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OPTIMAL RESERVATION PRICE IN THE VICKREY AUCTION

Jean-Jacques LAFFONT
University of Toulouse I, France

Eric MASKIN
MIT, Cambridge, MA 02139, USA

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The optimal reservation price in the Vickrey auction is shown to be independent of the number of buyers.

Consider a risk neutral seller who contemplates selling an object to n risk neutral buyers, using a Vickrey (1961) (or second price) auction with a reservation price b. We know that, in such an auction, bidding his true valuation is a dominant strategy for each buyer.

Let us assume that the seller believes that the individual valuations are drawn independently from the same distribution with cumulative distribution function $F(.)$ and density function $f(.)$.

Let us assume for simplicity of notation that the valuation of the seller is zero. Then, we have the following surprising result:

Theorem. The optimal reservation price b^* is independent of the number of buyers. [1]

Proof. Let X^1 and X^2 be the order statistics which denote the highest bid and the second highest bid (i.e., the highest valuation and the second highest valuation).

From David (1970, p. 9) we know that the joint distribution of X^1 and

[1] After writing this note we discovered that Riley and Samuelson (1979) have derived the same result in a more general set up, using a different argument.

X^2 has a density function

$$g(x^1,x^2)=n(n-1)\left[F(x^2)\right]^{n-2}f(x^2)f(x^1) \quad \text{for } x^1 \geqslant x^2,$$

$$= 0 \qquad \text{otherwise.} \tag{1}$$

The optimal reservation price b^* maximizes

$$\int_b^\infty \int_b^{x^1} x^2 g(x^1,X^2)\,\mathrm{d}x^2\,\mathrm{d}x^1 + b\int_b^\infty \int_0^b g(x^1,x^2)\,\mathrm{d}x^2\,\mathrm{d}x^1,$$

and therefore satisfies the first-order condition

$$\int_{b^*}^\infty \int_0^{b^*} g(x^1,x^2)\,\mathrm{d}x^2\,\mathrm{d}x^1 - b^*\int_0^{b^*} g(b^*,x^2)\,\mathrm{d}x^2 = 0,$$

or using (1):

$$n\left[F(b)\right]^{n-1}\left[1-F(b)\right] - bnf(b)\left[F(b)\right]^{n-1}=0,$$

or

$$1-F(b^*)=b^*f(b^*). \tag{2}$$

Q.E.D.

Actually, when $n=1$, the maximization problem of the seller reduces to

$$\max_b b(1-F(b)), \tag{3}$$

which yields immediately (2).

Using (3), and assuming for simplicity that the range of the distribution is [0,1], we provide a useful graphical representation (see fig. 1).

Consider, in the coordinate system AO, AB, the indifference curves $xy=c^{te}$. Since we maximize $b(1-F(b))$, the solutions are the highest (in the direction of 1) tangency points of this system of indifference curves with the function $F(.)$.

Clearly, multiple solutions are possible. A sufficient condition for uniqueness is that problem (3) be a strictly concave problem; i.e., $-F''/F'<2$. In the range where it is concave, the curvature of $F(.)$ must be such that 'the index of absolute risk aversion' is less than 2.

Given this graphical representation we can state without formal proofs the following obvious results.

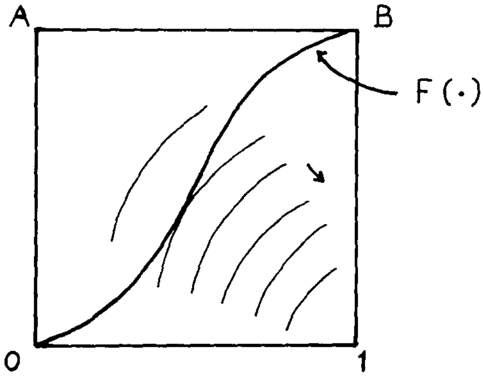

Fig. 1.

Proposition 1. If F(.) is the uniform distribution, the optimal reservation price is the mean of the distribution (see fig. 2).

Proposition 2. If F(.) is a mean preserving contraction (spread) of the uniform distribution, the optimal reservation value is smaller (larger) than the mean of the distribution (see fig. 3).

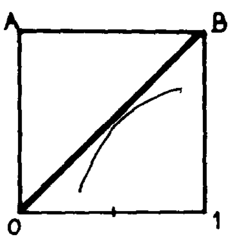

Fig. 2.

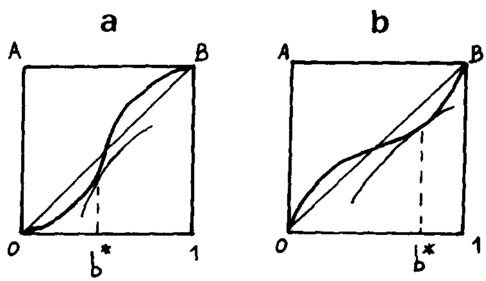

Fig. 3. (a) Contraction; (b) spread.

When $n = 1$, further results can be easily pictured.

Proposition 3. Let b^* be an optimal reservation price for the distribution F(.). If $b^* < 1/2$, the value of the auction for the seller increases for a mean preserving contraction of F(.). If $b^* > 1/2$, the value of the auction for the seller increases for a mean preserving spread of F(.), (see fig. 4).

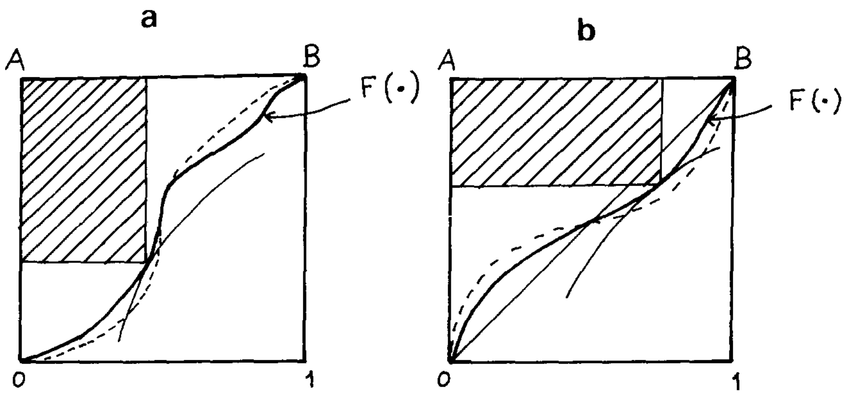

Fig. 4.

(a) Contraction; (b) spread.
The surface of the shaded area is the value of the auction for $F(.)$.

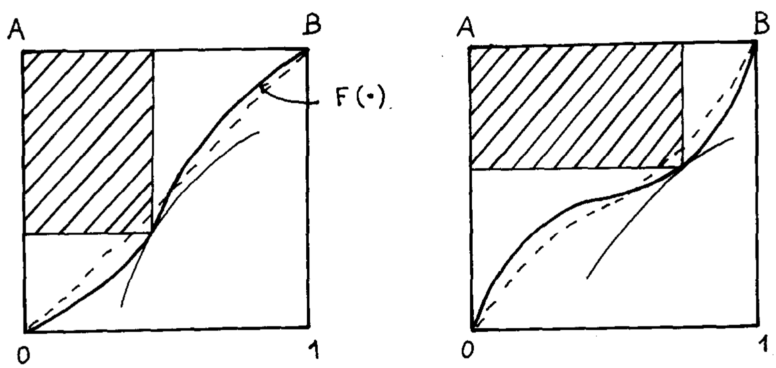

Fig. 5.

Proposition 4. Let b^* be the unique optimal reservation price for the distribution $F(.)$. If $b^* < 1/2$, the value of the auction for the seller decreases for a small enough mean preserving spread of $F(.)$. If $b^* > 1/2$, the value of the auction for the seller decreases for a small enough mean preserving contraction of $F(.)$, (see fig. 5).

References

David, H.A., 1970, Order statistics (Wiley, New York).

Riley, J. and W. Samuelson, 1979, Optimal auctions, D.P. 152 (Boston College, Chestnut Hill, MA).

Vickrey, W., 1961, Counterspeculation, auctions and competitive sealed tenders, Journal of Finance 16, 8–37.