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ABSTRACT

Expectations theories of asset returns may be interpreted either as stating that risk premia are zero or that they are constant through time. Under the former interpretation, different versions of the expectations theory of the term structure are inconsistent with one another, but I show that this does not necessarily carry over to the constant risk premium interpretation of the theory. I present a general equilibrium example in which different types of risk premium are constant through time and dependent only on maturity. Furthermore, I argue that differences among expectations theories are second-order effects of bond yield variability. I develop an approximate linearized framework for analysis of the term structure in which these differences disappear, and I test its accuracy in practice using data from the CRSP government bond tapes.

IN A WELL-KNOWN article in this *Journal*, Cox, Ingersoll, and Ross (CIR [4]) re-examine and find wanting certain hypotheses about the term structure of interest rates. A striking feature of CIR's re-examination is that it is entirely theoretical. CIR show that different versions of the pure expectations theory of the term structure, which traditionally were regarded as equivalent, are in fact inconsistent with one another when interest rates are random. Furthermore, in a fairly general continuous time arbitrage pricing framework, when interest rates are random, all versions of the theory except one are incompatible with equilibrium.

At first sight, CIR's results appear to be devastating to traditional empirical work on the term structure. They suggest that researchers must specify arbitrage pricing models with a small number of state variables before proceeding to empirical work. Such models must restrict not only the deterministic components of interest rate movements, but also the variance-covariance matrix of interest rate innovations and the information set of market participants.

The purpose of this paper is to defend traditional hypotheses about the term structure as a starting point for empirical research. Although these hypotheses may as a matter of fact be false, it is meaningful to test them against the data; useful empirical work can be done outside the confines of tightly specified arbitrage pricing models.

The defense has two parts. In the first, I argue that CIR's criticisms apply to a more restrictive type of expectations theory than is typically studied in the

* Department of Economics, Princeton University. This paper is based on Chapter 2 of my Yale Ph.D. Dissertation, *Asset Duration and Time-Varying Risk Premia.* I am grateful to Ed Kane, Pete Kyle, Huston McCulloch, Kermit Schoenholtz, Robert Shiller, and an anonymous referee for helpful correspondence and discussions, and particularly to Pete Kyle for showing me how to fill an important gap in the argument. I am responsible for any remaining errors.

empirical literature. In the second, I show that the inconsistencies pointed out by CIR are of "second order" in a precise mathematical sense, and I claim that they may often be ignored in empirical work.

Section I of the paper presents the first part of the defense. I begin by showing that it is natural to express a version of an expectations theory of the term structure as a statement about the expected difference between a random variable and a known one. Such an expected difference may be called a *term premium* or risk premium. CIR discuss versions of the *pure expectations theory* of the term structure, which states that term premia are zero.[1] But much of the literature is concerned with versions of a less restrictive theory, which states merely that term premia are constant through time. These are referred to here as versions of the *expectations theory* of the term structure.

CIR's basic point is that when interest rates are random, different term premia cannot simultaneously equal zero because of Jensen's Inequality; therefore, different versions of the pure expectations theory are inconsistent with one another. This is, of course, correct. But it turns out that different versions of the expectations theory, as opposed to the pure expectations theory, are not necessarily incompatible with each other or with arbitrage pricing equilibrium. I present a general equilibrium example in which expected excess holding returns on discount bonds are constant, and differences between forward rates and expected spot rates are also constant. That is, two versions of the expectations theory of the term structure hold simultaneously in this general equilibrium model. The example is of independent interest as a variant of the one in CIR [6].[2]

In Section II, I argue that in any case the differences among term premia are second-order effects of bond yield variability. I present an approximate linearized framework for the analysis of the term structure in which these differences disappear. The framework has a number of advantages: it states a linear relationship between the level and change of a bond yield and the holding return on the bond; it can easily be applied to coupon bonds as well as to discount bonds; and it suggests simple regression tests of the expectations theory. Finally, I briefly examine the empirical accuracy of the approximation, using data from the CRSP government bond tapes.

I. Expectations Theories: Zero versus Constant Risk Premia

Following CIR, I define $P(Y, t, T)$ as the time, t, price in consumption goods of a real discount bond, that is a claim to one unit of consumption goods at time, T. This price is a function of t, T, and some vector of state variables, Y, which summarizes the state of the economy at time, t. The corresponding *yield to maturity*, the continuously compounded rate of return on holding the bond to maturity, is

$$y(Y, t, T) = -[1/(T - t)] \ln P(Y, t, T). \tag{1}$$

At any time, t, and for given state, Y, the *term structure of interest rates* is the

[1] This terminology is due to Lutz [9].

[2] I am most grateful to Pete Kyle for supplying this example.

set of $P(Y, t, T)$ considered as a function of T. Assume that this function is differentiable. Then, the instantaneous *forward rate* on a loan at time, T, entered into at time, t, is

$$f(Y, t, T) = -[\partial P(Y, t, T)/\partial T]/P(Y, t, T)$$
$$= y(Y, t, T) + (T - t)[\partial y(Y, t, T)/\partial T]. \quad (2)$$

Equation (2) states a relation between the instantaneous forward rate and the yield which is analogous to the relation between marginal and average cost. Thus, for example, when the yield is rising with maturity, the forward rate is higher than the yield.

The instantaneous *spot rate* of interest at time t, $r(Y, t)$, is the limit as T approaches t of both $f(Y, t, T)$ and $y(Y, t, T)$.

The instantaneous *holding return* at time, t on a claim maturing at T is

$$h(Y, t, T) = dP(Y, t, T)/P(Y, t, T) \quad (3)$$

where dP is the change in P over an interval of time, dt.

I am now able to define two *term premia* which are the primitive objects of expectations theories. The *instantaneous holding premium* is the expected difference at t between the instantaneous holding return on a bond which matures at T and the spot rate at t: $\phi(Y, t, T) = E_t h(Y, t, T) - r(Y, t)$. The *instantaneous forward premium* is the expected difference at t between the forward rate at T and the spot rate at T: $\psi(Y, t, T) = f(Y, t, T) - E_t r(Y, T)$.[3]

Two versions of the pure expectations theory considered by CIR are $\phi(Y, t, T) = 0$ for all Y, t, and T (CIR's "Local Expectations Hypothesis") and $\psi(Y, t, T) = 0$ for all Y, t, and T (CIR's "Yield to Maturity Expectations Hypothesis"). The corresponding versions of the expectations theory are $\phi(Y, t, T) = H(T - t)$ and $\psi(Y, t, T) = F(T - t)$.

CIR use Jensen's Inequality to show that the theories $\phi = 0$ and $\psi = 0$ are inconsistent when interest rates are random. They also use an arbitrage argument to show that the theory $\psi = 0$ is incompatible with any rational expectations equilibrium. By contrast, the following example, which is a specialization of CIR [5] and a variant of CIR [6], shows that the conditions $\phi(Y, t, T) = H(T - t)$ and $\psi(Y, t, T) = F(T - t)$ may both hold simultaneously in general equilibrium.

The example provides a general equilibrium foundation for the model of Vasicek [14]. Its key feature is that the instantaneous variances of the underlying sources of uncertainty are constant through time rather than proportional to a state variable as in CIR [6]. This means that real interest rates periodically become negative, which they never do in CIR [6], but the parameter values may be chosen to make the occurrence of negative interest rates arbitrarily unlikely in any finite time interval.

Example: Consider an economy of identical individuals, each of whom

[3] CIR also discuss a third type of premium. This is defined as the difference between the gross, uncompounded return on holding a bond to maturity and the expected equivalent return on receiving the spot rate at each instant of time. I do not discuss this premium concept further here, as it seems more natural to consider a rate of return over an interval of time in the manner of Equation (1). The example given in which holding and forward premia are constant does not make this third type of premium constant.

maximizes

$$E_t \int_t^\infty \rho \exp(-\rho s)\log c(s)\, ds \tag{4}$$

by choosing consumption, $c(t)$, subject to the budget constraint, $W(s) \geq 0$, for all s, and

$$dW(t) = W(t)\, dn(t) - c(t)\, dt. \tag{5}$$

Here, $dn(t)$ is the instantaneous return on physical investment, which has stochastic constant returns to scale. It satisfies

$$dn(t) = (a + Y(t))\, dt + \sigma_n\, dz_n \tag{6}$$

where $Y(t)$ moves through time as

$$dY(t) = -gY(t)\, dt + \sigma_Y\, dz_Y \qquad g > -\rho. \tag{7}$$

$Y(t)$, the time-varying component of the expected return on physical investment, is the single state variable for the economy. However, there are two sources of uncertainty in the model, dz_n and dz_Y, which are scalar Wiener processes. They have some correlation which is indexed by a positive or negative number, θ: $E_t\, dz_n\, dz_Y = \theta\, dt$. Instantaneous investment in the productive process, $dn(t)$, is subject only to uncertainty arising from dz_n, but investment for more than an instant is also subject to uncertainty arising from random variation, dz_Y, in the expected instantaneous return on investment.

The theorems of CIR [5] can be applied to solve this model, as noted in CIR [6], footnote 5. The optimal consumption decision is $c(t) = \rho W(t)$, and the value function, the maximized value of (4), is $J(W, Y, t) = (\exp(-\rho t))[\log(\rho W) + (a - \rho - \sigma_n^2)/\rho + Y/(\rho + g)]$. The riskless instantaneous interest rate, $r(Y, t)$ is, from CIR [5] Theorem 1,

$$r(Y, t) = a + Y(t) - \sigma_n^2. \tag{8}$$

The dynamics of $r(Y, t)$ are just those of $Y(t)$. From (7) and (8),

$$dr = g(a - \sigma_n^2 - r)\, dt + \sigma_Y\, dz_Y \tag{9}$$

which is an Ornstein-Uhlenbeck process (a continuous-time AR-1) of the type studied by Vasicek [14].

Since consumption is proportional to wealth, a continuous time version of the Capital Asset Pricing Model holds (Ingersoll [8, Chapter 13]). This can be used to show that the "market price of interest rate risk" is just a constant, $\sigma_Y \sigma_n \theta$, as assumed by Vasicek [14]. The closed-form solution for the discount bond price is as in Vasicek:

$$\begin{aligned} P(Y, t, T) &= \exp\left[-\int_{s=t}^{T} f(Y, t, s)\right] \\ f(Y, t, s) &= a - \sigma_n^2 + Y \exp[-g(s - t)] \\ &\quad - \sigma_Y^2(1 - \exp[-g(s - t)])^2/2g^2 \\ &\quad - \sigma_Y \sigma_n \phi(1 - \exp[-g(s - t)])/g. \end{aligned} \tag{10}$$

The price, $P(Y, t, T)$, may be greater than unity (the yield may be negative), since, as previously noted, negative interest rates occur in the model.

The holding premium is constant and is found to be

$$\begin{aligned}\phi(Y, t, T) &= H(T - t) = \phi\sigma_n\sigma_Y((\partial P/\partial Y)/P) \\ &= -\theta\sigma_n\sigma_Y[1 - \exp(-g(T - t))]/g. \qquad (11)\end{aligned}$$

Discount bonds have positive instantaneous holding premia when θ is negative (since $\partial P/\partial Y$ is negative). Intuitively, when θ is negative, increases in wealth are associated with decreases in the interest rate and capital gains on bonds. Returns on bonds are thus positively correlated with the market, and a positive premium is required. Campbell [2] develops a similar characterization of holding premia in a discrete time exchange model.

The forward premium is also constant and smaller than the holding premium:

$$\begin{aligned}\psi(Y, t, T) &= F(T - t) = H(T - t) - \sigma_Y^2((\partial P/\partial Y)/P)^2/2 \\ &= H(T - t) - \sigma_Y^2[1 - \exp(-g(T - t))]^2/2g^2. \qquad (12)\end{aligned}$$

The forward premium is negative when the holding premium is zero (i.e., when $\theta = 0$). Equivalently, the expected continuously compounded return from t to T, on a bond which matures at T, is lower than the expected continuously compounded return on rolling over short bonds for this period. This can be explained heuristically as follows. The rollover strategy has high returns when $Y(t)$ is high on average between t and T. But with $\theta = 0$, wealth is high on average at T if $Y(t)$ has been high between t and T. Thus, the returns on the rollover strategy are positively correlated with the market and have a higher expected value, generating a negative forward premium.

The difference between $H(T - t)$ and $F(T - t)$ in Equation (12) is proportional to the instantaneous variance of the bond yield, while $H(T - t)$ itself is proportional to the instantaneous standard deviation. I now develop some implications of this observation.

II. An Approximate Linearized Framework for Study of the Term Structure

In this section, I present a set of linear approximations relating forward rates, holding returns, and yields to maturity. These approximations serve a double purpose. First, they show that the inconsistencies pointed out by CIR are second-order effects of bond yield variability. When bond yields are "not too" variable, the inconsistencies can be neglected. Secondly, linear approximations can be derived for coupon bonds as well as discount bonds, and thus allow an easy direct approach to the study of coupon bond data.[4] Linear approximations have previously been worked out for discrete time by Shiller, Campbell, and Schoenholtz [13] and Campbell and Shiller [3]; however, linearization in continuous time is new and requires somewhat different mathematical tools.

[4] Traditionally, researchers have followed McCulloch [12] and transformed a coupon bond yield curve into an implied discount bond yield curve before conducting their analysis. This procedure is elaborate and itself subject to error.

I begin by showing how a linear relationship between yields and forward rates, equivalent to that in Equation (2), can be derived for coupon bonds as an approximation. Define the yield to maturity on a coupon bond which pays a continuous coupon stream at rate C from t to T, and then a final payment of a dollar at time, T, by the implicit function

$$C \int_t^T \exp[-(s - t)y_C(Y, t, T)]\, ds + \exp[-(T - t)y_C(Y, t, T)]$$

$$= P_C(Y, t, T) = C \int_t^T P(Y, t, s)\, ds + P(Y, t, T).$$

This equation states that $y_C(Y, t, T)$ is that rate of return which discounts the coupon and principal payments of the bond to $P_C(Y, t, T)$, its time t price. The price is just the sum of those payments' present values—a coupon bond is equivalent to a portfolio of discount bonds—so $P_C(Y, t, T)$ can also be written in terms of discount bond prices $P(Y, t, T)$ or discount bond yields $y(Y, t, T)$.

It will be useful to apply the concept of "duration" in studying coupon bonds. Duration was defined by Macaulay [10] as the present-value-weighted average length of time before repayment of a loan, where the yield to maturity on the loan is used to compute present value. The duration $D_C(y_C, t, T)$ of a bond maturing at T with coupon C and yield y_C is

$$D_C(y_C, t, T)$$

$$= C \int_t^T (s - t)\exp[-(s - t)y_C]\, ds + (T - t)\exp[-(T - t)y_C]. \quad (14)$$

When $y_C = C = R$ for some R (i.e., when the bond is selling at par), this formula collapses to $D_R(T - t) = [1 - \exp[-(T - t)R]/R$. It is also worth noting that duration as defined in Equation (14) is just $(-1/y_C)$ times the elasticity of P_C with respect to y_C.

Equation (13) expresses a complicated nonlinear relationship between y_C, C, and the term structure of discount bond yields $\{y(Y, t, s)\}$. Write this as $K[y_C, C, \{y(Y, t, s)\}] = 0$, where K is a functional since one of its arguments is $\{y(Y, t, s)\}$, a function of s. A linear approximation is obtained by applying a first-order Volterra-Taylor expansion about a path (Volterra [15]), in a manner familiar from the calculus of variations and analogous to the Taylor expansion about a point for a function.[5] The linearization path is $y_C = C = \{y(Y, t, s)\} = R$ for some R, i.e., a flat-term structure with coupon bonds selling at par.

I obtain the following approximate relationship between the coupon bond yield and the term structure of forward rates. C does not appear in this relation, even though $C = R$ only at the path of linearization.

$$y_C(Y, t, T) \simeq [1/D_R(T - t)] \int_t^T \partial D_R(s - t)/\partial s\, f(Y, t, s)\, ds. \quad (15)$$

[5] Further details of the approximation procedure are explained in an Appendix available from the author.

This can be rearranged to express the forward rate $f(Y, t, T)$ as a function only of the level and slope of the term structure of coupon bond yields at the point (t, T):

$$f(Y, t, T) \simeq y_C(Y, t, T) + [D_R(T - t)/(\partial D_R(T - t)/\partial T)]\partial y_C(Y, t, T)/\partial T. \quad (16)$$

Equation (16) is directly analogous to Equation (2), expressing a marginal-average relation between the forward rate and the coupon bond yield. Wherever maturity appears in Equation (2), it is replaced by duration in Equation (16); for discount bonds duration and maturity are the same, and Equation (16) holds exactly.

Next, I apply the method of linear approximation to the holding return on a coupon bond. In a straightforward modification of Equation (3), this holding return is defined as

$$h_C(Y, t, T) = [dP_C(Y, t, T) + C\ dt]/P_C(Y, t, T). \quad (17)$$

I assume that the coupon bond yield, y_C, follows an Ito process[6] with parameters μ and σ:

$$dy_C(Y, t, T) = \mu(Y, t, T)\ dt + \sigma(Y, t, T)\ dz. \quad (18)$$

Since P_C is a function of y_C and C, I can apply Ito's lemma to express h_C as a nonlinear function of C, y_C, μ, and σ. This can then be linearized around the point $C = y_C = R$, $\mu = \sigma = 0$, yielding

$$h_C(Y, t, T) \simeq y_C(Y, t, T)\ dt - D_R(T - t)\ dy_C(Y, t, T). \quad (19)$$

Equation (19) says that, ceteris paribus, a high bond yield means a high instantaneous holding return. However, an increase in the yield causes a capital loss proportional to the duration of the bond (since the duration is the price elasticity with respect to the yield) and lowers the holding return accordingly.

Equation (19) can be rearranged to express the yield on a coupon bond as an approximate weighted sum of future holding returns:

$$y_C(Y, t, T) \simeq (1/D_R(T - t)) \int_t^T \partial D_R(s - t)/\partial s\ h_C(Y, s, T)\ ds. \quad (20)$$

Note that this equation holds in realization and not just in expectation.

Equations (15) and (16), and (19) and (20), make up a complete linearized framework for analysis of the term structure. It is easy to see that within this framework, there are no inconsistencies between expectations theories. Taking expectations and equating the right-hand sides of (15) and (20), we have

$$(1/D_R(T - t)) \int_t^T (\partial D_R(s - t)/\partial s)\psi(Y, t, s)\ ds$$

$$\simeq (1/D_R(T - t)) \int_t^T (\partial D_R(s - t)/\partial s)\ \phi(Y, s, T)\ ds. \quad (21)$$

[6] This is perfectly consistent with the assumption of CIR that the bond price follows an Ito process.

Within the linearized system, if $\phi(Y, t, T) = 0$ for all Y, t, and T, then $\psi(Y, t, T) = 0$ for all Y, t, and T: these two forms of the pure expectations theory imply one another. A fortiori, the corresponding forms of the expectations theory imply each other.

The linearized system also suggests simple tests of the expectations theory. Taking expectations of Equation (19), and substituting in the definition of $\phi(Y, t, T)$, we have

$$y_C(Y, t, T) - r(Y, t) = \phi(Y, t, T) + D_R(T - t)E_t\, dy_C(Y, t, T). \quad (22)$$

That is, the long-short spread, $y_C - r$, can be decomposed into a risk premium and a term which is the expected change in the long rate multiplied by duration. If the risk premium is zero or constant, an unusually high spread should predict a rise in long rates. Conversely, if expected changes in long rates vary little, the spread is a good proxy for the risk premium. These observations are the basis for empirical work by Fama [7], Mankiw and Summers [11], and Shiller, Campbell, and Schoenholtz [13], among others.

The usefulness of the linear approximations in this section depends on their accuracy in practice. I now present some summary statistics which are designed to help evaluate the accuracy of the approximate expression for the holding return on a long bond.[7]

Above, an approximation was derived only for the instantaneous holding return. This is easily extended to the return on holding a bond from time t to time t', $h_C(Y, t, t', T)$:

$$h_C(Y, t, t', T) \simeq (1/D_R(t' - t)) \int_t^{t'} \partial D_R(s - t)/\partial s\; h_C(Y, s, T)\, ds. \quad (23)$$

This equation is analogous to the expression (20) for the yield on a coupon bond: but here, the integral runs from t to t' rather than from t to T. It follows from (20) that the period holding return is a simple linear function of the yields at times t and t' on a bond maturing at time T:

$$h_C(Y, t, t', T) \simeq [D_R(T - t)y_C(Y, t, T) - (D_R(T - t) - D_R(t' - t))y_C(Y, t', T)]/D_R(t' - t). \quad (24)$$

In the empirical work of this section, I use the discrete time version of (24), developed by Shiller, Campbell, and Schoenholtz [13], in which

$$D_R(s - t) = (1 - \gamma^{s-t})/(1 - \gamma) \quad \text{and} \quad \gamma = 1/(1 + R).$$

The Center for Research in Securities Prices (CRSP) at the University of Chicago has a complete set of monthly data on individual government bonds from 1925. This offers an opportunity to evaluate the linear approximation (24) because the data set contains both yields and exact monthly holding returns on long bonds.

[7] Berger [1] examined a similar approximation for returns on consols; his work assumed that 20- and 30-year government bonds are effectively consols. Shiller, Campbell, and Schoenholtz [13] evaluated an approximation for forward rates.

In Tables I and II, I present summary statistics for exact holding returns and two different approximate holding returns on 24 bonds. Table I covers eight 10-year bonds and four 20-year bonds, while Table II covers 12 bonds of at least 30 years maturity at issue. All summary statistics are for the first 5 years (60 observations) after the bond was issued.

Table I

The Accuracy of Linear Approximations to Holding Returns on 10- and 20-Year Bonds

Bond	Coupon Rate	Issue Date	Maturity at Issue (Years)	Mean Error		Correlation of Approximate & Exact Returns		Ratio of Error Variance to Variance of Exact Return	
				(1)	(2)	(1)	(2)	(1)	(2)
1	4.000	1962:9	10	−0.051	0.020	0.9996	0.9996	0.001	0.001
2	4.000	1963:9	10	−0.061	0.018	0.9997	0.9997	0.001	0.001
3	4.125	1964:7	9½	−0.117	0.003	0.9998	0.9997	0.001	0.001
4	4.125	1964:12	9	−0.196	−0.045	0.9996	0.9995	0.002	0.001
5	4.250	1964:4	10	−0.091	0.001	0.9998	0.9997	0.001	0.001
6	7.000	1971:7	10	−0.001	0.014	0.9996	0.9997	0.001	0.007
7	6.375	1972:1	10	−0.037	−0.073	0.9996	0.9995	0.003	0.007
8	8.000	1976:8	10	−0.900	−1.482	0.9987	0.9987	0.008	0.027
9	4.000	1959:1	21	−0.036	−0.028	0.9998	0.9998	0.001	0.010
10	3.500	1960:9	20	−0.040	0.136	0.9997	0.9997	0.001	0.016
11	6.750	1973:1	20	−0.262	−0.459	0.9996	0.9995	0.004	0.027
12	7.500	1973:8	20	−0.214	−0.562	0.9968	0.9970	0.015	0.076

Table II

The Accuracy of Linear Approximations to Holding Returns on 30-Year Bonds

Bond	Coupon Rate	Issue Date	Maturity at Issue (Years)	Mean Error		Correlation of Approximate & Exact Returns		Ratio of Error Variance to Variance of Exact Return	
				(1)	(2)	(1)	(2)	(1)	(2)
1	3.250	1953:4	30	−0.032	−0.174	0.9984	0.9987	0.009	0.027
2	3.500	1958:1	32	−0.244	0.156	0.9998	0.9998	0.002	0.031
3	4.250	1962:7	30	−0.201	0.129	0.9995	0.9996	0.003	0.007
4	4.000	1963:1	30	−0.296	0.318	0.9996	0.9995	0.006	0.008
5	4.125	1963:4	31	−0.376	0.243	0.9995	0.9995	0.006	0.007
6	3.000	1955:1	40	−0.366	0.891	0.9996	0.9994	0.004	0.066
7	3.500	1960:9	38	−0.113	0.210	0.9998	0.9998	0.003	0.039
8	8.250	1975:5	30	−0.779	−1.984	0.9968	0.9971	0.043	0.292
9	7.625	1977:2	30	−3.366	−6.174	0.9949	0.9950	0.124	0.416
10	7.875	1977:11	30	−1.345	−2.979	0.9953	0.9954	0.110	0.434
11	8.375	1978:8	30	−1.813	−3.523	0.9954	0.9954	0.099	0.514
12	8.750	1978:11	30	−1.239	−2.996	0.9959	0.9960	0.089	0.570

The two approximations in the tables, (1) and (2), are constructed using Equation (24) and differ only in the point of linearization R. Approximation (1) takes the own coupon rate on the bond as the linearization point, while approximation (2) uses a common linearization point of 5.5% for all bonds. 5.5% was chosen because it is close to the average coupon rate of all bonds in Table II, and to the average long bond rate in the 1959–1979 period. Shiller, Campbell, and Schoenholtz [13] and Campbell and Shiller [3] linearized around similar points.

Three summary statistics are presented for each approximation. These are the mean error, the mean difference between the approximate holding return and the exact holding return; the correlation between the approximate and exact holding returns; and the ratio of the variance of the approximation error to the variance of the exact holding return.

The summary statistics of Tables I and II indicate that the linear approximations of this paper are reliable if used judiciously. Consider first the 10-year bonds in rows 1 through 8 of Table I. One would expect that the approximation (1) performs well so long as the interest rate remains close to its level at the issue date. The approximation (2) should perform well if, in addition, the bond coupon rate is fairly close to 5.5%. In rows 1 through 8 of Table I, the worst mean error is in row 8, for a bond with an 8% coupon in the period 1976:8 to 1981:7. 8% is further from 5.5% than any other coupon rate in the table, and the period was one of rapidly rising and volatile interest rates. The other mean errors never exceed absolute values of 0.196% for approximation (1) and 0.073% for approximation (2).

The correlations of exact and approximate holding returns are extraordinarily high. They exceed 0.999 for all 10-year bonds except row 8, where the correlations are 0.9987 for both approximations. The ratio of the error variance to the variance of the true return is very low, never exceeding 0.003 for (1) or 0.007 for (2) except in row 8, where the ratios are 0.008 and 0.027.

A similar pattern appears for the four 20-year bonds in rows 9 through 12 of Table I. The correlations and mean errors are comparable to those in the first part of Table I. For bond 12, the error variance ratio reaches 0.015 for approximation (1) and 0.076 for approximation (2), but all other statistics are favorable.

In Table II, the first 7 bonds were issued with low coupons in the relatively stable 1950's and 1960's; the last 5 bonds were issued with high coupons in the turbulent 1970's. The summary statistics reflect this distinction. The first 7 rows are comparable to those of Table I, but in the last 5 rows the linear approximations begin to break down. There are high mean errors, reaching 3.366% for (1) and 6.174% for (2) in row 9. The correlations remain very high, falling just below 0.995 in only one case, but the error variance ratios rise to more than 10% for approximation (1) and more than 50% for approximation (2).

In conclusion, linear approximations should be used with caution in describing the period of high and volatile interest rates in the late 1970's and 1980's, and in studying extremely long-term bonds. Even here, however, the high correlations of Table II show that the approximations capture the movements of returns well. The approximations (1) and (2) behave like the exact returns, but amplified and damped, respectively. For periods with less extreme interest rate movements, and for somewhat shorter bond maturities, the approximations are extremely accurate.

III. Conclusions

In this paper, I have tried to rehabilitate a unified view of the expectations theory of the term structure.[8] It is true that Jensen's Inequality places a wedge between different concepts of risk premia in the term structure; but this does not prevent different premia from being constant through time at different values, as I demonstrate with a simple general equilibrium example. Furthermore, the differences among risk premia are second-order effects of bond yield variability, which disappear in a framework of linear approximations to term structure concepts. These approximations track monthly movements in bond returns quite accurately in postwar U.S. data.

[8] This should not be confused with a rehabilitation of the theory itself, as a good empirical description of the behavior of interest rates. In Shiller, Campbell, and Schoenholtz [13] and Campbell and Shiller [3], I have helped to argue that the theory is strongly rejected in postwar U.S. data.

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