Gradual bidding in ebay-like auctions*

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This paper shows that in online auctions like ebay, if bidders are not continuously participating in the auction but can only place bids at random times, then many different equilibria arise besides truthful bidding, despite the option to leave proxy bids. These equilibria can involve gradual bidding, periods of inactivity, and waiting to start bidding towards the end of the auction - bidding behaviors common on ebay. For symmetric bidders in a complete information setting, we characterize a class of equilibria that include the best and worst Markovian equilibria for the seller. In specific cases we also identify the worst non-Markovian equilibrium for the seller. The revenue of the seller in these equilibria can be a small fraction of what could be obtained at a sealed-bid second-price auction, and it can paradoxically decrease in the value of the object for the buyers. We show that the existence of equilibria with similar features extends to settings with asymmetric bidders, time-dependent arrival rates, and asymmetric information.

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1 Introduction

A distinguishing feature of online auctions, relative to spot auctions, is that they typically last for a relatively long time.[1] However, this aspect is often suppressed in the related economics literature. In particular, for bidders with private evaluations, online auction mechanisms such as ebay, where bidders can leave a proxy bid and the highest bidder wins the object at a price equal to the second highest bid (plus a minimum bid increment) are commonly regarded as strategically equivalent to second-price sealed-bid auctions. Since bidding one's true evaluation in the latter context is a weakly dominant strategy, and placing a bid takes some effort, the argument establishes that a rational bidder at an ebay-like auction should only place one bid, equal to her true evaluation, at her earliest convenience.

In contrast with the above predictions, observed bidding behavior on ebay involves both a substantial amount of gradual bidding and last-minute bidding (commonly referred to as sniping). Ockenfels and Roth (2006) report that the average number of bids per bidder is 1.89 and 38 % of bidders submit more than one bid. In the field experiment of Hossain and Morgan (2006) 76 % of the auctions had at least one bidder placing multiple bids. Regarding sniping, Roth and Ockenfels (2002) report that 18% of auctions in their data received bids in the last minute, while Bajari and Hortacsu (2003) find that the median winning bid arrives after 98.3% of the auction time elapsed, while 25% of the winning bids arrive after 99.8% of the auction time elapsed.

While Roth and Ockenfels (2002) proposes a model in which last-minute bidding can be an equilibrium,[2] the existing literature typically considers gradual bidding to be a naive (irrational) behavior. Relatedly, Ku et al. (2005) explain bidding behavior on online auctions with a model of emotional decision-making and competitive arousal, Ely and Hossain (2009) describe incremental bidders as confused, mistaking eBay's proxy system for an ascending auction, while Hossain (2008) explains observed bidding behavior using behavioral buyers who learn about their own valuations through the process of placing bids.[3]

In this paper we show that if bidders are not present for the whole duration of the auction (a clearly unrealistic scenario for online auctions), instead they have periodic random opportunities to check the status of the auction and place

[1] On eBay sellers can specify durations ranging from 1 day to 30 days.

[2] However, Hasker et al. (2009), using ebay data, reject the hypothesis that bidders follow a war of snipe profile as in Roth and Ockenfels (2002). See also Ariely et al. (2005) for a related laboratory experiment.

[3] See also Compte and Jehiel (2004) and Rasmusen (2006) for bidders learning their true evaluations during the auction. Other explanations include the presence of multiple overlapping auctions for identical or close substitute objects as in Peters and Severinov (2006), Hendricks et al. (2009), and Fu (2009). However, gradual and last-minute bidding seems to be prevalent for rare or unique objects, too, not only for objects with many close substitutes being auctioned at any time (for example, they occur in the experiments of Ariely et al. (2005) despite there is no concurrent competing auction). Furthermore, as Hossain (2008) points out, this type of argument also has trouble explaining many bids by the same bidder in a short interval of time, which is quite common in ebay. Bajari and Hortacsu (2003) raise the possibility that all ebay auctions have some common value component.

bids,[4] then despite the possibility of proxy bids, there can be many different equilibria of the resulting game with perfectly rational bidders, in a private value context. The best equilibrium for the seller in this game still implies truthful bidding by the bidders, upon the first time they can place a bid. If the time horizon of the auction is long, the seller's revenue in this equilibrium is approximately what he could get in a second-price sealed bid auction. However, there are typically many other equilibria, in weakly undominated strategies, which imply incremental bidding, long periods of intentionally not placing bids, and potentially sniping. The expected revenue of the seller from these equilibria can be a very small fraction of the expected revenue from the best equilibrium, even when the time horizon for the auction is arbitrarily large and bidders get frequent opportunities to place bids.

To understand the intuition for the existence of such equilibria, consider two bidders, each with evaluation $v > 2$ and getting random chances to make bids (including potentially proxy bids exceeding what becomes the leading price) over the course of the auction according to a Poisson arrival process. Suppose that the initial price is 0. Clearly, there is an equilibrium in which whenever the current price is below v and a bidder has the opportunity to make a bid, he places a bid of v. However, there is another equilibrium in which a bidder, when she gets the chance, increases the price gradually, by the minimum bid increment. The key insight here is that gradual bidding is self-enforcing: if other bidders follow such a strategy then it is strictly in the interest of a bidder to do likewise. Increasing the price by more than the minimum increment does not increase the likelihood of eventually winning the object, only speeds up the increase of the leading price, reducing the surplus that the winning bidder gets. Hence, gradual bidding is a form of implicit collusion among bidders that is consistent with equilibrium in our model.

If the time horizon of the auction is long (relative to the arrival rates) then besides gradual bidding, it also becomes optimal for bidders to wait and pass on opportunities to increase the current price, for prolonged periods of time. In particular, we show that for certain prices there are cutoff points in time such that bidders pass on opportunities to increase the price further if they get arrivals before the cutoff, instead prefer to remain losing bidders at the current price. This implies that if the time horizon is long (relative to the arrival rates) then players are inactive for most of the duration of the auction, and only start incrementing bids near the deadline. For this reason, the expected revenue of the seller can be a small fraction of v no matter how long the auction is, or how frequently players get opportunities to bid. Another noteworthy feature of a gradual equilibrium is that it can prescribe placing a bid of 1 upon the first arrival, and then a long period of inactivity, followed by all bidders trying to incrementally overbid each other towards the end of the auction. This is consistent with the finding reported in Roth and Ockenfels (2002) and in Bajari

[4] It is important for our results that whenever a bidder gets the chance to check the status of the auction, she can place multiple bids. In particular, if she places a bid incrementing the current price but gets notified that this bid was not enough to take over the lead, she can place a subsequent higher bid.

and Hortacsu (2003) that the distribution of bid timing as fraction of auction duration is bimodal, with a small peak at the beginning and a lot of activity at the very end.

For symmetric bidders we characterize both the worst Markovian equilibrium for the seller, and for two symmetric bidders the worst non-Markovian equilibrium as well. The latter has the feature that players wait until the very end of the auction (with bids placed earlier triggering switching to a truthful bidding equilibrium) and then bid gradually. This bidding behavior is similar to sniping, as in Roth and Ockenfels (2002), with the difference that the snipers only overbid incrementally instead of truthfully. In fact, it can be shown that in our framework Roth and Ockenfels type strategy profiles, in which bidders wait till the end of the auction and then bid truthfully, cannot be an equilibrium. Such profiles in a continuous-time framework (with no special "last period" as in Roth and Ockenfels) unravel, with each bidder wanting to start sniping at least a little bit earlier than the others.

A paradoxical feature of the worst equilibrium is that as the value of the object for the bidders increases, gradual bidding starts later, *decreasing* the expected revenue of the seller. Increasing the number of bidders also delays the start of bidding, but over all increases the expected revenue of the seller.

We also show that equilibria involving gradual bidding and periods of waiting extend beyond the complete information symmetric bidders case where we can derive general analytical results. With both a low evaluation and a high evaluation bidder present, the new feature relative to the symmetric case is that the high evaluation bidder wants to opt for standard sniping by the end of the auction (if getting an arrival then), since a truthful proxy bid secures winning the object at the price of the other bidder's evaluation. However, at earlier times the high evaluation bidder can either engage in gradual bidding or just wait (depending on the exact specification of evaluations). Hence, in these equilibria gradual bidding can be followed by a jump bid. If there is uncertainty regarding the evaluation of one of the bidders, the opposite can happen: a high evaluation bidder might want to start with a jump bid corresponding to the evaluation of a low type, taking her out of the contest. But if the other bidder turns out to be a high evaluation type, she responds by incremental bidding, leading to a gradual bidding continuation profile.

Gradual bidding equilibria also exist when arrival rates change over time, such as when they increase over time as the end of the auction approaches. Relative to constant arrival rates, in the latter case players wait longer before they start placing bids, and (gradual) bidding might only take place for a short interval before the end of the auction. For this reason, even when arrival rates reach very high levels before the end of the auction, as long as they are bounded from above, the expected number of bids and the winning price can remain very low. In line with this point, in the survey of Roth and Ockenfels (2002), 90 percent of bidders reported that sometimes when they specifically planned to bid late, something came up that prevented them from being available at the end of the auction, and they could not place a bid as planned.[5]

We note that our work is part of a recent string of papers examining continuous time games with random discrete opportunities to take actions, in different contexts: Ambrus and Lu (2009) investigates multilateral bargaining with a deadline in a similar context, while Kamada and Kandori (2009), Kamada and Sugaya (2010) and Calgano and Lovo (2010) examine situations in which players can publicly modify their action plans before playing a normal-form game. An important difference between these models and the current one, leading to different types of predictions, is that in the latter models the actions players can take are unrestricted by previous history. In contrast, in an auction game like in the current paper previous bids restrict the set of feasible bids subsequently, as the leading price can only increase. There is also a recent string of papers in industrial organizations, on structural estimation of continuous-time models in which players can change their actions at discrete random times, but payoffs are accumulated continuously (Doraszelski and Judd (2010), Arcidiacono et al. (2010)).

2 The model and the benchmark truthful equilibrium

A continuous-time single-good auction Λ is defined by a set of n potential bidders with reservation values $v_1, v_2, ..., v_n$ and arrival rates $\lambda_1, \lambda_2, ..., \lambda_n$, and a start time $T < 0$. We normalize the end time of the auction to 0, the starting bid to 0, and the minimum bid increment to 1. We assume that $v_i \in \mathbb{Z}_{++}$ and $\lambda_i \in \mathbb{R}_{++}$ for every $i = 1, ..., n$.

Between times T and 0 bidders get random opportunities to place bids according to independent Poisson processes with arrival rates as above. We normalize the starting bid to 0, and minimum bid increment to 1. Bidders may make multiple bids during a single arrival and can observe the outcome of each bid. For simplicity we assume that all bids made during an arrival are carried out instantaneously.

We assume that bidders can leave proxy bids, hence we need to distinguish between current price P and current highest bid $B \geq P$. The set of available bids is given by $\{b \in \mathbb{Z}_{++} | b \geq B + 1\}$ for the bidder who holds the highest bid at time t and $\{b \in \mathbb{Z}_{++} | b \geq P + 1\}$ otherwise. When a bid b is made, the price adjusts as follows: if $b \geq B + 1$, then P becomes $B + 1$ and B becomes b. Otherwise, P becomes b and B remains the same. At the end of the auction ($t = 0$) the current high bidder wins the good and pays the current price. We assume that the evolution of P is publicly observed, but B is only known by the

[5] We also provide an example of gradual bidding equilibrium with two bidders with different arrival rates. The qualitative features of this equilibrium are similar to the gradual bidding equilibria with symmetric bidders. In particular, the cutoff points for incremental bidding at different prices are the same for the slow and the fast bidder. The example shows that very busy bidders who can only check the status of the auction infrequently do not necessarily have more incentive to leave truthful proxy bids.

bidder holding the highest bid (with the exception of $P = 0$, which necessarily implies $B = 0$).

The assumption that bidders can place multiple bids at an arrival opportunity enables them to bid incrementally, no matter what the current highest bid is. In particular, a bidder can achieve this by always bidding the current price plus one, until she becomes the winning bidder. This is an important component of our model.

Strategies of bidders specify bidding behavior (that is either placing an available bid or not placing a bid) upon arrival as a function of calendar time, public history (time path of P) and private history (previous arrival times of the player and previous actions chosen at those arrival times). In order for expected payoffs to be well defined for all strategy profiles, we restrict bidders' strategies to be measurable with respect to the natural topologies.[6]

We say that a bidder's strategy is Markovian if it only depends on payoff-relevant information, namely the current leading price P, high price B (if known), and calendar time t. The latter is payoff relevant as it determines the distribution of future arrival sequences by the bidders (in particular, the probability that the given bidder will not get another chance to place a bid).

As a solution concept we use perfect Bayesian Nash equilibrium on weakly undominated strategies (from now on, for the sake of exposition, referred to simply as *equilibrium*). Note that without the latter requirement, even in sealed-bid second-prize auctions typically there exist many equilibria, as low valuation bidders may place bids above their evaluations, influencing the winning price, as long as they do not win the object. The weak undomination requirement implies that no bidder i ever places a bid above v_i.[7]

It turns out the weak undomination requirement, although considerably less restrictive in our game than in a static auction, is still a relatively easy to check restriction. In particular, it rules out placing bids above one's evaluation after any history, and with the exception of one particular price level below it does not rule out placing any bid at or below the true evaluation, after any history.

Claim 1: A strategy of player i is weakly undominated iff it satisfies the following: (i) it never calls for placing a bid $b > v_i$ after any history; (ii) it never calls for taking over the lead with $B = v_i - 2$; (iii) if i is not the current winner and $P < v_i - 1$ then it never calls for not placing a bid at times $t > t^*$, where t^* is the infimum of times at which i would prefer to take over the lead at a price smaller than v_i even when the other bidders follow the trigger strategy that prescribes not placing any new bid until i places a bid and bid v_i afterwards.

The proof of Claim 1 is given in the appendix. It is easy to see that a strategy profile in which each bidder i bids v_i whenever arriving and current

[6] For the formal details, see Appendix A of Ambrus and Lu (2009) in a similar continuous-time game with random arrivals.

[7] A strategy in which at some history i places a bid above v_i is weakly dominated by a strategy which at this history specifies placing a bid v_i (or restraining from bidding if $P \geq v_i$), otherwise identical to the original strategy.

price being below this level, and otherwise not placing a bid, constitutes an equilibrium. Given other bidders' strategies, no bidder can gain at any history by deviating from this strategy, and Claim 1 implies that these strategies are weakly undominated. Furthermore, there cannot be any equilibrium giving a higher expected revenue to the seller, given that in equilibrium as defined above, no player ever places a bid above her evaluation. For this reason, and because it is analogous to the unique equilibrium in a second-price sealed bid auction, the above truthful equilibrium is a natural benchmark to compare all other equilibria to in the subsequent analysis.

3 Symmetric bidders

In this section we consider symmetric bidders, that is when $v_i = v$ and $\lambda_i = \lambda$ for every $i = 1, ..., n$. For this case we can analytically characterize a class of gradual bidding equilibria that contain the most gradual possible bidding equilibrium (that is when bidding always implies raising the price incrementally, with one caveat when the price gets close to v) on one extreme, and truthful equilibrium on the other. For the case of two bidders, we characterize both the worst over all equilibrium for the seller, and the worst symmetric Markovian equilibrium, for any time horizon. For more than two bidders we characterize these equilibria for short enough time horizons, and the worst symmetric Markovian equilibrium for any time horizon. We then compare the expected revenue of the seller in these equilibria to that in the benchmark truthful equilibrium. The latter is close to v for long auctions.

In 3.1 we provide an example of an incremental equilibrium, and explain the main features of the dynamic strategic interaction in such equilibrium. In subsections 3.2-3.4 we focus on Markovian equilibria, in short auctions, in long auctions with two bidders, and in long auctions with more than two bidders. In 3.5 we discuss non-Markovian equilibria.

3.1 A 2-Bidder Example

Consider an auction with 2 symmetric bidders with values and arrival rates given by $v = 5$ and $\lambda = 1$, and let $T = -1$. We would like to construct an equilibrium in which bidders make only the minimal bid necessary to hold the current high bid, whenever they arrive. We first note that at $p = 2$, bidding 3 is weakly dominated by bidding 4 or 5, therefore incremental bidding may only occur at prices $p = 0$ and $p = 1$. Claim 1 implies that incremental bidding at these prizes is not weakly dominated.

Formally, we consider a strategy profile in which a losing bidder bids $p + 1$ when $p \in \{0, 1, 3, 4\}$, bids 4 when $p = 2$ and restrains from bidding when $p \geq 5$. At the same time, a winning bidder restrains from increasing the current (proxy) price if she gets the chance to do so.

Let $W(p, t)$ and $L(p, t)$ denote the expected payoff of a winning bidder (the bidder holding the current high bid) and the losing bidder respectively at time

t when current price is p, conditional on the above prescribed profile. Note that we suppress the current high bid as this is uniquely determined by the price along the path of play.

Trivially, $W(5,t) = 0$ and $L(p,t) = 0$ for $p > 2$. At $p = 2$ and $p = 3$ the winning bidder gets a payoff of $v-p$ if the other bidder does not arrive before the end of the auction and 0 otherwise. Therefore $W(3,t) = 2e^t$ and $W(2,t) = 3e^t$. The expected value of being a losing bidder at $L(2,t)$ is given by the expectation over the likelihood of arriving and becoming the winning bidder at $p = 3$,

$$L(2,t) = \int_t^0 e^{-(\tau-t)} W(3,t) d\tau = -2te^t$$

Similarly, $L(1,t) = -3te^t$.The expected value of being a winning bidder at $p = 1$ is equal to 4 if the other bidder does not arrive and is equal to the expectation of being the losing bidder at $p = 2$ if the other bidder does arrive.

$$W(1,t) = \int_t^0 e^{-(\tau-t)} L(2,t) d\tau + 4e^t = t^2 e^t + 4e^t$$

At $p = 0$, neither bidder holds the high bid and so the expected payoff expectation over becoming either the winning bidder at $p = 1$ or the losing bidder at $p = 1$

$$L(0,t) = \int_t^0 e^{-2(\tau-t)} (L(1,t) + W(1,t)) d\tau = 3e^t - 3e^{2t} - te^t + t^2 e^t$$

Figure 1 depicts the expected continuation payoffs of winning and losing bidders at different prices, for the time horizon of the game. It is straightforward to check that for $t \geq -1$, all the incentive compatibility conditions hold for the above strategy profile to be an equilibrium. In particular, a losing bidder always prefers to take over the lead upon arrival, and being a winning bidder at a lower price is strictly better than at a higher price, providing an incentive for incremental bidding. In fact, at low prices bidders strictly prefer following the equilibrium strategy to taking any other action.

The intuition behind the above incremental bidding equilibrium is that if the opponent uses an incremental bidding strategy, a *losing bidder* faces a clear trade-off in her bidding decision. On one hand, placing a bid makes her the current high bidder which increases her chance of winning the auction. The downside is that placing a bid raises the expected price at which the good will sell and hence lowers the expected value from winning the auction through a further sequence of bids (e.g. if an opponent arrives and then she arrives again). Placing a bid greater than the increment proscribed in equilibrium increases the downside without affecting the upside and hence if she chooses to place a bid it will also be incremental. Further, the upside is increasing in t while the downside is decreasing in t. If an auction is short enough, it will support an incremental equilibrium in which bids are placed at every arrival by a losing bidder. This argument also hints that in longer auctions equilibrium also requires periods of

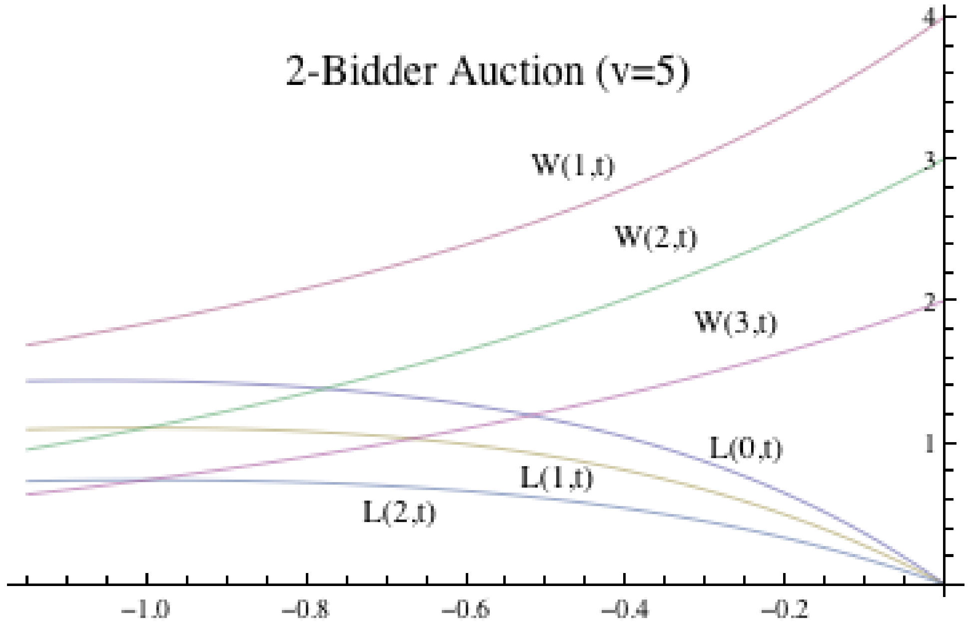

Figure 1: 2-Bidder example with $v = 5$.

waiting (losing bidders passing on opportunities to place a bid and take over the lead), as the incentive to slow down the increase of the current price might become stronger than the incentive to take over the lead. We discuss incremental equilibria with delay in long auctions in Subsection 3.3.

We conclude this subsection by noting that in the above equilibrium, a bidder's expected payoff is $L(0, -1) \approx 1.43$, and the expected revenue for the seller is $(1 - e^{-2})5 - 2L(0, -1) \approx 1.46$. These expected payoffs are considerably more favorable to the bidders than those in the benchmark equilibrium, in which the expected payoffs are roughly 0.93 and 2.46 for the bidders and seller respectively.

3.2 Short Auctions with Symmetric Bidders

We now generalize the existence of equilibria with incremental bidding behavior. In particular, we characterize a class of equilibria in which bidding behavior only depends on the current price and whether the bidder is currently winning the object.

Definition: A *bidding sequence* $S = \{b_1, ..., b_k\}$ is an integer-valued set that satisfies $0 < b_1 < ... < b_k = v$. A *proper bidding sequence* satisfies $b_i \neq v - 2$ for any $i \in \{1, ..., k\}$

Definition: A bidder follows an *incremental bidding strategy* over bidding sequence $S = \{b_1, ..., b_k\}$ by bidding $\min_r b_r$ such that $b_r > P$ whenever she arrives at the auction and is the losing bidder, and otherwise restrains from bidding.

Proposition 1 *In an auction with n symmetric bidders, for any proper bidding sequence S, there exists a $t^* < 0$ such that if $T \geq t^*$ the auction has an equilibrium in which bidders follow the incremental bidding strategy over S.*

Proof. The following proof is for the two bidder case where we can show that incremental equilibria exist iff $T \geq -\frac{1}{\lambda}$. The proof for n bidders is conceptually the same but notationally more demanding, and it is given in the Appendix. Let $W(b,t)$ and $L(b,t)$ denote the expected continuation value of the winning and losing bidder respectively at time t and current high bid b, conditional on bidders using an incremental bidding strategy over $S = \{b_1, ..., b_K\}$. Note that we suppress the current price as the current high bid uniquely determines the price in equilibrium.

We construct the expected continuation values recursively, with $L(b_K, t) = 0$ and $W(b_K, t) = (v - b_{K-1} + 1)e^{\lambda t}$ and for $0 < k < K$,

$$L(b_k, t) = \int_t^0 \lambda e^{-\lambda(\tau - t)} W(b_{k+1}, \tau) d\tau$$
$$W(b_k, t) = \int_t^0 \lambda e^{-\lambda(\tau - t)} L(b_{k+1}, \tau) d\tau + (v - b_{k-1} + 1)e^{\lambda t}$$

The following incentive conditions on the continuation value functions are sufficient to show that an incremental bidding strategy is a best response:

$$L(b_k, t) \geq L(b_{k+1}, t) \quad (1)$$
$$W(b_{k+1}, t) \geq L(b_k, t) \quad (2)$$

The first condition ensures that incremental bids are weakly better than higher bids; higher bids weakly reduce the expected continuation value from becoming a losing bidder without affecting the expected continuation value from remaining the winning bidder until the end of the auction. Note that this also implies that winning bidders will weakly prefer to not adjust their initial bid upon subsequent arrival. The second inequality implies that making an incremental bid is always weakly preferred to remaining a losing bidder.

We will prove the incentive conditions hold with an inductive proof.

$$\begin{aligned} W(b_{K-1}, t) &= \int_t^0 \lambda e^{-\lambda(\tau - t)} L(b_K, \tau) d\tau + (v - b_{K-2} + 1)e^{\lambda t} \\ &= (v - b_{K-2} + 1)e^{\lambda t} \end{aligned}$$

and

$$\begin{aligned}
L(b_{K-1},t) &= \int_t^0 \lambda e^{-\lambda(\tau-t)} W(b_K,\tau)d\tau \\
&= \int_t^0 \lambda e^{-\lambda(\tau-t)} \left((v-b_{K-1}+1)e^{\lambda\tau}\right) d\tau \\
&= -(v-b_{K-1}+1)\lambda t e^{\lambda t}
\end{aligned}$$

$$\begin{aligned}
L(b_{K-2},t) &= \int_t^0 \lambda e^{-\lambda(\tau-t)} W(b_{K-1},\tau)d\tau \\
&= \int_t^0 \lambda e^{-\lambda(\tau-t)} \left((v-b_{K-1}+1)e^{\lambda\tau}\right) d\tau \\
&= -(v-b_{K-2}+1)\lambda t e^{\lambda t}
\end{aligned}$$

From above, $L(b_K,t)=0$ and $W(b_K,t)=(v-b_{K-1}+1)e^t$, which implies that $W(b_K,t) \geq L(b_{K-1},t)$, $W(b_{K-1},t) \geq L(b_{K-2},t)$ when $t \geq -\frac{1}{\lambda}$. It is trivially the case that $L(b_{K-2},t) \geq L(b_{K-1},t) \geq L(b_K,t)$.

We now prove the inductive step; if $W(b_k,t) \geq L(b_{k-1},t)$ and $L(b_{k-1},t) \geq L(b_k,t)$ then $W(b_{k-2},t) \geq L(b_{k-3},t)$ and $L(b_{k-3},t) \geq L(b_{k-2},t)$.

$$\begin{aligned}
L(b_{k-3},t) \quad &= \int_t^0 \lambda e^{-\lambda(\tau-t)} W(b_{k-2},\tau)d\tau \\
&= \int_t^0 \lambda e^{-\lambda(\tau-t)} \left[\int_\tau^0 \lambda e^{-\lambda(s-t)} L(b_{k-1},s)ds + (v-b_{k-3}+1)e^{\lambda\tau}\right] d\tau \\
&= \int_t^0 \lambda e^{-\lambda(\tau-t)} \int_\tau^0 \lambda e^{-\lambda(s-t)} L(b_{k-1},s)dsd\tau + \int_t^0 \lambda e^{-\lambda(\tau-t)}(v-b_{k-3}+1)e^{\lambda\tau}d\tau \\
&\leq \int_t^0 \lambda e^{-\lambda(\tau-t)} \int_\tau^0 \lambda e^{-\lambda(s-t)} W(b_k,s)dsd\tau + \int_t^0 \lambda e^{-\lambda(\tau-t)}(v-b_{k-3}+1)e^{\lambda\tau}d\tau \\
&= \int_t^0 \lambda e^{-\lambda(\tau-t)} L(b_{k-1},\tau)d\tau - (v-p+2)\lambda t e^{t\lambda} \\
&= W(b_{k-2},t) - (v-b_{k-3}+1)e^{t\lambda} - (v-b_{k-3}+1)\lambda t e^{t\lambda} \\
&\leq W(b_{k-2},t)
\end{aligned}$$

The first inequality follows from our assumption that $W(b_k,t) \geq L(b_{k-1},t)$ and the second inequality follows from $t < \frac{1}{\lambda}$.

$$\begin{aligned}
L(b_{k-3}, t) &= \int_t^0 \lambda e^{-\lambda(\tau - t)} W(b_{k-2}, \tau) d\tau \\
&= \int_t^0 \lambda e^{-\lambda(\tau - t)} \left[\int_\tau^0 \lambda e^{-\lambda(s-t)} L(b_{k-1}, s) ds + (v - b_{k-3} + 1) e^{\lambda \tau} \right] d\tau \\
&> \int_t^0 \lambda e^{-\lambda(\tau - t)} \left[\int_\tau^0 \lambda e^{-\lambda(s-t)} L(b_k, s) ds + (v - b_{k-2} + 1) e^{\lambda \tau} \right] d\tau \\
&= \int_t^0 \lambda e^{-\lambda(\tau - t)} W(b_{k-1}, \tau) d\tau \\
&= L(b_{k-2}, t)
\end{aligned}$$

At the beginning of the auction both bidders are active until the first bid is placed, hence the continuation values at $p = 0$ must be treated separately. The expected continuation value at $p = 0$ is given by,

$$L(0, t) = \int_t^0 \lambda e^{-2\lambda(\tau - t)} (W(b_1, \tau) + L(b_1, \tau)) d\tau$$

To prove that $W(b_1, t) \geq L(0, t)$ for $t > -\frac{1}{\lambda}$, we show more generally,

$$W(b_k, t) \geq \int_t^0 \lambda e^{-2\lambda(\tau - t)} (W(b_k, \tau) + L(b_k, \tau)) d\tau$$

The argument at b_{K-1} and b_{K-2} is identical,

$$\begin{aligned}
&\int_t^0 \lambda e^{-2\lambda(\tau - t)} (W(b_{K-1}, \tau) + L(b_{K-1}, \tau)) d\tau \\
&= \int_t^0 \lambda e^{-2\lambda(\tau - t)} \left((v - b_{K-2} + 1) e^{\lambda \tau} - \lambda \tau (v - b_{K-1} + 1) \right) e^{\lambda \tau} d\tau \\
&< \int_t^0 \lambda e^{-2\lambda(\tau - t)} \left((v - b_{K-2} + 1) e^{\lambda \tau} - \lambda \tau (v - b_{K-2} + 1) \right) e^{\lambda \tau} d\tau \\
&= -(v - b_{K-2} + 1) \\
&\leq W(b_{k-1}, t)
\end{aligned}$$

and the inductive step is as follows,

$$
\begin{aligned}
&\int_t^0 \lambda e^{-2\lambda(\tau-t)}(W(b_k,\tau)+L(b_k,\tau))d\tau \\
=&\int_t^0 \lambda e^{-2\lambda(\tau-t)}\left(\int_\tau^0 \lambda e^{-\lambda(s-\tau)}(W(b_{k+1},s)+L(b_{k+1},s))ds+(v-b_{k-1}+1)e^{\lambda\tau}\right)d\tau \\
=&\int_t^0 \lambda e^{-2\lambda(\tau-t)}\left(\int_\tau^0 \lambda e^{-\lambda(s-\tau)}(W(b_{k+1},s)+L(b_{k+1},s))ds\right)d\tau+(1-e^{\lambda t})(v-b_{k-1}+1)e^{\lambda t} \\
=&\int_t^0 \lambda e^{-2\lambda(\tau-t)}\left(\int_\tau^0 \lambda e^{-\lambda(s-\tau)}\left(\int_s^0 \lambda e^{-\lambda(r-s)}((W(b_{k+2},r)+L(b_{k+2},r))dr\right)ds\right)d\tau \\
&+(e^{\lambda t}-1-(\lambda t)^2)(v-b_k+1)e^{\lambda t}+(1-e^{\lambda t})(v-b_{k-1}+1)e^{\lambda t} \\
=&\int_\tau^0 \lambda e^{-\lambda(s-\tau)}\left(\int_s^0 \lambda e^{-\lambda(r-s)}\left(\int_t^0 \lambda e^{-2\lambda(\tau-t)}((W(b_{k+2},r)+L(b_{k+2},r))d\tau\right)dr\right)ds \\
&+(e^{\lambda t}-1-(\lambda t)^2)(v-b_k+1)e^{\lambda t}+(1-e^{\lambda t})(v-b_{k-1}+1)e^{\lambda t} \\
=&\int_t^0 \lambda e^{-\lambda(s-\tau)}\left(\int_s^0 \lambda e^{-\lambda(r-s)}\left(\int_r^0 \lambda e^{-2\lambda(\tau-t)}((W(b_{k+2},\tau)+L(b_{k+2},\tau))d\tau\right)dr\right)ds \\
&+(e^{\lambda t}-1-(\lambda t)^2)(v-b_k+1)e^{\lambda t}+(1-e^{\lambda t})(v-b_{k-1}+1)e^{\lambda t} \\
<&\int_t^0 \lambda e^{-\lambda(s-\tau)}\left(\int_s^0 \lambda e^{-\lambda(r-s)}W(b_{k+2},r)dr\right)ds \\
&+(e^{\lambda t}-1-(\lambda t)^2)(v-b_k+1)e^{\lambda t}+(1-e^{\lambda t})(v-b_{k-1}+1)e^{\lambda t} \\
=&W(b_k,t)-(v-b_{k-1}+1)e^{\lambda t} \\
&+(e^{\lambda t}-1-(\lambda t)^2)(v-b_k+1)e^{\lambda t}+(1-e^{\lambda t})(v-b_{k-1}+1)e^{\lambda t} \\
=&W(b_k,t)-(1-(\lambda t)^2)(v-b_k+1)e^{\lambda t}-e^{2\lambda t})(b_k-b_{k-1} \\
<&W(b_k,t)
\end{aligned}
$$

■

The proof of Proposition 1 reveals that if $t > t^*$ then a losing bidder prefers overtaking the lead even when the other bidder's subsequent bid is sufficiently high to prevent the former bidder from obtaining any surplus from the auction. Intuitively, for short enough auctions there cannot be sufficient incentives to prevent losing bidders from overtaking the lead, since the probability that another bidder arrives is low enough that even the most severe punishment by the other bidders (switching to a truthful equilibrium) is not sufficient in preventing such behavior.

Another observation we make is that in a symmetric Markovian equilibrium bidding cannot stop until price reaches v, even though at $P = v - 1$ a bidder is indifferent between placing a bid of v and abstaining. This is because if with some positive probability players abstain from overbidding $P = v - 1$, a bid of $v - 1$ becomes strictly better at any point than bidding $v - 3$ (the next lower weakly undominated bid). Hence, players would never bid $v - 3$. But then the same argument can be used iteratively to establish that players would never bid $v-4$, $v-5$ and so on, leading to the unraveling of any gradual bidding.[8]

The above two observations imply that in short enough auctions the worst symmetric Markovian equilibrium is given by the most gradual incremental bidding equilibrium - the one over $S = \{1, 2, ..., v-3, v-1, v\}$.[9] For the over all worst equilibrium, which along the equilibrium path has a very similar structure, see Subsection 3.5.

3.3 Long Auctions with Two Symmetric Bidders

In this section we characterize incremental bidding in auctions with longer time horizons where incremental equilibria with no waiting are not supported. The failure of the latter equilibria in long auctions can be seen by extending the length of the 2-bidder auction example from the previous section such that $T = -2$. Figure 2 plots the non-trivial bidder value functions in the fully incremental equilibrium over the interval $[-2, 0]$. As we demonstrated previously, for all p and $t > -1$, placing a bid is optimal as $W(p+1, t) > L(p, t)$. However, at any time $t < -1$, a winning bidder's expected value at $p = 3$ is lower than a losing bidder's expected value at $p = 2$ and hence a losing bidder facing a price of 2 would find it profitable in expectation to wait until $t > -1$ to place a bid. Nonetheless, we can still construct equilibria with incremental bidding behavior in long auctions.

Sustaining incremental bidding in equilibrium requires intervals during which bidders abstain from bidding even though the price is below their value and they do not hold the current high bid. Bidders choose to wait when the cost of increasing the price outweighs the likelihood of winning the object with the current bid. In our example, the trade-off is straightforward. Bidding at $p = 3$, since it induces the other player trying to bid again, yields a positive payoff only in the event that the other bidder does not return to the auction. This occurs with decreasing likelihood as we extend the time remaining in the auction. On the other hand, the likelihood of returning to the auction at the same price but closer to the end of the auction, and thereby face a more favorable trade-off, is increasing in the time remaining in the auction. For these reasons, far enough from the deadline a losing bidder at $p = 3$ prefers waiting, while close enough to the deadline he prefers taking over the lead.

We refer to the point in time τ_p at which at which a bidder is indifferent between overtaking the current high bid at price p and waiting for the next opportunity, as the cutoff for price p. An incremental equilibrium with waiting is characterized by a bidding sequence and its corresponding set of cutoff points. In this subsection we provide a general result on the existence of such equilibria for two bidders. The next subsection discusses extension of the analysis to more

[8] There can be asymmetric Markovian equilibria in which one player stops bidding at $P = v-1$. Moreover, there can be asymmetric non-Markovian equilibria in which all players stop bidding at $P = v-1$.

[9] This equilibrium is outcome-equivalent to the incremental bidding equilibrium over $S = \{1, 2, ..., v-3, v\}$.

than two bidders.

Definition: Let $S = \{b_1, ..., b_k\}$ be a bidding sequence, a *cutoff sequence* is an ordered set $C_S = \{t_1, ..., t_k\}$ such that for all j, $t_j < 0$.
Definition: A bidder following an *Incremental Bidding Strategy with Delays* over bidding sequence $S = \{b_1, ..., b_k\}$ and cutoff sequence $C_S = \{t_1, ..., t_k\}$ places a bid of $\min_j\{b_j | p < b_j\}$ upon arriving at time t and price p if and only if she is a losing bidder and $t > t_j$ and otherwise does not place a bid.

In an equilibrium in which bidders follow a symmetric Incremental Bidding Strategy with Delays, bidder value functions are constructed in the same manner as for incremental equilibria with no waiting. When $C_S = \{1, 2, .., v-3, v\}$, the value functions are given by

$$\begin{aligned}
L(p,t) &= \int_s^0 \lambda e^{-\lambda(\tau-s)} W(p+1,\tau) d\tau \\
W(p,t) &= \int_s^0 \lambda e^{-\lambda(\tau-s)} L(p+1,\tau) d\tau + (v-p)e^{\lambda s} \\
L(0,t) &= \int_s^0 \lambda e^{-2\lambda(\tau-s)} W(1,\tau) d\tau + \int_s^0 \lambda e^{-2\lambda(\tau-s)} L(1,\tau) d\tau
\end{aligned}$$

where $s = \max\{t, \tau_p\}$. Non-trivial cutoffs satisfy $L(p, \tau_p) = W(p+1, \tau_p)$.

In our example, there are two relevant cutoffs (i.e. not equal to T); $\tau_0 = -\frac{1}{2}\log(6e + e^2)$ and $\tau_2 = -1$. Figure 3 plots the value functions for bidders following these cutoffs. The auction is divided into three periods; in the first period no bidding occurs and price remains 0 even if bidders get arrivals. In the second period an arriving player bids incrementally if $p \in \{0, 1\}$ but passes on opportunities to bid if $p \geq 2$. Finally, in the third period a losing bidder submits a bid until price reaches $p = 5$.

The equilibrium expected bidder payoff and seller revenue in this example are 1.35 and 2.09 respectively. The seller revenue compares favorably to that of the short auction but it is still significantly less than in the benchmark equilibrium. Further, extending the length of the auction has no further effect on the expected revenue as it serves only to lengthen the period in which no bidding occurs.

The existence of non-trivial cutoffs for every second bid is not specific to the above example, but a general feature of incremental bidding equilibria with waiting. Note that at any price, the winning bidder's expected value is greater than that of the losing bidder. Now suppose a losing bidder arrives at time t and faces a price $p-1$. If bidding at price p does not begin until $\tau_p > t$, a losing bidder cannot do better than to be the winning bidder at price p at time τ_p and hence he must at least weakly prefer placing a bid. The same intuition for the existence of the final cutoff for bidding $v-3$ implies that if there is no cutoff for bidding $p-1$, then there exists a cutoff for bidding $p-2$.

Proposition 2 generalizes the existence of incremental equilibria with delay in auctions of arbitrary length.

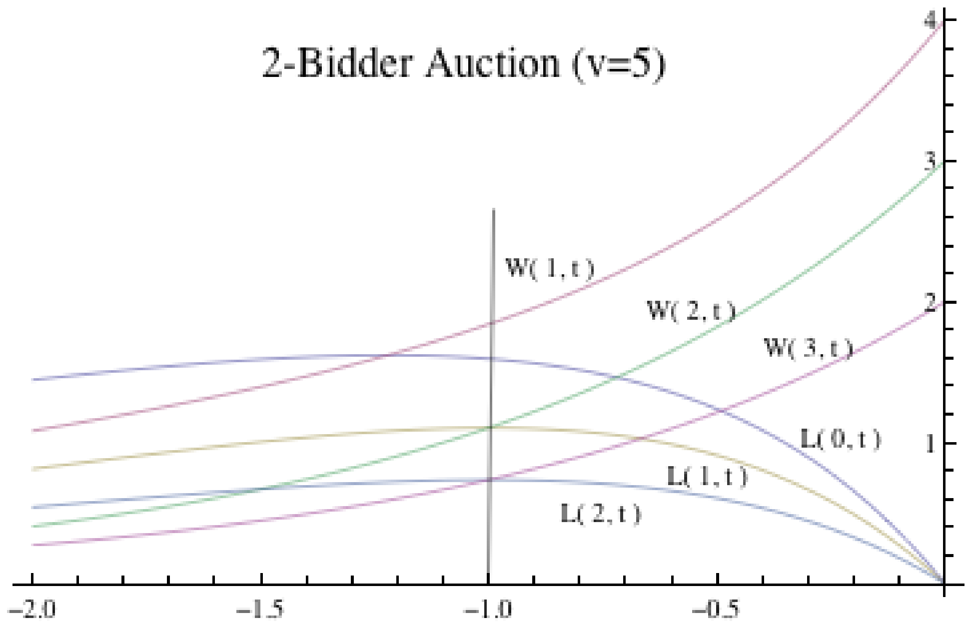

Figure 2: Fully Incremental Equilibrium value functions.

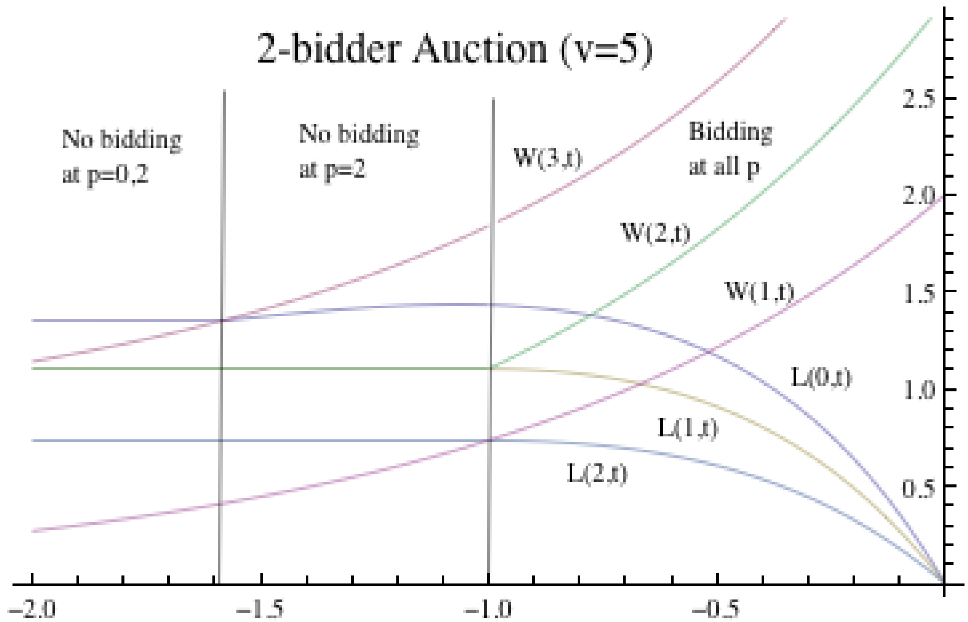

Figure 3: Incremental Equilibrium with Delay value functions.

Proposition 2: In symmetric 2-bidder auctions, for any proper bidding sequence S there exists a cutoff sequence C_S such that incremental bidding strategies with delay over S, C_S form an equilibrium.

The proof of this proposition, featured in the Appendix, consists of two parts. In the first part, we construct a set of cutoff points such that the resulting value functions satisfy the bidding incentive constraints. The final cutoff is given by $-\frac{1}{\lambda}$ and is independent of the bidding sequence. This holds because a bidder facing the decision to bid at the final cutoff (at $p = 1$ in the above example) can only win the auction with positive payoff at one price ($p = 2$ in the example) and hence the trade-off is only between the likelihood of having another arrival and the likelihood of the other bidder arriving. All other cutoffs may be constructed recursively. Suppose that we know the cutoff for price p, τ_p as well as the bidder value functions at τ_p. For any time t such that $\tau_{p-2} < t < \tau_p$, we can derive $W(p-2,t)$ and $L(p-3,t)$ as a functions of $L(p-3,t)$, $W(p-2,\tau_p)$, $L(p-1,\tau_p)$ and $t-\tau_p$. Setting $W(p-2,t)$ and $L(p-3,t)$ equal allows us to solve for $\tau_{p-2}-\tau_p$ in terms of $L(p-3,t)$, $W(p-2,\tau_p)$, $L(p-1,\tau_p)$.

The second part of the proof requires proving that overbidding is suboptimal. In short auctions it is easy to show that overbidding is dominated because it increases the price that a bidder would face if he were outbid and returned to the auction without providing any increased likelihood of winning the auction. However, when there are periods of inactivity in an auction, overbidding has more subtle consequences. Suppose that at τ_p the current price is $p-1$ and consider the outcomes at τ_{p+2}. Bidding p results in either $W(p,\tau_{p+2})$ or $L(p+1,\tau_{p+2})$, whereas bidding $p+1$ results in either $W(p+1,\tau_{p+2})$ or $W(p,\tau_{p+2})-k$ where k reflects the loss associated with overbidding after the cutoff. The loss from overbidding occurs when the other bidder does not arrive, hence the cutoffs cannot be too far apart or they will induce overbidding. We show that the cutoffs satisfy this condition.

The incremental equilibrium with $S = \{1,2,...,v-3,v-1,v\}$ and corresponding C_S constitutes the most gradual Markovian bidding equilibrium. However, it is not the revenue-minimizing equilibrium over all strategies. The latter is non-Markovian, and discussed in Subsection 3.5.

3.4 Long auctions with more than two bidders

Partially incremental equilibria in long n-player auctions are considerably more complicated to characterize than in the 2-bidder case. We present an example in which $v = 5$, $\lambda = 1$, and $T = -\infty$. As in the 2-bidder long auctions, we construct a Markovian equilibrium in which bidders bid incrementally according to a bidding sequence $S = \{1,2,5\}$ and a set of cutoffs $\{\tau_1,\tau_2,\tau_5\}$. Value functions for $p > 0$ are defined recursively by

$$W(p,t) = \int_t^0 \lambda e^{-(n-1)\lambda(\tau-t)}(n-1)L(p+1,\tau)d\tau + (v-p)e^{(n-1)\lambda t}$$

$$L(p,t) = \int_t^0 \lambda e^{-(n-1)\lambda(\tau-t)}\left(W(p+1,\tau) + (n-2)L(p+1,\tau)\right)d\tau$$

where $s = max\{t, \min_k \tau_k | k > p\}$. $W(3,t) = 2e^{-(n+1)t}$ and $L(3,t) = 0$ (as in the 2-bidder example, bidding 3 is weakly dominated and hence 2 is the highest incremental bid.) Finally, $L(0,t)$ is given by

$$L(0,t) = \int_t^0 \lambda e^{-n\lambda(\tau-t)}\left(W(1,\tau) + (n-1)L(1,\tau)\right)d\tau$$

The structure of the incremental equilibrium supported in this auction is sensitive to the number of bidders n. When n is large, the supported incremental equilibria resemble the Fully Incremental Equilibria of short auctions and when n is small the auction supports Partially Incremental Equilibria with cutoffs at alternating bids. More specifically, for n large, $\tau_1 > \tau_p$ for all p. As a result, no bidding occurs in equilibrium before τ_1 at which point bidders proceed to follow a fully incremental bidding strategy. Hence, bidder payoffs and seller revenue are independent of the length of the auction for $T < \tau_1$. In contrast, when n is small $\tau_1 > \tau_p$ no longer holds for all p. Bidders may prefer to wait early in the auction and when prices are low, and expected payoffs may dependent on the length of the auction.

We illustrate this point by considering the equilibria for two values of n in our example. Figures 4 and 5 show the expected payoffs at $p = 0$ and $p = 1$ for an auction with 10 bidders and 3 bidders respectively. In the auction with 10 bidders, bidding begins at $\tau_1 = -0.439 < \tau_2, \tau_3$. The expected payoff to a bidder is only 0.06, but the total bidder payoff is still greater than 10% of v. In the 3 bidder auction, $\tau_1 = -\infty$ and $\tau_2 = -0.79$, which imply that the first bidder to arrive will place a bid of 1. At $p = 1$, bidding only resumes at $t = -0.79$. The expected payoff to a bidder in this case is 0.77 and the seller's revenue is 2.68. Figure 6 plots the seller's expected revenue as a function of the number of bidders in the most gradual Markovian equilibrium. The non-monotonicity of expected seller revenues highlights the shift in equilibrium structure as the number of bidders changes from 8 to 9. We note that the non-monotonicity is mitigated as the length of the auction is reduced and the importance of the initial period prior to τ_1 declines.

3.5 Non-Markovian equilibria

History dependent strategies allow the construction of equilibria that further reduce the seller's expected revenue beyond the most gradual Markovian equilibria. In this section we characterize the seller-revenue minimizing equilibria in 2-bidder auctions. Strategies in the seller revenue-minimizing equilibrium

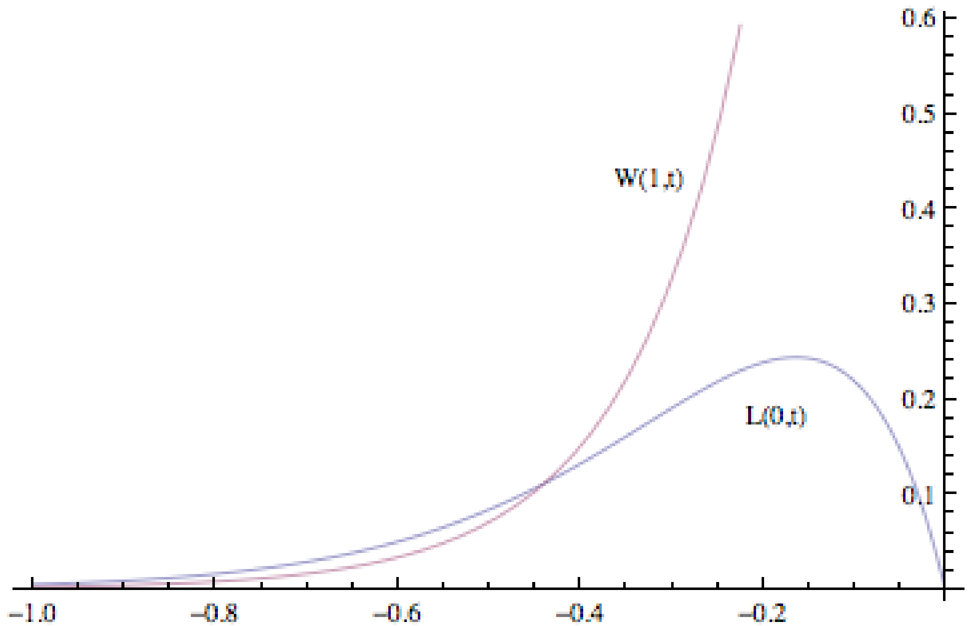

Figure 4: Expected Continuation Payoffs in a 10-bidder auction.

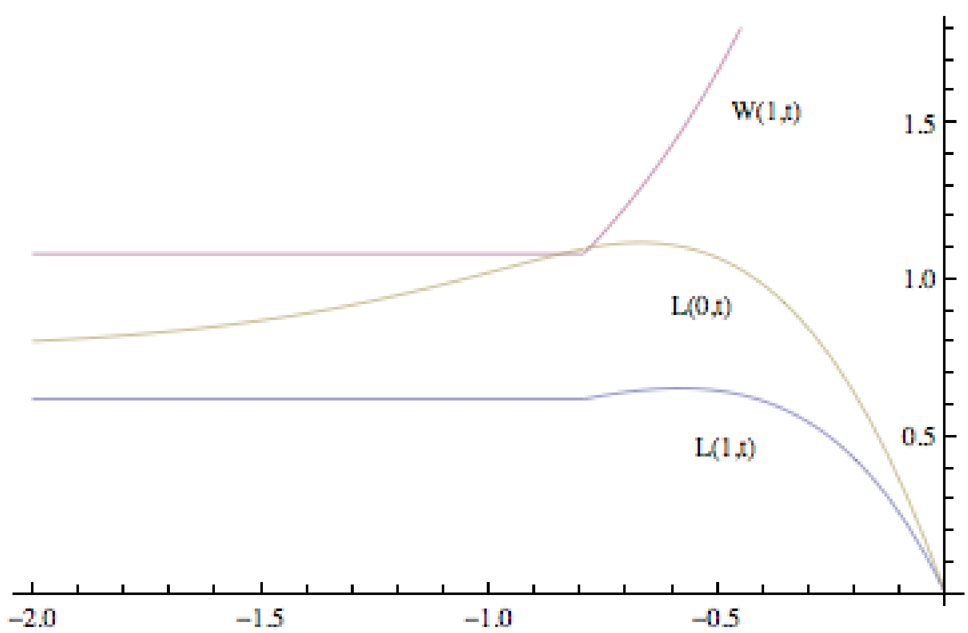

Figure 5: Expected Continuation Payoffs in a 3-bidder auction.

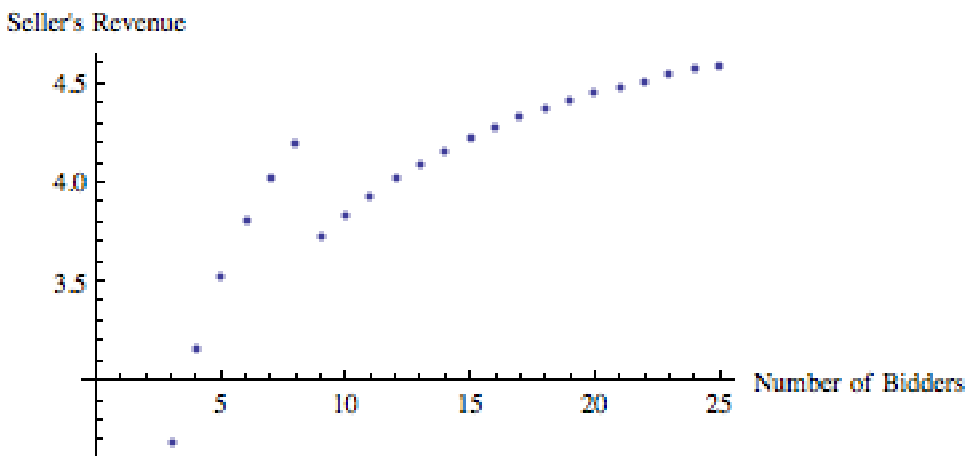

Figure 6: Seller Revenue in the Most Gradual Markovian Equilibrium.

take the following form: bidders pass on all bidding opportunities prior to $t^* < 0$ after which bidders engage in incremental bidding without delay over $S = \{1, 2, ...v-3, v-1\}$. Any deviation in this stage is punished by reverting to a truthful equilibrium, which is the most severe punishment a defector can face as it implies that any subsequent arrival by the other bidder reduces the deviator's payoff to 0.

We begin by noting that the threat of punishment allows the final bid to be $v-1$ without causing unravelling; the threat of punishment makes a bidder indifferent between bidding $v-3$ and $v-1$. Further, if t^* is greater than the final cutoff in the equilibrium without punishment then placing any bid strictly dominates waiting. In particular this means placing a bid of v dominates waiting and hence the threat of punishment cannot induce a bidder to wait.

In a symmetric value auction with 2 bidders, let $W(p,t)$ and $L(p,t)$ denote the expected payoffs of winning and losing bidders respectively at time t and price p, conditional on bidders following an incremental bidding strategy over $S = \{1, 2, ...v-3, v-1, v-1\}$. The cutoff t_v^* is the point at which reversion to truthful bidding no longer provides sufficient incentives for a bidder to refrain from bidding. Therefore, at t_v^* a bidder is indifferent between triggering truthful bidding, for an expected payoff of $(v-1)e^{(n+1)t_{v,n}^*}$, and waiting, which yields an expected payoff of $L(0, t_{v,n}^*)$. We begin by showing that $t_{v,n}^*$ is increasing in v. Suppose t_v^* satisfies

$$(v-1)e^{t_v^*} = \int_{t^*}^{0} e^{-(\tau - t^*)} \left(W(1,\tau) + L(1,\tau)\right) d\tau$$

then,

$$\int_{t_v^*}^{0} e^{-(\tau - t_v^*)} \left(W(1,\tau) + L(1,\tau)\right) d\tau - \int_{t_v^*}^{0} e^{-(\tau - t_v^*)} \left(W(2,\tau) + L(2,\tau)\right) d\tau$$
$$= \int_{t_v^*}^{0} e^{-n(\tau - t_v^*)} \left((W(1,\tau) - W(2,\tau)) + (L(1,\tau) - L(2,\tau))\right) d\tau$$
$$> \int_{t_v^*}^{0} e^{-n(\tau - t_v^*)} \frac{1}{v-2} \left((W(1,\tau) + (L(1,\tau))\right) d\tau$$
$$= \frac{v-1}{v-2} e^{t_v^*}$$
$$> (v-1)e^{t_v^*} - (v-2)e^{t_v^*}$$

which implies that

$$(v-2)e^{t_v^*} > \int_{t^*}^{0} e^{-(\tau - t_v^*)} \left(W(2,\tau) + L(2,\tau)\right) d\tau$$

and therefore $t_{v-1,n}^* < t_v^*$. over $S = \{1, 2, ...v-3, v-1, v-1\}$.

The first inequality follows from noting that the likelihood at time t, $p = 1$ that the auction will end at a price less than $v-1$ is given by $W(1,t) - W(2,t) + (n-1)(L(1,t) - L(2,t))$, which is greater than $\frac{1}{v-2}$ times the total expected value to the bidders at $p = 1$. A direct calculation show that t_5^* is greater than the final cutoff in the equilibrium without punishment and therefore the same holds for any t_v^* such that $v \geq 5$.

It now remains only to demonstrate that bidding is maximally delayed, which is a straightforward point. The sum of the expected continuation values of the two bidders in an auction of length t^* with arbitrary equilibrium bidding sequence R is bounded above by that of the same auction with equilibrium bidding sequence S. Hence, if a bidder is indifferent at time t between triggering punishment and waiting given the bidding sequence S, the bidder must weakly prefer triggering the punishment for any other bidding sequence T. Therefore, the proposed strategies form the seller minimizing equilibrium.

The cutoff value t^* is increasing in v, as at a higher v players can be induced to wait longer, through trigger strategies. Numerical calculations show that t_v^* is increasing in v fast enough so that the expected number of bids and therefore the seller's expected revenue is decreasing in v (larger t^* implies a smaller expected number of arrivals during the period when players are actively bidding, and this outweighs the effect that bidding can potentially reach higher prices if v is larger and a lot of arrival events occur towards the end). As a result, the ratio of the seller's expected revenue to the bidder's value v goes to zero as v grows large.

4 Extensions

When departing from the case of complete information and symmetric players, incremental equilibria like the ones featured in the previous section become infeasible to compute analytically, and it is also difficult to prove the existence of a certain type of incremental equilibrium in a completely general setting. However, it is relatively easy to identify incremental equilibria in concrete examples, and even to provide constructions yielding equilibria with incremental bidding in large classes of games. In this section we demonstrate this for cases in which bidders are heterogeneous in their evaluations or in their arrival rates, for time-variant arrival rates, and for a class of incomplete information games in which bidders' evaluations are drawn from a binary set. These results show that the existence of incremental equilibria is not tied to the analytically convenient case of symmetric bidders with perfect information, but a more general feature of continuous-time auctions with random arrivals. The analysis also reveals some new qualitative insights of incremental equilibria when buyers' valuations are asymmetric.

4.1 Asymmetric Values

We first show that for two bidders, there always exists an equilibrium in which the initial bid by a low type is incremental. In this equilibrium, low types win with non-trivial frequency for auctions of arbitrary length. In the Appendix we also provide an example in which both the low and high bidders make incremental bids and discuss the generalization of this example.

Proposition 3: For any 2-bidder auction with bidder values $v_H > v_L > 2$, symmetric arrival rates and $|T|$ sufficiently large, there exists an equilibrium that displays incremental bidding behavior.

Proof: We construct an equilibrium characterized by three cutoff points; t^L, t_0^H, and t_1^H that satisfy $t_1^H < t^L < t_0^H$. The low type's equilibrium strategy is to bid 1, whenever $p = 0$ and $t > t^L \geq T$ and to bid v_L whenever $p > 0$. The high type's equilibrium strategy is to bid v_L iff $p = 0$ and $t \geq t_0^H$ or $p = 1$ and $t \geq t_1^H$.

We begin by solving for the high type's cutoffs. Given the strategies described above, the high type's continuation values at $p = 1, 2$ are given by,

$$W_H(p,t) = v_H - v_L + (v_L - p)e^{\lambda t}$$
$$L_H(1,t) = \int_t^0 \lambda e^{-\lambda(\tau - t)} W_H(2,\tau) d\tau$$

When $p = 0$, the high type is indifferent between making a truthful bid and delaying at t_0^H. Therefore, t_0^H satisfies

$$W_H(1,t_0^H) = \int_{t_0^H}^0 \lambda e^{-2\lambda(\tau - t_0^H)} \left(W_H(2,\tau) + L_H(1,\tau)\right) d\tau$$

or $t_0^H = \frac{1}{\lambda} \log \frac{v_L - 2}{v_L + v_H - 3}$. Similarly, because the low type bidder does not raise his bid while holding the current high bid at $p = 1$,

$$W_H(2, t_1^H) = \int_{t_1^H}^{0} ve^{-\lambda(\tau - t_1^H)} W_H(2, \tau) d\tau$$

which yields $t_1^H = \frac{2-v_H}{\lambda(v_L-2)}$.

If $T < t^L$, the low type is indifferent between making a bid of 1 and waiting until the next arrival to place a bid. We first show that $t^L < t_0^H$. Consider the low type's decision at $t \geq t_0^H$. Because the high type's strategy going forward is independent of the low type's actions, the low type must strictly prefer to place a bid, as it strictly increases the likelihood of winning without affecting the expected payoff from winning. The low type's continuation value at $p = 0$ is given by

$$L_L(0, t) = (v_L - 1)e^{\lambda t}$$

and t^L satisfies

$$L_L(0, t^L) = \int_{t^L}^{t_0^H} \lambda e^{-\lambda(\tau - t^L)} L_L(0, \tau) d\tau + e^{\lambda(t^L - t_0^H)} \int_{t_0^H}^{0} e^{-2\lambda(\tau - t_0^H)} L_L(0, \tau) d\tau$$

which yields

$$t^L = \frac{1}{\lambda}\left(\log\left(\frac{v_L - 2}{v_H + v_L - 3}\right) - \frac{v_L - 2}{v_H + v_L - 3}\right)$$

The above computations imply that the strategy profile prescribed above constitutes an equilibrium. QED

There are several features of the equilibrium that are instructive to point out. First, let $r = v_L/v_h$ be the ratio of the two values and rewrite $t_1^H = -\frac{1-2/v_H}{r-2/v_H}$, $t_0^H = \log\frac{r-2/v_H}{r+1-3/v_H}$. This makes it clear that t_0^H, t_1^H are increasing in r. The high type is induced to wait by the potential to raise her payoff by $v_L - 1$ or $v_L - 2$. As v_L becomes large relative to the certain payoff from bidding truthfully, i.e. r approaches 1, the incentive becomes stronger and hence the high type is willing to wait longer. Likewise, holding r fixed, the cutoffs are also increasing in v_H.

A second feature is that even when the auction is arbitrarily long, the low type can win the auction with nontrivial probability and achieve a substantial payoff. To see this, consider an auction with $v_H = 6$, $v_L = 4$ and $t = -\infty$. In the equilibrium constructed above, cutoffs for the high type are $t_1^H = -3$ and $t_0^H = \log\frac{2}{7}$ and the low type's cutoff is at $t^L = \log\frac{2}{7} - \frac{2}{7}$.

In the benchmark truthful equilibrium, the high type and low type bid 6 and 4 respectively upon their first arrival. This implies that with probability 1, the high type wins and gets a payoff of 1 or 2 (depending on which bidder arrives first) and the seller's revenue is at least 4. In contrast to this, in the equilibrium constructed in Proposition 3, the low type's expected payoff, given by $L(0, t_L) = \frac{6}{7}e^{-\frac{2}{7}} = .64$. Because the low type can only win at price $p = 1$, this implies that the low type has approximately 21% chance of winning the auction. Further, the total expected payoff among both bidders is equal to 3.5 versus at most 2 in the benchmark equilibrium.

We are also interested in the likelihood that no bid will be placed in the auction, a source of inefficiency. This is straightforward to calculate as the likelihood that the low type does not arrive after t^L and the high type does not arrive after t_0^H, which is given by,

$$P = e^{-\frac{v_L - 2}{v_H + v_L - 3}} \left(\frac{v_L - 2}{v_H + v_L - 3} \right)^2$$

In the example above, P is approximately 6%. However, P is increasing in v_L/v_H and bounded at approximately 15%. For v_L, v_H sufficiently large and v_L/v_K close to 1, this equilibrium resembles a sniping equilibrium in that bidding occurs only close to the end of the auction, and the high type starts out bidding truthfully (however, the low type starts out by bidding incrementally).

4.2 Asymmetric Arrival Rates

Incremental bidding equilibria are also supported when bidders have asymmetric arrival rates. We return to our 2-bidder example with $v = 5$ and $\lambda_A = 1$, but now specify that $\lambda_B = \beta$ with $\beta > 1$. Equilibrium value functions and cutoffs for incremental equilibria can be computed the same way that we used in the symmetric bidders case. Below we focus on the most gradual incremental equilibrium ($S = \{1, 2, 5\}$).

Surprisingly, the cutoff points for both bidders turn out to be equivalent. The first cutoff,

$$\tau_2 = \frac{\log \beta}{1 - \beta}$$

is increasing in β. This is between -1, the corresponding cutoff in a game with two bidders with both $\lambda = 1$, and $-\frac{1}{\beta}$, the cutoff in a game with two bidders with both $\lambda = \beta$. Earlier cutoffs cannot be derived analytically. The cutoff for bidding at $p = 0$ is roughly -1.135. The value of the cutoff is again in between the value that obtains in a game with both bidders having arrival rate $\lambda = 1$ and in a game with both having arrival rate $\lambda = \beta$.

Expected payoffs are 0.83 and 1.89 for bidder A and B respectively and the seller's expected revenue is 2.11. Increasing the arrival rate of bidder B from 1 to 2 increases the seller's expected revenue, but the relationship is not monotonic; at $\beta = 10$ the seller's expected revenue is only 1.485. The intuition behind this is that while the likelihood that at least one bid is placed (the object is sold) is increasing in β, the number of arrivals by the low type is decreasing because equilibrium cutoffs are increasing in β.

4.3 Time-dependent arrival rates

Multiplying all arrival rates by a constant $\alpha > 0$ is equivalent to rescaling time by $\frac{1}{\alpha}$. In particular, if the original game has an incremental bidding strategy equilibrium over bidding sentence $\{b_1, ..., b_k\}$ and cutoff sentence $\{t_1, ..., t_k\}$ then the game where arrival rates are multiplied by α has an incremental bidding strategy equilibrium over bidding sentence $\{b_1, ..., b_k\}$ and cutoff sentence $\{\frac{1}{\alpha}t_1, ..., \frac{1}{\alpha}t_k\}$. Furthermore, expected payoffs with time horizon T in the original game are the same as with time horizon $\frac{T}{\alpha}$ in the game with the rescaled arrival rates. In particular, if $T \leq t_1$ then increasing arrival rates and keeping the time horizon the same does not change equilibrium expected payoffs: it only shifts all cutoffs to the right, in an inversely proportional manner. Intuitively, if bidders get frequent opportunities to place bids, it makes them postpone bidding at different price levels until later, in a way that exactly offsets the effect of increasing the arrival rates.

We get qualitatively similar conclusions when allowing for time-varying arrival rates, as long as they stay bounded. For example, it is natural to assume that towards the end of the auction bidders pay more attention and are much more likely to check the status of the auction than far away from the deadline. In our model this would lead to arrival rates increasing over time, reaching high levels immediately preceding the end of the game.[10]

We return to the symmetric 2-Bidder Auction example where $v = 5$ and $T = -2$, but now let the arrival rate be given by a strictly increasing function $\lambda(t)$. Let $\lambda(t)$ be of the form $\frac{a}{(1-bt)^2}$ where $a, b > 0$; the arrival rate at the end of the auction, $\lambda(0)$, is then given by the parameter a and the parameter b determines how steeply arrival rates increase at the end of the the auction. The average arrival rate over the auction is given by, $\bar{\lambda} = \int_{-2}^{0} \frac{a}{(1-bt)^2} dt$. We choose $a = 10$ and $b = 9/2$ illustrated in Figure 7, which gives an average arrival rate of 1 as in our original example. Figure 8 shows the effect of an increasing arrival rate on bidder value functions in the most gradual equilibrium. The structure of the equilibrium is identical, bidding begins after an initial cutoff and bidders wait to bid at $p = 2$ until after a second cutoff. However, with bidder's arrival rates increasing over the course of the auction, the cutoffs are much closer to the end of the auction, which re-enforces the high frequency of late-bidding in equilibrium. As long as arrival rates are bounded, incremental equilibria are robust to increasing arrival rates.

4.4 Asymmetric Information

Thus far we have considered auction environments in which each bidder's value is common knowledge. We now consider the case when each bidder knows their own value, but only the distribution from which the other bidder's value is drawn. For simplicity, we restrict attention to two bidders with iid evaluations

[10] Presumably endogenizing arrival rates would lead to such time patterns of arrival rates, although we do not pursue this direction formally here, due to its analytical complexity.

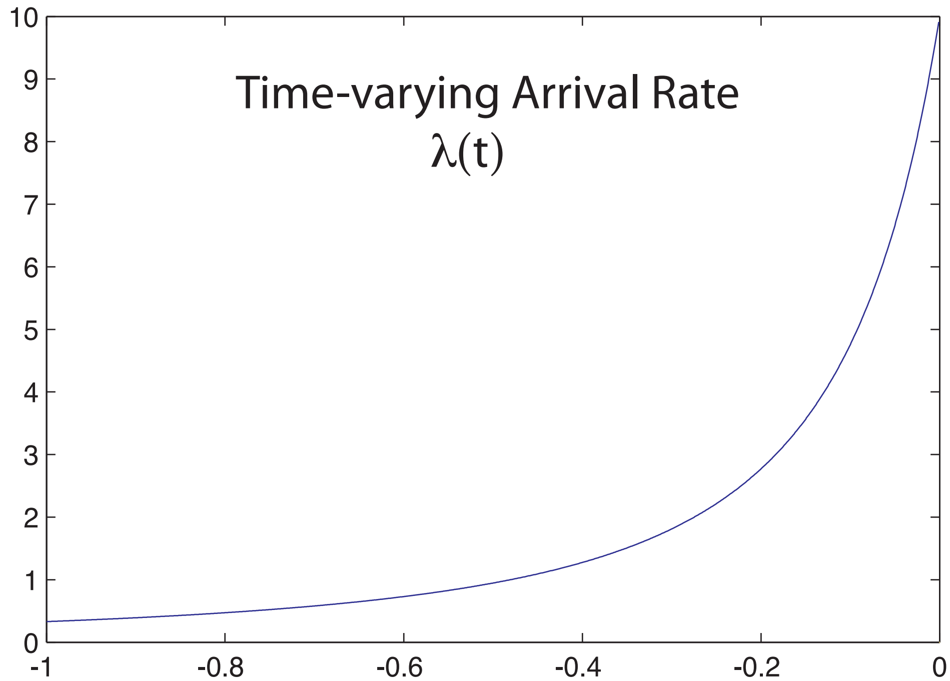

Figure 7: Time varying arrival rate $\lambda(t)$.

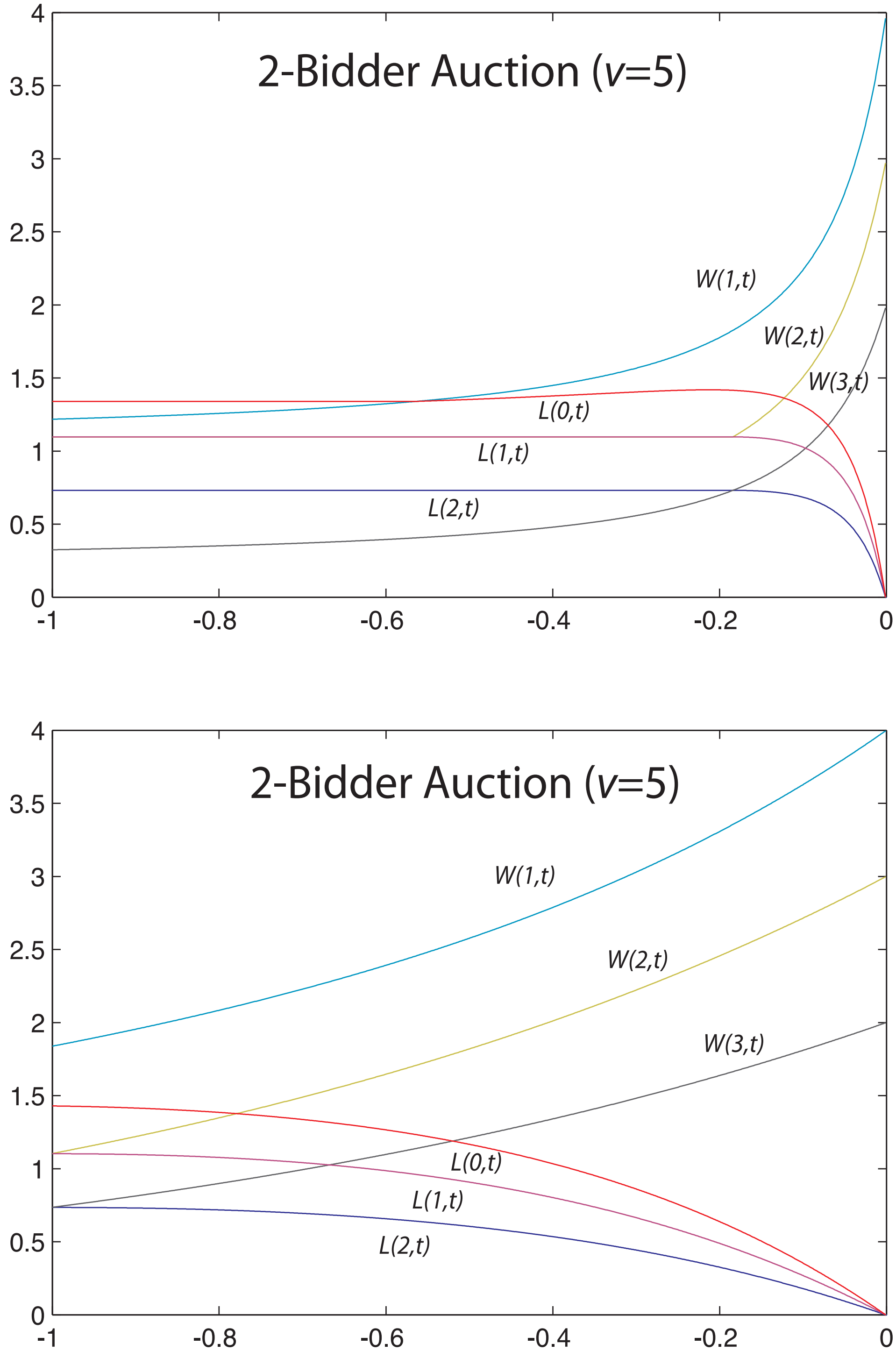

Figure 8: Continuation values with time-varying (top) and static arrival rates.

of binary support. In this environment it is possible to construct equilibria in which a bidder can only make inferences on the other bidder's type once the opponent's type is no longer relevant to her bidding decisions. This greatly simplifies the calculation of cutoff points and incentive constraints. However, we conjecture that the games at hand have many more complicated incremental equilibria in which bidders draw nontrivial inferences on each others' types along the equilibrium path.

In what follows, assume that bidders' valuations are drawn iid, taking value $v_L > 1$ with probability $q \in (0, 1)$ and $v_H > v_L$ with probability $1 - q$. A particularly simple equilibrium with some incremental bidding in this context is when low valuation types bid $v_L - 1$ whenever arrive and $p < v_L - 1$, bid v_L if $p = v_L - 1$, and restrain from bidding otherwise. Meanwhile, high valuation bidders bid v_L whenever they arrive and $p < v_L$, and follow the same bidding strategy as in an incremental equilibrium with waiting, as characterized in Subsection 3.3, of a complete information game with two bidders with valuation v_H. The idea behind this construction is that in equilibrium only high types would ever place a bid above v_L, hence $p = v_L$ implies that either no further bidding occurs in the game, or both of the bidders have high evaluation, validating a continuation equilibrium from a game in which both bidder valuations are commonly known to be v_H.

In the above equilibrium a high type overbids the low type upon first arrival, implying that gradual bidding only occurs if both bidders are high types, and that in a long auction the probability of a low type winning the auction if the other bidder is of a high type is close to 0. Below we provide an example in which there is an equilibrium such that a high type waits until near the end of the auction, and therefore a low type wins the object with significant probability even if the realized type of the other bidder is high.

For this example, let $v_L = 2$ and $v_H = 5$. Consider the following strategies: for $t < t^* < 0$, high types bid 2 whenever $p = 0$, they do not make a bid at $p = 1$ and bid 5 whenever $p > 1$. For $t > t^*$, high types that do not hold the high bid place a bid of 5 at any price and high types holding the high bid do not place bids. Low types bid 1 if $p = 0$ and $t < t^*$, and bid 2 if they do not hold the winning bid and $t \geq t^*$.

It follows immediately that the above strategies are best responses to each other for $t > t^*$. A high type bidder that holds the high bid does not benefit from increasing her bid as she has outbid the low type and cannot outbid a high type. Similarly, a low type with a high bid cannot benefit from raising their bid.

We now solve for t^* and show that strategies are optimal for $t < t^*$. Let $v(t, q)$ denote the expected value of a high type who decides to make a bid at time t and $p = 1$ given bidders follow the above strategy. The payoff will be 3 if the other bidder is a low type and 2 if the other type is a high type but does not arrive again before the end of the auction.

$$v(t, q) = 3q + 2e^t(1 - q)$$

$W(t, q)$ denote the expected continuation value of a high type at time t and price 1 who plans to bid upon next arrival.

$$\begin{aligned} w(t,q) &= \int_t^0 e^{-(\tau-t)} v(\tau,q) d\tau \\ &= \int_t^0 e^{-(\tau-t)} (3q + 2e^{\tau}(1-q)) d\tau \end{aligned}$$

$v(t^*, q) \geq w(t^*, q)$ iff $t \geq \frac{2+q}{2q-2} = t^*$ and therefore for high types bidding at $p = 1$, $t < t^*$ is strictly worse than not placing a bid. At $p = 1$, low types are indifferent between bidding decisions. At t^*, holding the winning bid is strictly preferred by both types of bidders, so both types strictly prefer to make a bid for all $t < t^*$.

The cutoff point for jump bidding is decreasing in q, the likelihood that a bidder is a low type. A high type risks less by outbidding early as the likelihood that her opponent is a low type increases. When $q = 0$, $t^* = -1$ and the game reduces to the symmetric complete information case. For long auctions, each bidder has a likelihood of $\frac{1}{2}$ of being the winning bidder at t^* and the likelihood of the other bidder not arriving subsequently is given by e^{t^*}, hence a low type bidder will win the auction with positive payoff with approximate likelihood of $0.5e^{t^*}$. For low values of q, this likelihood is quite high; at $q = .1$, a low type wins with probability 0.15.

5 Conclusion

This paper shows that on online auctions like ebay where bidders can leave proxy bids, if bidders get random chances to place bids then many different equilibria arise in weakly undominated strategies. Bidders can implicitly collude by bidding gradually or by waiting to place bids, in a self-enforcing manner, slowing down the increase of leading price. These features of our model are consistent with the empirical observations that both gradual bidding and sniping are common bidder behaviors on ebay.

Our investigation suggests that given a fixed set of bidders, running an ascending auction with a long time horizon (long enough that bidders cannot continuously participate) has the potential to affect the revenue of the seller very adversely, even when proxy bidding is possible, relative to running a prompt auction. Hence, introducing a time element can only be beneficial if it takes time for potential bidders to find out about the auction.[11] It is an open question what mechanism guarantees the highest possible revenue for the seller in such environments. In order to prevent implicit collusive equilibria, sellers might

[11] In a recent paper, Fuchs and Skrzypacz (2010) considers the arrival of new buyers over time, but in a dynamic bargaining context in which the seller cannot commit to a mechanism. Another difference compared to our setting is that in their model once a buyer arrives, she is continuously present until the end of negotiations.

want to set high reservation prices and/or minimum bid increments. They might also want to allow each bidder to submit at most one bid over the course of an auction, although in practice this might be difficult to enforce, given that the same person can have multiple online identities. We leave the formal investigation of these issues to future research.

6 Appendix

6.1 Proof of claim 1

First note that a strategy that prescribes bidding $b > v_i$ at any history is weakly dominated by a strategy that specifies bidding v_i if $P < v_i$ and not placing a bid otherwise.

Taking over the lead with $B = v_i - 2$ is weakly dominated by increasing B to $v_i - 1$ at the same arrival. To see this, note that opponents can only find out that $B = v_i - 2$ or $B = v_i - 1$ if one of them bids at least $v_i - 1$, in which case i cannot attain a positive payoff. Hence, increasing B to $v_i - 1$ cannot hurt player i. At the same time, it is strictly beneficial if the opponents follow a strategy of bidding $v_i - 1$ upon next arrival and then not bid again.

Placing a bid $b \leq v_i - 3$ is the unique best response to a strategy of the opponents which prescribe bidding b upon the next arrival if that happens at $t < \varepsilon$ and overbidding incrementally if $t < \varepsilon$, provided that $\varepsilon < 0$ is close enough to 0. Similarly, bidding $b = v_i - 1$ or $b = v_i$ are strictly better than either waiting or bidding $b < v_i - 1$ if the opponents bid $v_i - 1$ upon the next arrival and then restrains from further bidding, and they are both strictly better than bidding $p > v_i$ if the opponents bid $v_i + 1$ upon next arrival. Furthermore, bidding $v_i - 1$ or v_i (and not placing bids above v_i afterwards) yield the same payoff to i for any strategy of the opponent, hence neither of them is weakly dominated by the other one.

Clearly, not placing a bid when $P \geq v_i - 1$ is not weakly dominated, as placing a bid cannot increase i's payoff for any realization of the arrival process and any strategy followed by the others. If $P < v_i - 1$ and the strategy of others is not placing another bid until i places a new bid, but bid v_i afterwards then not placing a bid then not placing a bid is the unique optimal action for $t < t^*$. For $t > t^*$ not placing a bid is weakly dominated by incremental bidding, as for $B \geq v_i - 1$ they yield the same payoff, while for $B < v_i - 1$ incremental bidding yields a strictly higher expected payoff for any strategy of the other bidders. QED

6.2 Proof of Proposition 1 (n-bidders)

To simplify notation, let $m = n + 2$ where $n \geq 0$. Bidder value functions are given by,

$$W(b_k, t) = \int_t^0 \lambda e^{-(m+1)\lambda(\tau - t)}(m+1)L(b_{k+1}, \tau)d\tau + (v - b_{k-1} + 1)e^{(m+1)\lambda t}$$

$$L(b_k, t) = \int_t^0 \lambda e^{-(m+1)\lambda(\tau - t)} \left(W(b_{k+1}, \tau) + mL(b_{k+1}, \tau)\right) d\tau$$

We construct t^* as follows,

$$
\begin{aligned}
&W(b_k,t) - L(b_{k-1},t) \\
&= \int_t^0 \lambda e^{-(m+1)\lambda(\tau-t)} \left((m+1)L(b_{k+1},\tau) - mL(b_k,\tau) - W(b_k,\tau)\right) d\tau + (v - b_{k-1}+1)e^{(m+1)\lambda t} \\
&> -\int_t^0 \lambda e^{-(m+1)\lambda(\tau-t)} \left(m(L(b_{k+1},\tau) - L(b_k,\tau)) + W(b_k,\tau)\right) d\tau + (v - b_{k-1}+1)e^{(m+1)\lambda t} \\
&= \frac{\left(e^{(m+1)\lambda t}-1\right)\left(m(b_{k+1}-b_k) + (v-b_{k-1}+1)\right)}{m+1} + (v-b_{k-1}+1)e^{(m+1)\lambda t} \\
&\geq 0
\end{aligned}
$$

for all t,

$$
t_k = \frac{\log\left(\frac{m(b_{k+1}-b_k)+(v-b_{k-1}+1)}{m(b_{k+1}-b_k)+2(v-b_{k-1}+1)+m(v-b_{k-1}+1)}\right)}{m+1} \leq t < 0
$$

Let $t^* = \max_k t_k$ and we have $W(b_k,t) - L(b_{k-1},t) > 0$ for $t > t^*$. A straightforward induction argument proves that the other incentive constraints, $L(b_{k-1},t) - L(b_k,t) > 0$ also hold for $t > t^*$.

6.3 Proof of Proposition 2

We construct the equilibrium cutoff points such that if we restrict bids to be incremental, i.e. the set of available bids only includes $p+1$ for $p < v-3$ and v for $p \geq v-3$, partial incremental strategies that follow these cutoffs form an equilibrium. We will then relax the restriction on bids and show that bidders never prefer to overbid.

Let $\tau_{v-3} = \tau_{v-5} ... = -\infty$. We construct a sequence $0 > \tau_{v-2} > \tau_{v-4} > ... > -\infty$ such that for all k even

$$
W(v-k,t) \quad \begin{cases} \leq L(v-k-1,t) & \text{if } t < \tau_{v-k}. \\ \geq L(v-k-1,t) & \text{if } t \geq \tau_{v-k}. \end{cases} \tag{3}
$$

and for all $0 < p < v-k$,

$$
L(p,t) \geq L(p+1,t) \quad \text{if } t \geq \tau_{v-k} \tag{4}
$$
$$
W(p,t) > L(p-1,t) \quad \text{if } t \geq \tau_{v-k}, \tag{5}
$$

where $L(p,t)$ and $W(p,t)$ are the expected continuation values of the losing and winning bidder respectively, conditional on bidders following partial incremental strategies with respect to $\tau_v, \tau_{v-1}, ... \tau_0$.

$L(p,t)$ and $W(p,t)$ can be constructed recursively. $L(v,t) = W(v,t) = 0$ and for $p > 0$,

$$\begin{aligned} L(p,t) &= \int_s^0 \lambda e^{-\lambda(\tau-s)} W(p+1,\tau)d\tau \\ W(p,t) &= \int_s^0 \lambda e^{-\lambda(\tau-s)} L(p+1,\tau)d\tau + (v-p)e^{\lambda s} \\ L(0,t) &= \int_s^0 \lambda e^{-2\lambda(\tau-s)} W(1,\tau)d\tau + \int_s^0 \lambda e^{-2\lambda(\tau-s)} L(1,\tau)d\tau \end{aligned}$$

with $s = \max\{t, \tau_{p+1}\}$.

Let $\tau_{v-2} = -\frac{1}{\lambda}$. From Lemma 1 it follows that 6.1-6.3 hold for $k = 2$. Next we assume that for k even, there exists a set $\{\tau_{v-4}, ..., \tau_{v-k}\}$ such that 6.1-6.3 hold. Let τ_{v-k-2} be given by

$$\tau_{v-k-2} = \tau_{v-k} - \frac{1}{\lambda}\frac{W(v-k-2,\tau_{v-k}) - L(v-k-3,\tau_{v-k})}{W(v-k-2,\tau_{v-k}) - L(v-k-1,\tau_{v-k})}$$

By assumption $W(v-k-2,\tau_{v-k}) > L(v-k-3,\tau_{v-k})$ and $W(v-k-2,\tau_{v-k}) > L(v-k-1,\tau_{v-k})$, which ensures that $\tau_{v-k-2} < \tau_{v-k}$. Let $\tau_{v-k-2} < t < \tau_{v-k}$. By construction, $W(v-k-1,t) = W(v-k-1,\tau_{v-k})$ and by assumption $W(v-k-1,\tau_{v-k}) \geq L(v-k-2,\tau_{v-k})$ and hence $W(v-k-1,t) \geq L(v-k-2,\tau_{v-k})$.

$$\begin{aligned} W(v-k-2,t) = &\left(\int_t^{\tau_{v-k}} \lambda e^{-\lambda(\tau-t)}d\tau\right) L(v-k-1,\tau_{v-k}) + \left(1 - \int_t^{\tau_{v-k}} \lambda e^{-\lambda(\tau-s)}d\tau\right) W(v-k-2,\tau_{v-k}) \\ = &\left(1 - e^{-\lambda(\tau_{v-k}-t)}\right) L(v-k-1,\tau_{v-k}) + \left(e^{-\lambda(\tau_{v-k}-t)}\right) W(v-k-2,\tau_{v-k}) \\ = &L(v-k-1,\tau_{v-k}) + \lambda e^{-\lambda(\tau_{v-k}-t)} \left(L(v-k-1,\tau_{v-k}) - W(v-k-2,\tau_{v-k})\right) \end{aligned}$$

which we can rewrite as

$$W(v-k-2,t) - L(v-k-1,\tau_{v-k}) = e^{-\lambda(\tau_{v-k}-t)} \left(L(v-k-1,\tau_{v-k}) - W(v-k-2,\tau_{v-k})\right)$$

$$\begin{aligned} &L(v-k-3,t) - W(v-k-2,t) \\ &\quad= \int_t^{\tau_{v-k}} \lambda e^{-\lambda(\tau-t)} (W(v-k-2,\tau) - L(v-k-1,\tau_{v-k}))d\tau \\ &\quad\quad+ \left(1 - \int_t^{\tau_{v-k}} \lambda e^{-\lambda(\tau-s)}d\tau\right) \left(L(v-k-3,\tau_{v-k}) - W(v-k-2,\tau_{v-k})\right) \\ &\quad= \int_t^{\tau_{v-k}} \lambda e^{-\lambda(\tau-t)} (\lambda e^{-\lambda(\tau_{v-k}-\tau)} \left(L(v-k-1,\tau_{v-k}) - W(v-k-2,\tau_{v-k})\right))d\tau \\ &\quad\quad+ \left(1 - \int_t^{\tau_{v-k}} \lambda e^{-\lambda(\tau-s)}d\tau\right) \left(L(v-k-3,\tau_{v-k}) - W(v-k-2,\tau_{v-k})\right) \\ &\quad= \lambda(\tau_{v-k} - t)\left(e^{-\lambda(\tau_{v-k}-t)}\right) \left(W(v-k-2,\tau_{v-k}) - L(v-k-1,\tau_{v-k})\right) \\ &\quad\quad- \left(e^{-\lambda(\tau_{v-k}-t)}\right) \left(W(v-k-2,\tau_{v-k}) - L(v-k-3,\tau_{v-k})\right) \end{aligned}$$

Hence, $L(v-k-3,\tau_{v-k-2})-W(v-k-2,\tau_{v-k-2})=0$ and because $L(v-k-3,\tau_{v-k-2})-W(v-k-2,\tau_{v-k-2})$ is strictly increasing in t, $W(v-k-2,\tau_{v-k-2}) \geq L(v-k-3,\tau_{v-k-2})$ iff $t \geq \tau_{v-k-2}$.

To show that overbidding is suboptimal will require the following property of the equilibrium value functions.

Property 1. For all integers $k > 0$ and $t \leq t_1$

$$L(v-k-1,t)-L(v-k,t) \geq \frac{1}{e}\left(1+\frac{1}{3!}+\frac{1}{5!}+...+\frac{1}{(\bar{k}-2)!}\right)$$

$$W(v-k-1,t)-W(v-k,t) \leq \frac{1}{e}\left(1+\frac{1}{2!}+\frac{1}{4!}+...+\frac{1}{(\tilde{k}-2)!}\right)$$

where $\tilde{k}=k$ if k even and $k-1$ otherwise, $\bar{k}=k$ if k odd and $k-1$ otherwise. Equality holds when $t \leq t_1$.

Proof. Note that $L(v-k-1,t)-L(v-k,t)$ and $W(v-k-1,t)-W(v-k,t)$ are equal to the likelihoods of eventually winning at any price $p < v$ at time t conditional on a bidder losing and winning at price $v-k-1$ respectively. It follows immediately that $L(v-k-1,t)-L(v-k,t)$ is weakly decreasing in t and $(W(v-k-1,t)-W(v-k,t))$ is weakly increasing in t.

$W(v-k-1,t_1)-W(v-k,t_1)$

$$\begin{aligned} &= e^{t_1\lambda}+\int_{t_1}^{0}\lambda e^{-\lambda(\tau-t_1)}(-\tau\lambda e^{\tau\lambda})d\tau+\int_{t_1}^{0}\lambda e^{-\lambda(\tau-t_1)}(-\tau^3\lambda^3 e^{\tau^3\lambda^3})d\tau+... \\ &= \frac{1}{e}+\frac{1}{2}\left(\frac{1}{e}\right)+\frac{1}{24}\left(\frac{1}{e}\right)+... \end{aligned}$$

■

Consider the auction at time t, $T < t < 0$ with current price (and high bid) $p < v-k$ where k is the even integer satisfying $t_{k+2} < t < t_k$. If opponent strategies are given by candidate equilibrium, then a bidder strictly prefers having a winning bid of q to $q^{'}$ for any q, $q^{'}$ that satisfy $p \leq q < q^{'} \leq v$.

Case 1: $p+1 \leq q < q^{'} \leq v-(k+2)$

Given that the opponent plays according to the candidate strategies, the expected continuation value of bidding q and $q^{'}$ are

$$\int_{t}^{0}\lambda e^{-\lambda(\tau-t)}L(v-q,\tau)d\tau+(v-p-1)e^{\lambda t}$$

and

$$\int_{t}^{0}\lambda e^{-\lambda(\tau-t)}L(v-q^{'},\tau)d\tau+(v-p-1)e^{\lambda t}$$

respectively. It follows from Claim 1 that $q < q^{'}$ implies that the difference

$$\int_{t}^{0} \lambda e^{-\lambda(\tau - t)}(L(v - q, \tau) - L(v - q^{'}, \tau))d\tau > 0$$

is strictly positive.

Case 2: $p + 1 \leq q < q^{'} = v - (k + 1)$

We consider the trade-offs at time t_k. If the losing bidder does not arrive between t and t_k, any subsequent arrival by the losing bidder will result in the winning bidder being outbid. Hence the difference in expected continuation values is reflected in the different prices at which a winning bidder might become the losing bidder- with a higher bid resulting in a higher price. We can write the expected cost of 'overbidding' as follows:

$$e^{\lambda(t - t_k)} \int_{t_k}^{0} \lambda e^{-\lambda(\tau - t_k)}(L(q, \tau) - L(v - k - 1, \tau))d\tau$$

If the other bidder does arrive between t and t_k, then bidding $v - k - 1$ does not affect the current price, only the identity of the winning bidder at t_k. Hence the expected benefit of overbidding can be written as

$$(1 - e^{\lambda(t - t_k)}) \left(W(v - k - 1, t_k) - L(v - k - 1, t_k)\right)$$

It is clear that the benefit is maximized and the cost minimized at $t = t_{k+2}$ and when $q = v - k$, so it is sufficient to show that:

$$e^{\lambda(t_{k+2} - t_k)} \int_{t_k}^{0} \lambda e^{-\lambda(\tau - t_k)}(L(v - k - 1, \tau) - L(v - k, \tau))d\tau \qquad (6)$$
$$> (1 - e^{\lambda(t_{k+2} - t_k)}) \left((W(v - k - 1, t_k) - L(v - k - 1, t_k)\right)$$

iff

$$e^{\lambda(t_{k+2} - t_k)} \left[W(v - k - 2, t_k) - W(v - k - 1, t_k) - e^{\lambda t_k}\right]$$
$$> (1 - e^{\lambda(t_{k+2} - t_k)}) \left((W(v - k - 1, t_k) - L(v - k - 1, t_k)\right)$$

iff

$$e^{\lambda(t_{k+2} - t_k)} \left[W(v - k - 2, t_k) - W(v - k - 1, t_k) - e^{\lambda t_k}\right]$$
$$> (1 - e^{\lambda(t_{k+2} - t_k)}) \left((W(v - k - 1, t_k) - W(v - k, t_k)\right)$$

which we re-arrange and get,

$$e^{\lambda(t_{k+2} - t_k)} \left[W(v - k - 2, t_k) - W(v - k, t_k)\right] - e^{\lambda t_{k+2}}$$
$$> W(v - k - 1, t_k) - W(v - k, t_k)$$

We note that for real valued x, $1 + x \leq e^x$, hence it suffices to show that

$$(1+\lambda(t_{k+2}-t_k))\left[W(v-k-2,t_k)-W(v-k,t_k)\right]-e^{\lambda t_{k+2}}$$
$$> W(v-k-1,t_k)-W(v-k,t_k)$$

iff

$$W(v-k-2,\tau)-W(v-k-1,t_k) > e^{\lambda t_{k+2}}+\lambda(t_k-t_{k+2})\left[W(v-k-2,t_k)-W(v-k,t_k)\right]$$

and from above

$$\lambda(t_k-t_{k-2}) = \frac{W(v-k-2,t_k)-L(v-k-3,t_k)}{W(v-k-2,t_k)-W(v-k,t_k)}$$

therefore,

$$W(v-k-2,t_k)-W(v-k-1,t_k) >$$
$$e^{\lambda t_{k+2}}+\frac{W(v-k-2,t_k)-L(v-k-3,t_k)}{W(v-k-2,t_k)-W(v-k,t_k)}\left[W(v-k-2,t_k)-W(v-k,t_k)\right]$$

iff

$$W(v-k-2,t_k)-W(v-k-1,t_k) > e^{\lambda t_{k+2}}+W(v-k-2,t_k)-L(v-k-3,t_k)$$

iff

$$L(v-k-3,t_k)-W(v-k-1,t_k) > e^{\lambda t_{k+2}}$$

iff

$$(L(v-k-3,t_k)-L(v-k-1,t_k))-(W(v-k-1,t_k)-L(v-k-1,t_k)) > e^{\lambda t_{k+2}}$$

iff

$$(L(v-k-3,t_k)-L(v-k-2,t_k))+(L(v-k-2,t_k)-L(v-k-1,t_k))-(W(v-k-1,t_k)-W(v-k,t_k)) > e^{\lambda t_{k+2}}$$

Property 1 implies that it is sufficient that

$$\frac{2}{e}\left(1+\frac{1}{3!}+\frac{1}{5!}+...+\frac{1}{(\bar{k}-2)!}\right)-\frac{1}{e}\left(1+\frac{1}{2!}+\frac{1}{4!}+...+\frac{1}{(\bar{k}-2)!}\right) > e^{\lambda t_{k+2}}$$

or

$$1-\frac{1}{2!}+\frac{2}{3!}-\frac{1}{4!}+\frac{2}{5!}+... > e^{\lambda(t_{k+2}-t_1)}$$
$$1-\frac{1}{3!}-\frac{3}{5!}-\frac{5}{7!}... > e^{\lambda(t_{k+2}-t_1)}$$

And,

$$e^{\lambda(t_{k+2}-t_1)} < e^{\lambda(t_4-t_1)} < e^{-0.3} < \lim_{k\to\infty} 1-\frac{1}{3!}-\frac{3}{5!}-\frac{5}{7!}...-\frac{k-1}{(k+1)!}$$

Finally, it can be verified directly that (6) holds for $k = 2$.

Case 3: $v - k \leq q < q^{'}$

Let r be the smallest odd integer such that $r > v - q^{'}$. With either a bid of $q^{'}$ or $q^{'} - 1$, the bidder will be the winning bidder at time t_r. However, by the arguments made above the bidder would prefer to have a winning bid of $q^{'} - 1$ to a winning bid of $q^{'}$.

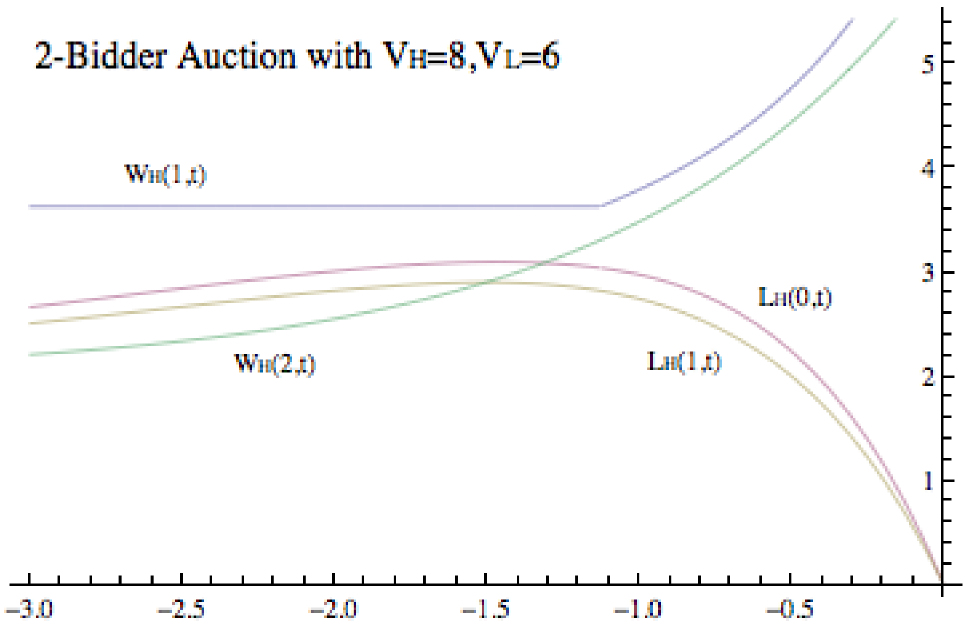

Figure 9: Expected Continuation Payoffs for the High Type Bidder

6.4 Asymmetric Values Example 2

We return to the 2-bidder case whose values are given by $v_H > v_L$ and construct an equilibrium in which both types place incremental bids. To illustrate why it is difficult to characterize these auctions more generally, we investigate an equilibrium with minimal incremental bidding by both types and demonstrate that the structure of this equilibrium depends on the values of v_H and v_L and that this dependency is difficult to analytically characterize.

In order for the high type to place an initial incremental bid of 1 in equilibrium, it must be the case that at $p = 1$, the low type's strategy calls for placing a bid of 2 beginning at some time t prior to the high type's final cutoff for $p = 1$. Low type's bidding behavior at $p = 0$ is pinned down by the high type's bidding behavior. We allow all bidding at $p \geq 2$ to be truthful. Working backwards from these assumptions, we construct the following example for $v_H = 8$ and $v_L = 6$.

Equilibrium bidding strategies are as follows; At $p = 0$, low types bid 1 for all t and high types bid 1 for $t < \tau_1^H = -1.13$ and v_H otherwise. At $p = 1$, low types bid 2 for $t > \tau_1^L = -1.4505$ and delays otherwise. High types place a bid v_H after τ_1^H and delay otherwise. At $p \geq 2$ both types bid their true value. Figures 9 and 10 plot the bidder's expected payoffs. In a sufficiently long auction, the high type bidder has an expected payoff of 2.65 and the low type bidder has an expected payoff of 1.03. The low type wins with probability 0.23 and the seller's expected revenue is only 3.86.

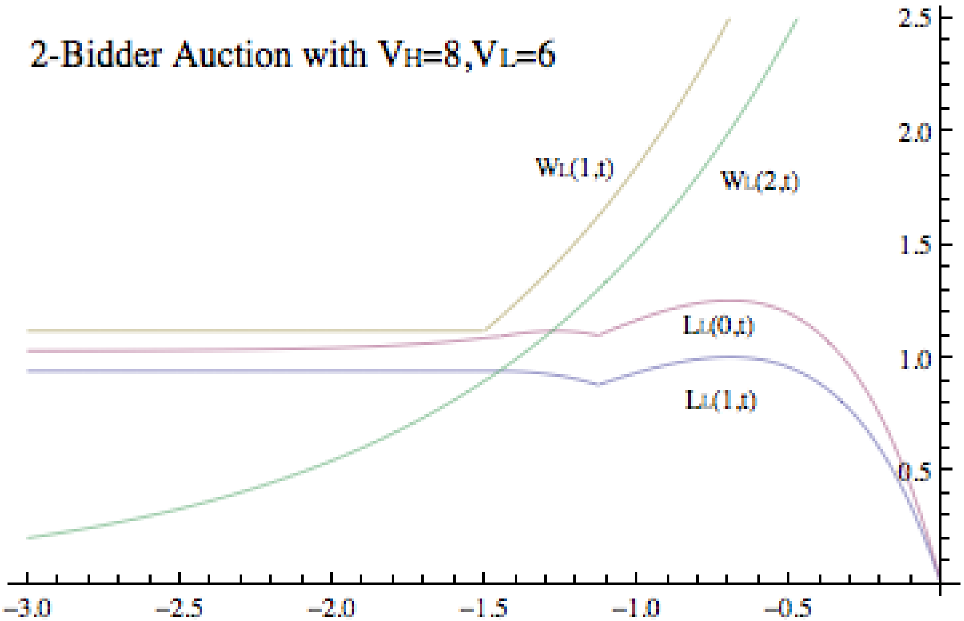

Figure 10: Expected Continuation Payoffs for the Low Type Bidder

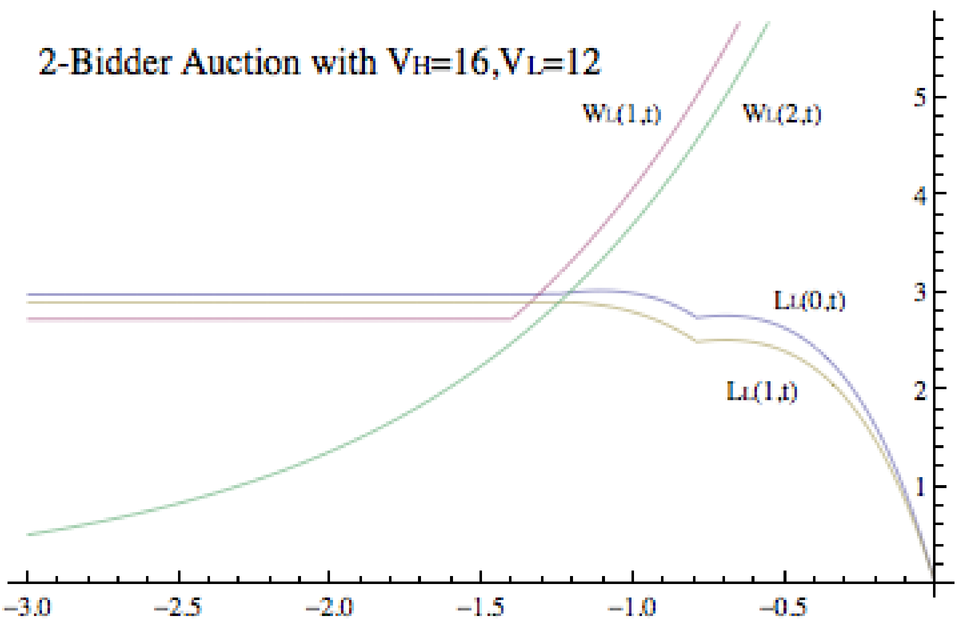

Figure 11: Expected Continuation Payoffs for the Low Type Bidder

Numerical investigations suggest to us that an equilibrium of this type, when initially both types bid incrementally, exists when $v_H - v_L$ is not too large, relative to v_L, but the condition on v_L and v_H guaranteeing existence of the equilibrium above is complicated. For example, when this ratio is increased such that $v_H = 16$ and $v_L = 8$, incremental bidding by the high type is not sustainable; in equilibrium the high type prefers a bid of v_H whenever she prefers bidding to delaying. However, holding the ratio fixed is not sufficient. Figure 11 illustrates the low types expected payoffs when both values from the original example are doubled to $v_H = 16$ and $v_L = 12$, which reveal that equilibrium strategies now involve the low type delaying at $p = 0$.

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