R}$ recursively

[7]We call this sort of payoff shocks "shocks to actions," as opposed to the case where payoff shocks apply only to a set of "immediate outcomes" Although this is a standard assumption in empirical work, the literature using the discounted logit model does recognize that there may sometimes be problems with adding shocks in this way. Note that the distinction between the two sorts of shocks parallels the difference between the initial model proposed by Hendel and Nevo (2006), where the payoff shocks apply to the value of consuming a durable good at different dates, and the model they use in estimation (the "simplified dynamic problem" defined on p. 1651), where shocks apply to the act of purchasing goods that will be stored for future consumption. They argue the simplified model is equivalent for their purposes.

defined by

$$U_t(z_t, A_{t+1}) = v_t(z_t) + \delta_t \left[\max_{q \in \Delta(A_{t+1})} \sum q(a_{t+1}) U_{t+1}(a_{t+1}) + \eta_{t+1} H(q) \right] \tag{3}$$

such that for all $t = 0, \ldots, T$ and A_t

$$\Phi_t[\cdot \mid A_t] = \arg\max_{q \in \Delta(A_t)} \sum q(a_t) U_t(a_t) + \eta_t H(q). \tag{4}$$

The representation is *stationary* iff $v_0 = \cdots = v_{T-1} = U_T$, $\delta_0 = \cdots = \delta_{T-1}$, and $\eta_0 = \cdots \eta_T$; a stationary representation is *impatient* iff $\delta < 1$.

Note that the entropy term H is non-negative, as are the η_t; for this reason the agent always at least weakly prefers larger choice sets as he can assign probability 0 to the added options. Note also that the entropy of the uniform distribution over n objects is $\log(n)$, which increases without bound in n; hence the agent will prefer a menu of many roughly similar objects to the singleton menu with just one of them. We elaborate on the consequences of this below.

3.2 Representations for Error-Averse Choice Rules

With the other kind of preferences we consider, the agent prefers a singleton menu to a larger menu formed by adding inferior choices. Such preferences provide indirect evidence that the random choices arise from errors, and we will interpret them that way. We show that the following two error-averse representations are equivalent, and characterize them in terms of the choices they generate.

Definition 4 (Discounted Logit with Menu Costs Representation). A dynamic stochastic choice rule has a *Discounted Logit with Menu Costs Representation* iff there exist surjective felicity functions $v_t : Z \to \mathbb{R}$, a *discount factor* $\delta_t > 0$, and value functions $U_t : \mathcal{A}_t \to \mathbb{R}$ recursively defined by

$$U_t(z_t, A_{t+1}) = v_t(z_t) + \delta_t \mathbb{E} \left[\max_{a_{t+1} \in A_{t+1}} U_{t+1}(a_{t+1}) + \epsilon_{a_{t+1}} - \eta_{t+1} \log |A_{t+1}| \right] \tag{5}$$

such that for all $t = 0, \ldots, T$, all A_t, and all $a_t \in A_t$

$$\Phi_t[a_t \mid A_t] = \text{Prob} \left(U_t(a_t) + \epsilon_{a_t} \geq \max_{b_t \in A_t} U_t(b_t) + \epsilon_{b_t} \right), \tag{6}$$

where ϵ_{a_t} are i.i.d. with the extreme value distribution $F(\theta) = \exp(-\exp(-\theta/\eta_t - \gamma))$, γ is Euler's constant and $\eta_t > 0$ are noise level parameters.

The representation is *stationary* iff $v_0 = \cdots = v_{T-1} = U_T$, $\delta_0 = \cdots = \delta_{T-1}$, and $\eta_0 = \cdots \eta_T$; a stationary representation is *impatient* iff $\delta < 1$.

In this representation, choice is derived from value in exactly the same way as in the discounted logit representation, the difference is that the value of a menu is decreasing in its size. These preferences are error-averse even though each new object added to the menu provides another chance for a good realization of the random shock ϵ_{z_t} because of the constant $\eta_t \log |A_{t+1}|$ that is subtracted from the overall value of the menu. Moreover, this constant is such that the agent is just indifferent about whether to add a duplicate to a singleton menu, though this indifference is easier to see in the next, equivalent, representation.

Definition 5. For any $q \in \Delta^n$, let $R(q) := \sum_{i=1}^n q_i \log(nq_i)$ be the *relative entropy* of q with respect to the uniform distribution.

Definition 6 (Discounted Relative Entropy Representation). A dynamic stochastic choice rule has a *Discounted Relative Entropy Representation* if and only there is $\eta_T > 0$ and $U_T : Z \to \mathbb{R}$ and for $t = 0, 1, \ldots, T-1$, there exists $\eta_t > 0$, surjective felicity functions $v_t : Z \to \mathbb{R}$, an *discount factor* $\delta_t > 0$, and value functions $U_t : \mathcal{A}_t \to \mathbb{R}$ recursively defined by

$$U_t(z_t, A_{t+1}) = v_t(z_t) + \delta_t \left[\max_{q \in \Delta(A_{t+1})} \sum q(a_{t+1}) U_{t+1}(a_{t+1}) - \eta_{t+1} R(q) \right] \tag{7}$$

such that for all $t = 0, \ldots, T$ and A_t

$$\Phi_t[\cdot \mid A_t] = \arg\max_{q \in \Delta(A_t)} \sum q(a_t) U_t(a_t) - \eta_t R(q). \tag{8}$$

The representation is *stationary* iff $v_0 = \cdots = v_{T-1} = U_T$, $\delta_0 = \cdots = \delta_{T-1}$, and $\eta_0 = \cdots \eta_T'$; a stationary representation is *impatient* iff $\delta < 1$.

With these preferences, the agent tends to prefer removing the lowest-ranked items from a menu, but is indifferent between adding duplicates to a singleton menu: When presented with a menu of two equally good items, the agent will choose to randomize uniformly, so that the relative entropy term is 0, and the realized utility will thus be the same as from a menu with only one of those two items. This indifference is consistent with our interpretation of stochastic choice as arising from error, but rules out preferences that incorporate only consideration costs based on the size of the menu. As the equivalent logit representation suggests, though, the error-averse preferences are consistent with a combination of consideration costs based on menu size and logit-type payoff shocks.

3.3 Illustrative Example

To illustrate these choice rules, consider the following example of a high school student's choice of whether or not to go to college, which we adapt from Train (2009, Chapter 7). There are two periods: the college years and the post-college years. In period 0 the student can either go to college, which gives immediate payoff $v(c)$, or take a job and work instead, which gives immediate payoff $v(w)$. Her choices in period 0 have consequences for the sets of options available in period 1; namely, if the student works in period 0, there will be only one job available for her in period 1 (job z), which gives her a payoff of $v(z)$. On the other hand, if the student goes to college in period 0, she will choose between two jobs x and y with payoffs $v(x)$ and $v(y)$. Thus, the decision tree that the student faces looks like the one in Figure 1.

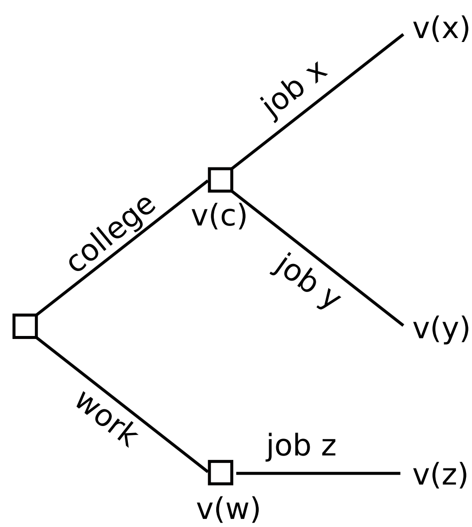

Figure 1: Choosing whether to go to college

To represent this decision tree as one of our *dynamic choice problems*, let $A_1 = \{x, y\}$ and $B_1 = \{z\}$ be the two possible continuation problems in period 1 (after choosing to go to college or not). Then the time zero choice problem is $A_0 = \{(c, A_1), (w, B_1)\}$. We write $\Phi_0[(c, A_1)|A_0]$ to denote the probability that the student chooses to go to college in period 0 and $\Phi_1[x|A_1]$ to denote the probability that in period 1 (conditional on having gone to college) the student chooses a job x.

Under the discounted logit model of stochastic choice (Definition 1), the value of the continuation choice problem A_1 is

$$\mathbb{E}\max\{v(z) + \epsilon_x, v(y) + \epsilon_y\} = \log\left(e^{v(x)} + e^{v(y)}\right) \tag{9}$$

and the probability of choosing job x from A_1 is

$$\frac{e^{v(x)}}{e^{v(x)} + e^{v(y)}}, \tag{10}$$

where both formulas follow from the assumption that all $\epsilon \sim$ i.i.d. extreme values with parameter $\eta = 1$ and the well known "log-sum" representation of the logit value function (see, e.g., Train, 2009, Chapter 3, or Lemma 6 in our Online Appendix). Thus, the student goes to college if

$$v(c) + \delta \log\left(e^{v(x)} + e^{v(y)}\right) + \epsilon_{(c,A_1)} \geq v(w) + \delta v(z) + \epsilon_{(w,B_1)}.$$

The probability that the student will go to college thus equals

$$\Phi_0[(c, A_1)|A_0] = \frac{\exp\Big(v(c) + \delta \log\left(e^{v(x)} + e^{v(y)}\right)\Big)}{\exp\Big(v(c) + \delta \log\left(e^{v(x)} + e^{v(y)}\right)\Big) + \exp\Big(v(w) + \delta v(z)\Big)}.$$

The calculation for the discounted entropy model, Definition 6, (see Lemma 4 in the Online Appendix) leads to identical choice probabilities. On the other hand, the discounted logit with menu costs model assigns value

$$\log\left(e^{v(x)} + e^{v(y)}\right) - \log 2$$

to A_1; and probability that the student will go to college thus equals

$$\Phi_0[(c, A_1)|A_0] = \frac{\exp\Big(v(c) + \delta \log\left(e^{v(x)} + e^{v(y)}\right) - \delta \log 2\Big)}{\exp\Big(v(c) + \delta \log\left(e^{v(x)} + e^{v(y)}\right) - \delta \log 2\Big) + \exp\Big(v(w) + \delta v(z)\Big)}.$$

(The calculations for the discounted relative entropy model lead to the same choice probabilities.) Thus, the probability that the student will go to college is lower when menu costs are present because the option value that the choice set A_1 carries is smaller. An extreme case is when $v(x) = v(y) = v(z) = v_1$ and $v(c) = v(w) = v_0$.

Then under discounted logit

$$\Phi_0[(c, A_1)|A_0] = \frac{e^{v_0 + \delta(v_1 + \log 2)}}{e^{v_0 + \delta(v_1 + \log 2)} + e^{v_0 + \delta v_1}} > \frac{1}{2},$$

while under discounted logit with menu costs

$$\Phi_0[(c, A_1)|A_0] = \frac{e^{v_0 + \delta v_1}}{e^{v_0 + \delta v_1} + e^{v_0 + \delta v_1}} = \frac{1}{2}.$$

One way to understand this difference is to recall the red-bus/blue-bus problem of Debreu (1960), where adding an identical copy of an item raises the probability of choosing that item (or its duplicate). With discounted logit preferences, this problem manifests itself not only by adding probability mass to the item, but also by adding value to future opportunity sets containing duplicates, while discounted logit with menu costs does not value the addition of duplicates. We analyze this problem further in Section 4.4.

4 Axioms

We present the axioms in four subsections. The axioms in the first subsection simply ensure that preferences reduce to the logit case in a static problem and that preferences are independent of the fixed continuation problem, which also implies that preferences over today's outcomes with a fixed continuation problem reduce to logit. The second group is a single axiom that relates choices at times t and $t+1$ when the continuation problem is a singleton. This restricted domain means that the axiom can avoid taking a stand on whether the agent prefers flexibility; we use it to help explain the axioms in the third group. The representations diverge with the third group of axioms: here we develop parallel axioms for the flexibility-preferring and error-averse cases. These axioms are sufficient to obtain recursive representations of the preferences, i.e., representations that express choice at time t in terms of the utility of time-t outcomes and a continuation value. (The equivalence between such representations and these axioms is established in Appendix A.) However, just as in deterministic dynamic choice (Koopmans (1960) and Fishburn (1970)), discounted representations require an additional separability assumption to ensure that the representation has the necessary additive separability. The discounted representations introduced above correspond to the condition "separable ratios" that we introduce in the fourth subsection, along with the assumptions that characterize stationarity and impatience. The discounted Luce representation of Section 7 corresponds instead to a "separable differences" condition that we define there.

4.1 Logit-esque Axioms

Axiom 1 (Positivity). For any t, $A_t \in \mathcal{M}_t$ and $a_t \in A_t$ we have $\Phi_t[a_t \mid A_t] > 0$.

As argued by McFadden (1973), a zero probability is empirically indistinguishable from a positive but small probability, and since keeping all probabilities positive facilitates estimation, the positivity axiom is usually assumed in econometric analysis of both static and dynamic discrete choice. In settings where the stochastic term arises from utility perturbations, positivity corresponds to the assumption that the utility perturbations have sufficiently large support that even a typically unattractive option is occasionally preferred. Positivity is implied by perturbed objective function representation for stochastic choice; it is motivated there by the fact that no deterministic rule can be Hannan (or "universally") consistent.

Axiom 2 (Stage IIA). For any $t \leq T$, $a_t, b_t \in \mathcal{A}_t$, and $A_t \in \mathcal{M}_t$ such that $a_t, b_t \in A_t$ we have

$$\frac{\Phi_t[a_t \mid \{a_t, b_t\}]}{\Phi_t[b_t \mid \{a_t, b_t\}]} = \frac{\Phi_t[a_t \mid A_t]}{\Phi_t[b_t \mid A_t]},$$

whenever the probabilities in the denominators are both positive.

Stage IIA says that the "choice ratio"- that is, the ratio of choice probabilities between two actions, does not depend on other actions in the menu; it reduces to the standard IIA axiom in period T by our assumption that choices do not depend on past history. Notice that positivity and IIA imply that the stochastic preference $\succsim_t$ is transitive (see, e.g., Luce, 1959).

As is well known, this axiom is very restrictive. As we noted in the introduction, it and the closely related logistic choice rule are widely used in empirical work for reasons of tractability. Assuming IIA lets us focus on other aspects of stochastic dynamic choice; we discuss some of the issues related to relaxing this assumption in Section 8.

Our primitive is a dynamic stochastic choice rule $\{\Phi_t\}_{t=0}^T$. The notion of stochastic preference $\succsim_t$ on A_t is derived from Φ_t as follows.

Definition 7 (Stochastic Preference). Action a_t is stochastically preferred to action b_t at time t, denoted $a_t \succsim_t b_t$, if and only if

$$\Phi_t\left[a_t \mid \{a_t, b_t\}\right] \geq \Phi_t\left[b_t \mid \{a_t, b_t\}\right].$$

Note that the relation $\succsim_t$ on M_t is complete and transitive (by Axiom 2). Given $\succsim_t$ we define induced stochastic preferences on Z and M_{t+1} as follows: z is stochastically preferred to w at time t, denoted $z \succsim_t w$, if and only if for any menu $A_{t+1} \in M_{t+1}$

$$(z, A_{t+1}) \succsim_t (w, A_{t+1}).$$

Menu A_{t+1} is stochastically preferred to B_{t+1} at time t, denoted $A_{t+1} \succsim_t B_{t+1}$, if and only if for any outcome $z \in Z$

$$(z, A_{t+1}) \succsim_t (z, B_{t+1}).$$

Note that the induced relations $\succsim_t$ are always transitive. Axiom 3 below will imply that they are complete. Anticipating that, we define the strict stochastic preference $\succ_t$ and stochastic indifference $\sim_t$ as the asymmetric and symmetric parts of $\succsim_t$.

Axiom 3 (Ordinal Time Separability). For all $t < T$, $z, z' \in Z$, and $A_{t+1}, A'_{t+1} \in \mathcal{M}_{t+1}$

1. $(z, A_{t+1}) \succsim_t (z, A'_{t+1})$ iff $(z', A_{t+1}) \succsim_t (z', A'_{t+1})$

2. $(z, A_{t+1}) \succsim_t (z', A_{t+1})$ iff $(z, A'_{t+1}) \succsim_t (z', A'_{t+1})$

This axiom says that preferences over future decision problems are independent of the outcome in the current period, and conversely that preferences over current outcomes do not depend on the choice problem to be confronted tomorrow. It is thus a stochastic version of Postulate 3 of Koopmans (1960), and corresponds to what Fishburn (1970, Chapter 4) calls independence. Notice that as its name suggests Axiom 3 applies only to the the ordinal stochastic preference; it does not require that the numerical values of the choice probabilities be equal. Our Axioms 12 and 16 strengthen Axiom 3 by ensuring the preservation of numerical values of choice probabilities (in a ratio or difference form respectively).

Axiom 3 together with either of the recursivity axioms of section 4.3 is sufficient for a history-independent recursive representation of the agent's preferences (see Theorems 5 and 6 in the Appendix.)[8] As in the case of deterministic choice, additively separable representations require stronger forms of independence. The discounted representations presented above correspond to "separable ratios;" the discounted Luce representation in Section 7 instead corresponds to "separable differences."

We use the following richness axiom, which is slightly weaker that the Strong Richness axiom of Gul, Natenzon, and Pesendorfer (2012). It implies that the set of choice probabilities is convex-ranged and (by taking $\lambda = 1$) that for each outcome z in each period t there are countably many of what we will later call "duplicates of z". We use this in the proofs of Theorems 1 and 2 where it helps us obtain additive time-separability and uniqueness.[9]

Axiom 4 (Richness). For any $t \leq T$, action $(z_t, A_{t+1}) \in Z \times \mathcal{M}_{t+1}$, finite set of outcomes $Z' \subseteq Z$, and $\lambda \in (0, \infty)$ there exists an outcome $z_t^\lambda \in Z \setminus Z'$, such that

$$\frac{\Phi_t[(z_t^\lambda, A_{t+1})|\{(z_t, A_{t+1}), (z_t^\lambda, A_{t+1})\}]}{\Phi_t[(z_t, A_{t+1})|\{(z_t, A_{t+1}), (z_t^\lambda, A_{t+1})\}]} = \lambda.$$

[8] Axiom 3 could be relaxed to allow time-t preferences to depend on past choices only through an observed state variable, as in many empirical applications, see, e.g., Aguirregabiria and Mira (2010). For example, even without Axiom 3, Axioms 1, 2 and 8 imply a history-dependent form of the "Recursive Logit" representation obtained in Theorem 5. Some of the empirical literature uses state-dependent preferences to capture the way current actions can influence future menus while holding the nominal set of actions constant: Instead of making an action infeasible, its utility is set to be minus infinity, as in Train (2009, Chapter 7). We do not need to use this modeling device, because we have explicitly modeled the way current actions influence future menus. Modeling history-dependent menus via payoffs seems innocuous in the usual discounted logit setting, but it does not make sense with error-averse preferences, as can be seen by considering the discounted logit with menu costs representation.

[9] As in the deterministic choice setting of of Koopmans (1960) and Fishburn (1970), the assumption of a rich set of alternatives gives our separability assumptions more bite. We suspect that one could obtain additive time-separability without uniqueness by replacing the richness assumption with some other conditions, as in Fishburn (1970)'s Theorem 4.1, but we have not explored this possibility.

4.2 Tying Choices in Different Time Periods

Now we introduce several axioms that relate choices in consecutive time periods. We have two reasons for interest in these axioms. One is the normative idea that the decision maker should base his choice of menus on the attractiveness of the menus' components, and these are reflected in the probabilities these components are selected in the next period. The other motivation to better understand the representations presented in Section 3, which imply all of the axioms of this subsection.

The least restrictive and perhaps simplest way to link consecutive decisions compares two binary choice problems: one at time t where both options involve the same instantaneous payoff and differ only in the future, and the other at time $t+1$ where both options are exactly the continuations of the options from the period t choice problem. The axiom requires that the stochastic preference between these two options is the same.

Axiom 5 (Singleton Recursivity). For all $t < T$ and all singleton menus $A_{t+1} = \{(z_{t+1}, A_{t+2})\}$, and $B_{t+1} = \{(w_{t+1}, B_{t+2})\} \in \mathcal{M}_{t+1}$

$$A_{t+1} \succsim_t B_{t+1} \quad \text{iff} \quad (z_{t+1}, A_{t+2}) \succsim_{t+1} (w_{t+1}, B_{t+2})$$

Below we use another way of writing the same axiom, which will also be more convenient for its strengthening.

Axiom 5'. For all $t < T$, and singleton menus $A_{t+1}, B_{t+1} \in \mathcal{M}_{t+1}$

$$A_{t+1} \succsim_t B_{t+1} \quad \text{iff} \quad \Phi_{t+1}\left[A_{t+1} \mid A_{t+1} \cup B_{t+1}\right] \geq \Phi_{t+1}\left[B_{t+1} \mid A_{t+1} \cup B_{t+1}\right].$$

Axiom 5 may not be as restrictive as it seems at first because it only pins down the period t preference on singleton continuation menus.

Notice that Singleton Recursivity is a form of monotonicity: raising the $t+1$ choice probability of an action makes the singleton choice set more attractive at time t. We now state a monotonicity condition that also applies to nonsingleton menus.

Axiom 6 (Monotone Recursivity). Let $\{a^1_{t+1}, \ldots, a^n_{t+1}\}$ and $\{\hat{a}^1_{t+1}, \ldots, \hat{a}^n_{t+1}\}$ be such that $\hat{a}^i_{t+1} \succ_{t+1} a^i_{t+1}$ for some i and $\hat{a}^i_{t+1} \succsim_{t+1} a^i_{t+1}$ for all i. Then

$$\{\hat{a}^1_{t+1}, \ldots, \hat{a}^n_{t+1}\} \succ_t \{a^1_{t+1}, \ldots, a^n_{t+1}\}$$

Note that this axiom rules out time-t preferences that evaluate menus only by their most or least preferred component.

In the discounted representations we study, the effect of the time $t+1$ choice probabilities on period-t choice is not only monotone but linear:

Axiom 7 (Linear Recursivity). For all $t < T$, and menus $A_{t+1}, B_{t+1} \in \mathcal{M}_{t+1}$ with the same cardinality

$$A_{t+1} \succsim_t B_{t+1} \quad \text{iff} \quad \Phi_{t+1}\left[A_{t+1} \mid A_{t+1} \cup B_{t+1}\right] \geq \Phi_{t+1}\left[B_{t+1} \mid A_{t+1} \cup B_{t+1}\right].$$

We show below that discounted logit implies linear recursivity. To see why this is so, recall from the illustrative example above that when $\eta = 1$ the logit model assigns value $\log\left(e^{v(x)} + e^{v(y)}\right)$ to the menu $A_1 = \{x, y\}$. Now let $B_1 = \{f, g\}$ and suppose that the agent has a period-0 choice between $\{(z, A_1)\}$ and $\{(z, B_1)\}$. The probability that the agent chooses (z, A_1) is then

$$\frac{e^{v(z)+\delta \log\left(e^{v(x)}+e^{v(y)}\right)}}{e^{v(z)+\delta \log\left(e^{v(x)}+e^{v(y)}\right)} + e^{v(z)+\log\left(e^{v(f)}+e^{v(g)}\right)}} = \frac{e^{\delta \log\left(e^{v(x)}+e^{v(y)}\right)}}{e^{\delta \log\left(e^{v(x)}+e^{v(y)}\right)} + e^{\log\left(e^{v(f)}+e^{v(g)}\right)}}$$

which exceeds 1/2 when $e^{v(x)} + e^{v(y)} > e^{v(f)} + e^{v(g)}$. This is exactly the condition that the sum of the choice probabilities of elements of A_1 exceed the sum for the elements of B_1, as, e.g., $\Phi_1[x|\{x, y, f, g\}] = \frac{e^{v(x)}}{e^{v(x)}+e^{v(y)}+e^{v(f)}+e^{v(g)}}$. Moreover, as the discounted entropy model is equivalent to logit, it too satisfies linear recursivity; the same follows for the two error-averse discounted representations from the fact that they are equivalent to their flexibility-preferring counterparts when comparing menus of the same size.

One way to think about the relationship between monotone and linear recursivity is that the latter amounts to successively imposing three additional conditions: The ranking of menus should a) depend in a separable way on the individual choice probabilities, b) depend on the elements of the menu only via their choice probabilities and c) be linear in the choice probabilities.[10]

The choice-theoretic implications of linearity seem interesting, but we have not worked out what they are, in large part because even linear recursivity imposes no restrictions on how the agent compares menus of different sizes and so is too weak to lead to useful predictions. In particular, would the agent rather have a menu with a single very good choice or a menu with more options, each of which is less appealing on its own? The answer to this question is what pins down the difference between flexibility-preferring and error-averse preferences, as we show below.

[10] As an example of a non-linear representation that satisfies a) and b), consider the case where menus are ranked by the product of the choice probabilities instead of the sum, i.e. $A_{t+1} \succsim_t B_{t+1}$ iff
$\prod_{a_{t+1} \in A_{t+1}} \Phi_{t+1}\left[a_{t+1} \mid A_{t+1} \cup B_{t+1}\right] \geq \prod_{b_{t+1} \in A_{t+1}} \Phi_{t+1}\left[b_{t+1} \mid A_{t+1} \cup B_{t+1}\right]$. An agent whose choice probabilities on $\{x, y, f, g\}$ are (.25, .25, .4, .1) would strictly prefer $A_1 = \{x, y\}$ to $B_1 = \{f, g\}$, while an agent with discounted logit (or any other preferences that satisfy linear recursivity) would be indifferent.

4.3 Flexibility-Preferring or Error Averse?

Now we turn to axioms that strengthen Linear Recursivity by considering choices between menus of different sizes. The next axiom says that a future choice problem A is more likely to be selected now than some other B if elements of A are more likely to be selected than elements of B when both are presented as an immediate decision next period. The axiom might at first seem to require no more than that the agent is sophisticated, as it is a stochastic version of the temporal consistency axiom of Kreps and Porteus (1978), which requires that a future choice problem A is selected now over some other B if there exists an element of A which is selected over any element of B when both are presented as an immediate decision next period.[11]

Axiom 8 (Aggregate Recursivity). For all $t < T$ and menus $A_{t+1}, B_{t+1} \in \mathcal{M}_{t+1}$

$$A_{t+1} \succsim_t B_{t+1} \quad \text{iff} \quad \Phi_{t+1}\left[A_{t+1} \mid A_{t+1} \cup B_{t+1}\right] \geq \Phi_{t+1}\left[B_{t+1} \mid A_{t+1} \cup B_{t+1}\right].$$

This axiom is satisfied by the discounted flexibility preferring logit choice rule. A one-line proof shows it implies the following stochastic version of Kreps (1979)' Preference for Flexibility.

Axiom 9 (Preference for Flexibility). For all $t < T$ and menus $A_{t+1}, B_{t+1} \in \mathcal{M}_{t+1}$

$$A_{t+1} \succsim_t B_{t+1} \quad \text{whenever} \quad A_{t+1} \supseteq B_{t+1}$$

Proposition 1. *Axiom 8 implies Axiom 9. Moreover, in the presence of Axiom 1, a strict version of Axiom 9 is implied.*

As we show in Propositions 3,4, and 6, this preference for flexibility implies (in the presence of our other maintained assumptions) that the agent has a preference for adding duplicates and near duplicates to a menu, and for making decisions early. The discounted logit and entropy representations satisfy aggregate recursivity and hence have a preference for flexibility.

The error averse logit and entropy representations introduced above satisfy all of the same axioms as the flexibility-prefering ones, except for Axiom 8. Instead of that axiom, they satisfy the following condition, which says that choice problem A is more likely to be selected now then some other B if the *average* of the choice probabilities of elements of A is higher than that of B when the choice set tomorrow is the union of A and B.

Axiom 10 (Average Recursivity). For all $t < T$ and menus $A_{t+1}, B_{t+1} \in \mathcal{M}_{t+1}$

$$A_{t+1} \succsim_t B_{t+1} \quad \text{iff} \quad \frac{1}{|A_{t+1}|}\Phi_{t+1}\left[A_{t+1} \mid A_{t+1} \cup B_{t+1}\right] \geq \frac{1}{|B_{t+1}|}\Phi_{t+1}\left[B_{t+1} \mid A_{t+1} \cup B_{t+1}\right].$$

[11] The axiom is also similar to Koopmans' Postulate 4, which combines the requirement of stationarity with dynamic consistency.

The axiom is especially easy to understand when A_{t+1} is a singleton that is disjoint from B_{t+1}. In this case it says that the singleton menu A_{t+1} is preferred if next period its element is selected with greater than uniform probability (greater than $1/(1+|A'_{t+1}|)$) from the menu $A_{t+1} \cup B_{t+1}$. The axiom rules out preference for flexibility of Kreps (1979) and Dekel, Lipman, and Rustichini (2001), and implies a form of Gul and Pesendorfer (2001)'s Preference for Commitment: Given any A_{t+1} with more than one element, the agent strictly prefers the subset $A_{t+1} \setminus \{a_{t+1}\}$ if and only if $\Phi_{t+1}(a_{t+1}|A_{t+1}) < 1/|A_{t+1}|$. The axiom also implies the following stochastic version of Gul and Pesendorfer (2001) Set Betweenness:

Axiom 11 (Disjoint Set Betweenness). For all $t < T$ and disjoint menus $A_{t+1}, B_{t+1} \in \mathcal{M}_{t+1}$

$$A_{t+1} \succsim_t B_{t+1} \quad \text{implies} \quad A_{t+1} \succsim_t A_{t+1} \cup B_{t+1} \succsim_t B_{t+1}$$

Remark 1. The aggregate and average recursivity conditions can be seen as special cases of a more general "α-recursivity" condition axiom that penalizes larger choice sets by dividing the associated choice probabilities by $|X|^\alpha$.[12] Several of our representation results extend to α-recursivity, but as we do not have motivation or intuition for the more general representations, we have chosen not to include them.

Proposition 2. *Axiom 10 implies Axiom 11.*

4.4 Duplicates

In this section we show how the Aggregate and Average Recursivity axioms differ in their predictions about the impact of adding duplicate choices to the menu.

Definition 8. We say that outcomes z and z' are *duplicates*[13] at time t iff $z \sim_t z'$.

We say that z and z' are ϵ-*duplicates* at time t, denoted $z \sim_t^\epsilon z'$ if for some $C_{t+1} \in \mathcal{M}_{t+1}$

$$\frac{\Phi_t[(z, C_{t+1}) \mid \{(z, C_{t+1}), (z', C_{t+1})\}]}{\Phi_t[(z', C_{t+1}) \mid \{(z, C_{t+1}), (z', C_{t+1})\}]} \in [1-\epsilon, 1+\epsilon].$$

The richness assumption guarantees there are duplicates and $\epsilon-$duplicates for each outcome.

We now show that Aggregate Recursivity (plus the previous axioms) implies the agent is "duplicate loving" while Average Recursivity (and the same previous axioms) implies the agent

[12] Formally, the preferences are α-recursive if there is $\alpha \in [0,1]$ such that for all $t < T$ and $A_{t+1}, B_{t+1} \in \mathcal{M}_{t+1}$

$$A_{t+1} \succsim_t B_{t+1} \quad \text{iff} \quad \frac{1}{|A_{t+1}|^\alpha} \Phi_{t+1}\left[A_{t+1} \mid A_{t+1} \cup B_{t+1}\right] \geq \frac{1}{|B_{t+1}|^\alpha} \Phi_{t+1}\left[B_{t+1} \mid A_{t+1} \cup B_{t+1}\right].$$

[13] Gul, Natenzon, and Pesendorfer (2012) define what it means for two outcomes to be duplicates in a static choice environment without IIA; in the presence of IIA our definition is equivalent to theirs.

is “duplicate averse.” By this we mean the following: suppose that in period $t+1$ $z^2, z^3, \ldots$ are duplicates of z^1. Then Axiom 8 combined with Axioms 1–3 implies the agent is duplicate loving in the sense that for any menu A_{t+1}, a menu of sufficiently many duplicates of any given outcome z^1 is preferred to A_{t+1}. On the other hand, under Axioms 1–3 and 10 the agent is duplicate averse, in the sense that if $z \succ_{t+1} y \succ_{t+1} z^1$ then $\{y\} \succ_t \{z, z^1, z^2, ...z^n\}$ for large enough n. Moreover, these results extend to ϵ-duplicates if ϵ is small enough.

Proposition 3. *Under Axioms 1–3 and 8*

1. *For any sequence of duplicates $z^1, z^2, \ldots$ at time $t+1$, any menu $A_{t+1} \in \mathcal{M}_{t+1}$, and any continuation menu $C_{t+2} \in \mathcal{M}_{t+2}$ there exists N such that for all $n \geq N$*

$$\{(z^1, C_{t+2}) \ldots, (z^n, C_{t+2})\} \succ_t A_{t+1}.$$

2. *For any sequence of outcomes such that, $z^1 \sim_T^\epsilon z^i$ for some $\varepsilon < 1$ and all i, any menu $A_{t+1} \in \mathcal{A}_{t+1}$, and any continuation menu $C_{t+2} \in \mathcal{M}_{t+2}$, there exists N such that for $n \geq N$,*

$$\{(z^1, C_{t+2}) \ldots, (z^n, C_{t+2})\} \succ_t A_{t+1}.$$

Proposition 4. *Under Axioms 1–3 and 10,*

1. *For any sequence of duplicates $z^1, z^2, \ldots$ at time $t+1$, and any continuation menu $C_{t+2} \in \mathcal{M}_{t+2}$, if $z \succ_{t+1} y \succ_{t+1} z^1$ then $\{(z, C_{t+2})\} \succ_t \{(z^1, C_{t+2}), \ldots, (z^n, C_{t+2})\}$ for all n, and there exists N such that for $n \geq N$, $\{(y, C_{t+2})\} \succ_t \{(z, C_{t+2}), (z^1, C_{t+2}), \ldots, (z^n, C_{t+2})\}\}$*

2. *If $z \succ_{t+1} y \succ_{t+1} z^1$, then there is an $\bar{\epsilon}$ such that for any sequence of outcomes with $z^1 \sim_{t+1}^\epsilon z^i$ for some $\epsilon < \bar{\epsilon}$ and all i, and any continuation menu $C_{t+2} \in \mathcal{M}_{t+2}$ we have $\{(z, C_{t+2})\} \succ_t \{(z^1, C_{t+2}), \ldots, (z^n, C_{t+2})\}$ for all n, and there is an N such that for $n \geq N$, $\{(y, C_{t+2})\} \succ_t \{(z, C_{t+2}), (z^1, C_{t+2}), \ldots, (z^n, C_{t+2})\}\}$.*

4.5 Additivity, Stationarity, and Impatience

The axioms we have stated so far are sufficient for the recursive representations presented in the Appendix, but to pin things down to the discounted form, we need to add a strong separability condition that ensure preferences are additively separable over time.

Axiom 12 (Separable Ratios). For any $A_{t+1} = \{a_{t+1}, b_{t+1}, c_{t+1}, d_{t+1}\}$ if

$$\frac{\Phi_{t+1}[a_{t+1}|A_{t+1}]}{\Phi_{t+1}[b_{t+1}|A_{t+1}]} = \frac{\Phi_{t+1}[c_{t+1}|A_{t+1}]}{\Phi_{t+1}[d_{t+1}|A_{t+1}]},$$

then for all z_t, w_t

$$\frac{\Phi_t[(z_t,\{a_{t+1}\})|A_t]}{\Phi_t[(z_t,\{b_{t+1}\})|A_t]} = \frac{\Phi_t[(w_t,\{c_{t+1}\})|A_t]}{\Phi_t[(w_t,\{d_{t+1}\})|A_t]},$$

where $A_t = \{(z_t,\{a_{t+1}\}),(z_t,\{b_{t+1}\}),(w_t,\{c_{t+1}\}),(w_t,\{d_{t+1}\})\}$

Remark 2. Axiom 12 ensures that the ratio of choice probabilities between $(z_t,\{a_{t+1}\})$ and $(z_t,\{b_{t+1}\})$ when choosing at time t is independent of the (common) prize z_t and is also independent of the particular choice of a_{t+1} and b_{t+1} but depends only on the choice ratio at time $t+1$.

To get a sense of why something like this condition is necessary for the logit and entropy representations, consider the discounted entropy representation with $\eta = 1$. Let $t = T-1$, so that the continuation menus A_{t+2} are absent. Then,

$$\frac{\Phi_t[(z_t,\{a_T\})|A_t]}{\Phi_t[(z_t,\{b_T\})|A_t]} = \frac{\exp(v_t(z_t)+\delta v_T(a_T))}{\exp(v_t(z_t)+\delta v_T(b_T))} = \frac{\exp(\delta v_T(a_T))}{\exp(\delta v_T(b_T))} = \left(\frac{\Phi_T[a_T|A_T]}{\Phi_T[b_T|A_T]}\right)^{\delta},$$

so the choice ratio at time t is independent of z_t and depends on a_T, b_T only through their choice ratio.

To obtain a stationary discounted model, we also need an axiom to ensure that the discount factor and felicity function are time invariant. The original form of stationarity introduced by Koopmans (1960) relies on an infinite horizon; we use a similar axiom introduced by (Fishburn, 1970, Chapter 7). The axiom is imposed only on choice in period zero. Note, that although the stochastic preference at time zero, $\succsim_0$ compares elements of the form (z_0, A_1), it induces a preference over consumption streams $(z_0, z_1, \ldots, z_T)$ by appropriately defining $A_1 := \{(z_1, A_2)\}$, $A_2 := \{(z_2, A_3)\}$, etc.

Axiom 13 (Stationarity over Streams). For any $z, z_1, \ldots, z_T, z'_1, \ldots, z'_T \in Z$

$$(z, z_1, \ldots, z_T) \succsim_0 (z, z'_1, \ldots, z'_T)$$

if and only if

$$(z_1, \ldots, z_T, z) \succsim_0 (z'_1, \ldots, z'_T, z).$$

Another form of stationarity will be needed to guarantee that the noise parameter η is time invariant. The following axiom captures the necessary and sufficient restrictions by ensuring that preferences over current-period outcomes are the same in every period.

Axiom 14 (Stationarity of Choices). For any $z, z' \in Z$, any $A_1 \in \mathcal{M}_1$, any $t = 0, \ldots, T$ and $A_t \in \mathcal{M}_t$ we have

$$\Phi_0[(z, A_1)|\{(z, A_1),(z', A_1)\}] = \Phi_t[(z, A_t)|\{(z, A_t),(z', A_t)\}].$$

Finally, to ensure that the discount factor is less than one, we impose the following impatience axiom.

Axiom 15 (Impatience). For any $z, z', z_0, \ldots, z_T \in Z$ if

$$(z, \ldots, z) \succ_0 (z', \ldots, z')$$

then

$$(z_0, \ldots, z_{t-1}, z, z', z_{t+2}, \ldots, z_T) \succsim_0 (z_0, \ldots, z_{t-1}, z', z, z_{t+2}, \ldots, z_T).$$

5 Representation Theorems

Theorem 1. *Suppose that Axiom 4 holds. Then:*

A. The following conditions are equivalent for a dynamic stochastic choice rule $\{\Phi_t\}$*:*

1. $\{\Phi_t\}$ *satisfies Axioms 1, 2, 3, 12, and 8*
2. $\{\Phi_t\}$ *has a Discounted Logit Representation*
3. $\{\Phi_t\}$ *has a Discounted Entropy Representation.*

B. If in addition Axioms 13 and 14 hold, then the representation is stationary and if in addition Axiom 15 holds, then the representation is impatient.

C. The stationary representation is unique in the following sense: if v, δ, η *and* v', δ', η' *are representations of* $\{\Phi_t\}$*, then* $\delta' = \delta$ *and* $v' = \frac{\eta'}{\eta} v + \beta$ *for some* $\beta \in \mathbb{R}$*.*

Theorem 2. *Suppose that Axiom 4 holds. Then:*

A. The following conditions are equivalent for a dynamic stochastic choice rule $\{\Phi_t\}$*:*

1. $\{\Phi_t\}$ *satisfies Axioms 1, 2, 3, 12, and 10*
2. $\{\Phi_t\}$ *has a Discounted Logit with Menu Costs Representation.*
3. $\{\Phi_t\}$ *has a Discounted Relative Entropy Representation.*

B. If in addition Axioms 13 and 14 hold, then the representation is stationary and if in addition Axiom 15 holds, then the representation is impatient.

C. The stationary representation is unique in the following sense: if v, δ, η *and* v', δ', η' *are representations of* $\{\Phi_t\}$*, then* $\delta' = \delta$ *and* $v' = \frac{\eta'}{\eta} v + \beta$ *for some* $\beta \in \mathbb{R}$*.*

5.1 Proof Sketch

The proofs of Theorems 1 and 2 have a similar structure. First, Lemma 1 in the Appendix shows that Axioms 1–3 are equivalent to what we call a "sequential Luce representation," where

$$\Phi_t[a_t \mid A_t] = \frac{W_t(a_t)}{\sum_{b_t \in A_t} W_t(b_t)}$$

and for any $a_t = (z_t, A_{t+1})$ the function $W_t(a_t)$ can be written as

$$W_t(z_t, A_{t+1}) = G_t\left(v_t(z_t), h_t(A_{t+1})\right),$$

for an appropriately chosen aggregator function G_t and functions v_t and h_t. The first formula above follows from Axioms 1 and 2; then Axiom 3 lets us determine the choice probabilities for pairs (z_t, A_{t+1}) in the previous periods as a function of the felicity $v_t(z_t)$ and "anticipated utility" $h_t(A_{t+1})$.

This representation is not recursive, because there need be no link between actual choice probabilities from the menu A_{t+1} in period $t+1$ and the "anticipated utility" of A_{t+1} in period t, $h_t(A_{t+1})$. The aggregate and average recursivity conditions are alternate ways of providing this link, and each leads to representations evaluate the continuation problem with an aggregator that depends on anticipated future payoff, as in Koopmans (1960), Kreps and Porteus (1978), Epstein and Zin (1989), and Stokey, Lucas, and Prescott (1989). Theorem 5 provides three equivalent recursive representations for flexibility-preferring choice: one for each of the static entropy, logit, and Luce representations. Then, as Theorem 1 shows, the richness condition of Axiom 4 and the separability condition in Axiom 8 pin the entropy and logit forms down to their discounted versions.[14] The proof of Theorem 2 is quite similar; here we impose average instead of aggregate recursivity (Axiom 10) to arrive at recursive relative entropy, logit with menu costs, and error-averse Luce representations, and again use Axiom 12 to specialize the relative entropy and logit with menu costs representations to their discounted forms. Finally it is straightforward to obtain the stationarity and impatience conclusions of part b of the theorems from the additional axioms of Section 4.5.

6 Timing of Decisions and Payoffs

The simple form of the stationary impatient representations makes it easy to explore their implications for issues related to the timing of decisions and payoffs. We explore two such

[14]Theorem 3 shows that analogous discounted version of recursive flexibility-preferring Luce corresponds to a different separability condition, that of Axiom 16.

scenarios here.

6.1 Indifference to Distant Consequences

Both the flexibility preferring and error averse forms of the impatient stationary choice rule imply that the agent becomes less concerned about a choice as its consequences recede into the future. This implies that choice over distant rewards is close to the uniform distribution. To see this, let A be a finite subset of Z. Suppose the agent will receive a fixed sequence $\bar{z} = (z_0, z_1, ..., z_{T-1})$ in periods 0 through $T-1$, and that the only non-trivial decision (non-singleton choice problem) that the agent faces is to decide in time 0 what $\widetilde{z}_T$ to receive at time T. In the formalism of the paper, the agent's time-0 decision problem is to choose between $|A|$ different continuation problems, one for each element of A. Let $A_0 := \{(z_0, z_1, ..., z_{T-1}, \tilde{z}_T) : \tilde{z}_T \in x\}$ be the choice set at time 0 and define $q^T(\tilde{z}_T) := \Phi_t[(z_0, z_1, ..., z_{T-1}, \tilde{z}_T)|A_0]$ to be the choice probability of a given element of that set.

Proposition 5. *For the stationary impatient entropy choice rule (equivalently the stationary impatient logit choice rule) as well as the stationary impatient relative entropy choice rule (equivalently the stationary impatient logit with menu costs choice rule) we have* $\lim_{T\to\infty} q^T(\tilde{z}_T) = \frac{1}{|A|}$.

6.2 Choosing When to Choose

Now we consider when the agent would like to make a choice from a given menu, with the outcome to be received at some later time. As we show, the flexibility-preferring representations imply a preference for early decision, while the error-averse representations imply a preference to postpone choice.

Let A be a finite subset of Z with a generic element $\tilde{z}_T$. Suppose that the agent must choose between $a_0 = (z_0, A_1)$ and $b_0 = (z_0, B_1)$ at time 0. Under either decision problem, he will receive the same sequence $(z_1, \ldots, z_{T-1})$ in periods 0 through $T-1$. Under A_1, he will face a choice in period 1 of which element $\tilde{z}_T \in X$ to receive at time T, while under B_1 he selects his time-T outcome $\tilde{z}_T \in A$ in period T. Figure 2 shows a simple problem of this kind. We are interested in the choice ratio

$$r_T := \frac{\Phi_0[a_0|\{a_0, b_0\}]}{\Phi_0[b_0|\{a_0, b_0\}]},$$

which reflects the strength of the preference for making an early decision.

Proposition 6. *For the stationary impatient entropy choice rule (equivalently the stationary impatient logit choice rule) the choice ratio* $r_T > 1$ *as long as* $|A| \geq 2$. *Moreover,* $\lim_{T\to\infty} r_T = |A|^\delta$.

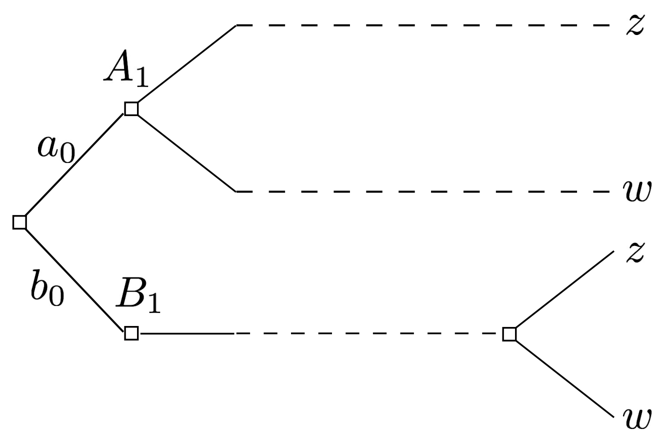

Figure 2: Choosing when to choose with $A = \{z, w\}$

Remark 3. Note that this preference for early choice holds even though the agent prefers larger menus and so satisfies the stochastic form of Kreps's (1979) "preference for flexibility."[15] This might at first seem surprising, as Kreps shows that such preferences over menus can arise when the agent is uncertain about his future preferences, and such uncertainty suggests the agent would prefer to delay the decision. Note, though, that Kreps' model is not rich enough to pose the question of when the agent would like to make a future selection from a menu of fixed size and that a preference for larger menus can arise for many other reasons.[16] With the stationary entropy choice rule, the agent derives a benefit (measured by the entropy function) from the simple act of choice, and impatience implies the agent would like to receive this benefit as early as possible. With stationary logit preferences, the reason the agent prefers early resolution is that the payoff shocks ϵ_t apply to pairs (z_t, A_{t+1}) of current action and continuation plan, and since the expected value of the shock of the chosen action is positive, the agent again prefers early choice.[17]

Proposition 7. *For the stationary impatient relative entropy choice rule (equivalently the stationary impatient logit with menu costs choice rule) the choice ratio $r_T \leq 1$ with equality if and only if all the elements of the set A are duplicates. Moreover, $\lim_{T\to\infty} r_T = 1$.*

Remark 4. The intuition for this result is that impatient agents prefer to postpone losses,

[15]The same preference for later decision holds when the times 1 and T in the example are replaced by any $t > 0$ and $t' > t$.

[16]For example, preferences where the agent judges menus solely by their size are consitent with Kreps' axioms, which use only preferences on menus and no information l about subsequent choice,

[17]As we noted earlier, our representations correspond to "shocks to actions" and not "shocks to consumption payoff." Note that a model with independent shocks each period to the utility associated with the current outcome z would have the property that the period-t choice between two actions with identical period-t outcomes would be deterministic. To get stochastic choice one could allow the agent to receive imperfect signals of future payoffs. We hope to consider such models in future work but it is not clear whether they will be tractable.

and with relative entropy preferences, the agent perceives the act of choice as a "bad" unless the choice distribution is uniform, i.e. unless the choices are duplicates. With the equivalent error-averse logit representation the implied menu costs again make choice a bad and so imply a preference for later choice.

7 Luce Representations

7.1 Discounted Luce Representations

Here we define discounted versions of the Luce representation. They are closely related to the entropy and logit representations introduced above but they have different time-separability properties.

The Discounted Flexibility-Preferring Luce Representation extends Luce's static representation to dynamic settings in a way that implies a preference for flexibility.

Definition 9. A dynamic stochastic choice rule has a *Discounted Flexibility-Preferring Luce Representation* if and only there exist felicity functions $v_t : Z \to \mathbb{R}_{++}$, a *discount factor* $\delta_t > 0$, and value functions $W_t : \mathcal{A}_t \to \mathbb{R}_{++}$ recursively defined by

$$W_t(z_t, A_{t+1}) = v_t(z_t) + \delta_t \sum_{a_{t+1} \in A_{t+1}} W_{t+1}\left(a_{t+1}\right), \tag{11}$$

such that for all t, all A_t, and all $a_{t+1} \in A_t$

$$\Phi_t[a_t \mid A_t] = \frac{W_t(a_t)}{\sum_{b_t \in A_t} W_t(b_t)}. \tag{12}$$

This choice rule exhibits a preference for flexibility because the function W_{t+1} takes non-negative values and it is independent of the particular menu opportunity set A_{t+1}; hence, adding elements to the set A_{t+1} increases the value of the sum in expression (11).

The Discounted Error-Averse Luce Representation extends Luce's static representation in a way that implies error aversion.

Definition 10. A dynamic stochastic choice rule has a *Discounted Error Averse Luce Representation* if and only there exist felicity functions $v_t : Z \to \mathbb{R}_{++}$, a *discount factor* $\delta_t > 0$, and value functions $W_t : \mathcal{A}_t \to \mathbb{R}_{++}$ recursively defined by

$$W_t(z_t, A_{t+1}) = v_t(z_t) + \delta_t \frac{1}{|A_{t+1}|} \sum_{a_{t+1} \in A_{t+1}} W_{t+1}\left(a_{t+1}\right), \tag{13}$$

such that for all $t = 0, \ldots, T$, A_t, and all $a_t \in A_t$

$$\Phi_t[a_t \mid A_t] = \frac{W_t(a_t)}{\sum_{b_t \in A_t} W_t(b_t)}. \tag{14}$$

Intuitively, this choice rule exhibits error aversion because adding elements to the set A_{t+1} can decrease the value of the average in expression (13).

7.2 Axiomatic Characterization of Luce Representations

These Luce representations do not satisfy the time-separability condition of Axiom 12; instead they satisfy the following separability condition.

Axiom 16 (Separable Differences). For any $t < T$, any $z, z' \in Z$, and any four distinct elements $a_{t+1}, a'_{t+1}, b_{t+1}, b'_{t+1} \in \mathcal{A}_{t+1}$:

$$\Phi_t[(z, \{a_{t+1}, b_{t+1}\})|A_t] - \Phi_t[(z, \{a_{t+1}, b'_{t+1}\})|A_t] = \Phi_t[(z', \{a'_{t+1}, b_{t+1}\})|A_t] - \Phi_t[(z', \{a'_{t+1}, b'_{t+1}\})|A_t],$$

where $A_t = \{(z, \{a_{t+1}, b_{t+1}\}), (z, \{a_{t+1}, b'_{t+1}\}), (z', \{a'_{t+1}, b_{t+1}\}), (z', \{a'_{t+1}, b'_{t+1}\})\}$.

The key difference between the logit and entropy representations on one hand and the Luce representations on the other is that with the Luce representation the terms $v_t(z_t)$ and W_{t+1} enter additively into the current choice probability, while with the entropy representation these terms enter multiplicatively (see Remark 2). For this reason the Luce representation satisfies the separable differences axiom, while the entropy representation has separable ratios.

Theorem 3. *Suppose that Axiom 4 holds. Then:*

A. The following conditions are equivalent for a dynamic stochastic choice rule $\{\Phi_t\}$*:*

1. $\{\Phi_t\}$ *satisfies Axioms 1, 2, 3, 16, and 8*

2. $\{\Phi_t\}$ *has a Discounted Flexibility-Preferring Luce Representation.*

B. If in addition Axiom 13 holds, then the representation is stationary and if in addition Axiom 15 holds, then the representation is impatient.

C. The stationary representation is unique in the following sense: suppose that v, δ *and* $\hat{v}, \hat{\delta}$ *are two representations of* $\{\Phi_t\}$*. Then* $\hat{\delta} = \delta$ *and* $\hat{v} = \alpha v$ *for some* $\alpha > 0$*.*

An analogous set of results holds for the error-averse choice rules.

Theorem 4. *Suppose that Axiom 4 holds. Then:*

A. The following conditions are equivalent for a dynamic stochastic choice rule $\{\Phi_t\}$*:*

1. $\{\Phi_t\}$ *satisfies Axioms 1, 2, 3, 16, and 10*

2. $\{\Phi_t\}$ *has a Discounted Error-Averse Luce Representation*

B. If in addition Axiom 13 holds, then the representation is stationary and if in addition Axiom 15 holds, then the representation is impatient.

C. The stationary representation is unique in the following sense: suppose that v, δ *and* $\hat{v}, \hat{\delta}$ *are two representations of* $\{\Phi_t\}$*. Then* $\hat{\delta} = \delta$ *and* $\hat{v} = \alpha v$ *for some* $\alpha > 0$*.*

Because the Luce representations satisfy different time-separability properties than Logit and Entropy, they are not equivalent in terms of choice behavior. However, there is a tight connection between them, which we formalize in Section A.2.5 of the Appendix. Roughly speaking, a choice rule has an additive Luce representation if and only if it also has a particular kind of non-additive entropy (or logit) representation, where the aggregation of the period t felicity and period $t+1$ anticipated value is similar to that of Epstein and Zin (1989). Similarly, a choice rule has an additively separable entropy (or logit) representation if and only if it also has a particular Epstein–Zin like Luce representation. In the special case where the only choice is at time 0, the additively separable Luce representation coincides with the non-separable Luce representation that is equivalent to discounted relative entropy. In general, though, these two Luce representations differ due to the non-linear aggregator in the non-separable representation. One interpretation of the nonlinear aggregator is that the agent has a preference for early or late resolution of uncertainty a la Kreps and Porteus (1978) when it comes to the randomization in his own subsequent choices. A formally identical situation arises with the Epstein and Zin (1989) preferences which are additively separable over deterministic consumption streams and where the nonseparabilities arise only when randomizations (by nature) are involved.

The analogy with the Kreps–Porteus–Epstein–Zin preferences suggests that we should be able to determine whether the discounted relative entropy choice rule has a preference for or against delaying choice by looking at its alternative Epstein–Zin like representation, and particularly, the convexity of the aggregator function, as suggested by Theorem 3 in Kreps and Porteus (1978). By inspecting equation (26) in Proposition 13 we notice that preference for late resolution of uncertainty will obtain if the function $\gamma \mapsto \exp(v_t(z_t) + \delta_t \frac{\eta_{t+1}}{\eta_t} \log(\gamma))$ is concave. Since in the stationary impatient model this function is concave we conclude that the preference for early decisions obtains, which is also confirmed by Proposition 6. Likewise, looking at equation (13) in the definition of discounted Luce preferences suggests that the choice rule will display indifference to timing, as the corresponding function is linear in γ. This is indeed the case, as the following Proposition confirms. As in Section 6, we are interested in the choice ratio

$$r_T := \frac{\Phi_0[a_0'|\{a_0', a_0''\}]}{\Phi_0[a_0''|\{a_0', a_0''\}]}$$

Proposition 8. *For the stationary impatient Error Averse Luce choice rule the choice ratio* $r_T = 1$.

The analogy with the Kreps-Porteus-Epstein-Zin preferences is not helpful in determining the timing attitudes of the flexibility-preferring versions of our choice rules because they involve sums and not averages (and KP-EZ have averages so those theorems don't apply). Nevertheless we can study the timing properties of these choice rules directly, as in the following proposition.

Proposition 9. *For the stationary impatient Flexibility Preferring Luce choice rule the choice ratio* $r_T \geq 1$ *with a strict inequality whenever* $|X| \geq 2$.

8 Summary and Discussion

This paper has provided axiomatic characterizations of two sorts of stochastic dynamic choice rules, namely flexibility-preferring and error-averse. As we saw, the key difference between them is what form of recursivity axiom links together choices in different periods: flexibility-preferring choice rules correspond to aggregate recursivity, while error-averse choice rules satisfy average recursivity. We pointed out that flexibility-preferring impatient preferences have a preference for early decision, even though they also satisfy the stochastic form of Kreps (1979)'s preference for flexibility; while error-averse stationary preferences prefer to act later. This highlights the facts that Kreps' two-period model is not rich enough to pose the question of when the agent would like to make a decision, and that a preference for larger menus can arise for many reasons. In addition, our results provide a foundation for the use of the discounted-logit-with-menu-costs representation in emprical work; it seems just as tractable as the usual discounted logit and may better describe behavior in at least some choice problems where the menu size varies.

The paper is related to quite a large number of papers, as it draws on and extends the literature on static stochastic choice pioneered by Luce (1959) and Harsanyi (1973a), the literature on discounting representations of deterministic dynamic choice (notably Koopmans (1960) and Fishburn (1970)), and the literature on choices over menus pioneered by Kreps (1979).

The work of Falmagne (1978), Barberá and Pattanaik (1986), Gul and Pesendorfer (2006) and Gul, Natenzon, and Pesendorfer (2012) provides axiomatic characterizations of static stochastic choice without the IIA assumption; this suggests that our dynamic representations could also be generalized beyond IIA, though obtaining a dynamic model that is both general and tractable seems challenging.[18] Furthermore, to do this we would first want to develop a parallel characterization of the static choice probabilities that can be generated when the entropy or relative entropy perturbation term is replaced by a more general function from the class or perturbations studied by Hofbauer and Sandholm (2002); we have reduced that to a problem in convex analysis but do not yet have a solution. We would also need to find appropriate extensions of average and aggregate recursivity.

In recent years there have been several generalizations of Koopmans (1960)'s characterization to forms of "behavioral" dynamic choice, as in Jackson and Yariv (2010) and Montiel Olea and Strzalecki (2011); in principle one could introduce stochastic choice into those setups but there does not seem to be a compelling reason to do so.

The most active related literature is that on choice between menus. Some of these papers develop representations motivated by "consideration costs" or "costs of thinking"; to the extent that this cost is increasing in the menu size it is related to our error-averse representations. Ergin and Sarver (2010), following Ergin (2003), develop a representation with a double maximization, in which "costly contemplation" corresponds to buying a signal about the second-period attractiveness of the various options. They motivate their assumptions with the idea that agents may prefer to make ex post choices and not a complete contingent plan; this motivation uses choice over lotteries of menus, which is not part of their formal model. Ortoleva (2011) does explicitly consider lotteries over menus. He develops a "cost of thinking" that resembles the consideration cost of Ergin and Sarver; one key difference is that Ortoleva's agent ranks menus as if she expected to choose the best option from each of them, despite the fact that doing so requires costly thinking that might not be ex-post optimal.

Other recent papers on choice from menus are of interest here primarily for how they impose recursivity or dynamic consistency. Ahn and Sarver (2012) is perhaps closest, as like this paper it treats both initial choice of a menu and subsequent choice from it as observable. They use recursivity axioms to pin down a unique state space and probabilities in the two-stage menu choice model; their Axiom 1 is similar in spirit to our aggregate recursivity condition 8, but as stated it is vacuously satisfied given our positivity assumption.[19] Ahn and Sarver assume a preference for flexibility and so rule out temptation. Dekel and Lipman (2012) impose consis-

[18]Emprical work on static choice uses preferences nested logit and BLP (Berry, Levinsohn, and Pakes, 1995) that avoid some of the starkest implications of IIA, but these specifications also seem quite complicated to work with in dynamic settings.

[19]Their Axiom 1 requires that if adding the lottery p makes a menu A more appealing, then p has a positive probability of being chosen from the menu A. They complement this with Axiom 2, which combines the converse to their Axiom 1 with a form of continuity assumption.

tency between the first period choice of a menu and second period choice from a menu at the level of the representation, and use choices in the two periods to distinguish between "random GP" and "random Strotz" representations in cases where temptation is present.[20] Krishna and Sadowski (2012) provide two representations for a decision maker who is uncertain about his future utility in an infinite-horizon decision problem. Their stationarity axiom corresponds to our Axioms 3 and 13, but neither average nor aggregate recursivity is consistent with the indifference required by their "continuation strategic rationality" axiom, and the same appears to be true with respect to the less restrictive axioms developed in their Section 4. Both of their representations imply a preference for flexibility in the sense of preferring larger menus. We conjecture that these representations exhibit a preference for later decision, in contrast to the impatient flexibility-preferring dynamic logit, although we have not worked out the details.

Another line of extensions would be to consider alternatives to Linear Recursivity, such as those currently discussed in the subsection that is work in progress. Finally, the menu-choice literature suggests interesting extensions of our work in addition to those already mentioned above: Specifically, one could try to model the stochastic choice of lotteries, and once the model can handle explicit exogenous uncertainty, one could then examine the way that agents respond to information. It would also be interesting to examine the empirical implications of dynamic logit preferences with menu costs; these logit preferences seem as tractable for estimation as the more conventional form.

[20] They also show that the random Strotz model can accommodate the non-linear cost of self control introduced by Fudenberg and Levine (2006) and further analyzed by Fudenberg and Levine (2011, 2012) and Noor and Takeoka (2010a,b).

Appendix

A.1 Overview of the Proofs

In Section 5.1 we gave a brief summary of the flow of the proofs of Theorems 1 and 2. Here we give a somewhat more detailed sketch, along with some intuitions.

Step 1: The first step in both proofs is Lemma 1, which shows that Axioms 1-3 are equivalent to a "sequential Luce representation": there are weights W_t for actions a_t such that

$$\Phi_t[a_t \mid A_t] = \frac{W_t(a_t)}{\sum_{b_t \in A_t} W_t(b_t)}.$$

Here our maintained assumption that Φ_t is history independent and Axioms 1 and 2 let us use Luce's original argument to conclude there are weights that describe period-T choice, and Axiom 3 then lets us mimic the proof of Koopmans' Proposition 3 and obtain a representation of W_t as $W_t(z_t, A_{t+1}) = G_t(v_t(z), h_t(A_{t+1}))$, where G_t is a strictly increasing function of both variables.

Step 2: The representation in Step 1 is not recursive, as there need not be a link between actual choice in period $t+1$ and the choice that is implicitly anticipated at period t. The next step in our argument is to relate choices in different time periods. As noted in the text, recursivity on singleton continuation menus is not sufficient, so we introduce the alternative notions of aggregate recursivity and average recursivity in order to handle decisions that compare menus of different sizes. Theorem 5 shows that Axioms 1–3 plus the Aggregate Recursivity assumption (Axiom 8) implies that preferences have a "recursive flexibility-preferring Luce" representation, where the weight assigned to (z_t, A_{t+1}) in period t, depends on the weights that will be used in period $t+1$ according to

$$W_t(z_t, A_{t+1}) = G_t\left(v_t(z_t), \sum_{(z_{t+1}, A_{t+2}) \in A_{t+1}} W_{t+1}(z_{t+1}, A_{t+2})\right),$$

for some function G_t strictly increasing in both variables (on an appropriately defined domain). The equivalence of this representation with the recursive flexibility-preferring entropy and logit representations then follows from the static equivalence of these representations and a recursion argument. For Theorem 6, we use Average Recursivity (Axiom 10) to prove an analagous equivalence with a recursive error-averse Luce representation, and from there prove equivalences with the related relative entropy and logit with menu costs forms.

Step 3: The final step in part a) of Theorems 1 and 2 is to use Axioms 4 and 10 to pin down the discounting form. The separability condition of Axiom 10 implies that the recursive Luce weights satisfy

$$\frac{W_t(z_t, \{(z_{t+1}^\lambda, A_{t+2})\})}{W_t(z_t, \{(z_{t+1}, A_{t+2})\})} = \frac{W_t(\hat{z}_t, \{(\hat{z}_{t+1}^\lambda, \hat{x}_{t+2})\}\})}{W_t(\hat{z}_t, \{(\hat{z}_{t+1}, \hat{x}_{t+2})\})},$$

where for any z and $\lambda > 0$, z^λ is the prize whose existence is guaranteed by the richness condition Axiom 4. We then show that $W_t(z_t, \{(z_{t+1}^\lambda, A_{t+2})\}) = \lambda W_t(z_t, \{(z_{t+1}, A_{t+2})\})$ and $W_t(\hat{z}_t, \{(\hat{z}_{t+1}^\lambda, \hat{x}_{t+2})\}\} = \lambda W_t(\hat{z}_t, \{(\hat{z}_{t+1}, \hat{x}_{t+2})\})$, and from there obtain a functional equation that implies the desired result; the richness condition of Axiom 4 ensures that Axiom 10 has enough bite to imply the desired conclusion.

Part b) of the theorems follows from the separability and stationarity assumptions of Section 4.5 in a straightforward way.

A.2 Recursive Representations

In this section we study recursive representations a la Koopmans (1960), Kreps and Porteus (1978), Epstein and Zin (1989), and Stokey, Lucas, and Prescott (1989). The representations we present here are also history-independent, but have fewer time-separability properties than a discounted sum, and correspond to a shorter list of axioms. We use these representations as intermediate steps towards the main theorems. In addition, because the difference between the logit and entropy representations on one hand and the Luce representations on the other arise only when we also ask for time separability, in the more general setting considered here all three representations satisfy the same set of axioms.

A.2.1 Sequential and Sophisticated Luce

Definition 11 (Sequential Luce). A dynamic stochastic choice rule has a *Sequential Luce Representation* if and only there exists functions $v_t : Z \to \mathbb{R}$, $h_t : \mathcal{M}_{t+1} \to \mathbb{R}$ with ranges Rv_t and Rh_t respectively, and $G_t : Rv_t \times Rh_t \to \mathbb{R}_{++}$, strictly increasing in both variables, and value functions $W_t : \mathcal{A}_t \to \mathbb{R}_{++}$ recursively defined by

$$W_t(z_t, A_{t+1}) = G_t\left(v_t(z_t), h_t(A_{t+1})\right), \tag{15}$$

such that for all A_t and all $a_t \in A_t$

$$\Phi_t[a_t \mid A_t] = \frac{W_t(a_t)}{\sum_{b_t \in A_t} W_t(b_t)}. \tag{16}$$

Definition 12 (Sophisticated Luce). A dynamic stochastic choice rule has a *Sophisticated Luce Representation* if and only it has a Sequential Luce Representation such that for all singletons $A_{t+1} = \{(z_{t+1}, A_{t+2})\}$

$$h_t(A_{t+1}) = W_{t+1}(z_{t+1}, A_{t+2})$$

A.2.2 Flexibility-Preferring Representations

Definition 13 (Recursive Flexibility-Preferring Luce Representation). A dynamic stochastic choice rule has a *Recursive Flexibility-Preferring Luce Representation* if and only it has a Sequential Luce Representation with

$$h_t(A_{t+1}) = \sum_{(z_{t+1}, A_{t+2}) \in A_{t+1}} W_{t+1}(z_{t+1}, A_{t+2}) \tag{17}$$

Definition 14 (Recursive Entropy Representation). A dynamic stochastic choice rule has a *Recursive Entropy Representation* if and only there is there exist parameters $\eta_t > 0$, a utility function $v_t : Z \to \mathbb{R}$, an *aggregator* $G_t : D_t \to \mathbb{R}_{++}$, strictly increasing in both variables (on an appropriately defined domain D_t) and value functions $U_t : \mathcal{A}_t \to \mathbb{R}$ recursively defined by

$$U_t(z_t, A_{t+1}) = G_t\left(v_t(z_t), \max_{q \in \Delta(A_{t+1})} \sum q(a_{t+1}) U_{t+1}(a_{t+1}) + \eta_{t+1} H(q)\right) \tag{18}$$

such that for all A_t, and all $(z, A_{t+1}) \in A_t$

$$\Phi_t[\cdot \mid A_t] = \arg\max_{q \in \Delta(A_t)} \sum q(a_{t+1}) U_t(a_{t+1}) + \eta_t H(q) \tag{19}$$

Definition 15 (Recursive Logit Representation). A dynamic stochastic choice rule has a *Recursive Logit Representation* iff

1. There are random variables ϵ_{a_t}, one for every $a_t \in \mathcal{A}_t$, all i.i.d. with the extreme value distribution $F(\theta) = \exp(-\exp(-\theta/\eta_t - \gamma))$, where γ is Euler's constant and $\eta_t > 0$ are noise level parameters.

2. There exists a utility function $v_t : Z \to \mathbb{R}$, an *aggregator* $G_t : D_t \to \mathbb{R}_{++}$, strictly increasing in both variables (on an appropriately defined domain D_t) and value functions

$U_t : \mathcal{A}_t \to \mathbb{R}$ recursively defined by

$$U_t(z_t, A_{t+1}) = G_t\left(v_t(z_t), \mathbb{E}\left[\max_{a_{t+1}\in A_{t+1}} U_{t+1}(a_{t+1}) + \epsilon_{a_{t+1}}\right]\right) \tag{20}$$

such that

$$\Phi_t[a_t \mid A_t] = \text{Prob}\left(U_t(a_t) + \epsilon_{a_t} \geq \max_{b_t\in A_t} U_t(b_t) + \epsilon_{b_t}\right) \tag{21}$$

A.2.3 Error-Averse Representations

Definition 16 (Recursive Error-Averse Luce Representation). A dynamic stochastic choice rule has a *Recursive Error-Averse Luce Representation* if and only if and only it has a Sequential Luce Representation with

$$h_t(A_{t+1}) = \frac{1}{|A_{t+1}|} \sum_{a_{t+1}\in A_{t+1}} W_{t+1}(a_{t+1})$$

Definition 17 (Recursive Logit with Menu Costs Representation). A dynamic stochastic choice rule has a *Recursive Logit with Menu Costs Representation* iff

1. There are random variables ϵ_{a_t}, one for every $a_t \in \mathcal{A}_t$, all i.i.d. with the extreme value distribution $F(\theta) = \exp(-\exp(-\theta/\eta_t - \gamma))$, where γ is Euler's constant and $\eta_t > 0$ are noise level parameters.

2. There exist utility functions $v_t : Z \to \mathbb{R}$, *aggregators* $G_t : D_t \to \mathbb{R}_{++}$, strictly increasing in both variables (on an appropriately defined domain D_t) and value functions $U_t : \mathcal{A}_t \to \mathbb{R}$ recursively defined by

$$U_t(z_t, A_{t+1}) = G_t\left(v_t(z_t), \mathbb{E}\left[\max_{a_{t+1}\in A_{t+1}} U_{t+1}(a_{t+1}) + \epsilon_{a_{t+1}} - \eta_t \log |A_{t+1}|\right]\right) \tag{22}$$

such that

$$\Phi_t[a_t \mid A_t] = \text{Prob}\left(U_t(a_t) + \epsilon_{a_t} \geq \max_{b_t\in A_t} U_t(b_t) + \epsilon_{b_t}\right) \tag{23}$$

Definition 18 (Recursive Relative Entropy Representation). A dynamic stochastic choice rule has a *Recursive Relative Entropy Representation* if and only there is there exist parameters $\eta_t > 0$, a utility function $v_t : Z \to \mathbb{R}$, an *aggregator* $G_t : D_t \to \mathbb{R}_{++}$, strictly increasing in both variables (on an appropriately defined domain D_t) and value functions $U_t : \mathcal{A}_t \to \mathbb{R}$

recursively defined by

$$U_t(z_t, A_{t+1}) = G_t\left(v_t(z_t), \max_{q\in\Delta(A_{t+1})} \sum q(a_{t+1})U_{t+1}(a_{t+1}) - \eta_{t+1}R(q)\right) \tag{24}$$

such that for all A_t, and all $(z, A_{t+1}) \in A_t$

$$\Phi_t[\cdot \mid A_t] = \arg\max_{q\in\Delta(A_t)} \sum q(a_t)U_t(a_t) - \eta_t R(q) \tag{25}$$

A.2.4 Representation Theorems for Recursive Stochastic Choice

Lemma 1. *The following conditions are equivalent for a dynamic stochastic choice rule* $\{\Phi_t\}$

1. $\{\Phi_t\}$ *satisfies Axioms 1–3*
2. $\{\Phi_t\}$ *has a Sequential Luce Representation*

Lemma 2. *If* $\{\Phi_t\}$ *satisfies Axioms 1–3 and 4, then it has a Sequential Luce Representation such that for any* $A_{t+1} \in \mathcal{M}_{t+1}$ *the set* $\{W_t(z_t, A_{t+1}) \mid z_t \in Z, A_{t+1} \in \mathcal{M}_{t+1}\} = (0, \infty)$.

Lemma 3. *The following conditions are equivalent for a dynamic stochastic choice rule* $\{\Phi_t\}$

1. $\{\Phi_t\}$ *satisfies Axioms 1–3 and 5*
2. $\{\Phi_t\}$ *has a Sophisticated Luce Representation*

Theorem 5. *The following conditions are equivalent for a dynamic stochastic choice rule* $\{\Phi_t\}$

1. $\{\Phi_t\}$ *satisfies Axioms 1–3 and 8*
2. $\{\Phi_t\}$ *has a Recursive Flexibility-Preferring Luce Representation*
3. $\{\Phi_t\}$ *has a Recursive Entropy Representation*
4. $\{\Phi_t\}$ *has a Recursive Logit Representation.*

Theorem 6. *The following conditions are equivalent for a dynamic stochastic choice rule* $\{\Phi_t\}$

1. $\{\Phi_t\}$ *satisfies Axioms 1–3 and 10*
2. $\{\Phi_t\}$ *has a Recursive Error-Averse Luce Representation*
3. $\{\Phi_t\}$ *has a Recursive Relative Entropy Representation*
4. $\{\Phi_t\}$ *has a Recursive Logit with Menu Costs Representation.*

A.2.5 Relationship between Logit-Entropy and Luce representations

In the following propositions we state the relationship between the Entropy and Luce representations. An analogous relationship exists between the Logit and Luce representations.

Proposition 10. *The following conditions are equivalent:*

1. *$\{\Phi_t\}$ has a Discounted Flexibility-Preferring Luce Representation*
2. *$\{\Phi_t\}$ has a representation defined by $\Phi_t[\cdot \mid A_t] = \arg\max_{q\in\Delta(A_t)} \sum q(a_t)U_t(a_t) + \eta_t H(q)$, where the functions U_t are defined recursively by*

$$U_t(z_t, A_{t+1}) = \eta_t \log \left(v_t(z_t) + \delta_t \exp \left(\frac{1}{\eta_{t+1}} \max_{q\in\Delta(A_{t+1})} \sum q(a_{t+1})U_{t+1}(a_{t+1}) + \eta_{t+1}H(q) \right) \right)$$

Proposition 11. *The following conditions are equivalent:*

1. *$\{\Phi_t\}$ has a Discounted Entropy Representation*
2. *$\{\Phi_t\}$ has a representation defined by $\Phi_t[a_t \mid A_t] = \frac{W_t(a_t)}{\sum_{b_t\in A_t} W_t(b_t)}$. where the functions W_t are defined recursively by*

$$W_t(z_t, A_{t+1}) = \exp \left(\frac{v_t(z_t)}{\eta_t} + \delta_t \frac{\eta_{t+1}}{\eta_t} \log \left(\sum_{a_{t+1}\in A_{t+1}} W_{t+1}(a_{t+1}) \right) \right)$$

Proposition 12. *The following conditions are equivalent:*

1. *$\{\Phi_t\}$ has a Discounted Error-Averse Luce Representation*
2. *$\{\Phi_t\}$ has a representation defined by $\Phi_t[\cdot \mid A_t] = \arg\max_{q\in\Delta(A_t)} \sum q(a_t)U_t(a_t) - \eta_t R(q)$, where the functions U_t are defined recursively by*

$$U_t(z_t, A_{t+1}) = \eta_t \log \left(v_t(z_t) + \delta_t \exp \left(\frac{1}{\eta_{t+1}} \max_{q\in\Delta(A_{t+1})} \sum q(a_{t+1})U(a_{t+1}) - \eta_{t+1}R(q) \right) \right)$$

Proposition 13. *The following conditions are equivalent:*

1. *$\{\Phi_t\}$ has a Discounted Entropy Representation*
2. *$\{\Phi_t\}$ has a Representation defined by $\Phi_t[a_t \mid A_t] = \frac{W_t(a_t)}{\sum_{b_t\in A_t} W_t(b_t)}$, where the functions W_t are defined recursively by*

$$W_t(z_t, A_{t+1}) = \exp \left(\frac{v_t(z_t)}{\eta_t} + \delta_t \frac{\eta_{t+1}}{\eta_t} \log \left(\frac{1}{|A_{t+1}|} \sum_{a_{t+1}\in A_{t+1}} W_{t+1}(a_{t+1}) \right) \right) \tag{26}$$

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Online Appendix to "Recursive Stochastic Choice"

by D. Fudenberg and T. Strzalecki

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This Online Appendix has all the missing proofs from the paper except for the proof of Lemma 3 from our Appendix A which is omitted as it is obvious.

B.1 Useful Lemmas from the Literature

The following two lemmas are well known, see, e.g., Dupuis and Ellis (1997).

Lemma 4. *For any vector* $(x_1, \ldots, x_n) \in \mathbb{R}^n$

$$\max_{p \in \Delta^n} \sum_{i=1}^{n} p_i x_i + \eta H(p) = \eta \log \left(\sum_{i=1}^{n} \exp\left(\frac{x_i}{\eta}\right) \right)$$

and the solution is

$$p_i = \frac{\exp(\frac{x_i}{\eta})}{\sum_{j=1}^{n} \exp(\frac{x_j}{\eta})}.$$

Lemma 5. *For any vector* $(x_1, \ldots, x_n) \in \mathbb{R}^n$

$$\max_{p \in \Delta^n} \sum_{i=1}^{n} p_i x_i - \eta R(p) = \eta \log \left(\sum_{i=1}^{n} \frac{1}{n} \exp\left(\frac{x_i}{\eta}\right) \right)$$

and the solution is

$$p_i = \frac{\exp(\frac{x_i}{\eta})}{\sum_{j=1}^{n} \exp(\frac{x_j}{\eta})}.$$

The next two lemmas are well known as well, see, e.g., Chapter 3 of Train (2009).

Lemma 6. *Suppose that* $\epsilon_1, \ldots, \epsilon_n$ *are i.i.d. random variables with the extreme value distribution* $F(\theta) = \exp(-\exp(-\theta/\eta - \gamma))$*, where* γ *is Euler's constant and* $\eta > 0$ *are noise level parameters.*

For any vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$

$$\mathbb{E}\left[\max_{i=1,\ldots,n} x_i + \epsilon_i\right] = \eta \log\left(\sum_{i=1}^{n} \exp\left(\frac{x_i}{\eta}\right)\right).$$

Moreover,

$$\text{Prob}\left(x_i + \epsilon_i \geq \max_{j=1,\ldots,n} x_j + \epsilon_j\right) = \frac{\exp(\frac{x_i}{\eta})}{\sum_{j=1}^{n} \exp(\frac{x_j}{\eta})}.$$

Lemma 7. *Suppose that* $\epsilon_1, \ldots, \epsilon_n$ *are i.i.d. random variables with the extreme value distribution* $F(\theta) = \exp(-\exp(-\theta/\eta - \gamma))$, *where* γ *is Euler's constant and* $\eta > 0$ *are noise level parameters.*

For any vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$

$$\mathbb{E}\left[\max_{i=1,\ldots,n} x_i + \epsilon_i - \eta \log n\right] = \eta \log\left(\sum_{i=1}^{n} \frac{1}{n} \exp\left(\frac{x_i}{\eta}\right)\right).$$

Moreover,

$$\text{Prob}\left(x_i + \epsilon_i - \eta \log n \geq \max_{j=1,\ldots,n} x_j + \epsilon_j - \eta \log n\right) = \frac{\exp(\frac{x_i}{\eta})}{\sum_{j=1}^{n} \exp(\frac{x_j}{\eta})}.$$

B.2 Proof of Lemmas 1 and 2

B.2.1 Proof of the implication $(i) \Rightarrow (ii)$ of Lemma 1

Step 1

Fix an arbitrary element $\bar{z} \in Z$ and define $W_T(\bar{z}) = 1$. For any other $z \in Z$ define

$$W_T(z) := \frac{\Phi_T[z \mid \{z, \bar{z}\}]}{\Phi_T[\bar{z} \mid \{z, \bar{z}\}]}.$$

For any $A_T = \{z^1, \ldots, z^n\} \in \mathcal{M}_T$, Axioms 1 and 2 imply that for all $i, j = 1 \ldots, n$

$$\frac{\Phi_T[z^i \mid A_T]}{\Phi_T[z^j \mid A_T]} = \frac{\Phi_T[z^i \mid \{z^i, z^j, \bar{z}\}]}{\Phi_T[z^j \mid \{z^i, z^j, \bar{z}\}]}.$$

Using Axioms 1 and 2 it is straightforward to show that

$$\frac{\Phi_T[z^i \mid A_T]}{\Phi_T[z^j \mid A_T]} = \frac{W_T(z^i)}{W_T(z^j)}. \tag{27}$$

This equation implies that

$$\begin{aligned}\frac{1}{\Phi_T[z^j \mid A_T]} &= \frac{1}{\Phi_T[z^j \mid A_T]}\sum_{i=1}^{n}\Phi_T[z^i \mid A_T] \\ &= \sum_{i=1}^{n}\frac{\Phi_T[z^i \mid A_T]}{\Phi_T[z^j \mid A_T]} = \frac{\sum_{i=1}^{n} W_T(z^i)}{W_T(z^j)};\end{aligned}$$

therefore equation (16) follows.

For $t = 0, \ldots, T-1$ we proceed analogously, to define $W_t(z, A_{t+1})$ that satisfies equation (16).

Step 2

Part (1) of Axiom 3 together with Step 1 imply that for all $z, z' \in Z$ and $A_{t+1}, A'_{t+1} \in \mathcal{M}_{t+1}$

$$W_t(z, A_{t+1}) \geq W_t(z, A'_{t+1}) \Longleftrightarrow W_t(z', A_{t+1}) \geq W_t(z', A_{t+1}). \tag{28}$$

Fix $\bar{z} \in Z$ and define $h_t(A_{t+1}) := W_t(\bar{z}, A_{t+1})$ for all $A_{t+1} \in \mathcal{M}_{t+1}$. Property 28 implies that for any $z \in Z$ there exists a strictly increasing function $f_t^z : Rh_t \to \mathbb{R}$ such that $W_t(z, A_{t+1}) = f_t^z(h_t(A_{t+1}))$ for all $A_{t+1} \in \mathcal{M}_{t+1}$. Define a real valued function $F_t : Z \times Rh_t \to \mathbb{R}$ by $F_t(z, h) = f_t^z(h)$ for any $z \in Z$ and $h \in Rh_t$. Note that the function F_t is strictly increasing in the second variable. With this notation we have

$$W_t(z, A_{t+1}) = F_t(z, h_t(A_{t+1})). \tag{29}$$

Step 3

Part (2) of Axiom 3 together with Step 1 and equation (29) imply that for all $z, z' \in Z$ and $A_{t+1}, A'_{t+1} \in \mathcal{M}_{t+1}$

$$F_t(z, h_t(A_{t+1})) \geq F_t(z', h_t(A_{t+1})) \Longleftrightarrow F_t(z, h_t(A'_{t+1})) \geq F_t(z', h_t(A'_{t+1})). \tag{30}$$

Fix $\bar{h} \in Rh_t$ and define $v_t(z) := F_t(z, \bar{h})$. Property 30 implies that for any $h \in Rh_t$ there exists a strictly increasing function $m_t^h : Rv_t \to \mathbb{R}$ such that $F_t(z, h) = m_t^h(v_t(z))$. Define a real valued function $G_t : Rv_t \times Rh_t \to \mathbb{R}$ by $G_t(v, h) = m_t^h(v)$ for any $v \in Rv_t$ and $h \in Rh_t$. Note that the function G_t is strictly increasing in both variables. This, together with equation (29), implies that equation (15) holds.[21]

B.2.2 Proof of the implication $(ii) \Rightarrow (i)$ of Lemma 1

Axiom 1 is true since the function W takes strictly positive values. Axiom 2 immediately follows from formula (16).

To prove that the first part of Axiom 3 holds we need to show that

$$\Phi_t\left[(z, A_{t+1}) \mid \{(z, A_{t+1}), (z, A'_{t+1})\}\right] \geq \Phi_t\left[(z, A'_{t+1}) \mid \{(z, A_{t+1}), (z, A'_{t+1})\}\right]$$

iff

$$\Phi_t\left[(z', A_{t+1}) \mid \{(z', A_{t+1}), (z', A'_{t+1})\}\right] \geq \Phi_t\left[(z', A'_{t+1}) \mid \{(z', A_{t+1}), (z', A'_{t+1})\}\right].$$

By formula (16) this is equivalent to $W_t(z, A_{t+1}) \geq W_t(z, A'_{t+1})$ iff $W_t(z', A_{t+1}) \geq W_t(z', A'_{t+1})$. Since by formula (15) $W_t(z_t, A_{t+1}) = G_t(v_t(z_t), h_t(A_{t+1}))$, where G_t is increasing in its second argument, we know that $W_t(z, A_{t+1}) \geq W_t(z, A'_{t+1})$ iff $h_t(A_{t+1}) \geq h_t(A'_{t+1})$ iff $W_t(z', A_{t+1}) \geq W_t(z', A'_{t+1})$.

To prove that the second part of Axiom 3 holds we need to show that

$$\Phi_t\left[(z, A_{t+1}) \mid \{(z, A_{t+1}), (z', A_{t+1})\}\right] \geq \Phi_t\left[(z', A_{t+1}) \mid \{(z, A_{t+1}), (z', A_{t+1})\}\right]$$

iff

$$\Phi_t\left[(z, A'_{t+1}) \mid \{(z, A'_{t+1}), (z', A'_{t+1})\}\right] \geq \Phi_t\left[(z', A'_{t+1}) \mid \{(z, A'_{t+1}), (z', A'_{t+1})\}\right].$$

By formula (16) this is equivalent to $W_t(z, A_{t+1}) \geq W_t(z', A_{t+1})$ iff $W_t(z, A'_{t+1}) \geq W_t(z', A'_{t+1})$. Since by formula (15) $W_t(z_t, A_{t+1}) = G_t(v_t(z_t), h_t(A_{t+1}))$, where G_t is increasing in its first argument, we know that $W_t(z, A_{t+1}) \geq W_t(z', A_{t+1})$ iff $v_t(z) \geq v_t(z')$ iff $W_t(z, A'_{t+1}) \geq W_t(z', A'_{t+1})$.

[21] Note that our Steps 2 and 3 essentially mimic Koopmans (1960) proof, which leads him to formula (7).

B.2.3 Proof of Lemma 2

Fix an arbitrary element $(z_t, A_{t+1}) \in \mathcal{A}_t$ and let $r := W_t(z_t, A_{t+1})$. For any $\hat{r} \in (0, \infty)$ let $\lambda := \frac{\hat{r}}{r}$. Axiom 4 implies that there exists $(\hat{z}_t, A_{t+1}) \in \mathcal{A}_t$ such that

$$\frac{\Phi_t[(\hat{z}_t, A_{t+1}) \mid \{(z_t, A_{t+1}), (\hat{z}_t, A_{t+1})\}]}{\Phi_t[(z_t, A_{t+1}) \mid \{(z_t, A_{t+1}), (\hat{z}_t, A_{t+1})\}]} = \lambda.$$

By Lemma 1,

$$\frac{\hat{r}}{r} = \frac{W_t(\hat{z}_t, A_{t+1})}{W_t(z_t, A_{t+1})} = \frac{W_t(\hat{z}_t, A_{t+1})}{r}$$

Thus, $W_t(\hat{z}_t, A_{t+1}) = \hat{r}$. Since $\hat{r}$ was chosen arbitrarily, the conclusion follows. □

B.3 Proof of Theorem 5

B.3.1 Proof of the implication $(i) \Rightarrow (ii)$

By Lemma 1, the choice rule Φ_t has a sequential Luce representation with associated W_t and G_t functions (see Definition 11). Axiom 8 implies that for $t = 0, \ldots, T-1$

$$\begin{gathered} W_t(z_t, A_{t+1}) \geq W_t(z_t, A'_{t+1}) \\ \text{iff} \\ \sum_{(z_{t+1}, A_{t+2}) \in A_{t+1}} W_{t+1}(z_{t+1}, A_{t+2}) \geq \sum_{(z_{t+1}, A_{t+2}) \in A'_{t+1}} W_{t+1}(z_{t+1}, A_{t+2}). \end{gathered} \tag{31}$$

By equation (15) in Definition 11, the first inequality is equivalent to

$$G_t(v_t(z_t), h_t(A_{t+1})) \geq G_t(v_t(z_t), h_t(A'_{t+1})).$$

As the function G_t is strictly increasing in the second variable, property (31) is equivalent to

$$h_t(A_{t+1}) \geq h_t(A'_{t+1})$$

iff

$$\sum_{(z_{t+1}, A_{t+2}) \in A_{t+1}} W_{t+1}(z_{t+1}, A_{t+2}) \geq \sum_{(z_{t+1}, A_{t+2}) \in A'_{t+1}} W_{t+1}(z_{t+1}, A_{t+2}).$$

Note that the two inequalities above define numerical representations of the same weak order on $\mathcal{M}_{t+1}$. Thus by appropriately transforming G_t we can take

$$h_t(A_{t+1}) = \sum_{(z_{t+1}, A_{t+2}) \in A_{t+1}} W_{t+1}(z_{t+1}, A_{t+2})$$

B.3.2 Proof of the implication $(ii) \Rightarrow (i)$

To prove that Axiom 8 holds we need to show that for all $t = 0, 1, \ldots, T-1$ all $z \in Z$ and all $A_{t+1}, A'_{t+1} \in \mathcal{M}_{t+1}$

$$A_{t+1} \succsim_t B_{t+1}$$

$$\text{iff}$$

$$\Phi_{t+1}\left[A_{t+1} \mid A_{t+1} \cup B_{t+1}\right] \geq \Phi_{t+1}\left[B_{t+1} \mid A_{t+1} \cup B_{t+1}\right].$$

By formula (16) the top inequality is equivalent to $W_t(z, A_{t+1}) \geq W_t(z, B_{t+1})$ for some $z \in Z$ and the bottom one is equivalent to $\sum_{a_{t+1} \in A_{t+1}} W_{t+1}(a_{t+1}) \geq \sum_{b_{t+1} \in B_{t+1}} W_{t+1}(b_{t+1})$.

The conclusion follows since by formula (15)

$$W_t(z, A_{t+1}) = G_t\left(v_t(z), \sum_{a_{t+1} \in A_{t+1}} W_{t+1}(a_{t+1})\right),$$

where the function G_t is strictly increasing in the second variable.

B.3.3 Proof of the implication $(ii) \Leftrightarrow (iii)$

Suppose that $\{\Phi_t\}$ has a Recursive Flexibility Preferring Luce Representation with W_t, v_t, and G_t. By Lemma 4 and equation (16), the choice probabilities Φ_t are the solution to

$$\max_{p \in \Delta^n} \sum_{i=1}^{n} p_i \eta_t \log W_t(z_t^j, A_{t+1}^j) - \eta_t R(p)$$

so equation (19) in Definition 14 is satisfied by setting $U_t = \eta_t \log W_t$.

Finally, $U_t = \eta_t \log W_t$ and formula (17) imply that

$$U_t(z_t, A_{t+1}) = \eta_T \log G_t \left(v_t(z_t), \sum_{a_{t+1} \in A_{t+1}} \exp\left(\frac{U_{t+1}(a_{t+1})}{\eta_{t+1}} \right) \right),$$

which by setting $\bar{G}_t(u, \gamma) = \eta_t \log G_t \left(v, \exp\left(\frac{\gamma}{\eta_{t+1}} \right) \right)$ is equivalent to

$$U_t(z_t, A_{t+1}) = \bar{G}_t \left(v_t(z_t), \eta_{t+1} \log \left(\sum_{a_{t+1} \in A_{t+1}} \exp\left(\frac{U_{t+1}(a_{t+1})}{\eta_{t+1}} \right) \right) \right),$$

which by Lemma 4 is equivalent to

$$U_t(z_t, A_{t+1}) = \bar{G}_t \left(v_t(z_t), \max_{q \in \Delta(A_{t+1})} \sum q(a_{t+1}) U_{t+1}(a_{t+1}) + \eta_{t+1} H(q) \right),$$

which is equation (18) of Definition 14. The proof of the converse follows immediately.

B.3.4 Proof of the equivalence $(iii) \Leftrightarrow (iv)$

Because the recursive logit representation described in (iv) uses the same functions U_t, v_t, and G_t as the recursive entropy representation in (iii), the equivalence follows from Lemmas 4 and 6.

B.4 Proof of Theorem 6

The proof is entirely analogous to the proof of Theorem 5, except that every instance of Axiom 8, Lemma 4, and Lemma 6 is replaced with an instance of Axiom 10, Lemma 5, and Lemma 7, respectively.

B.5 Proof of Theorem 1 and Propositions 10 and 11

Section B.5.1 proves part A of Theorem 1. Moreover, Step 1 in that section is a proof of Proposition 10. Proposition 11 follows immediately from Step 1 as well. The subsequent sections prove parts B and C of Theorem 1.

B.5.1 Proof of part A of Theorem 1

B.5.1.1 Proof of the implication $(i) \Rightarrow (ii)$

Step 1: By Theorem 5, $\{\Phi_t\}$ has a Recursive Flexibility Preferring Luce Representation, i.e., there exist functions $W_0, \ldots, W_T$ that represent the choice rules $\Phi_0 \ldots, \Phi_T$ according to

$$\Phi_t[a_t \mid A_t] = \frac{W_t(a_t)}{\sum_{b_t \in A_t} W_t(b_t)}. \tag{32}$$

where

$$W_t(z, A_{t+1}) = G_t\left(\bar{v}_t(z), \sum_{a_{t+1} \in A_{t+1}} W_{t+1}(a_{t+1})\right) \tag{33}$$

where $\bar{v}_t : Z \to \mathbb{R}$ and $G_t : \text{Range}(\bar{v}_t) \times (0, \infty) \to \mathbb{R}$ is a function that is strictly increasing in both variables. (Note that the domain of the second argument of G_t is $(0, \infty)$, which follows from Axiom 4 and Lemma 2.)

We want to obtain a representation where

$$\Phi_t[\cdot \mid A_t] = \operatorname*{arg\,max}_{q \in \Delta(A_t)} \sum q(a_t) U_t(a_t) + \eta_t H(q) \tag{34}$$

and

$$U_t(z_t, A_{t+1}) = v_t(z_t) + \delta_t \left[\max_{q \in \Delta(A_{t+1})} \sum q(a_{t+1}) U_{t+1}(a_{t+1}) + \eta_{t+1} H(q)\right] \tag{35}$$

for some function $v_t : Z \to \mathbb{R}$ and $\delta_t > 0$.

By Lemma 4, equation (34) follows from (32) by setting $U_t(a_t) := \eta_t \log W_t(a_t)$. Moreover, with this definition of U_t equation (35) is equivalent to

$$W_t(z_t, A_{t+1}) = \exp\left(\frac{v_t(z_t)}{\eta_t} + \delta_t \frac{\eta_{t+1}}{\eta_t} \log\left(\sum_{a_{t+1} \in A_{t+1}} W_{t+1}(a_{t+1})\right)\right).$$

which is equivalent to

$$W_t(z_t, A_{t+1}) = \exp\left(\hat{v}_t(z_t) + \hat{\delta}_t \log\left(\sum_{a_{t+1} \in A_{t+1}} W_{t+1}(a_{t+1})\right)\right) \tag{36}$$

Thus it is sufficient to show that there are $\hat{v}_t : Z \to \mathbb{R}$ and $\hat{\delta}_t > 0$ so that (36) holds.

Step 2: From Lemma 2 it follows that

$$\{W_t(z_t, A_{t+1}) \mid z_t \in Z, A_{t+1} \in \mathcal{M}_{t+t}\} = (0, \infty). \tag{37}$$

Axiom 12 together with (32) imply that for any $t = 0, \dots, T-1$, any $z, z' \in Z$, any $\lambda \in (0, \infty)$ and any elements $(z_{t+1}, A_{t+2}), (\hat{z}_{t+1}, \hat{A}_{t+2}) \in Z \times \mathcal{M}_{t+2}$:

$$\frac{W_t(z_t, \{(z^\lambda_{t+1}, A_{t+2})\})}{W_t(z_t, \{(z_{t+1}, A_{t+2})\})} = \frac{W_t(\hat{z}_t, \{(\hat{z}^\lambda_{t+1}, \hat{A}_{t+2})\}\})}{W_t(\hat{z}_t, \{(\hat{z}_{t+1}, \hat{A}_{t+2})\})},$$

which can be written using (33) as

$$\frac{G_t(\bar{v}_t(z_t), W_{t+1}(\{(z^\lambda_{t+1}, A_{t+2})\}))}{G_t(\bar{v}_t(z_t), W_{t+1}(\{(z_{t+1}, A_{t+2})\}))} = \frac{G_t(\bar{v}_t(\hat{z}_t), W_{t+1}(\{(\hat{z}^\lambda_{t+1}, \hat{A}_{t+2})\}\}))}{G_t(\bar{v}_t(\hat{z}_t), W_{t+1}(\{(\hat{z}_{t+1}, \hat{A}_{t+2})\}))}.$$

Define functions $f^v_t : \mathbb{R} \to \mathbb{R}$ by $f^v_t(\gamma) := G_t(v, \gamma)$. Then by setting $v := \bar{v}_t(z_t)$ and $\hat{v} := \bar{v}_t(\hat{z}_t)$ the above equation is

$$\frac{f^v_t(W_{t+1}(\{(z^\lambda_{t+1}, A_{t+2})\}))}{f^v_t(W_{t+1}(\{(z_{t+1}, A_{t+2})\}))} = \frac{f^{\hat{v}}_t(W_{t+1}(\{(\hat{z}^\lambda_{t+1}, \hat{A}_{t+2})\}\}))}{f^{\hat{v}}_t(W_{t+1}(\{(\hat{z}_{t+1}, \hat{A}_{t+2})\}))}.$$

From Axiom 4 it follows that $W_{t+1}(\{(z^\lambda_{t+1}, A_{t+2})\}) = \lambda W_{t+1}(\{(z_{t+1}, A_{t+2})\})$ and $W_{t+1}(\{(\hat{z}^\lambda_{t+1}, \hat{A}_{t+2})\}\}) = \lambda W_{t+1}(\{(\hat{z}_{t+1}, \hat{A}_{t+2})\})$; thus,

$$\frac{f^v_t(\lambda W_{t+1}(\{(z_{t+1}, A_{t+2})\}))}{f^v_t(W_{t+1}(\{(z_{t+1}, A_{t+2})\}))} = \frac{f^{\hat{v}}_t(\lambda W_{t+1}(\{(\hat{z}_{t+1}, \hat{A}_{t+2})\}))}{f^{\hat{v}}_t(W_{t+1}(\{(\hat{z}_{t+1}, \hat{A}_{t+2})\}))}.$$

Since (z_{t+1}, A_{t+2}) and $(\hat{z}_{t+1}, \hat{A}_{t+2})$ were arbitrary, and in the light of (37), we have that for all $r, \hat{r} > 0$

$$\frac{f^v_t(\lambda r)}{f^v_t(r)} = \frac{f^{\hat{v}}_t(\lambda \hat{r})}{f^{\hat{v}}_t(\hat{r})}. \tag{38}$$

Step 3: Let $v = \hat{v}$ and $\hat{r} = 1$. Define $f(r) := f^v_t(r)$ and $g(\lambda) := f(\lambda)/f(1)$. Then (38) implies that

$$f(\lambda r) = g(\lambda) f(r)$$

This is a special case of the Pexider functional equation. [22] Its nonzero solutions are of the form (see Theorem 4 Section 3.1 of Aczél, 1966) $f(r) = \beta r^\delta$ and $g(r) = r^\delta$ for some $\beta, \delta > 0$.

Thus, for any v and t there exist $\beta_t^v, \delta_t^v > 0$ such that $f_t^v(r) = \beta_t^v r^{\delta_t^v}$. Substituting this into equation (38) we obtain

$$\frac{\beta_t^v (\lambda r)^{\delta_t^v}}{\beta_t^v (r)^{\delta_t^v}} = \frac{\beta_t^{\hat{v}} (\lambda \hat{r})^{\delta_t^{\hat{v}}}}{\beta_t^{\hat{v}} (\hat{r})^{\delta_t^{\hat{v}}}},$$

which after simplifying is $\lambda^{\delta_t^v} = \lambda^{\delta_t^{\hat{v}}}$. Since λ is arbitrary, this implies that the value of δ_t^v does not depend on v. To summarize, we conclude that for any v and t there exist $\beta_t^v, \delta_t > 0$ such that $f_t^v(r) = \beta_t^v r^{\delta_t}$.

Step 4: By Step 3 and the definition of f_t^v we have that

$$\begin{aligned} W_t(z_t, A_{t+1}) &= G_t\left(\bar{v}_t(z_t), \sum_{a_{t+1} \in A_{t+1}} W_{t+1}(a_{t+1})\right) = \beta_t^{\bar{v}_t(z_t)} \left(\sum_{a_{t+1} \in A_{t+1}} W_{t+1}(a_{t+1})\right)^{\delta_t} \\ &= \exp\left(\hat{v}_t(z_t) + \hat{\delta}_t \log\left(\sum_{a_{t+1} \in A_{t+1}} W_{t+1}(a_{t+1})\right)\right), \end{aligned}$$

where $\hat{v}_t(z_t) := \log \beta_t^{\bar{v}_t(z_t)}$ and $\hat{\delta}_t = \delta_t$. This is exactly (36). □

Step 5: To prove that functions v_t are surjective, note that Lemma 2 implies that for any $A_{t+1} \in \mathcal{X}_{t+1}$ and any $r \in (0, \infty)$ there exists $z_t \in Z$ such that $W_t(z_t, A_{t+1}) = r$. By (36), this implies that for any $A_{t+1} \in \mathcal{M}_{t+1}$ and any $r' \in \mathbb{R}$ there exists $z_t \in Z$ such that

$$\hat{v}_t(z_t) + \hat{\delta}_t \log\left(\sum_{a_{t+1} \in A_{t+1}} W_{t+1}(a_{t+1})\right) = r'. \tag{39}$$

Fix $r'' \in R$. We will show that there exists $z_t \in Z$ such that $\hat{v}_t(z_t) = r''$ (since the function $v_t = \eta_t \hat{v}_t$ this proves the claim). By Lemma 2 there exists $a_{t+1} \in \mathcal{A}_{t+1}$ such that $W_{t+1}(a_{t+1}) = \exp(\frac{r' - r''}{\hat{\delta}_t})$. To conclude, apply (39) with $A_{t+1} = \{a_{t+1}\}$.

[22] Other forms of this functional equation are used in Klibanoff, Marinacci, and Mukerji (2005) and in Strzalecki (2011).

B.5.1.2 Proof of the implication $(ii) \Rightarrow (i)$

The necessity of Axioms 1–3 and 6 follows from Theorem 5. To see that Axiom 12 holds, fix $t \in \{0, \ldots, T-1\}$, $z_t, \hat{z}_t \in Z$, $\lambda \in (0, \infty)$ and $(z_t, \{(z_{t+1}, A_{t+2})\}), (\hat{z}_t, \{(\hat{z}_{t+1}, \hat{x}_{t+2})\}) \in Z \times \mathcal{M}_{t+2}$. Let

$$A_t = \{(z_t, \{(z_{t+1}, A_{t+2})\}), (z_t, \{(z^{\lambda}_{t+1}, A_{t+2})\}), (\hat{z}_t, \{(\hat{z}_{t+1}, \hat{x}_{t+2})\}), (\hat{z}_t, \{(\hat{z}^{\lambda}_{t+1}, \hat{x}_{t+2})\})\}.$$

By definition of $(z^{\lambda}_{t+1}, A_{t+2})$, in Axiom 4 and expression (4) in the definition of the representation, and by Lemma 4 we have

$$\lambda = \frac{\Phi_{t+1}[(z_t, \{(z^{\lambda}_{t+1}, A_{t+2})\})|\{(z_t, \{(z_{t+1}, A_{t+2})\}), (z^{\lambda}_{t+1}, A_{t+2})\}]}{\Phi_{t+1}[(z_t, \{(z_{t+1}, A_{t+2})\})|\{(z_t, \{(z_{t+1}, A_{t+2})\}), (z^{\lambda}_{t+1}, A_{t+2})\}]} = \frac{\exp\left(\frac{U_{t+1}(z_t, \{(z^{\lambda}_{t+1}, A_{t+2})\})}{\eta_{t+1}}\right)}{\exp\left(\frac{U_{t+1}(z_t, \{(z_{t+1}, A_{t+2})\})}{\eta_{t+1}}\right)},$$

thus,

$$U_{t+1}(z_t, \{(z^{\lambda}_{t+1}, A_{t+2})\}) = U_{t+1}((z_t, \{(z_{t+1}, A_{t+2})\})) + \eta_{t+1} \log \lambda. \tag{40}$$

By expression (4) and by Lemma 4

$$\frac{\Phi_t[(z_t, \{(z^{\lambda}_{t+1}, A_{t+2})\})|Y]}{\Phi_t[(z_t, \{(z_{t+1}, A_{t+2})\})|Y]} = \frac{\exp\left(\frac{U_t(z_t, \{(z^{\lambda}_{t+1}, A_{t+2})\})}{\eta_t}\right)}{\exp\left(\frac{U_t(z_t, \{(z_{t+1}, A_{t+2})\})}{\eta_t}\right)}$$

by expression (3) applied to the singletons $\{(z^{\lambda}_{t+1}, A_{t+2})\}$ and $\{(z_{t+1}, A_{t+2})\}$ and expression (40)

$$= \exp\left(\frac{v_t(z) + \delta_t\left(U_{t+1}(z_{t+1}, A_{t+2}) + \eta_{t+1} \log \lambda\right)}{\eta_t}\right) \div \exp\left(\frac{v_t(z) + \delta_t U_{t+1}((z_{t+1}, A_{t+2}))}{\eta_t}\right)$$
$$= \exp\left(\delta_t \frac{\eta_{t+1} \log \lambda}{\eta_t}\right)$$

which is independent of (z_{t+1}, A_{t+2}). Thus, the value of $\frac{\Phi_t[(\hat{z}_t, \{(\hat{z}^{\lambda}_{t+1}, \hat{x}_{t+2})\})|Y]}{\Phi_t[(\hat{z}_t, \{(\hat{z}_{t+1}, \hat{x}_{t+2})\})|Y]}$ is the same. $\square$

B.5.1.3 Proof of the equivalence $(ii) \Leftrightarrow (iii)$

Because the recursive logit representation described in (iv) uses the same functions U_t, v_t, and G_t as the recursive entropy representation in (iii), the equivalence follows from Lemmas 4 and 6.

B.5.2 Proof of part B of Theorem 1

B.5.2.1 Stationarity over Streams

Recall that the stochastic preference $\succsim_0$ induces a preference on *consumption streams* $\tilde{z} = (z_0, z_1, \ldots, z_T)$. Moreover, given (35), this preference is represented by $\tilde{z} \mapsto \sum_{t=0}^{T} \prod_{s=0}^{t-1} \delta_s v_t(z_t)$. Let $v := v_0$ and $\delta := \delta_0$. From the proof of Theorem 7.5 of Fishburn (1970)[23] it follows that if Axiom 13 is satisfied, then $v_t \equiv v$ for all t and $\delta_t = \delta$ for all t.

B.5.2.2 Stationarity of Choices

By (32), Axiom 14 is equivalent to

$$\frac{W_0(z, A_1)}{W_0(z, A_1) + W_0(z', A_1)} = \frac{W_t(z, A_t)}{W_t(z, A_t) + W_t(z', A_t)}.$$

Let $\gamma_i := \log\left(\sum_{a_{i+1} \in A_{i+1}} W_{i+1}(a_{i+1})\right)$ for $i = 0, t$. By (36), this equation is equivalent to

$$\frac{\exp(\hat{v}_0(z) + \hat{\delta}_0\gamma_0)}{\exp(\hat{v}_0(z) + \hat{\delta}_0\gamma_0) + \exp(\hat{v}_0(z') + \hat{\delta}_0\gamma_0)} = \frac{\exp(\hat{v}_t(z) + \hat{\delta}_t\gamma_t)}{\exp(\hat{v}_t(z) + \hat{\delta}_t\gamma_t) + \exp(\hat{v}_t(z') + \hat{\delta}_t\gamma_t)},$$

where $\hat{v}_i = \frac{v}{\eta_i}$ and $\hat{\delta}_i = \delta\frac{\eta_{i+1}}{\eta_i}$ which simplifies to

$$\frac{\exp(\hat{v}_0(z))}{\exp(\hat{v}_0(z)) + \exp(\hat{v}_0(z'))} = \frac{\exp(\hat{v}_t(z))}{\exp(\hat{v}_t(z)) + \exp(\hat{v}_t(z'))}.$$

Thus by letting $v := v(z)$ and $v' := v(z')$ we have

$$\frac{\exp\left(\frac{v}{\eta_0}\right)}{\exp\left(\frac{v}{\eta_0}\right) + \exp\left(\frac{v'}{\eta_0}\right)} = \frac{\exp\left(\frac{v}{\eta_t}\right)}{\exp\left(\frac{v}{\eta_t}\right) + \exp\left(\frac{v'}{\eta_t}\right)}.$$

Thus

$$\frac{v}{\eta_0} + \frac{v'}{\eta_t} = \frac{v}{\eta_t} + \frac{v'}{\eta_0},$$

so $(\eta_t - \eta_0)(v - v') = 0$, which implies that $\eta_t = \eta_0$.

[23] Our induced preference on consumption streams may not satisfy the continuity property that Fishburn requires. However, the cardinal uniqueness of additive representations invoked in his proof holds due to the surjectivity of the v_t functions.

B.5.2.3 Impatience

Given (35), and the stationarity of v and δ, the induced preference on *consumption streams* $\tilde{z} = (z_0, z_1, \ldots, z_T)$ is represented by $\tilde{z} \mapsto \sum_{t=0}^{T} \delta^{t-1} v(z_t)$,with $(z, \ldots, z) \succ_0 (z', \ldots, z')$ if and only if $v(z) > v(z')$ and

$$(z_0, \ldots, z_{t-1}, z, z', z_{t+2}, \ldots, z_T) \succsim_0 (z_0, \ldots, z_{t-1}, z', z, z_{t+2}, \ldots, z_T)$$

if and only if $v(z)+\delta v(z') > v(z')+\delta v(z)$. Thus, Axiom 15 implies that $(v(z)-v(z'))(1-\delta) > 0$, which implies that $\delta < 1$.

B.5.3 Proof of part C of Theorem 1

Suppose that v, δ, η and v', δ', η' represent a stationary $\{\Phi_t\}$ Given (35) and stationarity, the induced preference on consumption streams is represented by $\tilde{z} \mapsto \sum_{t=0}^{T} \delta^{t-1} v(z_t)$ and by $\tilde{z} \mapsto \sum_{t=0}^{T} \delta'^{t-1} v'(z_t)$. Thus, given that both v and v' are surjective, the uniqueness result for additive representations, e.g., Theorems 5.2 and 5.3 of Fishburn (1970) implies that there are $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $v' \equiv \alpha v + \beta$ and $\delta' = \delta$.

By (4) and Lemma 4, for any $z, \check{z} \in Z$

$$\Phi_T[z|\{z, \check{z}\}] = \frac{\exp\left(\frac{v(z)}{\eta}\right)}{\exp\left(\frac{v(\check{z})}{\eta}\right)} = \frac{\exp\left(\frac{v'(z)}{\eta'}\right)}{\exp\left(\frac{v'(\check{z})}{\eta'}\right)},$$

so

$$\frac{v(z)}{\eta} + \frac{v'(\check{z})}{\eta'} = \frac{v'(z)}{\eta'} + \frac{v(\check{z})}{\eta},$$

which implies that $\eta'[v(z) - v(\check{z})] = \eta[v'(z) - v'(\check{z})]$. Since $v' \equiv \alpha v + \beta$, this implies that $(\eta' - \alpha\eta)[v(z) - v(\check{z})] = 0$, and since z and $\check{z}$ were arbitrary, $\alpha = \frac{\eta'}{\eta}$.

B.6 Proof of Theorem 2 and Propositions 13 and 12

The proof is analogous to the entropy case with the exception that any expression $H(q)$ should be replaced with $R(q)$ and $\sum_{a_{t+1} \in A_{t+1}} W_{t+1}(a_{t+1})$ should be replaced with $\frac{1}{|A_{t+1}|} \sum_{a_{t+1} \in A_{t+1}} W_{t+1}(a_{t+1})$ and Lemma 5 can then be invoked instead of Lemma 4. $\square$

B.7 Proof of Theorem 3

By Lemma 1, there exist a Sequential Luce Representation and by Lemma 2 it follows that $\left\{\sum_{(z_t,A_{t+1})\in A_t} W_t(z_t,A_{t+1}) \mid A_t \in \mathcal{M}_t\right\} = (0,\infty)$.
Step 1: By Theorem 5

$$W_t(z, A_{t+1}) = G_t\left(v_t(z), \sum_{a_{t+1}\in A_{t+1}} W_{t+1}(a_{t+1})\right) \tag{41}$$

where $v_t : Z \to \mathbb{R}$ and $G_t : \text{Range}(v_t) \times (0,\infty) \to \mathbb{R}$ is a function that is strictly increasing in both variables. We want to show that

$$W_t(z, A_{t+1}) = \hat{v}_t(z) + \delta_t \sum_{a_{t+1}\in A_{t+1}} W_{t+1}(a_{t+1}) \tag{42}$$

for some function $\hat{v}_t : Z \to \mathbb{R}$ and $\delta_t > 0$. Toward obtaining (42), Steps 2 and 3 show that

$$G_t(v,\gamma) = \alpha_t(v) + \delta_t\gamma \tag{43}$$

for some $\delta_t > 0$ and function $\alpha_t : \text{Range}(v_t) \to \mathbb{R}$. To show that (42) follows from (43), set $\hat{v}_t(z) := \alpha_t(v_t(z))$.
Step 2: Define the strictly increasing function $h_t^v : (0,\infty) \to \mathbb{R}$ by $h_t^v(\gamma) := G_t(v,\gamma)$. Let z, z' be such that $v_t(z) = v_t(z') = v$; then Axiom 16 implies that

$$\begin{aligned} &h_t^v\left(W_{t+1}(a_{t+1}) + W_{t+1}(b_{t+1})\right) - h_t^v\left(W_{t+1}(a_{t+1}) + W_{t+1}(b'_{t+1})\right) \\ &\quad = h_t^v\left(W_{t+1}(a'_{t+1}) + W_{t+1}(b_{t+1})\right) - h_t^v\left(W_{t+1}(a'_{t+1}) + W_{t+1}(b'_{t+1})\right) \end{aligned} \tag{44}$$

Axiom 4 guarantees that for any $\mathbf{c}, \mathbf{d} \in (0,\infty)$ there exist elements $c_{t+1}, d_{t+1} \in \mathcal{A}_{t+1}$ such that $W_{t+1}(c_{t+1}) = \mathbf{c}$ and $W_{t+1}(d_{t+1}) = \mathbf{d}$. Moreover, Axiom 4 guarantees that there exist distinct elements $a_{t+1}, a'_{t+1}, b_{t+1}, b'_{t+1} \in \mathcal{A}_{t+1}$ such that $a_{t+1} \sim_{t+1} b_{t+1} \sim_{t+1} c_{t+1}$ and $a'_{t+1} \sim_{t+1} b'_{t+1} \sim_{t+1} d_{t+1}$. Thus, $W_{t+1}(a_{t+1}) = W_{t+1}(b_{t+1}) = \mathbf{c}$ and $W_{t+1}(a'_{t+1}) = W_{t+1}(b'_{t+1}) = \mathbf{d}$. This, together with equation (44) implies that for any $\mathbf{c}, \mathbf{d} \in (0, \infty)$

$$h_t^v(\mathbf{c}+\mathbf{d}) = \frac{h_t^v(2\mathbf{c}) + h_t^v(2\mathbf{d})}{2}$$

Since the domain of h_t^v is $(0, \infty)$, this is equivalent to

$$h_t^v\left(\frac{\mathbf{c}+\mathbf{d}}{2}\right) = \frac{h_t^v(\mathbf{c}) + h_t^v(\mathbf{d})}{2} \tag{45}$$

for all $\mathbf{c}, \mathbf{d} \in (0, \infty)$. Equation (45) is Jensen's equation. Since h_t^v is monotone, standard arguments (see, e.g., Theorem 1 of Section 2.14 together with arguments from the proof of Theorem 1 of Section 2.1 of Aczél, 1966) show that its solution is $h_t^v(\gamma) = \delta_t^v \cdot \gamma + \alpha_t^v$ for some constants $\delta_t^v, \alpha_t^v \in \mathbb{R}$. Moreover, $\delta_t^v > 0$ since the function h_t^v is strictly increasing. It follows that $G_t(v, \gamma) = \delta_t^v \gamma + \alpha_t^v$.

Step 3: Fix $v \neq v' \in \text{Range}(v_t)$ and let $z, z' \in Z$ be such that $v_t(z) = v$ and $v_t(z') = v'$. Fix $\mathbf{c}, \mathbf{d} \in (0, \infty)$ such that $\mathbf{c} \neq \mathbf{d}$. By Axiom 4, there exist distinct elements $a, b, a', b' \in\in Z \times \mathcal{M}_{t+2}$ such that $W_{t+1}(a) = W_{t+1}(b) = \mathbf{c}$ and $W_{t+1}(a') = W_{t+1}(b') = \mathbf{d}$. Axiom 16 implies that

$$\begin{aligned}
[\delta_t^v \cdot 2\mathbf{c} + \alpha_t^v] - [\delta_t^v \cdot (\mathbf{c}+\mathbf{d}) + \alpha_t^v] &= \left[\delta_t^{v'} \cdot (\mathbf{c}+\mathbf{d}) + \alpha_t^{v'}\right] - \left[\delta_t^{v'} \cdot 2\mathbf{d} + \alpha_t^{v'}\right] \\
\delta_t^v \cdot (\mathbf{c}-\mathbf{d}) &= \delta_t^{v'} \cdot (\mathbf{c}-\mathbf{d}) \\
\delta_t^v &= \delta_t^{v'}
\end{aligned}$$

Let δ_t be the common value of δ_t^v. Then $G_t(v(z), \gamma) = \delta_t \gamma + \alpha_t^v$. □

B.8 Proof of Results on Duplicates

B.8.1 Proof of Proposition 3

Proof of statement 1. Fix an arbitrary menu $A_{t+1} \in \mathcal{M}_{t+1}$. Let $z^1, \ldots, z^n, \ldots$ be duplicates at time $t+1$, and let $B_{t+1}^n = \{(z^1, C_{t+2}), \ldots, (z^n, C_{t+2})\}$. We need to show that there is an N such that $B_{t+1}^n \succ_t A_{t+1}$ for all $n > N$.

Since $z^1 \sim_{t+1} z^i$, Definitions 7 and 8 and Axiom 3 imply that $(z^1, C_{t+2}) \sim_{t+1} (z^i, C_{t+2})$, which by Definition 1 and Axiom 1 implies that

$$\Phi_{t+1}[(z^1, C_{t+2})|A_{t+1} \cup B^n_{t+1}] = \Phi_{t+1}[(z^i, C_{t+2})|A_{t+1} \cup B^n_{t+1}].$$

By Definition 1 and Axiom 6 it suffices to show that

$$\Phi_{t+1}[B^n_{t+1}|A_{t+1} \cup B^n_{t+1}] > \Phi_{t+1}[A_{t+1}|A_{t+1} \cup B^n_{t+1}],$$

i.e., $\Phi_{t+1}[B^n_{t+1}|A_{t+1} \cup B^n_{t+1}] > \frac{1}{2}$. Let

$$k := \frac{\Phi_{t+1}[A_{t+1}|A_{t+1} \cup \{(z^1, C_{t+2})\}]}{\Phi_{t+1}[(z^1, C_{t+2})|A_{t+1} \cup \{(z^1, C_{t+2})\}]} = \frac{\Phi_{t+1}[A_{t+1}|A_{t+1} \cup B^n_{t+1}]}{\Phi_{t+1}[(z^1, C_{t+2})|A_{t+1} \cup B^n_{t+1}]},$$

where the last equality follows from Axiom 1. We have

$$\begin{aligned} 1 &= \Phi_{t+1}[A_{t+1}|A_{t+1} \cup B^n_{t+1}] + \sum_{t=1}^{n} \Phi_{t+1}[(z^i, C_{t+2})|A_{t+1} \cup B^n_{t+1}] \\ &= \Phi_{t+1}[A_{t+1}|A_{t+1} \cup B^n_{t+1}] + n\Phi_{t+1}[(z^1, C)|A_{t+1} \cup B^n_{t+1}], \end{aligned}$$

so $\Phi_{t+1}[(z^1, C_{t+2})|A_{t+1} \cup B^n_{t+1}] = \frac{1}{k+n}$ and therefore $\Phi_{t+1}[B^n_{t+1}|A_{t+1} \cup B^n_{t+1}] = \frac{n}{k+n}$ which for n large enough is greater than $\frac{1}{2}$ as desired.

Proof of statement 2. Let $B^n_{t+1} := \{(z^1, C_{t+2}) \ldots, (z^n, C_{t+2})\}$. Definition 8 and Axiom 1 imply that

$$\frac{\Phi_{t+1}[(z^1, C_{t+2})|A_{t+1} \cup B^n_{t+1}]}{\Phi_{t+1}[(z^i, C_{t+2})|A_{t+1} \cup B^n_{t+1}]} \leq 1 + \epsilon$$

for all i. Analogous to statement 1, it suffices to show $\Phi_{t+1}[A_{t+1}|A_{t+1} \cup B^n_{t+1}] < \frac{1}{2}$. Let k be defined as in statement 1. Then

$$1 = \Phi_{t+1}[A_{t+1}|A_{t+1} \cup B^n_{t+1}] + \sum_{t=1}^{n} \Phi_{t+1}[(z^i, C_{t+2})|A_{t+1} \cup B^n_{t+1}]$$

thus dividing by $\Phi_{t+1}[(z^1, C_{t+2})|A_{t+1} \cup B^n_{t+1}]$

$$\frac{1}{\Phi_{t+1}[(z^1, C_{t+2})|A_{t+1} \cup B^n_{t+1}]} = k + 1 + \sum_{t=2}^{n} \frac{\Phi_{t+1}[(z^i, C_{t+2})|A_{t+1} \cup B^n_{t+1}]}{\Phi_{t+1}[(z^1, C_{t+2})|A_{t+1} \cup B^n_{t+1}]} \geq k + 1 + \frac{n-1}{1+\epsilon}$$

so we have

$$\Phi_{t+1}[(z^1, C_{t+2})|A_{t+1} \cup B^n_{t+1}] \leq \frac{1+\epsilon}{n+k+(k+1)\epsilon}$$

and thus

$$\Phi_{t+1}[A_{t+1}|A_{t+1} \cup B^n_{t+1}] \leq \frac{k+k\epsilon}{n+k+(k+1)\epsilon},$$

which for n large enough is smaller than $\frac{1}{2}$ as desired. □

B.8.2 Proof of Proposition 4

Proof of statement 1: Fix a continuation menu $C_{t+2} \in \mathcal{A}_{t+2}$. Let $z^1, z^2, \ldots$ be duplicates at time $t+1$ and let $B^n_{t+1} = \{(z^1, C_{t+2}) \ldots, (z^n, C_{t+2})\}$. Let $A_{t+1} = (z, C_{t+2})$ and let $D_{t+1} = (y, C_{t+2})$. We need to show that if $z \succ_{t+1} y \succ_{t+1} z^1$ then $A_{t+1} \succ_t B^n_{t+1}$ for all n, and there exists N such that for $n \geq N$, $D_{t+1} \succ_t A_{t+1} \cup B^n_{t+1}$. Let

$$k := \frac{\Phi_{t+1}[A_{t+1}|A_{t+1} \cup B^n_{t+1}]}{\Phi_{t+1}[(z^1, C_{t+2})|A_{t+1} \cup B^n_{t+1}]}.$$

Since $z \succ_{t+1} z^1$ we know that $k > 1$, so from the proof of statement 1 of the previous result

$$\Phi_{t+1}[A_{t+1}|A_{t+1} \cup B^n_{t+1}] = k/(k+n) > 1/(1+n).$$

Hence, Axiom 10 implies that $A_{t+1} \succ_t B^n_{t+1}$ for any n.

For the second claim let

$$\frac{\Phi_{t+1}[A_{t+1} \mid A_{t+1} \cup D_{t+1}]}{\Phi_{t+1}[D_{t+1} \mid A_{t+1} \cup D_{t+1}]} = \gamma > 1$$

and

$$\frac{\Phi_{t+1}[D_{t+1} \mid D_{t+1} \cup B^n_{t+1}]}{\Phi_{t+1}[(z^1, C_{t+2})|D_{t+1} \cup B^n_{t+1}]} = \lambda > 1.$$

Then

$$\Phi_{t+1}[A_{t+1} \mid A_{t+1} \cup D_{t+1} \cup B^n_{t+1}] + \Phi_{t+1}[D_{t+1} \mid A_{t+1} \cup D_{t+1} \cup B^n_{t+1}] + n\Phi_{t+1}[(z^1, C_{t+2}) \mid A_{t+1} \cup D_{t+1} \cup B^n_{t+1}] = 1,$$

so

$$\Phi_{t+1}[D_{t+1} \mid A_{t+1} \cup D_{t+1} \cup B^n_{t+1}] = \lambda/[n + \lambda + \lambda\gamma].$$

By Axiom 10 $D_{t+1} \succ_t A_{t+1} \cup B^n_{t+1}$if

$$\Phi_{t+1}[D_{t+1} \mid A_{t+1} \cup D_{t+1} \cup B^n_{t+1}] > \frac{1}{n+1}\Phi_{t+1}[A_{t+1} \cup B^n_{t+1} \mid A_{t+1} \cup D_{t+1} \cup B^n_{t+1}]$$

which is equivalent to $\lambda > \frac{1}{n+1}(n + \lambda\gamma)$, which is true for sufficiently large n because $\lambda > 1$.

Proof of Statement 2: Now let $z^2, z^3, \ldots$ be $\varepsilon-$duplicates of z^1 at time $t+1$ with $\varepsilon < \bar{\varepsilon}$, and let $B^n_{t+1} = \{(z^1, C_{t+2}) \ldots, (z^n, C_{t+2})\}$. Since

$$\frac{\Phi_{t+1}[D_{t+1} \mid D_{t+1} \cup \{(z^1, C_{t+2})\}]}{\Phi_{t+1}[(z^1, C_{t+2}) \mid D_{t+1} \cup \{(z^1, C_{t+2})\}]} > 1$$

there is an $\bar{\varepsilon} > 0$ such that

$$\lambda = \min_i \frac{\Phi_{t+1}[D_{t+1} \mid D_{t+1} \cup \{(z^i, C_{t+2})\}]}{\Phi_{t+1}[(z^i, C_{t+2}) \mid D_{t+1} \cup \{(z^i, C_{t+2})\}]} > 1.$$

Let $G^n_{t+1} = A_{t+1} \cup D_{t+1} \cup B^n_{t+1}$, and note that

$$\frac{1}{|\{z^1, ...z^n\}|}\Phi_{t+1}\left[B^n_{t+1} \mid G^n_{t+1}\right] = \frac{1}{n}\sum_{i=1}^{n}\Phi_{t+1}\left[(z^i, C_{t+2}) \mid G^n_{t+1}\right] \le \frac{1}{\lambda}\frac{1}{n}n\Phi_{t+1}\left[D_{t+1} \mid G^n_{t+1}\right]$$

and

$$\frac{1}{|\{z\}|}\Phi_{t+1}\left[A_{t+1} \mid G^n_{t+1}\right] = \gamma\frac{1}{|\{y\}|}\Phi_{t+1}\left[D_{t+1} \mid G^n_{t+1}\right].$$

Since $\gamma > 1 > \frac{1}{\lambda}$, we have

$$\frac{1}{|\{z\}|}\Phi_{t+1}\left[A_{t+1} \mid G^n_{t+1}\right] > \frac{1}{|\{z^1, ...z^n\}|}\Phi_{t+1}[B_{t+1} \mid G^n_{t+1}];$$

hence, by Stage IIA

$$\frac{1}{|\{z\}|}\Phi_{t+1}\left[A_{t+1} \mid A_{t+1} \cup B_{t+1}\right] > \frac{1}{|\{z^1, ...z^n\}|}\Phi_{t+1}[B_{t+1} \mid A_{t+1} \cup B_{t+1}]$$

and by Axiom 10 it follows that $A_{t+1} \succ_t B_{t+1}$,which proves the first claim in statement 2.

For the second claim, note that $\Phi_{t+1}[A_{t+1} \cup B_{t+1} \mid G^n_{t+1}] + \Phi_{t+1}[D_{t+1} \mid G^n_{t+1}] = 1$, and that we need to show that $\Phi_{t+1}[D_{t+1} \mid G^n_{t+1}] \geq \frac{1}{n+2}$. Note that $\Phi_{t+1}[A_{t+1} \mid G^n_{t+1}] + \Phi_{t+1}[D_{t+1} \mid G^n_{t+1}] + n\Phi_{t+1}[(z^1, C_{t+2}) \mid G^n_{t+1}] = 1$, and

$$\frac{\Phi_{t+1}[A_{t+1} \mid A_{t+1} \cup D_{t+1}]}{\Phi_{t+1}[D_{t+1} \mid A_{t+1} \cup D_{t+1}]} = \gamma$$

so

$$\Phi_{t+1}[D_{t+1} \mid G^n_{t+1}](1+\gamma) = 1 - n\Phi_{t+1}[(z^1, C_{t+2}) \mid G^n_{t+1}] \geq 1 - n\Phi_{t+1}[D_{t+1} \mid G^n_{t+1}]/\lambda,$$

where the last inequality follows from the definition of λ. Thus

$$\Phi_{t+1}[D_{t+1} \mid G^n_{t+1}](1 + \gamma + n/\lambda) \geq 1$$

or

$$\Phi_{t+1}[D_{t+1} \mid G^n_{t+1}] \geq \lambda/(\lambda + \lambda\gamma + n)$$

Thus $\Phi_{t+1}[D_{t+1} \mid G^n_{t+1}] > 1/(n+2)$ if $n > \lambda(\gamma-1)/(\lambda-1)$ as desired. □

B.9 Proof of Theorem 4

By Lemma 1, there exist a Sequential Luce Representation and by Lemma 2 it follows that $\left\{\frac{1}{|A_t|}\sum_{(z_t, A_{t+1})\in A_t} W_t(z_t, A_{t+1}) \mid A_t \in \mathcal{M}_t\right\} = (0, \infty)$.

Step 1: By Theorem 6, there exists a function $v_t : Z \to \mathbb{R}$ and a function $G_t : \text{Range}(v_t) \times (0, \infty) \to \mathbb{R}$, strictly increasing in both variables, such that

$$W_t(z, A_{t+1}) = G_t\left(v_t(z), \frac{1}{|A_{t+1}|}\sum_{a_{t+1}\in A_{t+1}} W_{t+1}(a_{t+1})\right). \tag{46}$$

We want to show that

$$W_t(z, A_{t+1}) = \hat{v}_t(z) + \delta_t \frac{1}{|A_{t+1}|}\sum_{a_{t+1}\in A_{t+1}} W_{t+1}(a_{t+1}) \tag{47}$$

for some function $\hat{v}_t : Z \to \mathbb{R}$ and $\delta_t > 0$. Toward obtaining (47), Steps 2 and 3 show that

$$G_t(v, \gamma) = \alpha_t(v) + \delta_t \gamma \tag{48}$$

for some $\delta_t > 0$ and function $\alpha_t : \text{Range}(v_t) \to \mathbb{R}$. To show that (47) follows from (48), set $\hat{v}_t(z) := \alpha_t(v_t(z))$.

Step 2: For any $t = 0, \dots, T-1$ and $v \in \text{Range}(v_t)$ define the strictly increasing function $h_t^v : (0, \infty) \to \mathbb{R}$ by $h_t^v(\gamma) := G_t(v, \gamma)$.

Let z, z' be such that $v_t(z) = v_t(z') = v$; then Axiom 16 implies that

$$\begin{aligned} h_t^v\left(\frac{W_{t+1}(a_{t+1}) + W_{t+1}(b_{t+1})}{2}\right) - h_t^v\left(\frac{W_{t+1}(a_{t+1}) + W_{t+1}(b'_{t+1})}{2}\right) \\ = h_t^v\left(\frac{W_{t+1}(a'_{t+1}) + W_{t+1}(b_{t+1})}{2}\right) - h_t^v\left(\frac{W_{t+1}(a'_{t+1}) + W_{t+1}(b'_{t+1})}{2}\right) \end{aligned} \tag{49}$$

Axiom 4 guarantees that for any $\mathbf{c}, \mathbf{d} \in (0, \infty)$ there exist elements $c_{t+1}, d_{t+1} \in \mathcal{A}_{t+1}$ such that $W_{t+1}(c_{t+1}) = \mathbf{c}$ and $W_{t+1}(d_{t+1}) = \mathbf{d}$. Moreover, Axiom 4 guarantees that there exist distinct elements $a_{t+1}, a'_{t+1}, b_{t+1}, b'_{t+1} \in \mathcal{A}_{t+1}$ such that $a_{t+1} \sim_{t+1} b_{t+1} \sim_{t+1} c_{t+1}$ and $a'_{t+1} \sim_{t+1} b'_{t+1} \sim_{t+1} d_{t+1}$. Thus, $W_{t+1}(a_{t+1}) = W_{t+1}(b_{t+1}) = \mathbf{c}$ and $W_{t+1}(a'_{t+1}) = W_{t+1}(b'_{t+1}) = \mathbf{d}$. This, together with equation (49) implies that for any $\mathbf{c}, \mathbf{d} \in (0, \infty)$

$$h_t^v\left(\frac{\mathbf{c} + \mathbf{d}}{2}\right) = \frac{h_t^v(\mathbf{c}) + h_t^v(\mathbf{d})}{2} \tag{50}$$

Equation (50) is Jensen's equation. Since h_t^v is monotone, standard arguments (see, e.g., Theorem 1 of Section 2.14 together with arguments from the proof of Theorem 1 of Section 2.1 of Aczél, 1966) show that its solution is $h_t^v(\gamma) = \delta_t^v \cdot \gamma + \alpha_t^v$ for some constants $\delta_t^v, \alpha_t^v \in \mathbb{R}$. Moreover, $\delta_t^v > 0$ since the function h_t^v is strictly increasing. It follows that $G_t(v, \gamma) = \delta_t^v \gamma + \alpha_t^v$.

Step 3: Fix $v \neq v' \in \text{Range}(v_t)$ and let $z, z' \in Z$ be such that $v_t(z) = v$ and $v_t(z') = v'$. Fix $\mathbf{c}, \mathbf{d} \in (0, \infty)$ such that $\mathbf{c} \neq \mathbf{d}$. By Axiom 4, there exist distinct elements $a, b, a', b' \in Z \times \mathcal{M}_{t+2}$ such that $W_{t+1}(a) = W_{t+1}(b) = \mathbf{c}$ and $W_{t+1}(a') = W_{t+1}(b') = \mathbf{d}$. Axiom 16 implies that

$$
\begin{aligned}
[\delta_t^v \cdot \mathbf{c} + \alpha_t^v] - \left[\delta_t^v \cdot \frac{\mathbf{c}+\mathbf{d}}{2} + \alpha_t^v\right] &= \left[\delta_t^{v'} \cdot \frac{\mathbf{c}+\mathbf{d}}{2} + \alpha_t^{v'}\right] - \left[\delta_t^{v'} \cdot \mathbf{c} + \alpha_t^{v'}\right] \\
\delta_t^v \cdot \frac{\mathbf{c}-\mathbf{d}}{2} &= \delta_t^{v'} \cdot \frac{\mathbf{c}-\mathbf{d}}{2} \\
\delta_t^v &= \delta_t^{v'}
\end{aligned}
$$

Let δ_t be the common value of δ_t^v. Then $G_t(v(z), \gamma) = \delta_t \gamma + \alpha_t^v$. □

B.10 Proof of Proposition 1

Because $\Phi_{t+1}\left[A_{t+1} \mid A_{t+1} \cup A'_{t+1}\right] = 1$ when $A_{t+1} \supseteq A'_{t+1}$, this follows immediately from the definition of aggregate recursivity. □

B.11 Proof of Proposition 2

Fix $t \in \{0, \ldots, T-1\}$, $z \in Z$, and disjoint $A_{t+1}, A'_{t+1} \in \mathcal{M}_{t+1}$. Suppose that

$$(z, A_{t+1}) \succsim_t (z, A'_{t+1})$$

or equivalently that

$$\Phi_t\left[(z, A_{t+1}) \mid \{(z, A_{t+1}), (z, A'_{t+1})\}\right] \geq \Phi_t\left[(z, A'_{t+1}) \mid \{(z, A_{t+1}), (z, A'_{t+1})\}\right].$$

First we prove that

$$(z, A_{t+1}) \succsim_t (z, A_{t+1} \cup A'_{t+1}).$$

To see that, observe that Axiom 10 implies that

$$\frac{1}{|A_{t+1}|}\Phi_{t+1}\left[A_{t+1} \mid A_{t+1} \cup A'_{t+1}\right] \geq \frac{1}{|A'_{t+1}|}\Phi_{t+1}\left[A'_{t+1} \mid A_{t+1} \cup A'_{t+1}\right].$$

Rearranging this inequality and recalling that A_{t+1} and A'_{t+1} are disjoint, we obtain

$$|A'_{t+1}|\Phi_{t+1}\left[A_{t+1} \mid A_{t+1} \cup A'_{t+1}\right] \geq |A_{t+1}|\left(1 - \Phi_{t+1}\left[A_{t+1} \mid A_{t+1} \cup A'_{t+1}\right]\right),$$

which is

$$|A_{t+1} \cup A'_{t+1}|\Phi_{t+1}\left[A_{t+1} \mid A_{t+1} \cup A'_{t+1}\right] \geq |A_{t+1}|,$$

which, since $\Phi_{t+1}\left[A_{t+1} \cup A'_{t+1} \mid A_{t+1} \cup A'_{t+1}\right] = 1$, is

$$\frac{1}{|A_{t+1}|}\Phi_{t+1}\left[A_{t+1} \mid A_{t+1} \cup A'_{t+1}\right] \geq \frac{1}{|A_{t+1} \cup A'_{t+1}|}\Phi_{t+1}\left[A_{t+1} \cup A'_{t+1} \mid A_{t+1} \cup A'_{t+1}\right],$$

which by Axiom 10 implies that

$$\Phi_t\left[(z, A_{t+1}) \mid \{(z, A_{t+1}), (z, A_{t+1} \cup A'_{t+1})\}\right] \geq \Phi_t\left[(z, A_{t+1} \cup A'_{t+1}) \mid \{(z, A_{t+1}), (z, A_{t+1} \cup A'_{t+1})\}\right],$$

which is the first part of the conclusion.

Second, prove that

$$(z, A_{t+1} \cup A'_{t+1}) \succsim_t (z, A'_{t+1}).$$

To see that, observe that Axiom 10 implies that

$$\frac{1}{|A_{t+1}|}\Phi_{t+1}\left[A_{t+1} \mid A_{t+1} \cup A'_{t+1}\right] \geq \frac{1}{|A'_{t+1}|}\Phi_{t+1}\left[A'_{t+1} \mid A_{t+1} \cup A'_{t+1}\right],$$

Rearranging this inequality and recalling that A_{t+1} and A'_{t+1} are disjoint, we obtain

$$|A'_{t+1}|\left(1 - \Phi_{t+1}\left[A'_{t+1} \mid A_{t+1} \cup A'_{t+1}\right]\right) \geq |A_{t+1}|\left(\Phi_{t+1}\left[A'_{t+1} \mid A_{t+1} \cup A'_{t+1}\right]\right),$$

which is

$$|A'_{t+1}| \geq |A_{t+1} \cup A'_{t+1}|\Phi_{t+1}\left[A'_{t+1} \mid A_{t+1} \cup A'_{t+1}\right],$$

which, since $\Phi_{t+1}\left[A_{t+1} \cup A'_{t+1} \mid A_{t+1} \cup A'_{t+1}\right] = 1$, is

$$\frac{1}{|A_{t+1} \cup A'_{t+1}|}\Phi_{t+1}\left[A_{t+1} \cup A'_{t+1} \mid A_{t+1} \cup A'_{t+1}\right] \geq \frac{1}{|A'_{t+1}|}\Phi_{t+1}\left[A'_{t+1} \mid A_{t+1} \cup A'_{t+1}\right],$$

which by Axiom 10 implies that

$$\Phi_t\left[(z, A_{t+1}\cup A'_{t+1}) \mid \{(z, A_{t+1}), (z, A_{t+1}\cup A'_{t+1})\}\right] \geq \Phi_t\left[(z, A'_{t+1}) \mid \{(z, A_{t+1}), (z, A_{t+1}\cup A'_{t+1})\}\right],$$

which is the second part of the conclusion. $\square$

B.12 Proofs of the results in Section 6

B.12.1 Proof of Proposition 5

With the (stationary impatient) discounted entropy representation, if the agent picks $\widetilde{z}_T$, his period-T utility is $v(\widetilde{z}_T)$, and his value in period 0 is $U_0(z_0, z_1, \ldots, z_{T-1}, \widetilde{z}_T) = \sum_{t=0}^{T-1}\delta^t v(z_t) + \delta^T v(\widetilde{z}_T)$. Thus, the agent's problem at time 0 is $\max_{q\in\Delta(x)}\sum_{\tilde{z}} U_0(z_0, z_1, \ldots, z_{T-1}, \widetilde{z}_T)q(\tilde{z})+H(q)$, with solution

$$\begin{aligned}
q(\widetilde{z}_T) &= \frac{\exp(U_0(z_0, z_1, \ldots, z_{T-1}, \widetilde{z}_T))}{\sum_{z'\in x}\exp(U_0(z_0, z_1, \ldots, z_{T-1}, z'))} = \frac{\exp\left(\sum_{t=0}^{T-1}\delta^t v(z_t) + \delta^T v(\widetilde{z}_T)\right)}{\sum_{z'\in x}\exp\left(\sum_{t=0}^{T-1}\delta^t v(z_t) + \delta^T v(z')\right)} \\
&= \frac{\exp\left(\delta^T v(\widetilde{z}_T)\right)}{\sum_{z'\in x}\exp\left(\delta^T v(z')\right)} \to_{T\to\infty} \frac{1}{|X|},
\end{aligned}$$

where we have set $\eta = 1$ which is without loss of generality from Theorem 1. The stationary impatient relative entropy model has the same implication, because the relative entropy and entropy forms both generate the logistic choice in one-shot decision problems such as this one. $\square$

B.12.2 Proof of Proposition 6

Notice that by Lemma 4

$$\begin{aligned}
U_0(z_0, A'_1) &= v(z_0) + \delta\log\left(\sum_{\tilde{z}_T\in x}\exp\left(U_1(z_1, \ldots, z_{T-1}, \tilde{z}_T)\right)\right) \\
&= v(z_0) + \delta\log\left(\sum_{\tilde{z}_T\in x}\exp\left(\sum_{t=1}^{T-1}\delta^{t-1}v(z_t) + \delta^{T-1}v(\tilde{z}_T)\right)\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}U_0(z_0, A_1'') &= v(z_0) + \delta v(z_1) + \ldots + \delta^{T-1} v(z_{T-1}) + \delta^T \log\left(\sum_{\tilde{z}_T \in x} \exp v(\tilde{z}_T)\right) \\ &= v(z_0) + \delta\left(\sum_{t=1}^{T-1} \delta^{t-1} v(z_t) + \delta^{T-1} \log\left(\sum_{\tilde{z}_T \in x} \exp v(\tilde{z}_T)\right)\right).\end{aligned}$$

Let $F(T) := \sum_{t=1}^{T-1} \delta^{t-1} v(z_t)$. Then with $\eta = 1$ (which again is without loss of generality) we have

$$\begin{aligned}r_T &= \frac{\exp\left(v(z_0) + \delta \log\left(\sum_{\tilde{z}_T \in x} \exp\left(F(T) + \delta^{T-1} v(\tilde{z}_T)\right)\right)\right)}{\exp\left(v(z_0) + \delta\left(F(T) + \delta^{T-1} \log\left(\sum_{\tilde{z}_T \in x} \exp v(\tilde{z}_T)\right)\right)\right)} \\ &= \frac{\exp(v(z_0))\left(\sum_{\tilde{z}_T \in x} \exp\left(F(T) + \delta^{T-1} v(\tilde{z}_T)\right)\right)^\delta}{\exp(v(z_0)) \exp\left(F(T) + \delta^{T-1} \log\left(\sum_{\tilde{z}_T \in x} \exp v(\tilde{z}_T)\right)\right)^\delta} \\ &= \left[\frac{\sum_{\tilde{z}_T \in x} \exp(F(T)) \exp\left(\delta^{T-1} v(\tilde{z}_T)\right)}{\exp(F(T)) \exp\left(\delta^{T-1} \log\left(\sum_{\tilde{z}_T \in x} \exp v(\tilde{z}_T)\right)\right)}\right]^\delta \\ &= \left[\frac{\sum_{\tilde{z}_T \in x} \exp\left(\delta^{T-1} v(\tilde{z}_T)\right)}{\exp\left(\delta^{T-1} \log\left(\sum_{\tilde{z}_T \in x} \exp v(\tilde{z}_T)\right)\right)}\right]^\delta \\ &= \left[\frac{\sum_{\tilde{z}_T \in x} \exp\left(v(\tilde{z}_T)\right)^{\delta^{T-1}}}{\left(\sum_{\tilde{z}_T \in x} \exp v(\tilde{z}_T)\right)^{\delta^{T-1}}}\right]^\delta > 1,\end{aligned}$$

where the last inequality follows from the facts that $|X| \geq 2$,that with $\delta < 1$ the function $z \mapsto z^{\delta^{T-1}}$ is strictly concave, and that strictly concave functions that pass through zero are strictly subadditive.[24].

For the limit result, note

$$r_T = \left[\frac{|X| \sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp\left(v(\tilde{z}_T)\right)^{\delta^{T-1}}}{|X|^{\delta^{T-1}}\left(\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp v(\tilde{z}_T)\right)^{\delta^{T-1}}}\right]^\delta = |X|^{\delta - \delta^T}\left[\frac{\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp\left(v(\tilde{z}_T)\right)^{\delta^{T-1}}}{\left(\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp v(\tilde{z}_T)\right)^{\delta^{T-1}}}\right]^\delta$$

[24]This is a standard result, we include a proof here for completeness: Suppose f is strictly concave and $f(0) = 0$. Then for $t \in (0, 1)$, $f(tx) = f(tx + (1 - t)0) > tf(x) + (1 - t)0 = tf(x)$.
Hence $f(x) + f(y) = f\left((x + y)\frac{x}{(x+y)}\right) + f\left((x + y)\frac{y}{(x+y)}\right) >$
$\left(\frac{x}{x+y}\right) f(x + y) + \left(\frac{y}{x+y}\right) f(x + y) = f(x + y)$.

Note that $\lim_{T\to\infty} |X|^{\delta-\delta^T} = |X|^\delta$. Observe that

$$\lim_{T\to\infty} \log \left[\frac{\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp\left(v(\tilde{z}_T)\right)^{\delta^{T-1}}}{\left(\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp v(\tilde{z}_T)\right)^{\delta^{T-1}}} \right]^\delta = \delta \lim_{T\to\infty} \log \left[\frac{\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp\left(v(\tilde{z}_T)\right)^{\delta^{T-1}}}{\left(\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp v(\tilde{z}_T)\right)^{\delta^{T-1}}} \right]$$

$$= \delta \log \left[\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \lim_{T\to\infty} \exp\left(v(\tilde{z}_T)\right)^{\delta^{T-1}} \right] - \lim_{T\to\infty} \delta^T \sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp v(\tilde{z}_T) = 0. \quad \square$$

B.13 Proofs of the remaining results from Section 7

B.13.1 Proof of Proposition 7

Notice that by Lemma 5

$$\begin{aligned} U_0(z_0, A_1') &= v(z_0) + \delta \log \left(\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp\left(U_1(z_1, \ldots, z_{T-1}, \tilde{z}_T)\right) \right) \\ &= v(z_0) + \delta \log \left(\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp\left(\sum_{t=1}^{T-1} \delta^{t-1} v(z_t) + \delta^{T-1} v(\tilde{z}_T) \right) \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} U_0(z_0, A_1'') &= v(z_0) + \delta v(z_1) + \ldots + \delta^{T-1} v(z_{T-1}) + \delta^T \log \left(\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp v(\tilde{z}_T) \right) \\ &= v(z_0) + \delta \left(\sum_{t=1}^{T-1} \delta^{t-1} v(z_t) + \delta^{T-1} \log \left(\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp v(\tilde{z}_T) \right) \right). \end{aligned}$$

Let $F(T) := \sum_{t=1}^{T-1} \delta^{t-1} v(z_t)$. Then with $\eta = 1$ (which again is without loss of generality) we have

$$
\begin{aligned}
r_T &= \frac{\exp\left(v(z_0) + \delta \log\left(\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp\left(F(T) + \delta^{T-1} v(\tilde{z}_T)\right)\right)\right)}{\exp\left(v(z_0) + \delta\left(F(T) + \delta^{T-1} \log\left(\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp v(\tilde{z}_T)\right)\right)\right)} \\
&= \frac{\exp(v(z_0)) \left(\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp\left(F(T) + \delta^{T-1} v(\tilde{z}_T)\right)\right)^{\delta}}{\exp(v(z_0)) \exp\left(F(T) + \delta^{T-1} \log\left(\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp v(\tilde{z}_T)\right)\right)^{\delta}} \\
&= \left[\frac{\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp\left(F(T) + \delta^{T-1} v(\tilde{z}_T)\right)}{\exp\left(F(T) + \delta^{T-1} \log\left(\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp v(\tilde{z}_T)\right)\right)}\right]^{\delta} \\
&= \left[\frac{\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp(F(T)) \exp\left(\delta^{T-1} v(\tilde{z}_T)\right)}{\exp(F(T)) \exp\left(\delta^{T-1} \log\left(\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp v(\tilde{z}_T)\right)\right)}\right]^{\delta} \\
&= \left[\frac{\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp\left(\delta^{T-1} v(\tilde{z}_T)\right)}{\exp\left(\delta^{T-1} \log\left(\sum_{\tilde{z}_T \in x} \frac{1}{|X|} \exp v(\tilde{z}_T)\right)\right)}\right]^{\delta} \\
&= \left[\frac{\sum_{\tilde{z}_T \in x} \exp \frac{1}{|X|} \left(v(\tilde{z}_T)\right)^{\delta^{T-1}}}{\left(\sum_{\tilde{z}_T \in x} \exp \frac{1}{|X|} v(\tilde{z}_T)\right)^{\delta^{T-1}}}\right]^{\delta} \leq 1,
\end{aligned}
$$

with equality if and only if all the elements of A are duplicates. This follows from the Jensen's inequality since $\delta < 1$ and the function $z \mapsto z^{\delta^{T-1}}$ is strictly concave.

The limit result follows from the proof of Proposition 6 □

B.13.2 Proof of Proposition 9

Let $F(T) := \sum_{t=1}^{T-1} \delta^{t-1} v(z_t)$ and notice that $F(T) > 0$ because v takes positive values only. Then with $\eta = 1$ (which again is without loss of generality) by the defn of Luce (see the right eqn number) we have

$$
r_T = \frac{\Phi_0[(z_0, A_1')|\{(z_0, A_1'), (z_0, A_1'')\}]}{\Phi_0[(z_0, A_1'')|\{(z_0, A_1'), (z_0, A_1'')\}]} = \frac{W_0(z_0, A_1')}{W_0(z_0, A_1'')},
$$

which by the repeated application of the other eqn in the definition equals

$$= \frac{v(z_0) + \delta\left(\sum_{\tilde{z}_T \in x} F(T) + \delta^{T-1} v(\tilde{Z}_T)\right)}{v(z_0) + \delta\left(F(T) + \sum_{\tilde{z}_T \in x} \delta^{T-1} v(\tilde{Z}_T)\right)} = \frac{v(z_0) + \delta|X|F(T) + \delta^T \sum_{\tilde{z}_T \in x} v(\tilde{Z}_T)}{v(z_0) + \delta F(T) + \delta^T \sum_{\tilde{z}_T \in x} v(\tilde{Z}_T)} > 1.$$

B.13.3 Proof of Proposition 8

Let $F(T) := \sum_{t=1}^{T-1} \delta^{t-1} v(z_t)$ and notice that $F(T) > 0$ because v takes positive values only. Then with $\eta = 1$ (which again is without loss of generality) by the defn of Luce (see the right eqn number) we have

$$r_T = \frac{\Phi_0[(z_0, A_1')|\{(z_0, A_1'), (z_0, A_1'')\}]}{\Phi_0[(z_0, A_1'')|\{(z_0, A_1'), (z_0, A_1'')\}]} = \frac{W_0(z_0, A_1')}{W_0(z_0, A_1'')},$$

which by the repeated application of the other eqn in the definition equals

$$= \frac{v(z_0) + \delta\left(\sum_{\tilde{z}_T \in x} \frac{1}{|X|}[F(T) + \delta^{T-1} v(\tilde{Z}_T)]\right)}{v(z_0) + \delta\left(F(T) + \sum_{\tilde{z}_T \in x} \frac{1}{|X|} \delta^{T-1} v(\tilde{Z}_T)\right)} = \frac{v(z_0) + \delta F(T) + \delta^T \sum_{\tilde{z}_T \in x} v(\tilde{Z}_T)}{v(z_0) + \delta F(T) + \delta^T \sum_{\tilde{z}_T \in x} v(\tilde{Z}_T)} = 1.$$