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# A THEORY OF DYNAMIC OLIGOPOLY, III

## Cournot Competition

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We study the Markov perfect equilibrium (MPE) of an alternating move, infinite horizon duopoly model where the strategic variable is quantity. We exhibit a pair of difference–differential equations that, when they exist, differentiable MPE strategies satisfy. For quadratic payoff functions, we solve these equations in closed form and demonstrate that the MPE corresponding to the solution is the limit of the finite horizon equilibrium as the horizon tends to infinity. We conclude with a discussion of adjustment costs and endogenization of the timing.

## 1. Introduction

In Maskin andTirole (1982, 1985), we presented a theory of how oligopolistic firms behave over time. We studied an explicitly temporal model stressing the idea of reactions based on short-run commitments. When we say that a firm is committed to a particular action, we mean that it cannot change that action for a certain finite (although possibly short) period, during which time other firms might act. By a firm's reaction to another firm, we mean the response it makes, possibly after some lag, to the other's chosen action.

To formalize the ideas of commitment-based reactions, we introduced a class of infinite-horizon sequential duopoly games. In the simplest version of these games, the two firms move alternatingly. Thus, when a firm picks its action, it has perfect information about the current action of the rival. The fact that after choosing an action a firm cannot change it for two periods is meant to capture the idea of short-run commitment.

A firm maximizes the present discounted value of its profits. Its strategy is assumed to depend only on the physical state of the system (i.e., to be Markov). In our model, the state is simply the other firm's current action. Hence, a strategy is simply a dynamic reaction function. A *Markov Perfect*

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*Equilibrium* (MPE) is a pair of reaction functions that form a perfect equilibrium. See section 2 below for a brief discussion and Maskin–Tirole (1982) for a more extended motivation of the MPE concept.

Maskin and Tirole (1982) applied this framework to a natural monopoly situation and provided a link between the older literature on fixed costs as an entry barrier and the more recent discussion of contestability. Maskin and Tirole (1985) examined dynamic price competition and developed equilibrium explanations for kinked demand curves, price cycles, excess capacity and market sharing.

In this paper, we adapt the alternating model to classic Cournot competition. Although we shall refer to a firm's choosing a 'quantity', one should think of this choice as that of capital or capacity [cf. Kreps and Schienkman (1983) or Maskin and Tirole (1982)]. That is, the quantity-setting game is the reduced form of a more complicated game in which long-run competition is conducted through capital and short-run competition through prices.

Section 2 recalls our infinite horizon duopoly model with alternating moves and develops the first-order conditions for differentiable reaction functions to form a MPE. These lead to a pair of difference–differential equations in the two reaction functions. Payoff functions are thereafter assumed to be quadratic. Section 3 shows that, given the quadratic assumption, there exists a unique MPE with linear reaction functions. This equilibrium is dynamically stable. Moreover, when firms discount the future heavily, the equilibrium strategies coincide with standard Cournot reaction functions. However, as firms grow more patient, competition becomes increasingly intense.

In their pioneering contribution, Cyert and de Groot (1970) analyzed the *finite* horizon analogue of the model studied in this paper. Even in the quadratic case, the finite horizon equilibrium cannot be exhibited in closed form, and so Cyert and de Groot were compelled to use numerical methods to solve it. In section 4, we use a contraction mapping argument to show that, as the horizon lengthens, the finite horizon equilibrium strategies converge to their infinite horizon counterparts discussed in section 3. Thus, the nature of equilibrium is robust to the length of the horizon.

Section 5 alters the model by imposing a cost on a firm for changing its quantity. The cost is assumed to grow with the size of the change in output. In this case, we show that, for any discount factor, the steady-state equilibrium converges to the static Cournot outcome as adjustment costs become large.

One restrictive feature of our analysis is that it relies on a model where firms' relative timing is exogenous. Both alternating and simultaneous move timings are, a priori, arbitrary. Asynchronism forces firms to react to one another; simultaneity, by contrast, does not permit reactions in our sense, since all firms' commitments expire at the same time. In our previous papers, we argued that the timing of the game should not be imposed but instead result from the adjustment technology and the strategic interactions. In section 6 we discuss the issue of endogenous timing for the Cournot framework.

## 2. The model and Markov Perfect Equilibrium

In this section we describe the main features of the exogenous timing duopoly model [for further discussion of this model, see Maskin and Tirole (1982)]. Competition between the two firms ($i=1,2$) takes place in discrete time with an infinite horizon. Time periods are indexed by $t$ ($=0,1,2,\ldots$). The time between two consecutive periods is $T$. At time $t$, firm $i$'s instantaneous profit $\Pi^i$ is a function of the two firms' current quantities (as we mentioned in the introduction, these quantities are really a 'shorthand' for technological scale) $q_{1,t}$ and $q_{2,t}$, but not of time: $\Pi^i = \Pi^i(q_{1,t}, q_{2,t})$. We assume that $\Pi^i$ is twice continuously differentiable, with $\Pi^i_{ii}<0$, $\Pi^i_j<0$ and $\Pi^i_{ij}<0$, where subscripts denote partial differentiation (for instance, $\Pi^i_{ij}$ is the cross partial derivative of $\Pi^i$ with respect to $q_{1,t}$ and $q_{2,t}$). Thus, firm $i$'s profit is concave in $i$'s output and, like marginal profit, decreases with $i$'s competitor's output. Static reaction curves are consequently well-defined and downward sloping.

Firms discount the future with the same interest rate $r$; thus, their discount factor is $\delta = \exp(-rT)$. Firm $i$'s intertemporal profit at time $t$ is

$$\sum_{s=0}^{\infty} \delta^s \Pi^i(q_{1,t+s}, q_{2,t+s}).$$

Let us now consider the timing of quantity setting. In odd-numbered periods $t$, firm 1 chooses its quantity, which remains unchanged until period $t+2$. That is, $q_{1,t+1} = q_{1,t}$ if $t$ is odd. Similarly, firm 2 chooses quantities only in even-numbered periods, so that $q_{2,t+1} = q_{2,t}$ if $t$ is even.

Firm $i$'s strategy is assumed to depend only on the payoff-relevant state, those variables that directly enter its profit function. That is, we suppose that strategies are *Markov*. In period $2k+1$, when it is firm 1's turn to pick a quantity, the payoff-relevant state is simply firm 2's current output: $q_{2,2k+1} = q_{2,2k}$. Firm 1's choice of quantity is, therefore, contingent only on $q_{2,2k}$. That is, its reaction function takes the form $q_{1,2k+1} = R_1(q_{2,2k})$. Similarly, firm 2 reacts to firm 1's quantity according to a reaction function $R_2(\cdot)$, where $q_{2,2k+2} = R_2(q_{1,2k+1})$. We shall call these Markov strategies *dynamic reaction functions*. In this paper, we consider only deterministic reaction functions.

As we argue in Maskin and Tirole (1982), the Markov assumption has appeal because it entails the simplest behavior on the part of firms that is consistent with rationality. Moreover, it gives rise to reactions that are closer in spirit to those of the informal industrial organization literature than do those of the supergame approach to oligopoly [e.g., Friedman (1977)], where, typically, a firm necessarily reacts not only to its competitors but to itself. Finally, as we shall see in section 5, it enables us to ignore the distinction between a horizon that is literally infinite and one that is long but finite (by contrast, again, with the supergame literature, where the distinction remains important).

We are interested in pairs of dynamic reaction functions $(R_1, R_2)$ that form perfect equilibria. Perfection requires that, starting in any state, a firm's dynamic reaction function maximizes its present discounted profit given the other firm's reaction function. We call such a pair of strategies a Markov Perfect Equilibrium (MPE). Note that because each firm has an incentive to use non-payoff-relevant history only if the other firm does, a MPE remains a perfect equilibrium when strategy spaces are unconstrained.

If $(R_1, R_2)$ is an MPE, then clearly, at any time $2k+1$ and given any time $2k$ move $q_{2,2k}$ by firm 2, the move $q_{1,2k+1} = R_1(q_{2,2k})$ maximizes firm 1's present discounted profit given that thereafter both firms move according to $(R_1, R_2)$. The analogous condition holds for firm 2. Indeed, from dynamic programming, these two conditions *suffice* for $(R_1, R_2)$ to constitute a MPE. That is, it is enough to rule out profitable one-shot deviations. Hence, $\{R_1, R_2\}$ is a MPE, if and only if there exist valuation functions $\{(V_1, W_1), (V_2, W_2)\}$ such that for any quantities $\{q_1, q_2\}$

$$V_1(q_2) = \max_q \{\Pi^1(q, q_2) + \delta W_1(q)\}, \tag{1}$$

$$R_1(q_2) \varepsilon \ \arg\max_q \{\Pi^1(q, q_2) + \delta W_1(q)\}, \tag{2}$$

$$W_1(q_1) = \Pi^1(q_1, R_2(q_1)) + \delta V_1(R_2(q_1)),^1 \tag{3}$$

and similarly, for firm 2. $V_1(q_1)$ is firm 1's valuation (present discounted profit) if (a) it is about to move, (b) the other firm's current quantity is $q_2$, and (c) firms use $\{R_1, R_2\}$ forever more.

We first show that equilibrium dynamic reaction functions are necessarily downward sloping. This property relies only on the assumption that the cross partial derivative $\Pi^i_{ij}$ is negative.

*Lemma 1. When they exist, equilibrium reaction functions are downward sloping, i.e., $R_i(q) \leqq R_i(\hat{q})$ if $q > \hat{q}$, $i = 1, 2$.*

[1]Notice that we have dropped the time subscripts from the quantities, because the dynamic reaction functions themselves are time independent.

*Proof.* Assume, to the contrary, that $q_1 > \hat{q}_1$, but $R_2(q_1) > R_2(\hat{q}_1)$. By definition, $R_2(q_1)$ is a best response to $q_1$. Thus,

$$\Pi^2(q_1, R_2(q_1)) + \delta W_2(R_2(q_1)) \geqq \Pi^2(q_1, R_2(\hat{q}_1)) + \delta W_2(R_2(\hat{q}_1)). \tag{4}$$

Similarly, $R_2(\hat{q}_1)$ is a best response to $\hat{q}_1$:

$$\Pi^2(\hat{q}_1, R_2(\hat{q}_1)) + \delta W_2(R_2(\hat{q}_1)) \geqq \Pi^2(\hat{q}_1, R_2(q_1)) + \delta W_2(R_2(q_1)). \tag{5}$$

Subtracting (5) from (4), we obtain

$$\Pi^2(q_1, R_2(q_1)) - \Pi^2(q_1, R_2(\hat{q}_1)) + \Pi^2(\hat{q}_1, R_2(\hat{q}_1)) - \Pi^2(\hat{q}_1, R_2(q_1)) \geqq 0, \tag{6}$$

which is equivalent to

$$\int_{\hat{q}_1}^{q_1} \int_{R_2(\hat{q}_1)}^{R_2(q_1)} \Pi^2_{12}(x, y)\, \mathrm{d}y\, \mathrm{d}x \geqq 0. \tag{7}$$

But, by assumption, $\Pi^2_{12} < 0$. Thus, (7) implies that $R_2(q_1) \leqq R_2(\hat{q}_1)$. Q.E.D.

Given the differentiability of the payoff functions, we can derive convenient differential conditions that characterize differentiable equilibrium reaction functions (if these exist). The first-order condition for the optimization problem in (1) is

$$\Pi^1_1(R_1(q_2), q_2) + \delta \frac{\mathrm{d}W_1}{\mathrm{d}q}(R_1(q_2)) = 0, \tag{8}$$

or, because $q_1 = R_1(q_2)$,

$$\Pi^1_1(q_1, R_1^{-1}(q_1)) + \delta \frac{\mathrm{d}W_1}{\mathrm{d}q}(q_1) = 0. \tag{9}$$

Similarly, substituting $q_2 = R_2(q_1)$ in (8), we obtain

$$\Pi^1_1(R_1(R_2(q_1)),\ R_2(q_1)) + \delta \frac{\mathrm{d}W_1}{\mathrm{d}q}(R_1(R_2(q_1))) = 0. \tag{10}$$

From (1), (2), and (3), we obtain a simple equation for $W_1$:

$$W_1(q_1) = \Pi^1(q_1, R_2(q_1)) + \delta \Pi^1(R_1(R_2(q_1)), R_2(q_1)) + \delta^2 W_1(R_1(R_2(q_1))). \tag{11}$$

Differentiating (11) yields

$$\frac{\mathrm{d}W_1}{\mathrm{d}q}(q_1)=\Pi_1^1(q_1,R_2(q_1))+\Pi_2^1(q_1,R_2(q_1))\frac{\mathrm{d}R_2}{\mathrm{d}q_1}(q_1)$$

$$+\delta\Pi_1^1(R_1(R_2(q_1)),R_2(q_1))\frac{\mathrm{d}R_2}{\mathrm{d}q_1}(q_1)\frac{\mathrm{d}R_1}{\mathrm{d}q_2}(R_2(q_1))$$

$$+\delta\Pi_2^1(R_1(R_2(q_1)),R_2(q_1))\frac{\mathrm{d}R_2}{\mathrm{d}q_1}(q_1)$$

$$+\delta^2\frac{\mathrm{d}W_1}{\mathrm{d}q}(R_1(R_2(q_1)))\frac{\mathrm{d}R_2}{\mathrm{d}q_1}(q_1)\frac{\mathrm{d}R_1}{\mathrm{d}q_2}(R_2(q_1)). \tag{12}$$

Substituting (9) and (10) in (12), we get

$$\frac{\mathrm{d}R_2}{\mathrm{d}q_1}(q_1)=\frac{-\Pi_1^1(q_1,R_1^{-1}(q_1))-\delta\Pi_1^1(q_1,R_2(q_1))}{\delta\Pi_2^1(q_1,R_2(q_1))+\delta^2\Pi_2^1(R_1(R_2(q_1)),R_2(q_1))}. \tag{13}$$

By symmetry,

$$\frac{\mathrm{d}R_1}{\mathrm{d}q_2}(q_2)=\frac{-\Pi_2^2(R_2^{-1}(q_2),q_2)-\delta\Pi_2^2(R_1(q_2),q_2)}{\delta\Pi_1^2(R_1(q_2),q_2)+\delta^2\Pi_1^2(R_1(q_2),R_2(R_1(q_2)))}. \tag{14}$$

The difference–differential eqs. (13) and (14) have a simple interpretation. Consider, for example, firm 2's optimal decision at time $t$ when firm 1 has chosen $q_1$ at time $t-1$. From optimality, a small change $\Delta q_2$ does not affect firm 2's present discounted profit. Firm 2's profit at time $t$ changes by $\Pi_2^2(q_1,R_2(q_1))\,\Delta q_2=\Pi_2^2(R_2^{-1}(q_2),q_2)\,\Delta q_2$. Its discounted profit at time $t+1$ is changed in two ways. There is the direct effect $\delta\Pi_2^2(R_1(q_2),q_2)\,\nabla q_2$ but also the indirect effect due to firm 1's marginal reaction $(\mathrm{d}R_1/\mathrm{d}q_2)(q_2)\Pi_1^2(R_1(q_2),q_2)\,\Delta q_2$. At time $t+2$ firm 1's time $t+1$ marginal reaction changes firm 2's discounted profit by $\delta^2(\mathrm{d}R_1/\mathrm{d}q_2)(q_2)\Pi_1^2(R_1(q_2),R_2(R_1(q_2)))\,\Delta q_2$. From the envelope theorem, there are no additional first order effects from $\Delta q_2$ on firm 2's profit. Thus, the derivative of firm 1's reaction function must satisfy (14).

Eqs. (13) and (14) are not sufficient for $(R_1,R_2)$ to form a Markov perfect equilibrium because they are only first order conditions. In the next section, we observe, however, that, for quadratic payoffs, the second order conditions are satisfied automatically for a linear solution.

## 3. The dynamics of Cournot competition

Henceforth, we will assume that the profit functions are quadratic and symmetric. Specifically,

$$\Pi^i = q_i(d - q_i - q_j) \quad \text{where} \quad d > 0. \tag{15}$$

The quantity $\Pi^i$ can be thought of as firm $i$'s profit in Cournot competition when the demand function and the production costs are linear. Then, $d$ represents the difference between the intercept of the demand curve and the marginal cost $c$.[2] We note that the 'reaction' functions[3] of the standard static Cournot model are

$$R_j^s(q_j) = (d - q_j)/2, \qquad i = 1, 2 \tag{16}$$

and that the static equilibrium is given by

$$q_1^s = q_2^s = d/3.$$

Differentiating (15) and (16), we obtain partial derivatives:

$$\Pi_i^i = d - 2q_i - q_j \tag{17}$$

and

$$\Pi_j^i = -q_i. \tag{18}$$

The linearity of these partial derivatives leads us to look for linear dynamic reaction functions:

$$R_i(q_j) = a_i - b_i q_j, \tag{19}$$

where from Lemma 1, $a_i > 0$ and $b_i > 0$.

Substituting (19) into (13) and (14) gives us the following two conditions in $b_i$ and $b_j$:

$$\delta^2 b_i^2 b_j^2 + 2\delta b_i b_j - 2b_i[1 + \delta] + 1 = 0, \qquad i = 1, 2, \quad j \neq i.$$

But from these two equations, it is clear that $b_1 = b_2$. Hence, we can drop the subscripts to obtain

$$\delta^2 b^4 + 2\delta b^2 - 2(1 + \delta)b + 1 = 0. \tag{20}$$

[2]One can work with either of two specifications of the model. In the first, quantities and price are unconstrained (i.e., can be negative) and payoffs are given by (15). In the second (more economic) specification, quantities must be non-negative and if $q_i + q_j \geqq d$, firm $i$'s profit is equal to $-cq_i$ (the price cannot be negative). The two alternatives yield the same MPE, and so we need not choose between them.

[3]These are, of course, not truly reaction functions because, in a one-period model, there is no opportunity to react.

Eq. (20), which determines the slope of the reaction function, has two real roots:[4] one in the interval $(0,\frac{1}{2})$ and the other in the interval $(1/\sqrt{\delta}, 1/\delta)$. As we will see below, only the former is relevant for our purposes. This root leads to dynamic reaction functions for which there is a steady-state output $q^e = 1/(1+b)$ and that are *dynamically stable*, i.e., starting from any production level $(q_1, q_2)$, production converges over time to the steady-state $(q^e, q^e)$. The other root gives rise to a dynamically unstable path.[5]

From (13), (14), and (19), we have

$$\delta^2 b^3 a_i - \delta^2 b^2 a_j + \delta b a_i + a_j = (1+\delta)db, \qquad i=1,2, \quad j \neq i.$$

Subtracting one of these equations from the other, we obtain

$$(\delta^2 b^3 + \delta^2 b^2 + \delta b - 1)(a_1 - a_2) = 0.$$

But as may be readily verified, the coefficient of $a_1 - a_2$ does not vanish in the interval $(0,\frac{1}{2})$, and so $a_1 = a_2$. We may, therefore, drop the subscripts from the $a$'s to obtain

$$a = \frac{(1+\delta)b}{\delta^2 b^3 - \delta^2 b^2 + \delta b + 1} d = \frac{1+b}{3-\delta b} d, \tag{21}$$

[6]

where the second equation in (21) is obtained using (20).

*Proposition 1. For any discount factor $\delta$:* (1) *there exists a unique linear MPE* [*given by* (19)–(21)], (2) *this MPE is dynamically stable*; *i.e., for any history of the game, the production levels converge to steady state outputs* $(q^e, q^e)$, (3) *each firm's steady state output $q^e$ is equal to the static Cournot equilibrium output $d/3$ when $\delta = 0$* [*in which case, the dynamic reaction functions coincide with their static counterparts, given by* (16)] *and grows with the discount factor.*

[4]This follows because the left-hand side of (20) is negative for $b = 1/2$ and tends to infinity as $b$ goes to either plus or minus infinity and because its second derivative is everywhere positive.

[5]In our first specification (unconstrained quantities and price; see footnote 2), the two firms' intertemporal payoffs fail to converge for this root; their instantaneous payoffs 'chatter' between negative and positive values that tend to $-\infty$ and $+\infty$. This root can also be ruled out in our second specification (at least for discount factors that are not too small), because, starting from a point where the other firm sets $q = a/b$, a firm earns zero intertemporal profit, although it could make a strictly positive profit by playing the (unstable) steady-state output. By contrast, our stable root gives a piece-wise linear solution ($R(q) = a - bq$ for $0 \leq q \leq a/b$, 0 for $q \geq a/b$) that is an MPE over the whole positive quadrant (not only in the subspace $[0, a/b]^2$) as can easily be checked.

[6] In this case where the firms' marginal costs differ, the slopes of the reaction functions are the same and given by (20), but the intercepts differ.

*Proof.* (1) If reaction functions are linear, then because the profit functions are quadratic, so are the valuation functions. By definition of the reaction function $R_1(q_2)$,

$$\Pi^1_1(R_1(q_2),q_2)+\delta\frac{\mathrm{d}W_1}{\mathrm{d}q_1}(R_1(q_2))=0,$$

and so

$$\left[\Pi^1_{11}(R_1(q_2),q_2)+\delta\frac{\mathrm{d}^2W_1}{\mathrm{d}q_1^2}(R_1(q_2))\right]\frac{\mathrm{d}R_1}{\mathrm{d}q_2}(q_2)+\Pi^1_{12}(R_1(q_2),q_2)=0.$$

Now, $\Pi^1_{12}$ is negative. Hence if reaction functions are downward sloping the above equation implies that the bracketed expression is negative. Thus, if $W_1$ is quadratic, we conclude that $\Pi^1(q_1,q_2)+\delta W_1(q_1)$ is concave, and so the necessary conditions (13) and (14) are also sufficient.

As we observed above, the two real solutions to eq. (20) lie in the intervals $(0,\frac{1}{2})$ and $(1/\sqrt{\delta},1/\delta)$. Footnote 5 demonstrates that the dynamics associated with the latter root are not consistent with an MPE. Those associated with the former root, however, *are* consistent with equilibrium (under either specification of the model). Thus because the objective function is concave, we conclude that the symmetric reaction functions given by (19)–(21) form an MPE.

(2) The unique steady state $(q^e,q^e)$ is given by

$$q^e=a-bq^e \tag{22}$$

or, from (21),

$$q^e=d/(3-\delta b). \tag{23}$$

This steady state is dynamically stable because the slopes of the reaction functions are less than one in absolute value.

(3) To study the behavior of the equilibrium reaction function and of the steady state when the discount factor varies, let us write the slope and intercept of the reaction function as functions of $\delta$: $a(\delta)$ and $b(\delta)$.

*Lemma 2. $a(\delta)$ and $b(\delta)$ are differentiable and satisfy*

$$\frac{\mathrm{d}}{\mathrm{d}\delta}(b(\delta))<0,\qquad \frac{\mathrm{d}}{\mathrm{d}\delta}(\delta b(\delta))>0,\qquad \frac{\mathrm{d}}{\mathrm{d}\delta}(a(\delta))<0.$$

Lemma 2, which is proved in Appendix 1, implies that, as the discount factor grows, the slope and the intercept of the reaction function decrease, and the steady state output increases [from eq. (23)].

When $\delta=0$, the firms do not care about their future payoffs; they move according to their *static* (Cournot) reaction functions, given by (16). This can be seen from eqs. (20) and (21): $a(0)=d/2$, $b(0)=1/2$. As $\delta$ grows, each firm takes its opponent's reaction more seriously; $a$ and $b$ decrease and $q^e$ increases. A numerical analysis of (20), (21) and (23) shows that when $\delta$ converges to 1, the limiting values of $a$, $b$ and $q^e$ are: $a \cong 0.48d$, $b \cong 0.30$, $q^e \cong 0.37d$. These values are the same as those found by Cyert and de Groot when they numerically computed the perfect equilibrium solution for the no-discounting finite-horizon game and took the limit as the horizon grows (see the convergence theorem, Proposition 2, below for an explanation of this coincidence). Q.E.D.

*Remark.* The convergence toward the static Cournot reaction functions when $\delta$ tends to 0 is a special case of a very general result obtained by Dana and Montrucchio (1986). They study MPE's of the alternating move game with strategies spaces that are compact and convex subsets of Euclidean spaces and with continuous payoff functions that are concave in a player's own action. They show that equilibrium dynamic reaction functions exist for all discount factors $\delta$, that valuation functions are upper semicontinuous, and that equilibrium converges to the Cournot reaction functions as $\delta$ tends to 0.[7]

The dynamic reaction functions for $\delta$ in $(0,1)$ are depicted in fig. 1. The dotted lines represent the static reaction functions $R_1^s$ and $R_2^s$ (which correspond to $\delta=0$), and the solid lines the dynamic reaction functions $R_1$ and $R_2$. $E$ denotes the steady state allocation, and $C$ the Cournot outcome. An example of a dynamic path is also provided.

That the outcome in our dynamic model is more 'competitive' than in the static case should not surprise us. In each period of the dynamic game, the firm about to move, say firm 1, takes two considerations into account: its short-run profit and the reaction it will induce in firm 2. Suppose that firm 2 is currently at the Cournot level. Then a slight increase in output above the Cournot level by firm 1 will have no effect (to the first order) on short–run profit. However, because reaction functions are downward–sloping, the increase will induce firm 2 to reduce its output below the Cournot level the following period, thereby increasing firm 1's long-term profit. This argument suggests that firm 1 has an incentive to choose a higher output in a dynamic rather than in a static setting because, each time it moves, it acts as a 'Stackelberg leader'. Since this is true of firm 2 as well, the end result is output above the Cournot level (i.e., more competitive behavior) by both firms.

[7]They also show that any pair of twice differentiable functions is an MPE of some game for a small enough discount factor.

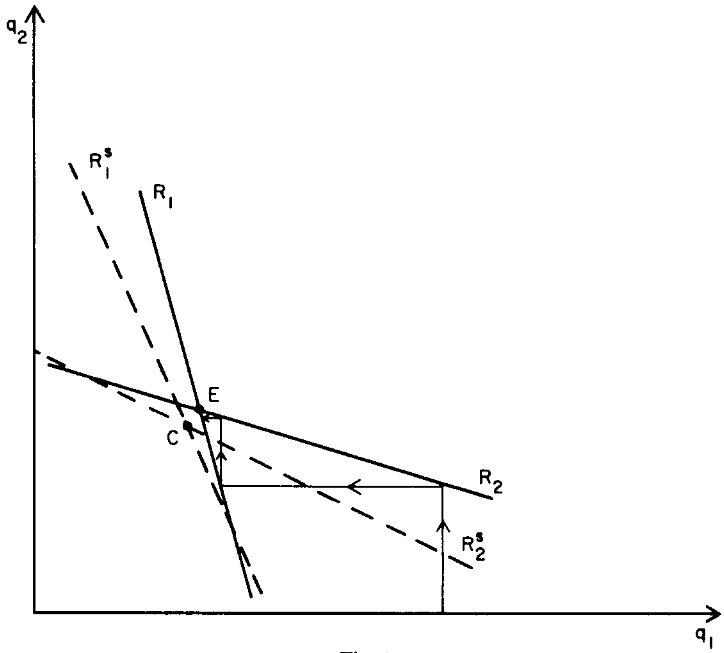


Fig. 1

This result contrasts with our findings when firms compete instead in prices. In that case, reaction functions can be upward-sloping, and thus dynamic equilibrium typically entails a *less* competitive outcome than its static counterpart [see Maskin and Tirole (1985)].

An increase in the discount factor means either that firms have become more patient ($r$ has fallen) or that the reaction lag $T$ has shrunk. A fall in the interest rate makes a firm more willing to forego current profits to induce the other firm to curtail its output. Thus, competition is enhanced. A decrease in the reaction lag $T$ also fosters competition because it means that the period of time before the other firm reacts becomes decreasingly significant relative to the future. As $T$ tends to 0, $\delta$ tends to 1, and the steady-state output diverges increasingly from the Cournot output. This result implies that the relative timing of firms' moves matters crucially, even in the limit when firms react very quickly. To see this, contrast our results with those of the *simultaneous* move game, where the (unique) MPE involves static Cournot outputs in every period. We conclude that the distinction between simultaneous and alternating moves remains important even when $T$ is very small.

## 4. Finite and infinite horizons

We have been unable to show that the equilibrium of Proposition 1 is the unique MPE of the infinite horizon game (it is, of course, unique within the linear class). It nonetheless possesses another attractive property, viz., it is the limit of the (unique) perfect equilibrium of the truncated finite horizon game when the horizon tends to infinity.

The finite horizon game is obtained from our infinite horizon model by truncating payoffs. In some time period, the game terminates, and all subsequent profits are zero. One computes the equilibrium of the finite horizon game by backward induction. In the last period, the firm about to move, say firm 1, plays according to its static reaction function. A period earlier, firm 2 moves knowing that the first firm will respond on its static reaction function, etc. (see below for a more detailed description of the induction). These considerations define reaction functions that depend on the period of play and the length of the horizon (they actually depend only on the difference between these two numbers). For a given finite horizon, perfect equilibrium is readily shown to be unique. Therefore, the equilibrium reaction functions depend only on the payoff-relevant state (any game of complete information admits an MPE. Hence, if perfect equilibrium is unique, it is necessarily Markov); a firm's action at time $t$ depends only on its competitor's output at $t-1$.

To compute the finite horizon reaction functions analytically appears intractable. We can demonstrate, however, that for any fixed time period, a firm's finite horizon perfect equilibrium reaction function for that period converges to the (infinite horizon) MPE reaction function as the horizon tends to infinity.

*Proposition 2. Fix a date t. In the model with horizon $v(>t)$, a firm's perfect equilibrium reaction function at date t converges uniformly to the infinite MPE reaction function given by* (19)–(21) *as $v\to\infty$.*

*Proof.* Consider a horizon of length $v$. It will prove convenient in our argument to index periods counting backwards from the end. Thus, the last period is indexed 0, and the first period is $v-1$. Suppose, for example, that firm 2 moves at time 0. As we already noticed, the best that it can do at time 0 is to play according to its static reaction function: $R^0(q)=R^s(q)$ (where $q$ denotes firm 1's output at time 1, and $R^s$ is the static reaction function). Consider the decision problem of firm 1 at time 1. Its reaction $R^1(q)$ is given by

$$R^1(q)=\arg\max_{\tilde{q}}\{\Pi^1(\tilde{q},q)+\delta\Pi^1(\tilde{q},R^0(\tilde{q}))\}. \tag{24}$$

Similarly, firm 2's optimal strategy $R^2(q)$ at time 2 satisfies

$$R^2(q)=\arg\max_{\tilde{q}}\{\Pi^2(q,\tilde{q})+\delta\Pi^2(R^1(\tilde{q}),\tilde{q})+\delta^2\Pi^2(R^1(\tilde{q}),R^0(R^1(\tilde{q})))\}. \tag{25}$$

Thus, the reaction function $R^2$ is determined by the succeeding two reaction functions, $R^1$ and $R^0$. More generally, the reaction function $R^\tau$ of a firm moving at time $\tau$ is determined by $R^{\tau-1}$ and $R^{\tau-2}$. Specifically, let

$$H^\tau(\tilde{q},q) \equiv \tilde{q}(d-q-\tilde{q}) + \delta\tilde{q}(d-\tilde{q}-R^{\tau-1}(\tilde{q})) + \delta^2 H^{\tau-2}(R^{\tau-2}(\tilde{q})), R^{\tau-1}(\tilde{q})),$$

where $H^0(\tilde{q},q) \equiv \tilde{q}(d-q-\tilde{q})$. By definition, $R^\tau(q) = \arg\max_{\tilde{q}} H^\tau(\tilde{q},q)$. Thus, from the envelope theorem,

$$[d-2\tilde{q}-q] + \delta\left[d-2\tilde{q}-R^{\tau-1}(\tilde{q}) - \tilde{q}\frac{\mathrm{d}R^{\tau-1}}{\mathrm{d}\tilde{q}}(\tilde{q})\right]$$

$$+\delta^2\left[-R^{\tau-2}(R^{\tau-1}(\tilde{q}))\frac{\mathrm{d}R^{\tau-1}}{\mathrm{d}\tilde{q}}(\tilde{q})\right] = 0.$$

Thus, $R^\tau$ is determined by $R^{\tau-1}$ and $R^{\tau-2}$.

Although it seems impossible to derive $R^\tau$ analytically, one can characterize it by induction. In particular, Appendix 2 demonstrates that $R^\tau$ is linear and has a slope between $-\frac{1}{2}$ and 0. Let $L$ be the set of linear functions with slope between $-\frac{1}{2}$ and 0 and intercept between 0 and $d$. Consider the mapping $m$ defined on $L \times L$ by

$$(X,Z) = m(R,S), \tag{26}$$

where

$$Z(q) = \arg\max_{\tilde{q}} \{\Pi^2(q,\tilde{q}) + \delta\Pi^2(R(\tilde{q}),\tilde{q}) + \delta^2\Pi^2(R(\tilde{q}), S(R(Z(q))))\}$$

and

$$X(q) = \arg\max_{\tilde{q}} \{\Pi^1(\tilde{q},q) + \delta\Pi^1(\tilde{q}, Z(\tilde{q})) + \delta^2\Pi^1(R(Z(X(q))), Z(\tilde{q}))\}.$$

Note that in the definition of $Z$ and $X$, we take the reaction two periods hence as given. Then, from the envelope theorem, the maximizations on the right-hand sides of these definitions are equivalent to those of the full intertemporal objective function. Note also that

$$(R^{2k+1}, R^{2k}) = m(R^{2k-1}, R^{2k-2}). \tag{27}$$

We consider the following metric on the space $L \times L$. Take

$$\|(R,S)-(\tilde{R},\tilde{S})\| = \max\{|a_R-\tilde{a}_R|, |a_S-\tilde{a}_S|, |b_R-\tilde{b}_R|, |b_S-\tilde{b}_S|\},$$

where $R(q) = a_R - b_R q$ and $\tilde{R}(q) = \tilde{a}_R - \tilde{b}_R q$, $S(q) = a_S - b_S q$, $\tilde{S}(q) = \tilde{a}_S - \tilde{b}_S q$.

Appendix 2 shows that, for $d$ sufficiently small, $m$ is a contraction mapping on $L \times L$ with this metric. Thus, there exists $K<1$ such that if $(X,Z)=m(R,S)$ and $(\tilde{X},\tilde{Z})=m(\tilde{R},\tilde{S})$, then $\|(X,Z)-(\tilde{X},\tilde{Z})\| \leqq K\|(R,S)-(\tilde{R},\tilde{S})\|$.

The contraction mapping property has a useful consequence [see, eg., Smart (1974, theorem 1.1.2)], namely, the sequence $\{(R^{2k+1},R^{2k})\}_{k=0}^{\infty}$ converges uniformly to a fixed point $(R_1,R_2)$ of $m$:

$$(R_1,R_2)=m(R_1,R_2) \tag{28}$$

[$(R_1,R_2)$ is, moreover, the *unique* fixed point of $m$ in $L \times L$]. Given that $(R_1,R_2)$ is a fixed point,

$$R_1(q)=\arg\max_{\tilde{q}}\{\Pi^1(\tilde{q},q)+\delta\Pi^1(\tilde{q},R_2(\tilde{q}))+\delta^2\Pi^1(R_1(R_1(q)),R_2(\tilde{q}))\}$$

and

$$R_2(q)=\arg\max_{\tilde{q}}\{\Pi^2(q,\tilde{q})+\delta\Pi^2(R_1(\tilde{q}),\tilde{q})+\delta^2\Pi^2(R_1(\tilde{q}),R_2(R_2(q)))\}.$$

Thus, $(R_1,R_2)$ is a Markov perfect equilibrium of the infinite horizon game. Because the infinite horizon linear MPE given by (19)–(21) belongs to $L \times L$, it must therefore coincide with $(R_1,R_2)$.

The preceding argument applies when $d$ is sufficiently small. But if $R_d^k$ is the $k$-period equilibrium reaction function for given $d$ and $R_d^k(q)=a_d^k-b_d^kq$, then $R_{\hat{d}}^k(q)=(\hat{d}/d)a_d^k-b_d^kq$. Thus the same conclusion obtains for *all d*. Q.E.D.

## 5. Adjustment costs

We have formalized inertia in decision making by assuming a two period commitment to output decisions. Inertia arises because technology decisions take time, contracts with suppliers cannot readily be changed, and so forth. A shortcoming of our two-period commitment structure is that it fails to make the implicit cost of changing output sensitive to the current output level (although it does make this cost time-contingent). One would expect that often small adjustments are cheaper than large ones. In this section, we introduce output-related adjustment costs to our model. We show that our techniques for studying differentiable MPE's can be extended to this more elaborate construct (which has a two-dimensional state space).

We assume that, if firm $i$ moves at time $t$, it incurs a cost (beyond the variable cost already embodied in $\Pi^i$) depending on its current choice, $q_{i,t}$, and previous output, $q_{i,t-2}$. Let $A_i(q_{i,t},q_{i,t-2})$ represent this adjustment cost, which we assume is differentiable. Because $q_{i,t-2}$ influences firm $i$'s current profit at time $t$, the payoff-relevant state is $(q_{j,t-1},q_{i,t-2})$. Thus, a Markov strategy at time $t$ takes the form

$$q_{i,t}=R_i(q_{j,t-1},q_{i,t-2}). \tag{29}$$

The definition and properties of a Markov perfect equilibrium are the same as before. To solve for such an equilibrium, one could use the valuation function approach of section 2. Instead, we will derive the difference–differential equations using marginal reasoning. Assume that firm 1 chooses $q_{1,t}$ at time $t$. A small increase above the optimal $q_{1,t}$ does not affect firm 1's present discounted payoff to the first order. Thus, we have

$$\begin{aligned}0=&\Pi_1^1(q_{1,t},q_{2,t-1})\Delta q-\frac{\partial A_1}{\partial q_{1,t}}(q_{1,t},q_{1,t-2})\Delta q\\&+\delta\left[\Pi_1^1(q_{1,t},q_{2,t+1})\Delta q+\Pi_2^1(q_{1,t},q_{2,t+1})\frac{\partial R_2}{\partial q_{1,t}}\Delta q\right]\\&+\delta^2\Pi_2^1(q_{1,t+2},q_{2,t+1})\frac{\partial R_2}{\partial q_{1,t}}\Delta q,\end{aligned} \tag{30}$$

where

$$q_{1,t}=R_1(q_{2,t-1},q_{1,t-2}), \tag{31}$$

$$q_{2,t+1}=R_2(q_{1,t},q_{2,t-1}),\quad\text{and} \tag{32}$$

$$q_{1,t+2}=R_1(q_{2,t+1},q_{1,t}). \tag{33}$$

As before, terms corresponding to later periods in (30) vanish due to the envelope theorem. The difference–differential equation corresponding to firm 2's move can be written similarly.

We now consider symmetric and quadratic profit functions and adjustment costs:

$$\Pi^i=q_i(d-q_i-q_j), \tag{34}$$

$$A_i(q_{i,t},q_{i,t-2})=\frac{\alpha}{2}(q_{i,t}-q_{i,t-2})^2. \tag{35}$$

The following proposition is proved in Appendix 3:

*Proposition 3. There exist parameter values $(a,b,\beta)$, depending on $(\delta,d,\alpha)$ such that the dynamic reaction functions $R_i(q_{j,t-1},q_{i,t-2})=a-bq_{j,t-1}+\beta q_{i,t-2}$ form* *a Markov perfect equilibrium of the market with adjustment costs. Furthermore, for any discount factor, $\delta$, the steady-state output $q^e = a/(1+b-\beta)$ tends to the static Cournot output $q^s$ as the adjustment cost parameter, $\alpha$, becomes large.*

Thus, as modelled, adjustment costs tend to weaken competition (remember that, for $\alpha=0$, $\beta=0$ and the steady-state output exceeds the Cournot level). Roughly speaking, this occurs because when a firm decreases its output (a 'cooperative' gesture), the other firm is less tempted to take advantage of this reduction by increasing its own output if adjustment costs are high. Of course, adjustment costs will also influence future decisions, but Proposition 3 shows that these effects do not interfere with the intuitive story.

To see why high adjustment costs lead to a steady state near the Cournot outcome, suppose that the firms start at the Cournot allocation. When a firm increases its output above its Cournot level, the other firm curtails its own output. But the higher the adjustment costs, the smaller the decrease in output, and so the smaller the gain to the aggressive firm. As adjustment costs grow, therefore, the short-run loss associated with the increase in output becomes more and more important.

Of course, the modelling of adjustment costs here, although an improvement over previous sections, remains crude. There are, no doubt, more subtle forms of inertia that do not fall conveniently into our time-and-output-contingent framework.

## 6. Endogenous timing

As we suggested in the introduction, the alternating moves feature of our model is a property that ought to be *derived* from more primitive assumptions rather than simply *imposed*. Indeed, in our earlier work on quantity competition with large fixed costs [Maskin and Tirole (1982)] and price competition [Maskin and Tirole (1985)], we proposed two rather different ways of 'endogenizing' the relative timing of firms' moves. In both cases, however (and in both the quantity and price settings) firms end up in equilibrium behaving exactly as in the fixed timing framework, i.e., alternating.

In this section, we briefly review the two endogenous timing models. We observe, however, that, in contrast with our previous results, they provide conflicting answers when applied to the Cournot setting. Although one gives rise to an equilibrium formally identical to that explored above, the other predicts that, in the steady state, firms move simultaneously. This conflict suggests that a more detailed study of the micro-foundations of timing in firms' decision-making is called for, an ambitious task that will have to be deferred to the future.

Probably the easiest way to obtain alternating moves endogenously is to posit a continuous time model where commitments are of *random* length. Suppose that, as before, when a firm chooses a quantity level, it remains committed to that level for some period of time, but now assume that the period is uncertain. The simplest such hypothesis is that the probability that a commitment will lapse during a (short) interval $\Delta t$ is proportional to the interval's length. That is, commitment terminations occur according to a Poisson process [for a more detailed presentation of this model, see Maskin and Tirole (1982)]. In this model, the mean length of time before some firm moves corresponds precisely to a single period in the fixed timing framework. Because, moreover, the probability that firms' commitments will lapse simultaneously is zero, equilibrium retains the feature that firms move alternately.

Turning to a somewhat different model, we revert to discrete time but now abandon the assumption that firm 1 can move only in odd-numbered periods and firm 2 only in the even-numbered ones. Instead we suppose that a firm can, in principle, move in any period but, once it does so, remains committed for two periods. Thus in any period where a firm is not already committed, it can choose a quantity level. Alternatively, it can refrain from moving at all, that is, it can produce nothing (in which case, it is free to move at any future date).

The considerations affecting the payoff of a firm about to move are (i) whether the other firm is currently committed to a quantity level, and (ii) if so, which quantity. Thus, Markov strategies (on which we continue to concentrate) now depend on a two-dimensional payoff-relevant state.

Despite the freedom firms have in this model to vary their relative timing, one can show that, at least for discount factors near 1, any MPE has the properties that (i) starting from any point, a steady state is reached with probability one, and (ii) a steady state involves firms' choosing Cournot quantities and moving simultaneously.

We will omit the formal demonstration of this result, but the intuition is not hard to provide. We have seen that when firms move alternatingly, they have a tendency to produce higher quantities leading to lower profits than were they to move simultaneously. Thus, the firms have a joint incentive to be in a simultaneous rather than alternating 'mode'. This, of course, is a cooperative, rather than individual, consideration. Just because firms would be better off moving simultaneously does not mean that an individual firm will make the first move to bring this about. It may well prefer that the other firm move first, and so, given the other firm's reciprocal attitude, firms may get 'stuck' moving alternatingly. When there is little discounting, however, the ultimate gain from moving simultaneously swamps the individual cost of engineering the transition from the alternating mode. Thus simultaneity ultimately prevails.

## Appendix 1

*Proof of Lemma 2* ($\mathrm{d}(\delta b)/\mathrm{d}\delta > 0$, $\mathrm{d}b/\mathrm{d}\delta < 0$, $\mathrm{d}a/\mathrm{d}\delta < 0$). Totally differentiating (20), we obtain

$$\mathrm{d}b(4\delta^2 b^3 + 4\delta b - 2(1+\delta)] + \mathrm{d}\delta[2\delta b^4 + 2b^2 - 2b] = 0. \tag{A.1}$$

Since $b \in (0, \frac{1}{2})$, (A.1) implies that $\mathrm{d}b/\mathrm{d}\delta < 0$. From (A.1),

$$\frac{\mathrm{d}(b\delta)}{b\delta} = \frac{\mathrm{d}b}{b} + \frac{\mathrm{d}\delta}{\delta} = \frac{\mathrm{d}\delta}{\delta}\left[1 - \frac{2\delta^2 b^3 + 2\delta b - 2\delta}{4\delta^2 b^3 + 4\delta b - 2(1+\delta)}\right]$$

$$= \frac{\mathrm{d}\delta}{\delta}\left[\frac{2\delta^2 b^3 + 2\delta b - 2}{4\delta^2 b^3 + 4\delta b - 2(1+\delta)}\right]. \tag{A.2}$$

For $b\varepsilon(0, \frac{1}{2})$ both the numerator and the denominator of (A.2) are negative. Thus, $\mathrm{d}(b\delta)/\mathrm{d}\delta > 0$. To show that $\mathrm{d}a/\mathrm{d}\delta < 0$, recall that

$$a = \frac{1+b}{3-\delta b} d. \tag{21}$$

Letting '$\propto$' indicate 'proportional to', we obtain

$$\frac{\mathrm{d}a}{\mathrm{d}\delta} \propto (3-\delta b)\frac{\mathrm{d}b}{\mathrm{d}\delta} + (1+b)\left(b + \delta\frac{\mathrm{d}b}{\mathrm{d}\delta}\right)$$

$$\propto b + b^2 + [3+\delta]\frac{\mathrm{d}b}{\mathrm{d}\delta}$$

$$\propto [b+b^2] - (3+\delta)\frac{2b - 2b^2 - 2\delta b^4}{2(1+\delta) - 4\delta b - 4\delta^2 b^3}$$

$$\propto [1+b][1+\delta - 2\delta b - 2\delta^2 b^3 - (3+\delta)(1-b-\delta b^3)]$$

$$\propto (-2 + 3b - \delta^2 b^3 - \delta b + 3\delta b^3). \tag{A.3}$$

But this last expression is negative for $b\varepsilon(0, \frac{1}{2})$. Hence, $\mathrm{d}a/\mathrm{d}\delta < 0$. Q.E.D.

## Appendix 2: *m* is a contraction mapping on $L \times L$ for *d* small

Let $(X, Z) = m(R, S)$ for $(R, S) \in L \times L$. By definition $Z(q)$ is derived from $R$

and $S$ according to

$$Z(q)=\arg\max_{\tilde{q}}\{\tilde{q}[d-q-\tilde{q}]+\delta\tilde{q}[d-\tilde{q}-R(\tilde{q})]$$
$$+\delta^2 S(R(Z(q))[d-S(R(Z(q))-R(\tilde{q})]\}. \tag{A.4}$$

Because $R$ and $S$ belong to $L$, they can be written as

$$\begin{aligned} R(x)&=a_R-b_R x \quad \text{with} \quad a_R \varepsilon (0,d) \quad \text{and} \quad b_R \varepsilon (0,\tfrac{1}{2}) \\ S(x)&=a_S-b_S x \quad \text{with} \quad a_S \varepsilon (0,d) \quad \text{and} \quad b_S \varepsilon (0,\tfrac{1}{2}). \end{aligned} \tag{A.5}$$

From (A.4) and (A.5),

$$Z(q)=\frac{d-q+\delta(d-a_R)+\delta^2 b_R(a_S-a_R b_S)}{2+2\delta(1-b_R)-\delta^2 b_R^2 b_S}. \tag{A.6}$$

The denominator of (A.6) exceeds 2. Thus, $Z$ is linear and has a slope $-b_Z$ between $-\frac{1}{2}$ and 0. Moreover,

$$0<d(1+\delta)-\delta a_R+\delta^2 b_R(a_S-a_R b_S)\leqq d(1+\delta),$$

implying that the intercept of $Z(q)$, $a_Z$, lies in $(0,d)$. Thus $Z$ belongs to $L$. Similarly, $X$, which is determined by $Z$ and $R$, also belongs to $L$. Hence, $m$ maps elements of $L\times L$ into elements of $L\times L$.

Now, choose $(\tilde{R},\tilde{S})\in L\times L$ such that

$$\|(R-\tilde{R},S-\tilde{S})\|<\varepsilon, \tag{A.7}$$

where $\varepsilon$ is small. Write

$$\tilde{R}(q)=\tilde{a}_R-\tilde{b}_R q \quad \text{and} \quad \tilde{S}(q)=\tilde{a}_S-\tilde{b}_S q.$$

Take $(\tilde{X},\tilde{Z})=M(\tilde{R},\tilde{S})$ and let

$$\tilde{Z}(q)=\tilde{a}_Z-\tilde{b}_Z q.$$

By our choice of metric,

$$\|Z-\tilde{Z}\|=\max\{|a_Z-\tilde{a}_Z|,|b_Z-\tilde{b}_Z|\}$$
$$\leqq\max\left\{\left|\frac{\partial a_Z}{\partial a_R}\right|\varepsilon+\left|\frac{\partial a_Z}{\partial a_S}\right|\varepsilon+\left|\frac{\partial a_Z}{\partial b_R}\right|\varepsilon+\left|\frac{\partial a_Z}{\partial b_S}\right|\varepsilon,\left|\frac{\partial b_Z}{\partial b_R}\right|\varepsilon+\left|\frac{\partial b_Z}{\partial b_S}\right|\varepsilon\right\}. \tag{A.8}$$

From (A.6),

$$\left|\frac{\partial b_Z}{\partial b_R}\right| = \left|\frac{-2\delta - 2\delta^2 b_R b_S}{(2+2\delta(1-b_R)-\delta^2 b_R^2 b_S)^2}\right| \leqq \tfrac{5}{8},$$

$$\left|\frac{\partial b_Z}{\partial b_S}\right| = \left|\frac{-\delta^2 b_R^2}{(2+2\delta(1-b_R)-\delta^2 b_R^2 b_S)^2}\right| \leqq \tfrac{1}{16}.$$

Hence, from (A.8)

$$|b_Z - \tilde{b}_Z| \leqq \tfrac{11}{16}\varepsilon. \tag{A.9}$$

Also from (A.10),

$$\left|\frac{\partial a_Z}{\partial a_R}\right| = \left|\frac{-\delta - \delta^2 b_R b_S}{2+2\delta(1-b_R)-\delta^2 b_R^2 b_S}\right| \leqq \tfrac{5}{8},$$

$$\left|\frac{\partial a_Z}{\partial a_S}\right| = \left|\frac{\delta^2 b_R}{2+2\delta(1-b_R)-\delta^2 b_R^2 b_S}\right| \leqq \tfrac{1}{4},$$

$$\left|\frac{\partial a_Z}{\partial b_R}\right| = [2\delta^2 a_S - 2\delta^2 a_R b_S + 2\delta^3 a_S - 2\delta^3 a_R b_S + 2\delta d + 2\delta^2 \mathrm{d} b_R b_S$$
$$+ 2\delta^2 d + 2\delta^3 \mathrm{d} b_R b_S - 2\delta^2 a_R + \delta^4 b_R^2 b_S a_S - 2\delta^3 b_R a_R b_S$$
$$- \delta^4 b_R^2 b_S^2 a_R]/(2+2\delta(1-b_R)-\delta^2 b_R^2 b_S)^2$$
$$\leqq \frac{11d}{4}$$

given that $a_R$ and $a_S$ belong to $(0,d)$, and

$$\left|\frac{\partial a_Z}{\partial b_S}\right| = [-2\delta^2 b_R a_R - 2\delta^3 b_R a_R + \delta^3 b_R^2 a_R + \delta^2 \mathrm{d} b_R^2$$
$$+ \delta^3 b_R^2 d + \delta^4 b_R^3 a_S]/(2+2\delta(1-b_R)-\delta^2 b_R^2 b_S)^2$$
$$\leqq \frac{11d}{16}.$$

Thus, for $d$ sufficiently small, (A.8) implies that

$$|a_Z - \tilde{a}_Z| \leqq \tfrac{15}{16}\varepsilon. \tag{A.10}$$

We conclude, from (A.9) and (A.10), that

$$\|Z-\tilde{Z}\| \leqq \tfrac{15}{16}\|(R-\tilde{R}, S-\tilde{S})\|.$$

Applying the same argument for $X$, we deduce that $m$ is a contraction mapping on $L \times L$.[8]

## Appendix 3: Adjustment costs

To solve for linear MPE reaction functions

$$R_i(q_{j,t-1}, q_{i,t-2}) = a - bq_{j,t-1} + \beta q_{i,t-2},$$

we substitute

$$q_{1,t} = a - bq_{2,t-1} + \beta q_{1,t-2},$$

$$q_{2,t+1} = (a-ab) + (b^2+\beta)q_{2,t-1} - b\beta q_{1,t-2}, \quad \text{and}$$

$$q_{1,t+2} = (a-ab+ab^2+a\beta) + (-b^3-2b\beta)q_{2,t-1} + (b^2\beta+\beta^2)q_{1,t-2}$$

into (30) to obtain

$$d-2a-\alpha a+\delta(d-3a+ab)+b(\delta a+\delta^2(a-ab+ab^2+a\beta))=0, \tag{A.11}$$

$$2b-1+\alpha b+\delta(2b-b^2-\beta)-\delta b^2-\delta^2(b^4+2b^2\beta)=0, \tag{A.12}$$

$$-2\beta+\delta(-2\beta+b\beta)-\alpha(\beta-1)+b(\delta\beta+\delta^2(b^2\beta+\beta^2))=0. \tag{A.13}$$

Eqs. (A.11)–(A.13) determine the equilibrium values of $a$, $b$ and $\beta$.

As the adjustment parameter $\alpha$ goes to infinity, we obtain, from (A.11)–(A.13), the following limiting values of $a$, $b$, $\beta$ and the steady-state output $q^e$:

$$a \approx d(1+\delta)/\alpha, \qquad b \approx (1+\delta)/\alpha,$$

$$\beta \approx 1-2(1+\delta)/\alpha, \qquad q^e \approx d/3 = q^s \quad \text{(the static Cournot quantity).}$$

[8]Actually, we have shown only that $m$ contracts points that are sufficiently close together. But the distance between any two points can be subdivided into such small intervals, and so $m$ consequently contracts *all* pairs of points.

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