

Legislative Bargaining with Long Finite Horizons

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Abstract

Institutional rules provide natural deadlines for negotiations in legislative bargaining. In the continuous-time bargaining model framework of Ambrus and Lu (2010) we show that as the time horizon of the bargaining increases, equilibrium payoffs with deadline converge to stationary equilibrium payoffs of the infinite-horizon bargaining game. We provide a characterization of these limit payoffs, and show that under a K -majority rule, the payoffs of the K legislators with the lowest relative recognition probabilities have to be equal to each other. Hence, by varying recognition probabilities, possible limit equilibrium payoffs are constrained to a lower-dimensional subset of the set of all possible allocations. This contrasts with the result of Kalandrakis (2006) that in the infinite-horizon Baron and Ferejohn (1989) framework, for any discount factor, any division of the surplus can be achieved as a stationary equilibrium payoff through some choice of recognition probabilities.

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1 Introduction

The standard workhorse model of legislative bargaining, introduced in Baron and Ferejohn (1989) and used extensively in many applications, assumes an infinite-horizon environment.¹ However, in practice, legislatures face various deadlines that put an end to possible negotiations.

One firm deadline that applies to all negotiations is the end of the mandate of the legislature. There are various other institutional deadlines applying to specific negotiations, mandated by constitutional law or international organizations. These include time limits on government formation after general elections,² European Union deadlines for member countries to modify their laws to reflect a common directive,³ and deadlines imposed by constitutional courts on legislatures to resolve conflicting laws.⁴

In most of the above cases, the deadline is far away relative to how frequently legislators can make proposals and counter-proposals. However, it is not clear whether the qualitative properties of legislative bargaining with long finite horizon (relative to the frequency of proposals) are similar to those with infinite horizon. In fact, Norman (2002) provides a negative result, showing that in discrete-time legislative bargaining games, expected subgame perfect equilibrium payoffs of finite-horizon games do not converge to stationary subgame perfect equilibrium payoffs of the infinite-horizon game.

We establish a connection between finite and infinite legislative bargaining, using the continuous-time finite-horizon model framework of Ambrus and Lu (2010). In this model, players get the chance to make proposals at random times, according to a Poisson process. A higher arrival rate for a given player means that, in expectation, she can make proposals more frequently. The paper shows that this model, for general payoff specification, has a unique Markov perfect equilibrium (MPE) payoff vector.⁵

In this paper, we show that in legislative bargaining games, if the time horizon of the game relative to the arrival rates increases, continuous-time MPE

¹Applications in political science and economics include Baron (1991, 1996), Winter (1996), Chari *et al.* (1997), Diermeier and Feddersen (1998), Banks and Duggan (2000) and LeBlanc *et al.* (2000).

²In the devolved legislatures in Scotland and Wales, the First Minister must be chosen within 28 days of the election. This rule is regarded as an effective limit on the time available to the parties to form a government (see Blackburn *et al.* (2010)).

³The Council of the EU set a 19 July 2003 deadline for the legislative bodies of member states to transpose the Racial Equality Directive (see Council Directive 2000/43/EC) and a 2 December 2003 deadline to transpose the Employment Equality Directive (see Council Directive 2000/78/EC).

⁴Article 46, Paragraph 1 of Hungary's *Act XXXII of 1989 on the Constitutional Court* states: "If the Constitutional Court establishes that a law at a higher level than the law promulgating the international treaty conflicts with the international treaty, it invites the legislative organ which concluded the international treaty to resolve the conflict, setting a deadline based on the consideration of circumstances."

⁵It is also shown that any sequence of subgame perfect equilibria of discrete-time games converging in a formal sense to the continuous-time bargaining game converges to the MPE of the latter game. This provides a justification for focusing on the MPE of the continuous-time game.

payoffs converge to stationary subgame-perfect equilibrium (SSPE) payoffs of the infinite-horizon game. Simply put, if the game is long, the expected division of the surplus is close to what would be an equilibrium result in case of open-ended negotiations.

We provide a simple algorithm that obtains the above limit division of the surplus for any vector of relative arrival rates. This characterization is particularly simple when we only consider strictly positive arrival rates: if K votes are needed for accepting a proposal, it requires that the expected payoffs of the K legislators with the lowest relative arrival rates be equal to each other. Conversely, any division of the surplus in which the lowest K shares are equalized can be achieved as limit equilibrium payoffs for some vector of arrival rates. We get a slightly more complex characterization when allowing for zero arrival rates: a division of the surplus (x_1, \dots, x_n) , where $x_1 \leq \dots \leq x_n$, is feasible as a limit equilibrium payoff if and only if there is $k < K$ such that $x_1 = \dots = x_k = 0$, and $x_{k+1} = \dots = x_K$.

The results imply that with a long but finite time horizon, the set of equilibrium surplus divisions that can be attained by varying arrival rates is a lower dimensional subset of the set of all possible divisions. Moreover, the restriction that our model places on the payoff division is a particularly simple one.

This finding contrasts with the main result in Kalandrakis (2006), which shows that in a discrete infinite-horizon bargaining framework, if recognition probabilities can be varied, there is no testable implication of SSPE with respect to expected payoffs: for any discount factor, any division of the surplus can be attained by appropriately chosen proposal probabilities.⁶ We obtain a different result because there is a multiplicity of equilibria in the infinite-horizon model when there are legislators with recognition probability 0. This multiplicity disappears in a game with a long finite time horizon, where the payoffs of such legislators are pinned down to 0. We consider it an empirical question whether our modeling approach, leading to different implications regarding possible divisions of surplus than in an infinite-horizon game, is valid.

For ease of exposition, in the main text, we focus on a model in which players receive their payoffs at a pre-specified time at or after the deadline, independently of when they reach an agreement. For example, this pre-specified time can represent the beginning of a new fiscal year. In this version of the model, the urgency to reach an agreement stems only from the approaching deadline and the randomness of proposal opportunities. In Appendix B, we extend the results to the more standard case where players divide the surplus at the time of reaching an agreement. In this case, we show that the set of limit equilibria is exactly the same as in the previous model if the time horizon of the game goes to infinity and the discount rate of the legislators goes to 0.

⁶Kalandrakis (2006) establishes this not only for quota rules, but also for a large class of voting rules that satisfy a monotonicity requirement.

2 The model

Our model of legislative bargaining builds on the framework introduced in Ambrus and Lu (2010). In particular, we consider a set of players $N = \{1, 2, \dots, n\}$, and a value function $V : 2^N \rightarrow \mathbb{R}_+$ with the property that there is $K > \frac{n}{2}$ such that for every $C \subseteq N$, $V(C) = 1$ if $|C| \geq K$, and $V(C) = 0$ otherwise. We refer to elements of 2^N as *coalitions*. Coalitions consisting of at least K members can pass a legislation regarding how to distribute a pie of size 1 among different members. We refer to K as the quota required to pass a legislation.

The noncooperative bargaining game we investigate in this setting is defined as follows. The game is set in continuous time, starting at $T < 0$. There is a Poisson arrival process associated with each player i , with arrival rate $\lambda_i \geq 0$. The processes are independent from each other. For future reference, we define $\lambda \equiv \sum_{i=1}^n \lambda_i$. Whenever the process realizes for a player i , she can make an offer $x = (x_1, x_2, \dots, x_n)$ to a coalition $C \subseteq N$ satisfying $i \in C$. The offer x must have the following characteristics:

1. $x_j \geq 0$ for all $1 \leq j \leq n$;
2. $\sum_{j=1}^n x_j \leq V(C)$.

Players in $C \setminus \{i\}$ immediately and sequentially accept or reject the offer (the order in which they do so turns out to be unimportant). If everyone accepts, the game ends, and all players in N are paid their shares according to x . If an offer is rejected by at least one of the respondents, it is taken off the table, and the game continues with the same Poisson arrival rates. If no offer has been accepted at time 0, the game ends, and all players receive payoff 0.

We assume that the payoffs are received at time 0, even if an agreement is reached at time $t < 0$. See Appendix B for how the results extend to the model in which players receive their payoffs immediately after an agreement was reached.

We are interested in MPE of the resulting game, that is subgame-perfect equilibria in which a proposer's strategy only depends on how much time is left before the deadline, and a respondent's strategy only depends on the offer on the table and previous responses to the proposal.

3 Results

For ease of exposition, we assume in this section that all players have strictly positive arrival rates: $\lambda_i > 0$ for all i . Extending the results to the case where some of the arrival rates can be zero is straightforward and discussed in the next section.

Our first result establishes that if a player has a higher arrival rate (can propose more frequently in expectation) than another player, then the former player's continuation value is always weakly above the latter one's. For the proofs of all formal results, see Appendix A.

Claim 1: If $\lambda_i \leq \lambda_{i'}$, then $w_i(t) \leq w_{i'}(t) \forall t \leq 0$.

We show below that the limit expected MPE payoffs as the time horizon goes to infinity converge to the unique SSPE payoff vector of the infinite-horizon game, and that these limits can be obtained by a simple procedure, which we present next.⁷

Without loss of generality, assume that players are ordered such that $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$, and that $\lambda = 1$ (the latter can be obtained by renormalizing time).

For every $j \in \{1, \dots, n\}$ define $\underline{x}^j = (x_1^j, \dots, x_n^j)$ as follows:

Let x be the solution to the equation $jx + (1 - (K - 1)x) \sum_{i=j+1}^n \lambda_i = 1$.

Then for every $i \in \{1, \dots, j\}$ let $x_i^j = x$, and for every $i \in \{j + 1, \dots, n\}$ let $x_i^j = (1 - (K - 1)x)\lambda_i$. If $j \geq K$ and players $1, \dots, j$ are tied for the lowest payoff x , \underline{x}^j is the stationary expected payoff vector with no discounting: players $j + 1, \dots, n$ would receive payoff $1 - (K - 1)x$ if they are the first to get a chance to propose, and would not be approached when another player proposes.

Let j^* be the smallest $j \in \{1, \dots, n\}$ such that $x_{j+1}^j \geq x_j^j$ (assume this holds trivially for $j = n$). It is easy to verify that $j \geq K$. Note that by Claim 1, payoffs (x_1^j, \dots, x_n^j) for $j < j^*$ cannot be candidates for limit MPE payoffs.

To simplify notation, let $x_i^* = x_i^{j^*} \forall i \in \{1, \dots, n\}$.

Theorem 1: $\lim_{T \rightarrow -\infty} w(T) = (x_1^*, \dots, x_n^*)$, the SSPE payoff vector of the infinite-horizon game. Moreover, there exists $T^* < 0$ such that for all $T < T^*$, $w_i(T) = w_{i'}(T) \forall i, i' \in \{1, \dots, j^*\}$.

The above theorem implies that if the deadline is sufficiently far away, then the continuation values of the j^* players (where $j^* \geq K$) with the lowest arrival rates are equal, and each of these players is approached with positive probability.⁸ The continuation values of the rest of the players are greater, and ordered according to the relative arrival rates. Far away from the deadline, these players are not approached by any other player.

The proof of Theorem 1 first notes that far enough away from the deadline, the K players with the lowest arrival rates must have equal continuation values. This is easy to see when considering stationary payoffs: if fewer than K players are tied for the lowest payoff, then player 1 would always be approached when another player has an arrival (and thus receive her continuation payoff), but she would also get more than her continuation payoff when she receives an arrival, a contradiction. The proof also shows that if the q weakest legislators' values (where $q > K$) are equal at some $t \neq 0$, then at all times before t , they are equal as well (and decreasing as time moves toward the deadline). As a result,

⁷Eraslan (2002) shows the uniqueness of stationary SPNE payoffs in the Baron and Ferejohn (1989) legislative bargaining model. The same techniques can be used in our continuous-time framework to show the uniqueness of stationary SPNE payoffs in the infinite-horizon version of the model. We skip the details here.

⁸Not with equal probability though. Players with lower arrival rates are approached by others more frequently - this is what keeps the continuation values of the j^* players equal.

the number of legislators tied for the lowest continuation value must converge as $t \rightarrow -\infty$, and therefore every player's payoff converges as well.

We conclude this section by observing that taking the time horizon of the game to infinity, given a fixed vector of arrival rates, is equivalent to scaling up arrival rates to infinity, for a fixed time horizon. Therefore, our result concerning limit payoffs applies to the case when the time horizon is fixed but legislators can make proposals more and more frequently.

Observation: Under a rescaling of time, for any $\alpha > 0$, a game with arrival rate vector and time horizon $(\lambda, \alpha T)$ is equivalent to a game with corresponding parameters $(\alpha\lambda, T)$.

Corollary: Let $\vec{\lambda}$ be a vector in \mathbb{R}_{++}^2 , and consider games with arrival rate vector $c\vec{\lambda}$. Then for any $T < 0$, $\lim_{c \rightarrow \infty} w(T) = (x_1^*, \dots, x_n^*)$, the SSPE payoff vector of the infinite-horizon game. Moreover, there exists $c^* > 0$ such that for all $c > c^*$, $w_i(T) = w_{i'}(T) \forall i, i' \in \{1, \dots, j^*\}$.

4 Discussion

4.1 Zero arrival rates

In our model, $\lambda_i = 0$ for some $i \in \{1, \dots, n\}$ implies that independently of T , in every MPE, other players offer 0 to i any time they can make a proposal. For this reason, and because i never proposes, $w_i(t) = 0$ for all t . Assume now that $m > 0$ of the players have zero arrival rates. If $m < K - 1$, the payoff functions of players $m + 1, \dots, n$ are the same as in a game without players $1, \dots, m$, and with quota $K - m$. Hence the results of the previous section can be directly applied. If $m \geq K - 1$, the game is trivial: upon the first arrival, which can only happen for a player $i > m$, player i approaches $K - 1$ of players in $\{1, \dots, m\}$ and proposes 1 for herself. The limit expected payoffs as $T \rightarrow \infty$ are then $(0, \dots, 0, \lambda_{m+1}, \dots, \lambda_n)$.

4.2 Possible surplus divisions

Theorem 1, together with the observations regarding zero arrival rates made in the previous subsection, imply that the main result of Kalandrakis (2006), namely that equilibrium expected payoffs are not restricted in legislative bargaining once the bargaining protocol (probabilities of being selected as a proposer) can be freely chosen, does not hold in our model.⁹ In particular, let (x_1, \dots, x_n) denote limit expected payoffs in (weakly) increasing order, as $T \rightarrow \infty$, given arrival rate vector λ . Then our results imply that there is $k \in \{1, \dots, K\}$ such that $x_i = 0$ for every $i < k$, and $x_i = x_{i'}$ for every $i, i' \in \{k + 1, \dots, K\}$.

⁹See also Eraslan (2002), Snyder *et al.* (2005) and Montero (2006) for further investigations of stationary SPNE payoffs in discrete-time legislative bargaining games.

The set of payoff divisions satisfying these restrictions is a lower dimensional subset of the set of all possible payoff divisions, which is the unit simplex in \mathbb{R}^n .

The root of this discrepancy lies in the potential presence of players with zero proposal probability in an infinite-horizon versus a finite-horizon model. For example, in 3-player legislative bargaining with equal weights and simple majority rule, the payoff division $(0.5, 0.3, 0.2)$ cannot be achieved as the limit of SPNE payoffs as the time horizon goes to infinity, given our results. This payoff vector can only be achieved in the Baron and Ferejohn model as a stationary equilibrium payoff if one of the proposal probabilities is 0. However, in our model, if some player has a 0 arrival rate, then her payoff has to be 0, and if all players have positive arrival rates, then the two lowest limit payoffs have to be equal.

4.3 Shorter time horizons

Although, in this paper, we focus on the case when the time horizon of negotiations is long relative to the expected frequency of proposals, here we make a few simple observations concerning shorter time horizons.

First, close to the deadline, expected payoffs are low, as there is a high probability that the deadline is reached. Moreover, payoffs are approximately proportional to the arrival rates: if the surplus that a proposer has to offer to other players is low, the relative magnitudes of payoffs are dominated by the relative magnitudes of arrival rates.

Second, we point out that Claim 1 applies to any time horizon. That is, the expected payoff of a legislator with higher arrival rate is at least weakly higher, no matter how much time is left before the deadline.

Finally, we provide an example showing that a strong legislator's payoff might be nonmonotonic in the time remaining before the deadline.

Consider a legislative bargaining game with $n = 3$ and $K = 2$. Suppose also that $\lambda_1 = 0.15$, $\lambda_2 = 0.25$, and $\lambda_3 = 0.6$. As Figure 1 shows, close to the deadline, $w_1 < w_2 < w_3$. In this region, both player 2 and player 3, upon arrival, form a coalition with player 1. At the same time, player 2 is only approached by player 1, not by player 3. Ultimately, this effect dominates the fact that player 2 has a higher arrival rate, and at some point the expected MPE payoff of player 1 becomes equal to that of player 2. Before this time, player 3 approaches both player 1 and player 2 with probability less than 1, in a way that keeps their continuation values equal. The expected continuation payoff of player 3, although she is never approached by any of the other players, remains bounded away from the other players' continuation payoffs, no matter how long the time horizon of the game is. This is due to the high relative arrival rate of player 3, and to the proposer surplus staying bounded away from 0 in this game.

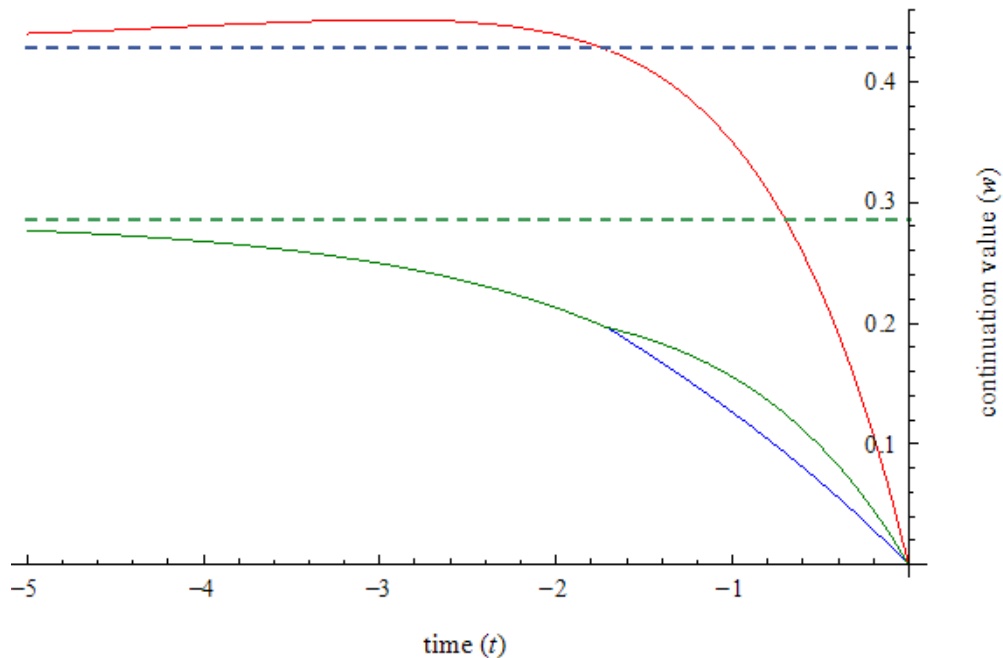


Figure 1.

Note that player 3's continuation payoff is strictly higher for intermediate values of T than in the limit as $T \rightarrow -\infty$. Therefore, if player 3 is the member of the legislature who can set a deadline for negotiations, she would choose an intermediate time horizon even if she were arbitrarily patient. The reason why a strong player might prefer a shorter deadline is that the expected payoffs of weak players increase in the time horizon of negotiations (in fact, weak players always prefer longer deadlines). Therefore, if the deadline is too far out, weak players have to be offered a relatively high share of the surplus.

5 Conclusion

In this paper, we find that in legislative bargaining with a deadline, the relative strength of weak legislators is only payoff-relevant when the deadline is looming. When the deadline is far away, their payoffs are equal. This implies a strong and potentially testable restriction on the set of possible equilibrium payoff divisions for different recognition probabilities.

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7 Appendix A: Proofs

Proof of Claim 1: Claim 3 in Ambrus and Lu (2010) implies that at any t in MPE, any player j proposes to the cheapest coalition of size K that includes her. Suppose $w_i(t) > w_{i'}(t)$. Then if any player $j \in N \setminus \{i, i'\}$ approaches i with positive probability, she must approach i' with probability 1. Moreover, if i' approaches i with positive probability, then i approaches i' with probability 1. Since $\lambda_i \leq \lambda_{i'}$, the probability of being included in a proposal at time t is weakly greater for i' than for i .

Also, $w_i(t) > w_{i'}(t)$ implies that the surplus from proposing for i' ($\max_{D \ni i'} \{V(D) - \sum_{j \in D} w_j(t)\}$) is weakly greater than for i .

Combining the above two facts, we have that $w_i(t) - w_{i'}(t)$ must weakly decrease, traveling back in time (away from the deadline), whenever it is positive. Since continuation value functions are continuous, and $w_i(0) = w_{i'}(0)$, we conclude that such t cannot exist. QED

Proof of Theorem 1: First, we establish the following lemma:

Lemma 1: There exists a time $t' \leq 0$ such that $w_i(t') = w_{i'}(t') \forall i, i' \in \{1, \dots, K\}$.

Proof: Note that if there is no t' as in the lemma, then at any time, player 1 is approached by every player with probability 1 at all times. Let $\hat{w}_j(t) = \min_{C \subset N \setminus \{j\}: |C|=K-1} \sum_{i \in C} w_i(t) + w_j(t)$. Since player 1 always has the lowest continuation value by Claim 1, $\hat{w}_1(t) = \min_{C \subset N: |C|=K} \sum_{i \in C} w_i(t) \leq \frac{K}{n}$. Thus $w_1(t)$ is bounded below by $W(t) < 1$, where $W'(t) = -\lambda_1(1 - \frac{K}{n}) < 0$. But then for t far enough away from the deadline, $W(t) < 1$ has to be violated. QED

Proof of Theorem: Suppose first that $\lambda_1 < \lambda_K$. Let $t^1 = \max_{t < 0} \{t | w_i(t) = w_{i'}(t) \forall i, i' \in \{1, \dots, K\}\}$. By Lemma 1 and the continuity of continuation values, t^1 is well-defined. Define j^1 such that $w_i(t^1) = w_{i'}(t^1) \forall i, i' \in \{1, \dots, j^1\}$, and $w_{j^1}(t^1) < w_{j^1+1}(t^1)$ (if $j^1 < N$). Define $w^*(t^1) \equiv w_1(t^1) = \dots = w_{j^1}(t^1)$. Note that since $j^1 \geq K$, every player at t^1 only approaches players in $\{1, \dots, j^1\}$, approaches exactly $K - 1$ of them, and offers $w^*(t^1)$ to each of them.

Consider now the auxiliary continuation payoff path $w^a = (w_1^a, \dots, w_{j^1}^a)$, which is the unique solution of the differential equation $\frac{\partial w_i^a(t)}{\partial t} = -\frac{1}{j^1} [w^a(K - 1) \sum_{i > j^1} \lambda_i + \sum_{i \leq j^1} \lambda_i] + \lambda w^a$ with terminal condition $w_i^a(0) = 0 \forall i \in \{1, \dots, j^1\}$. It corresponds to a payoff path resulting from all players at any time approaching players in $\{1, \dots, j^1\}$ in a way that keeps the payoff of all players in $\{1, \dots, j^1\}$ the same. This path is not necessarily feasible in the sense that in order to keep payoffs within $\{1, \dots, j^1\}$ the same, player 1 may need to be approached more frequently than if everyone approached her with probability 1, in order to keep her on par with player j^1 . At the same time, along this path, the continuation payoff of j^1 might need to grow more slowly than if all players

approached her with 0 probability. That is, along the auxiliary path, we ignore the constraint that at any point of time, the probability with which a player approaches another one has to be between 0 and 1. It is easy to see that $w_1^a(t) = \dots = w_{j^1}^a(t)$ is continuous and strictly monotonically decreasing in t , and converges to a limit higher than $w_1(t^1)$ (and therefore $w^*(t^1)$) as $t \rightarrow -\infty$. Moreover, as t decreases, the frequency at which player 1 (resp. player j^1) needs to be approached in order to keep her at the same continuation payoff as other players in $\{1, \dots, j^1\}$ is decreasing (resp. increasing), since the payoff of any proposer at t is $1 - (K - 1)w_1^a(t)$, which is strictly increasing in t (implying that players who propose more frequently have a greater advantage later in the game).

Note that for all $\varepsilon > 0$, there must be some interval within $(t_1, t_1 + \varepsilon)$ where $w_{j^1}(t) - w_1(t)$ increases in MPE. All the proposal probabilities must of course be feasible in MPE, so at t_1 , there must exist a set of feasible proposal probabilities keeping w_{j^1} and w_1 (and thus also w_2, \dots, w_{j^1-1}) the same.

Consider now t^a such that $w_1^a(t^a) = \dots = w_{j^1}^a(t^a) = w^*(t^1)$. Since $w_1(t^1) = \dots = w_{j^1}(t^1) = w^*(t^1)$, along the auxiliary path at t^a , all players have to be approached with feasible probabilities by every player. As noted earlier, maintaining $w_1^a = \dots = w_{j^1}^a$ becomes progressively easier as we move back in time, so the auxiliary path employs feasible probabilities to the left of t^a . Thus, there is an MPE of the original game for which there is an interval on the left of t^1 where all players propose only to players in $\{1, \dots, j^1\}$ in a way that keeps these players' continuation payoffs equal throughout the interval. Moreover, over this interval, continuation values $w_1(t) = \dots = w_{j^1}(t)$ are strictly decreasing in t . By Theorem 2 of Ambrus and Lu (2010), all MPEs have the same continuation value functions over the interval.

Now suppose that $\lambda_1 = \lambda_i < \lambda_{i+1}$, where $i \geq K$. In this case, let $j^1 = i$. Clearly, there is an interval on the left of 0 where all players propose only to players in $\{1, \dots, j^1\}$. Moreover, over this interval, continuation values $w_1(t) = \dots = w_{j^1}(t)$ are strictly decreasing in t . As above, by Theorem 2 of Ambrus and Lu (2010), all MPEs have the same continuation value functions over the interval.

Case 1: $w_1(t) = \dots = w_{j^1}(t)$ is strictly below $w_{j^1+1}(t) \forall t < t^1$. Then players $\{j^1 + 1, \dots, n\}$ are never approached in MPE. Since $w_1(t) = \dots = w_{j^1}(t)$ is monotonic in t and bounded within the interval $[0, 1]$, it converges as $t \rightarrow -\infty$. Since at any $t < t^1$, a proposer has to offer $(K - 1)w_1(t)$ to other players, the fact that players in $\{j^1 + 1, \dots, n\}$ are never approached by any player and that $w_1(t)$ converges as $t \rightarrow -\infty$ together imply that $w_i(t)$ converges for all $i \in \{j^1 + 1, \dots, n\}$ as well. Note that if there is an arrival at any time t , independently of the proposal, the value of the approached coalition is 1, and the proposal is accepted with probability 1. Hence, as $t \rightarrow -\infty$, $\sum_{i \in N} w_i \rightarrow 1$.

Let x_i^j and j^* be defined as in the main text. The above results imply that the limit of continuation values as $t \rightarrow -\infty$ is equal to $(x_1^{j^1}, x_2^{j^1}, \dots, x_n^{j^1})$. The

definition of j^* together with Claim 1 then imply $j^1 \geq j^*$. Now we show that $j^1 > j^*$ is not possible (unless $x_i^{j^1} = x_i^{j^*} \equiv x_i^*$ for all i). Suppose $j^1 > j^*$. Along an auxiliary path where all players at all times approach only players in $\{1, \dots, j^*\}$ in a way that keeps the latter players' payoffs equal, the payoff of any player $i \in \{1, \dots, j^*\}$ has to converge to a limit that is weakly greater than her limit MPE payoff $x_i^{j^1}$ (since they are being approached at least as often as in the MPE at all times). But since $j^1 > j^*$, $x_i^{j^1} \geq x_i^*$ for all $i \in \{1, \dots, j^*\}$, which means that $x_i^{j^1} = x_i^*$ for all $i \in \{1, \dots, j^*\}$, and therefore for all i .

Case 2: At some time $t < t^1$, $w_1(t) = \dots = w_{j^1}(t) = w_{j^1+1}(t)$. Let $t^2 = \max_{t < t^1} \{t \mid w_{j^1}(t) = w_{j^1+1}(t)\}$ and let j^2 such that $j^2 = \max_{i \in N} \{i \mid w_{j^1}(t) = w_i(t)\}$. Analogous arguments as above establish that there is an interval on the left of t^2 on which $w_1 = \dots = w_{j^2}$, and $w_1(t) = \dots = w_{j^2}(t)$ is strictly decreasing in t .

Since n is finite, continuing the same argument establishes that there are $t^k < \dots < t^1 \leq 0$ (for some $k \geq 1$) and $j^k > \dots > j^1 \geq K$ such that on (∞, t^k) , $w_1 = \dots = w_{j^k}$, $w_1(t) = \dots = w_{j^k}(t)$ is strictly decreasing in t , and that $w_i \rightarrow x_i^*$ as $t \rightarrow -\infty$. QED

8 Appendix B: Incorporating endogenous timing of surplus division and discounting

Consider the modification of the model mentioned in Section 2: if a proposal is accepted at time t , players divide the surplus according to the proposal right at t . Assume also that all legislators discount future payoffs using discount rate r .

First we note that Claim 1 applies to the new setting, with exactly the same proof as before.

Next, we show that the set of possible equilibrium payoff vectors in the limit when both the time horizon is taken to infinity and the legislators become infinitely patient is exactly the same as characterized in Theorem 1.

Theorem 2: There exists $\bar{r} > 0$ such that $r \in (0, \bar{r})$ implies that expected MPE payoffs, as $T \rightarrow \infty$, converge to a stationary SPNE payoff vector of the infinite-horizon game. Moreover, if $\lambda_i > 0$ for all i , $\lim_{r \searrow 0} \lim_{T \rightarrow \infty} w(-T) = (x_1^*, \dots, x_n^*)$, and there exist $r^* > 0$ and $t^* < 0$ such that $r \in (0, r^*)$ and $t < t^*$ imply $w_i(t) = w_{i'}(t) \forall i, i' \in \{1, \dots, j^*\}$.

Lemma 2: If $r < \lambda_1(1 - \frac{K}{n})$, there exists a time $t' \leq 0$ such that $w_i(t') = w_{i'}(t') \forall i, i' \in \{1, \dots, K\}$.

Proof: Note that if there is no t' as in the lemma, then at any time, player 1 is approached by every player with probability 1 at all times. Let $\hat{w}_j(t) = \min_{C \subset N \setminus \{j\}: |C|=K-1} \sum_{i \in C} w_i(t) + w_j(t)$. Since player 1 always has the lowest continuation value by Claim 1, $\hat{w}_1(t) = \min_{C \subset N: |C|=K} \sum_{i \in C} w_i(t) \leq \frac{K}{n}$. Thus $w_1(t)$ is bounded below by $W(t) < 1$, where $W'(t) = -\lambda_1(1 - \frac{K}{n}) + rW(t) < 0$. But then for t far enough away from the deadline, $W(t) < 1$ has to be violated. QED

Proof of Theorem: Suppose first that $\lambda_1 < \lambda_K$. Let $t^1 = \max_{t < 0} \{t | w_i(t) = w_{i'}(t) \forall i, i' \in \{1, \dots, K\}\}$. By Lemma 2 and the continuity of continuation values, t^1 is well-defined for low enough r . Define j^1 such that $w_i(t^1) = w_{i'}(t^1) \forall i, i' \in \{1, \dots, j^1\}$, and $w_{j^1}(t^1) < w_{j^1+1}(t^1)$ (if $j^1 < N$). Define $w^*(t^1) \equiv w_1(t^1) = \dots = w_{j^1}(t^1)$. Note that since $j^1 \geq K$, every player at t^1 only approaches players in $\{1, \dots, j^1\}$, approaches exactly $K - 1$ of them, and offers $w^*(t^1)$ to each of them.

Consider now the auxiliary continuation payoff path $w^a = (w_1^a, \dots, w_{j^1}^a)$, which is the unique solution of the differential equation $\frac{\partial w_i^a(t)}{\partial t} = -\frac{1}{j^1} [w^a(K - 1) \sum_{i > j^1} \lambda_i + \sum_{i \leq j^1} \lambda_i] + (\lambda + r)w^a$ with terminal condition $w_i^a(0) = 0 \forall i \in \{1, \dots, j^1\}$.

It corresponds to a payoff path resulting from all players at any time approaching players in $\{1, \dots, j^1\}$ in a way that keeps the payoff of all players in $\{1, \dots, j^1\}$ the same. This path is not necessarily feasible in the sense that in order to keep payoffs within $\{1, \dots, j^1\}$ the same, player 1 may need to be approached

more frequently than if everyone approached her with probability 1, in order to keep her on par with player j^1 . At the same time, along this path, the continuation payoff of j^1 might need to grow more slowly than if all players approached her with 0 probability. That is, along the auxiliary path, we ignore the constraint that at any point of time, the probability with which a player approaches another one has to be between 0 and 1. It is easy to see that $w_1^a(t) = \dots = w_{j^1}^a(t)$ is continuous and strictly monotonically decreasing in t , and converges to a limit higher than $w_1(t^1)$ (and therefore $w^*(t^1)$) as $t \rightarrow -\infty$. Moreover, as t decreases, the frequency at which player 1 (resp. player j^1) needs to be approached in order to keep her at the same continuation payoff as other players in $\{1, \dots, j^1\}$ is decreasing (resp. increasing), since the payoff of any proposer at t is $1 - (K - 1)w_1^a(t)$, which is strictly increasing in t (implying that players who propose more frequently have a greater advantage later in the game).

Note that for all $\varepsilon > 0$, there must be some interval within $(t_1, t_1 + \varepsilon)$ where $w_{j^1}(t) - w_1(t)$ increases in MPE. All the proposal probabilities must of course be feasible in MPE, so at t_1 , there must exist a set of feasible proposal probabilities keeping w_{j^1} and w_1 (and thus also w_2, \dots, w_{j^1-1}) the same.

Consider now t^a such that $w_1^a(t^a) = \dots = w_{j^1}^a(t^a) = w^*(t^1)$. Since $w_1(t^1) = \dots = w_{j^1}(t^1) = w^*(t^1)$, along the auxiliary path at t^a , all players have to be approached with feasible probabilities by every player. As noted earlier, maintaining $w_1^a = \dots = w_{j^1}^a$ becomes progressively easier as we move back in time, so the auxiliary path employs feasible probabilities to the left of t^a . Thus, there is an MPE of the original game for which there is an interval on the left of t^1 where all players propose only to players in $\{1, \dots, j^1\}$ in a way that keeps these players' continuation payoffs equal throughout the interval. Moreover, over this interval, continuation values $w_1(t) = \dots = w_{j^1}(t)$ are strictly decreasing in t . By Theorem 2 in Ambrus and Lu (2010), all MPEs have the same continuation value functions over the interval.

Now suppose that $\lambda_1 = \lambda_i < \lambda_{i+1}$, where $i \geq K$. In this case, let $j^1 = i$. Clearly, there is an interval on the left of 0 where all players propose only to players in $\{1, \dots, j^1\}$. Moreover, over this interval, continuation values $w_1(t) = \dots = w_{j^1}(t)$ are strictly decreasing in t . As above, by Theorem 2 of Ambrus and Lu (2010), all MPEs have the same continuation value functions over the interval.

Case 1: $w_1(t) = \dots = w_{j^1}(t)$ is strictly below $w_{j^1+1}(t) \forall t < t^1$. Then players $\{j^1 + 1, \dots, n\}$ are never approached in MPE. Since $w_1(t) = \dots = w_{j^1}(t)$ is monotonic in t and bounded within the interval $[0, 1]$, it converges as $t \rightarrow -\infty$. Since at any $t < t^1$, a proposer has to offer $(K - 1)w_1(t)$ to other players, the fact that players in $\{j^1 + 1, \dots, n\}$ are never approached by any player and that $w_1(t)$ converges as $t \rightarrow -\infty$ together imply that $w_i(t)$ converges for all $i \in \{j^1 + 1, \dots, n\}$ as well. Note that if there is an arrival at any time t , independently of the proposal, the value of the approached coalition is 1, and the proposal is

accepted with probability 1. Hence, as $t \rightarrow -\infty$, $\sum_{i \in N} w_i \rightarrow \frac{1}{1+r}$.

Define $x_{i,r}^j$ analogously to x_i^j in the main text, but with the equation instead being $jx + \frac{1-(K-1)x}{1+r} \sum_{i=j+1}^n \lambda_i = \frac{1}{1+r}$, and with $x_{i,r}^j = \frac{1-(K-1)x}{1+r} \lambda_i$ for $i > j$. Also analogously define j_r^* . The above results imply that the limit of continuation values as $t \rightarrow -\infty$ is equal to $(x_{1,r}^{j^*}, x_{2,r}^{j^*}, \dots, x_{n,r}^{j^*})$. The definition of j_r^* together with Claim 1 then imply $j^1 \geq j_r^*$. Now we show that $j^1 > j_r^*$ is not possible (unless $x_i^{j^1} = x_{i,r}^{j_r^*}$ for all i). Suppose $j^1 > j_r^*$. Along an auxiliary path where all players at all times approach only players in $\{1, \dots, j_r^*\}$ in a way that keeps the latter players' payoffs equal, the payoff of any player $i \in \{1, \dots, j_r^*\}$ has to converge to a limit that is weakly greater than her limit MPE payoff $x_{i,r}^{j^1}$ (since they are being approached at least as often as in the MPE at all times). But by definition, $x_{i,r}^{j^1} \geq x_{i,r}^{j_r^*}$ for all $i \in \{1, \dots, j_r^*\}$, which means that $x_i^{j^1} = x_{i,r}^{j_r^*}$ for all $i \in \{1, \dots, j_r^*\}$, and therefore for all i .

Case 2: At some time $t < t^1$, $w_1(t) = \dots = w_{j^1}(t) = w_{j^1+1}(t)$. Let $t^2 = \max_{t < t^1} \{t | w_{j^1}(t) = w_{j^1+1}(t)\}$ and let j^2 such that $j^2 = \max_{i \in N} \{i | w_{j^1}(t) = w_i(t)\}$. Analogous arguments as above establish that there is an interval on the left of t^2 on which $w_1 = \dots = w_{j^2}$, and $w_1(t) = \dots = w_{j^2}(t)$ is strictly decreasing in t .

Since n is finite, continuing the same argument establishes that there are $t^k < \dots < t^1 \leq 0$ (for some $k \geq 1$) and $j^k > \dots > j^1 \geq K$ such that on (∞, t^k) , $w_1 = \dots = w_{j^k}$, $w_1(t) = \dots = w_{j^k}(t)$ is strictly decreasing in t , and that $w_i \rightarrow \underline{x}_{i,r}^*$ as $t \rightarrow -\infty$.

Finally, note that the equation defining \underline{x}_r^j is continuous in r . This implies that:

- (i) if $x_{j^*+1}^{j^*} > x_{j^*}^{j^*}$, for low enough r , $j_r^* = j^*$;
- (ii) if $x_{j^*+1}^{j^*} = x_{j^*}^{j^*}$, then it may be that for all r in some interval $(0, \widehat{r})$, $j_r^* = j^* + 1$. But note that in this case, $\underline{x}^{j^*} = \underline{x}^{j^*+1}$.

Either way, as $r \rightarrow 0$, $x_{i,r}^* \rightarrow x_i^*$. QED