Review of Economic Dynamics **3,** 365–395 (2000)
doi:10.1006/redy.2000.0098, available online at http://www.idealibrary.com on IDEAL®

Optimal Intertemporal Consumption under Uncertainty[1]

Gary Chamberlain

Department of Economics, Harvard University, Cambridge, Massachusetts 02138

and

Charles A. Wilson[2]

Department of Economics, New York University, 269 Mercer Street, New York, New York 10003

Received November 30, 1999

We analyze the optimal consumption program of an infinitely lived consumer who maximizes the discounted sum of utilities subject to a sequence of budget constraints where both the interest rate and his income are stochastic. We show that if the income and interest rate processes are sufficiently stochastic and the long run average rate of interest is greater than or *equal* to the discount rate, then consumption eventually grows without bound with probability one. We also establish conditions under which the borrowing constraints must be binding and examine how the income process affects the optimal consumption program. *Journal of Economic Literature* Classification Number: D91.
Key Words: uncertainty; consumption; permanent income hypothesis.

1. INTRODUCTION

We shall consider the following problem. In each period t, a consumer receives income x_t. After receiving his income, he must decide how much to consume in that period, c_t, and how much to save for future consumption. His savings earn a gross rate of return, r_{t+1}, so that the value of his

[1]Support from the National Science Foundation and the Alfred P. Sloan Foundation is gratefully acknowledged. We have benefitted from discussions with William Brock, Ed Green, José Scheinkman, and Itzak Zilcha.

[2]To whom correspondence should be addressed. E-mail: Charles.Wilson@nyu.edu.

1094-2025/00 $35.00

assets at the beginning of period $t + 1$ (after he has received income x_{t+1}) is given by the relation $a_{t+1} = r_{t+1}(a_t - c_t) + x_{t+1}$. The consumer's problem is to choose a consumption plan to maximize the expectation of $\sum_{t=0}^{\infty} u(c_t)\beta^t$ subject to a budget constraint, where u is an increasing, strictly concave function. His only source of uncertainty is that x_t and r_t follow some stochastic process. The question we address is: What happens to the levels of c_t and a_t as t goes to infinity?

The motivation of the paper lies in the work of several authors who attempt to formalize the permanent income hypothesis of Friedman (1957). The first of these papers, Yaari (1976) and Schechtman (1976), considers the case where utility is not discounted ($\beta = 1$), the interest rate is zero ($r_t = 1$), and the horizon is finite; i.e., the consumer maximizes $\sum_{t=0}^{T} u(c_t)$. Assuming that income x_t is identically and independently distributed and imposing only a solvency constraint on lifetime consumption, Yaari shows that as $T \to \infty$, the optimal consumption plan requires c_t^T, the consumption in period t with horizon T, to converge with probability one to $E[x_t]$ for all t. Schechtman tightens the solvency constraint to forbid borrowing and obtains the weaker result that, as both T and t go to infinity, c_t^T converges to $E[x_t]$ with probability one. He also establishes that a_t converges to infinity. Bewley (1977) retains the restriction on borrowing but considers a more general case where both x_t and u_t follow a stationary stochastic process. He also allows for discounting ($\beta < 1$). He obtains the analogous result that $u_t'(c_t^T)$ converges to a constant as $T \to \infty$, $t \to \infty$, and $\beta \to 1$, so that, asymptotically, the consumer becomes insulated from risk. Some of the implications of this result are developed in Bewley (1980a, 1980b).

In this paper we investigate the asymptotic properties of the optimal consumption program for the infinite horizon model when income and interest rates are stochastic and the consumer discounts a bounded utility function. We put no additional restrictions on the form of the stochastic process (such as stationarity), and we allow for arbitrary restrictions on the permissible level of borrowing in each period. Under these assumptions, we show that if the discount rate is smaller than the long run average rate of interest, then c_t will converge to infinity almost surely. When the long run interest rate is equal to the discount rate, the asymptotic properties of c_t depend on how stochastic the income stream is. If the income stream is certain, c_t is nondecreasing and converges to the supremum of the maximum sustainable consumption level starting from the minimum permissible level of wealth in any given period. However, if the income stream is suitably stochastic, c_t must converge to infinity almost surely. This result extends some of the results of Sotomayor (1984), who examines the case where income is identically and independently distributed (but allows for utility to be unbounded).

Our analysis is based on the following observation. For simplicity, suppose that the interest factor is a constant, γ. Let $v(a_t, z^t)$ be the expected discounted utility from following the optimal policy, given asset level a_t and information state z^t at time t. Then the first order conditions for utility maximization require $v^+(a_t, z^t) \geq \beta\gamma E[v^+(a_{t+1}, z^{t+1})|z^t]$, where v^+ denotes the right-hand derivative of v with respect to a_t. When $\beta\gamma \geq 1$, this implies that $v^+(a_t, z^t)$ is a supermartingale. By a theorem of Doob (1953), this implies a finite random variable d_∞ such that $v^+(a_t, z^t)$ converges to d_∞ with probability 1. If $\beta\gamma > 1$, $v^+(a_t, z^t) \geq \beta\gamma E[v^+(a_{t+1}, z^{t+1})|z^t]$ implies $v^+(a_t, z^t) \to 0$ from which it is easy to show that c_t must converge to infinity. If $\beta\gamma = 1$, this argument can still be used to show that c_t converges with probability 1 to some extended random variable c_∞, but in this case, c_∞ need not equal infinity. In fact, as we show in Theorem 3, if x_t is not stochastic, c_∞ is generally *not* infinite. However, the following argument shows that c_∞ must be infinite if the discounted value of future income is suitably stochastic.

Suppose there is positive probability that $c_\infty < \infty$. Then we can choose an arbitrarily small interval $[b, b + \varepsilon]$ and a $\tau < \infty$ such that there is positive probability that $c_\tau \in [b, b + \varepsilon]$, and, with probability greater than $1 - \varepsilon$, $c_t \in [b, b + \varepsilon]$ for all $t \geq \tau$ whenever $c_\tau \in [b, b + \varepsilon]$. But if $c \cong c_\tau$ for $t \geq \tau$, then a_t will diverge to $\pm\infty$ unless $(\gamma/(\gamma - 1))c_\tau = a_\tau + \sum_{j=\tau+1}^{\infty} x_j \gamma^{\tau - j}$. We can show that it is not optimal to have $c \cong c_\tau$ if $a_t \to \infty$. So conditional on $c_\tau \cong b$, it follows that $\sum_{j=\tau+1}^{\infty} x_j \gamma^{\tau - j} \cong (\gamma/(\gamma - 1))b - a_\tau$. But since a_τ is known at time τ, this implies that the conditional variance of $\sum_{j=\tau+1}^{\infty} x_j \gamma^{\tau - j}$ can be made arbitrarily small for ε sufficiently small. Therefore, unless there is perfect foresight, savings must diverge. We conclude that if the income stream is suitably stochastic, then c_t converges to infinity with probability 1. In the body of the paper, these arguments are generalized to allow for a stochastic interest rate. In particular, if the discount rate is equal to the long run average rate of interest, then c_t grows without bound if there is sufficient uncertainty in the joint distribution of income and interest rates. This case arises in the treatment of the optimum quantity of money by Bewley (1980c, 1983).

We see that with discounting and a positive interest rate, the only counterpart to the Yaari–Schechtman–Bewley result is a zero limit for marginal utility. If $\beta\gamma \geq 1$, then consumption does converge, but it converges to bliss, not the expected value of discounted income.[3] We should also note that although both Schechtman and Bewley use the convergence theorem for supermartingales, their argument requires either the law of

[3]An example of Schechtman and Escudero (1977) shows that convergence to bliss with probability 1 can occur when $\beta\gamma < 1$ and income is independent and identically distributed. They also provide a condition on the utility function that rules this out.

large numbers or the ergodic theorem, which in turn requires a stationarity assumption. By itself, however, the supermartingale theorem requires no additional assumptions. It is the stochastic generalization of the result that a bounded, monotone sequence converges to a finite limit. Consequently, when we eliminate the need for a law of large numbers by introducing discounting, we are able to considerably weaken our assumptions on the stochastic process and still obtain convergence of c_t, even with a positive interest rate.

If the underlying process is stationary, there are alternative conditions on the interest rate sequence which ensure that consumption converges to infinity. In Subsection 4.4, we show that if the interest rate sequence is stationary and ergodic, then consumption will grow without bound if $E[\log \beta r_t] > 0$. If $E[\log \beta r_t] = 0$ and an additional condition is satisfied which guarantees that the interest rate has sufficient variance and its dependence on the past dies out sufficiently fast, then some subsequence of consumption will grow without bound. In both cases, the result follows from the fact that $\beta^t \prod_{j=1}^{t} r_j$ (or some subsequence) converges to infinity with probability 1.

Section 5 deals with some implications of the budget constraint. Rather than restrict ourselves to a no borrowing constraint at the outset, we allow the lower bound on borrowing in any period to be an arbitrary function of the consumer's information in that period. We then note that by redefining income and wealth, the original problem is equivalent to a problem where income is nonnegative in each period and no borrowing is permitted. Making this translation explicit emphasizes that these assumptions are not really substantive restrictions on the model, but are merely a simpler representation of a more general model. In particular, they are consistent with the possibility that the only constraint on borrowing is an intertemporal budget constraint. With this interpretation, we establish some conditions under which the borrowing constraint is never binding as well as conditions under which it must be binding in some periods. We also show how the optimal program changes as the borrowing constraints are relaxed.

In the final section of the paper, we show how some of our results can be extended to the case where the consumer chooses a portfolio of several risky assets. The main complication introduced by this extension is that the interest rate becomes endogenous.

2. ASSUMPTIONS AND NOTATION

$\mathbb{R}$ is the real line, and $\mathbb{R}^n$ is n-dimensional Euclidean space. $\mathbb{R}_+$ and $\mathbb{R}_+^n$ refer to the corresponding subsets of nonnegative elements. Nonnegative integers are denoted by t, τ, i, j. Other lower-case Roman letters generally refer to random variables or their realizations, and lower-case Greek letters (particularly, ε, δ, and α) generally refer to real numbers.

Let $\mathbf{z} \equiv (\dots, z_{-1}, z_0, z_1, \dots)$ be a stochastic process with transition probabilities $p(dz_t|z^{t-1})$ used to define conditional expectation, where $z_t \in \mathbb{R}^n$ and $z^t \equiv (z^{t-1}, z_t) \equiv (\dots z_{-1}, z_0, z_1, \dots, z_t)$. Interpret z_t as new information the consumer receives at time t and z^t as the information state at time t. Let $Z^t(z^0) = \{z^t : z_j^t = z_j^0,\ j \le 0\}$ be the set of states at time t which are consistent with state z^0. Except for Section 4.4 where we consider the case where $\mathbf{z}$ is stationary, we will take z^0 as fixed.

We assume that the income of the consumer at time $t \ge 0$ is a continuous function x_t: $Z^t(z^0) \to \mathbb{R}$.[4] At each time $t \ge 1$, the consumer may borrow and lend between time $t-1$ and time t at interest factor r_t which is assumed to be a positive, continuous function r_t: $Z^t(z^0) \to \mathbb{R}_+$. For $t \ge 0$, let $R_{tt} \equiv 1$. For $j > t \ge 0$, let $R_{tj} \equiv \prod_{k=t+1}^{j} r_k$ denote the interest factor between periods t and j, and let R_{tj}^{-1} denote its inverse.

The borrowing constraint of the consumer at time $t \ge 0$ is a continuous function, k_t: $Z^t(z_0) \to \mathbb{R}$ that satisfies an intertemporal consistency constraint,

$$P\left(r_t k_{t-1} + x_t \ge k_t,\ t \ge 0 | z^0\right) = 1, \tag{1}$$

where $r_0 k_{-1} \equiv 0$. Let $\mathbf{x} \equiv (x_0, x_1, \dots)$ and $\mathbf{k} \equiv (k_0, k_1, \dots)$. Interpret k_t as the minimum amount of wealth, measured in terms of period t consumption, that the individual may hold at the end of period t (after he receives his period t income and spends his period t consumption). Equation (1) requires that the lower bound on current net wealth be consistent with the borrowing constraint the consumer will face in the next period. It implies that current wealth can never be so low that it may become impossible for the individual to satisfy his borrowing constraint in the next period even if nothing is consumed in the current period. Defining $r_0 k_{-1} \equiv 0$ is simply a convention which implies that the consumer's initial wealth is derived solely from his period 0 income.

Since the consumer's decision at each information state depends only on what he knows at that state, a *consumption program*, $\mathbf{c} \equiv (c_0, c_1, \dots)$, is a sequence of Borel measurable functions, c_t: $Z^t(z^0) \to \mathbb{R}_+$, $t \ge 0$. Let u: $\mathbb{R}_+ \to \mathbb{R}$ be an increasing, continuous, strictly concave function with $0 \le u \le M \equiv \sup_{c \to \infty} u(c) < \infty$. For any consumption program $\mathbf{c}$, $u(c_t)$ is the undiscounted utility to the consumer from his consumption at time t. We suppose the consumer discounts utility by a factor β $(0 < \beta < 1)$ in each period. Therefore, at state z^0, the problem of the consumer is to choose a

[4]The continuity requirement is without loss of generality since we may always include a variable as one of the coordinates of z_t.

consumption program **c** to maximize $E[\sum_{j=0}^{\infty} u(c_j)\beta^j | z^0]$ subject to $P(\sum_{j=0}^{t}(x_j - c_j)R_{jt} \geq k_t,\ t \geq 0 | z^0) = 1$.

Some of our results require some regularity assumptions on the transition probabilities. We assume that for any z^0 the Feller property holds: if $f\colon \mathbb{R}^n \to \mathbb{R}$ is a bounded continuous function, then $g(z^t) = \int f(w)p(dw|z^t)$ is a continuous function from $Z^t(z^0)$ to $\mathbb{R}$. This condition combined with the requirement that x_t, r_t, and k_t be continuous functions is shown in Theorem A.1 (of the Appendix) to guarantee that the consumer's problem has a solution.

Before proceeding with our analysis, it will be useful to redefine our income variable so that the problem may be reduced to a simpler form. As stated, the problem of the consumer is to choose a consumption program that maximizes the discounted sum of utility subject to a (possibly random) borrowing constraint. Aside from technical considerations, there are no restrictions on either the income stream or the borrowing constraint other than the requirement that the borrowing constraint satisfy Eq. (1). For instance, if we assume that x_t is nonnegative and we wish to prohibit borrowing, we may set $k_t = 0$ for all t. Alternatively, if we wish to impose only an intertemporal solvency constraint, the appropriate constraint is $k_t(z^t) \equiv \inf\{\kappa \in \mathbb{R}\colon P(\kappa + \sum_{j=t+1}^{\infty} x_j R_{tj}^{-1} \geq 0 | z^t) = 1\}$.[5] However, we also allow for intermediate cases as well as cases where $\lim_{t\to\infty} k_t R_{0t}^{-1} \neq 0$.

Whatever additional assumptions we impose on x_t and k_t, however, we may always translate the variables so that the problem is equivalent to one in which income is nonnegative and borrowing is not permitted. The key is to redefine income at time t to be the increase in available purchasing power the consumer receives at time t. This is the difference between x_t, the income he receives in period t, and $k_t - r_t k_{t-1}$, the change in the minimum allowable wealth from period $t-1$ to period t, measured in units of income at time t. Letting $\hat{x}_t \equiv x_t - k_t + r_t k_{t-1}$ and letting $\hat{k}_t \equiv 0$ for all $t \geq 0$, Eq. (1) may be stated as $P(\hat{x}_t \geq 0,\ t \geq 0 | z^0) = 1$, and the borrowing constraint may be stated as $P(\sum_{j=0}^{t}(\hat{x}_j - c_j)R_{0j}^{-1} \geq 0,\ t \geq 0 | z^0) = 1$. Note that since x_t, k_t, and k_{t-1} are all continuous functions of z^t, $\hat{x}_t$ is also a continuous function of z^t. Consequently, the technical requirements and information restrictions on $\hat{x}_t$ and $\hat{k}_t$ are satisfied. Unless we indicate otherwise, we shall assume for the remainder of the paper that this translation has already been made so that we may restrict attention to the case $x_t \geq 0$ and $k_t = 0$. However, it is occasionally useful to emphasize the interpretation of our results in the original framework.

[5]To ensure the existence of some feasible consumption program consistent with $k_{-1} = 0$, we then require $P(\sum_{j=0}^{\infty} x_j R_{0j}^{-1} \geq 0 | z^0) = 1$, and to ensure that $-k_t < \infty$ so that an optimal consumption program exists, we then require $P(\sum_{j=t}^{\infty} x_j R_{tj}^{-1} < \infty | z^t) > 0$ for all z^t.

To establish the results that follow, it is convenient to work with the value function of future discounted utility defined over the level of accumulated assets. For any state z^t and wealth level $a \geq 0$, define

$$v(a, z^t) \equiv \max_{\mathbf{c}} E\left[\sum_{j=t}^{\infty} u(c_j)\beta^{j-t}|z^t\right]$$

subject to

$$P\left(a - c_t + \sum_{j=t+1}^{\tau} (x_j - c_j) R_{tj}^{-1} \geq 0,\ \tau \geq t|z^t\right) = 1$$

to denote the expected utility of the consumer starting at time t with information z^t, given purchasing power a at time t. Following the arguments of Blackwell (1965), Strauch (1966), and Maitra (1968), we establish as Theorem A.1 in the Appendix that $v(a, z^t) = \max_{0 \leq c \leq a}\{u(c) + \beta E[v((a-c)r_{t+1} + x_{t+1}, z^{t+1})|z^t]\}$.[6]

With the exception of Subsection 4.4, which uses the stationary assumptions, we shall assume the initial information state z^0 is fixed so that unconditional expectations should be understood to be conditional on z^0. It is also frequently convenient to suppress explicit reference to the state z^t, so that for a given state z^t, we sometimes write $v(a, z^t) \equiv v_t(a)$ and regard $v_t(a)$ as a random variable. Also, to simplify the statement of some of the theorems, we sometimes state properties of the optimal program as if they hold for all z^t, although we prove our results on a set of probability 1.

3. PRELIMINARY RESULTS

The existence and uniqueness of a solution to the consumer's maximization problem is established in the Appendix as Theorem A.1. We also establish there that v_t is a bounded, strictly concave function. Therefore, right- and left-hand derivatives exist. For any function f: $[0, \infty] \to \mathbb{R}$, let $f^+(x)$ denote the right-hand derivative of $f(x)$, let $f^-(x)$ denote its left-hand derivative when $x > 0$, and let $f^+(0) = \lim_{x \downarrow 0} f^+(x)$. Let $\mathbf{c}^* = (c_0^*, c_1^*, \ldots)$ denote the consumption program which solves the consumer's

[6]The result is actually established for the more general model with many assets discussed in Section 6.

maximization problem, and let $\mathbf{a}^* \equiv (a_0^*, a_1^*, \dots)$ denote the corresponding sequence of random variables representing the level of the consumer's wealth in each period after current income has been received but before consumption has been spent. The wealth sequence $\mathbf{a}^*$ is defined recursively by $a_0^* \equiv x_0$ and $a_t^* \equiv r_t(a_{t-1}^* - c_{t-1}^*) + x_t = x_t + \sum_{j=0}^{t-1}(x_j - c_j)R_{jt}$, for $t > 0$.

With these definitions, we may state some conditions which an optimal program must satisfy. The proof is quite standard and will be omitted.

LEMMA 1. (a) $v_t^+(a_t^*) = \max\{u^+(c_t^*), \beta E[r_{t+1}v_{t+1}^+(a_{t+1}^*)|z^t]\}$. (b) *Suppose $a_t^* > 0$. Then*

$$v_t^-(a_t^*) = \begin{cases} u^-(c_t^*) & \text{if } a_t^* - c_t^* = 0 \\ \beta E\left[r_{t+1}v_{t+1}^-(a_{t+1}^*)|z^t\right] & \text{if } c_t^* = 0 \\ \min\left\{u^-(c_t^*), \beta E\left[r_{t+1}v_{t+1}^-(a_{t+1}^*)|z^t\right]\right\} & \text{if } 0 < a_t^* - c_t^* < a_t^*. \end{cases}$$

For $t \geq 0$, let $\theta_t \equiv \beta^t R_{0t}$. If the level of consumption in period 0 is equal to the level of consumption in period t, then θ_t measures the rate at which the consumer can transform discounted utility in period t for utility in period 0. Condition (a) establishes that $\theta_t v_t^+(a_t^*)$ is a supermartingale. Our first theorem uses the supermartingale convergence theorem to establish that $\theta_t v_t^+(a_t^*)$ must converge to some random variable e_∞. The only problem is that to apply the convergence theorem, $v_t^+(a_t^*)$ must be finite which is not true if $x_0 = 0$ and $v_0^+(0) = \infty$. This difficulty can be avoided if we assume that the discounted value of the entire income stream is always positive. In this case, there is a stopping time τ such that $a_\tau^* > 0$, which allows us to apply the convergence theorem to the sequence $(\theta_\tau v_\tau^+(a_\tau^*))^{-1}\theta_{\tau+t}v_{\tau+t}^+(a_{\tau+t}^*)$.

THEOREM 1. *If $P(\mathbf{x} = 0) = 0$, then there is a (real-valued) random variable e_∞ such that $P(\lim_{t\to\infty}\theta_t v_t^+(a_t^*) = e_\infty) = 1$.*

Proof. $P(\mathbf{x} = 0) = 0$ implies a stopping time τ such that $x_\tau > 0$. Therefore, $a_\tau^* \geq x_\tau > 0$ and the concavity of v_τ imply that $v_t^+(a_t^*)$ is finite. Let $d_t \equiv (\theta_\tau v_\tau^+(a_\tau^*))^{-1}\theta_{\tau+t}v_{\tau+t}^+(a_{\tau+t}^*)$ for $t \geq 0$. Since τ is a stopping time, Lemma 1 implies that $(d_0, d_1, \dots)$ is a nonnegative supermartingale (Meyer, 1966, p. 66). But since $d_0 = 1$, there is a random variable d_∞ with

$E[d_\infty \leq 1]$ such that $P(\lim_{t\to\infty} d_t = d_\infty) = 1$ (Doob, 1953, p. 324). Then if we define $e_\infty \equiv \theta_\tau v_\tau^+(a_\tau^*) d_\infty$, it follows that $P(\lim_{t\to\infty} \theta_t v_t^+(a_t^*) = e_\infty) = 1$. ∎

In the next section, we use Theorem 1 to establish some conditions under which consumption must converge to infinity. To establish these conditions we use the fact that whenever assets grow without bound, consumption also grows without bound. This result is established as Lemma 2.

LEMMA 2. *For any $\alpha_1 > 0$, there is an $\alpha_2 > 0$ such that $a_t^* \geq \alpha_2$ implies $c_t^* \geq \alpha_1$.*

Proof. From Lemma A.2, $0 \leq v_t(a_t^*) < \frac{M}{1-\beta}$. Therefore, the concavity of v_t implies $a_t^* v_t^+(a_t^*) \leq v_t(a_t^*) - v_t(0) \leq \frac{M}{1-\beta}$. Choose $\alpha_2 = M/(1-\beta)u^+(\alpha_1)$. Then if $a_t^* \geq \alpha_2$, Lemma 1 implies $u^+(c_t^*) \leq v_t^+(a_t^*) \leq v_t^+(\alpha_2) \leq M/(1-\beta)\alpha_2 = u^+(\alpha_1)$. The concavity of u then implies $c_t^* \geq \alpha_1$. ∎

With these results in hand, we are prepared to examine the limiting behavior of the optimal consumption sequence.

4. CONVERGENCE THEOREMS

Recall that Theorem 1 establishes that the function $\theta_t v_t^+(a_t^*)$ must converge to some random variable e_∞. In this section we demonstrate that the implications of this result for the limiting behavior of the optimal consumption sequence depend primarily on the limiting value of θ_t. Roughly, our results may be summarized as follows. If $\lim_{t\to\infty} \theta_t = \infty$, the optimal consumption sequence of the consumer must grow without bound, regardless of the properties of the income and interest rate sequences. If the limiting value (or values) of θ_t is bounded above and away from zero and the income stream is suitably stochastic, then consumption still grows without bound. However, if the income sequence is not stochastic, then the consumption sequence generally converges to a finite limit.

These results may also be interpreted in terms of the relationship between the rate at which the consumer discounts future utility and the long run rate of interest. In Section 2, we defined r_t to be the one period interest factor between period $t-1$ and period t and $R_{tj} \equiv \prod_{k=t+1}^{j} r_k$ to be the accumulated interest factor between periods t and j. Therefore, $R_{tj}^{1/(j-t)}$ represents the *average* interest factor between periods t and j. If $\lim_{t\to\infty} R_{0t}^{1/t}$ exists, we call it the *long run interest factor*; otherwise, we say that the long run interest rate *fluctuates*. Now consider the meaning of $\theta_t \equiv \beta^t R_{0t}$. Rewriting the expression, we obtain $\beta R_{0t}^{1/t} = \theta_t^{1/t}$. Therefore, the long run rate of interest is equal to the discount rate if and only if $\lim_{t\to\infty}\theta_t^{1/t} = 1$. Clearly, if the limiting values of θ_t are positive and finite, then the long run interest rate exists and is equal to the discount rate. (Although this is not a necessary condition, we will sometimes blur this distinction for expositional convenience.)

We may translate our conclusions as follows. The limiting behavior of consumption depends primarily on the relation between the long run rate of interest and the rate at which the consumer discounts utility. If the long run rate of interest is greater than the discount rate, then consumption grows without bound as long as the consumer earns a positive income in some period. If the long run rate of interest is equal to the discount factor, then consumption generally converges to infinity only if there is sufficient uncertainty in either the income or interest rate sequences.

All of our results are established for the case where both the income and interest rate sequences may be stochastic. In many cases, however, the intuition behind our results and the meaning of the assumptions become more apparent when only the income sequence is stochastic. Throughout this section, therefore, we will frequently focus the discussion on the special case where the interest rate is a constant. We conclude the section with an interpretation of our results in the context of a stationary distribution of interest rates.

We turn first to the case where the long run interest rate exceeds the rate at which the consumer discounts utility.

4.1. Lim $\theta_t = \infty$

Our main result for the case where the long run rate of interest exceeds the discount rate is summarized in the following theorem.

THEOREM 2. *Suppose* $P(\mathbf{x} = 0) = 0$. *Then* (i) $P(\limsup_{t\to\infty}\theta_t = \infty) = 1$ *implies* $P(\limsup_{t\to\infty}c_t^* = \infty) = 1$, *and* (ii) $P(\lim_{t\to\infty}\theta_t = \infty) = 1$ *implies* $P(\lim c_t^* = \infty) = 1$.

Proof. Theorem 1 implies that if $P(\limsup_{t\to\infty}\theta_t = \infty) = 1$ and $P(\mathbf{x} = 0) = 0$, then $P(\liminf_{t\to\infty}v_t^+(a_t^*) = 0) = 1$. Lemma 1 then implies that $P(\liminf_{t\to\infty}u^+(c_t^*) = 0) = 1$, and therefore, that $P(\limsup_{t\to\infty}c_t^* = \infty) = 1$. This proves part (i). The proof of part (ii) is similar. ∎

Roughly, Theorem 2 says that consumption grows without bound so long as the long run rate of interest always exceeds the discount rate. If the long run interest rate fluctuates, but some subsequence always exceeds the discount rate, then some subsequence of consumption grows without bound. When the interest rate is constant, the theorem may be restated as follows.

COROLLARY 1. *Suppose that* $r_t = \gamma$ *for* $t \geq 1$. *If* $P(\mathbf{x} = 0) = 0$ *and* $\beta\gamma > 1$, *then* $P(\lim_{t\to\infty} c_t^* = \infty) = 1$.

4.2. *The Case of Certainty*: $\beta\gamma = 1$

For the remainder of this section, we are concerned primarily with the case where the long run interest rate is equal to the rate at which the consumer discounts future utility. To underline the importance of the role of uncertainty in the main results which follow, we first concentrate on the case where the income sequence is nonstochastic and the interest rate in each period is equal to the rate at which the consumer discounts utility. Again denote the constant interest factor by γ.

When $\beta\gamma = 1$, the first-order conditions for utility maximization imply that the consumer tries to equalize consumption in each period. As long as such a program does not violate the borrowing constraint, this condition characterizes the solution. If such a program does violate the borrowing constraint, however, the consumer must choose a consumption stream with unequal levels of consumption over time. Nevertheless, consumption never decreases over time. Otherwise, by saving more in an earlier period and consuming more later, lifetime utility can be increased. This observation implies that the consumption level approaches a limit (possibly equal to ∞), which we characterize in the next theorem.

Define $y_t \equiv ((\gamma - 1)/\gamma)\sum_{j=t}^{\infty} x_j \gamma^{t-j}$ to be the supremum of those consumption levels that can be sustained indefinitely when we consider only the borrowing constraints in the distant future, given that the borrowing constraint is binding in period $t-1$. Our next theorem states that c_t^* converges to the supremum of these maximum sustainable consumption levels.

THEOREM 3. *If* $\mathbf{x}$ *is not stochastic*, *then* $\beta\gamma = 1$ *implies* $\lim_{t\to\infty} c_t^* = \sup_t y_t$.

Proof. We show first that for $t \geq 1$, either (a) $c_{t-1}^* = c_t^*$, or (b) $c_{t-1}^* < c_t^*$ and $c_{t-1}^* = a_{t-1}^*$ (in which case, $a_t^* = x_t$). Suppose $c_{t-1}^* < a_{t-1}^*$. If $c_{t-1}^* = 0$, then Lemma 1 and the strict concavity of u and v_{t-1} imply $v_{t-1}^+(0) > v_{t-1}^+(a_{t-1}^*) \geq u^+(0)$, which violates Lemma A.3. So suppose $0 < c_{t-1}^* < a_{t-1}^*$. Then, on one hand, Lemma 1 implies $u^-(c_{t-1}^*) \geq v_{t-1}^-(a_{t-1}^*) \geq v_{t-1}^+(a_{t-1}^*) \geq v_t^+(a_t^*) \geq u^+(c_t^*)$, and therefore, $0 \leq c_{t-1}^* \leq c_t^*$. On the other hand, Lemma 1 also implies $u^+(c_{t-1}^*) \leq v_{t-1}^+(a_{t-1}^*) \leq v_{t-1}^-(a_{t-1}^*) \leq v_t^-(a_t^*) \leq u^-(c_t^*)$, and therefore, $c_t^* \leq c_{t-1}^*$.

Let $\bar{c} \equiv \lim_{t\to\infty} c_t^*$ and let $\bar{y} \equiv \sup_t y_t$. We show first that $\bar{c} \leq \bar{y}$. Suppose not and let t be the smallest $\tau \geq 0$ such that $c_\tau^* > \bar{y}$. Then, condition (b) above implies $a_t^* = x_t$. From the definition of a_t^*, it then follows that

$$\sum_{j=0}^{t-1} (x_j - c_j^*)\gamma^{-j} = 0. \tag{2}$$

Conditions (a) and (b) also imply that c_τ^* is nondecreasing in t. Therefore, $c_j^* > y_t$ for all $j \geq t$, so that $((\gamma - 1)/\gamma)\sum_{j=t}^{\infty} c_j^* \gamma^{t-j} \geq c_t^* > y_t = ((\gamma - 1)/\gamma)\sum_{j=t}^{\infty} x_j \gamma^{t-j}$, which implies a τ sufficiently large such that

$$\sum_{j=t}^{\tau} (x_j - c_j^*)\gamma^{t-j} < 0. \tag{3}$$

Combining (2) and (3) then yields $\sum_{j=0}^{\tau}(x_j - c_j^*)\gamma^{-j} < 0$, violating the budget constraint of the consumer's maximization problem.

To show that $\bar{c} \geq \bar{y}$, again assume the contrary. Then there is a y_t such that $c_j^* < y_t$ for all $j \geq 0$, which implies

$$\sum_{j=t}^{\infty} c_j^* \gamma^{t-j} < \sum_{j=t}^{\infty} x_j \gamma^{t-j}. \tag{4}$$

But if c_t^* is feasible, then

$$\sum_{j=0}^{t-1} (x_j - c_j^*)\gamma^{-j} \geq 0. \tag{5}$$

It follows from (4) and (5) that there is an $\varepsilon > 0$ and $\hat{\tau} \geq t$ such that, for all $\tau \geq \hat{\tau}$, $\sum_{j=0}^{\tau}(x_j - c_j^*)\gamma^{-j} > \varepsilon$. But then $\hat{\mathbf{c}}$, defined by $\hat{c}_j = c_j^*$ for $j \neq \hat{\tau}$ and $\hat{c}_{\hat{\tau}} = c_{\hat{\tau}}^* + \varepsilon$, also satisfies the consumer's budget constraint. But then $\sum_{j=0}^{\infty} u(\hat{c}_j)\beta^j > \sum_{j=0}^{\infty} u(c_j^*)\beta^j$ implies that $\mathbf{c}^*$ is not an optimal consumption program. ∎

For our purposes the main implication of this theorem is that when the income stream is certain, the consumption sequence generally converges to a finite limit. Consumption grows without bound only if the discounted value of future income is not bounded. However, the theorem also has a noteworthy implication in the context of the original formulation of the model.

Suppose that $\mathbf{x}$ is an arbitrary income sequence and $\mathbf{k}$ is an intertemporally consistent sequence of borrowing constraints from which we derive $\hat{x}_t \equiv x_t - k_t + \gamma k_{t-1} \geq 0$. Then, $y_t \equiv ((\gamma - 1)/\gamma)\sum_{j=t}^{\infty} \hat{x}_j \gamma^{t-j} = ((\gamma - 1)/\gamma)\lim_{\tau \to \infty}(\sum_{j=t}^{\tau} x_j \gamma^{t-j} - \gamma^{t-\tau} k_\tau) + (\gamma - 1)k_{t-1}$. Now suppose we start with a sequence of borrowing constraints which generates an optimal consumption program $\mathbf{c}^1$. Then if the $\mathbf{k}^1$ constraint is replaced by a tighter constraint, $\mathbf{k}^2 \geq \mathbf{k}^1$ (i.e., $k_t^2 \geq k_t^1$ for all t), but which allows for the same discounted value of consumption (i.e., $\lim_{\tau \to \infty} \gamma^{-\tau}(k_\tau^1 - k_\tau^2) = 0$), then the tighter constraint $\mathbf{k}^2$ generates a limiting value of consumption at least as large as did the $\mathbf{k}^1$ constraint (i.e., $\lim_{t \to \infty} c_t^2 \geq \lim_{t \to \infty} c_t^1$). An even stronger result can be established about the behavior of the sequence of accumulated wealth in each period, but this is left to Section 5, where the stochastic case is treated as well.

4.3. *The Stochastic Case*

As noted above, the analysis of the certainty case yields little insight into the limiting behavior of consumption when we introduce uncertainty into either the income sequence or the interest rate sequence. In this subsection, we formulate a general "uncertainty" condition under which we show that consumption grows without bound even when the long run interest rate equals the discount rate.

The meaning of our uncertainty condition is most transparent when the interest factor is fixed at $\gamma = \frac{1}{\beta}$. In this case, we require:

Condition (Uγ). There is an $\varepsilon > 0$ such that for any $\alpha \in \mathbb{R}$,

$$P\left(\alpha \leq \sum_{j=t}^{\infty} x_j \gamma^{t-j} \leq \alpha + \varepsilon | z^t\right) < 1 - \varepsilon$$

for all $z^t, t \geq 0$.

Condition (Uγ) says that starting at any information state, there is a fixed probability that the discounted value of future income lies outside any sufficiently small range. The key implication of this condition is when consumption stays within a sufficiently small range in each period, assets must diverge with some fixed probability from any information state. However, the proof of our main result requires a fixed probability that assets diverge whenever the consumption program keeps the *marginal utility* of consumption approximately constant. Consequently, when we allow for a stochastic interest rate, the uncertainty condition requires a slight reformulation that uses the utility function to restrict the relation between the income and interest rate sequences. Define the "inverse" of u^+ by $h(0) \equiv \infty$ and, for $y > 0$, $h(y) \equiv \inf\{c \in \mathbb{R}_+ \colon u^+(c) \leq y\}$.

Condition (U). There is an $\varepsilon > 0$ such that for any $\alpha \in \mathbb{R}$ and any $\phi > 0$,

$$P\left(\alpha \leq \sum_{j=t}^{\infty} \left(x_j - h\left(\frac{\phi}{\theta_j}\right)\right) R_{tj}^{-1} \leq \alpha + \varepsilon | z^t\right) < 1 - \varepsilon$$

for all $z^t, t \geq 0$.

Suppose the consumption program were chosen so that, in any information state, an increase in the present value of consumption by one unit generates an increase in discounted utility equal to ϕ. Then the consumption level in each information state would be determined by the relation $c_t^* = h(\phi/\theta_t)$, and Condition (U) would imply a nontrivial probability that the present value of current assets plus future income is bounded away from the present value of current plus future consumption, starting at any state z^t. Roughly speaking, the condition says that the actual income stream is stochastic relative to the hypothetical income stream required to make the marginal discounted utility of a "present value" unit of consumption equal in all information states. When the interest factor is constant, Condition (Uγ) implies Condition (U).

Our results are based on Lemma 4 below. Roughly, it says that if θ_t is bounded away from zero and infinity (the long run interest rate is equal to the discount rate), then Condition (U) implies that the marginal utility of assets $v_t^+(a_t^*)$ must converge to 0. The argument goes as follows. Suppose $v_t^+(a_t^*)$ does not converge to zero. Then we may choose $\tau < \infty$ and an arbitrarily small interval $[b, b+\varepsilon]$ with $b > 0$ such that (i) there is positive probability that $\theta_\tau v_\tau^+(a_\tau^*) \in [b, b+\varepsilon]$, and (ii) if $\theta_\tau v_\tau^+(a_\tau^*) \in [b, b + \varepsilon]$, then, with probability greater than $1-\varepsilon$, $\theta_t v_t^+(a_t^*) \in [b, b+\varepsilon]$ for all $t \geq \tau$. Now consider the (nonnull) set of paths for which $\theta_\tau v_\tau^+(a_\tau^*) \in [b, b+\varepsilon]$ for all $t \geq \tau$. Since the first-order conditions for utility maximization imply that $c_t^* = h(v_t^+(a_t^*))$, Condition (U) implies that whenever $\theta_t v_t^+(a_t^*) \in [b, b+\varepsilon]$ for all $t \geq \tau$, a_t^* must diverge with probability at least ε. But if a_t^* diverges, Lemma 2 implies that c_t^* must converge to infinity, which if θ_t is bounded above, implies that $\theta_t v_t^+(a_t^*)$ converges to 0. This contradiction establishes the result.

We proceed now with the formal analysis. To use Theorem 1 we must first establish that Condition (U) implies a nonzero income stream.

LEMMA 3. *Condition* (U) *implies* $P(\mathbf{x} = 0) = 0$.

Proof. We show first that Condition (U) implies that $P(\mathbf{x} = 0|z^t) < 1 - \varepsilon$ for all z^t. For this, it is sufficient to show that for any z^t we may choose ϕ sufficiently large so that $P(\sum_{j=t}^{\infty} h(\phi/\theta_j) R_{tj}^{-1} < \varepsilon | z^t) = 1$. Since u is concave and bounded between 0 and M, we have $cu^-(c) < M$ for all $c \in \mathbb{R}_+/\{0\}$. By definition, $u^-(h(\phi/\theta_j)) \geq \phi/\theta_j$. Therefore, $(\phi/\theta_j) h(\phi/\theta_j) \leq M$. Then $R_{tj}^{-1} \equiv (\theta_t/\phi)((\phi/\theta_j)\beta^{j-t})$ implies $\sum_{j=t}^{\infty} h(\phi/\theta_j) R_{tj}^{-1} = (\theta_t/\phi)\sum_{j=t}^{\infty}(\phi/\theta_j) h(\phi/\theta_j)\beta^{j-t} \leq (\theta_t/\phi)(M/(1-\beta))$. Setting $\phi > \theta_t M/\varepsilon(1-\beta)$ establishes the result.

All that remains is to show that $P(\mathbf{x} = 0|z^t) < 1 - \varepsilon$ for all z^t implies $P(\mathbf{x} = 0) = 0$. We establish this by contradiction. Suppose $P(\mathbf{x} = 0) > 0$. Let $A_t = \{\mathbf{x}^t = 0\}$ and $A_\infty = \{\mathbf{x} = 0\}$. Then $P(A_t) > 0$ for all $t \geq 0$, and $A_j \downarrow A_\infty$. Then, since $P(\mathbf{x} = 0|z^t) < 1 - \varepsilon$ for all z^t and A_t is measurable

z^t, we have $P(A_j|A_t) \to P(A_\infty|A_t) < 1 - \varepsilon$. Therefore, there is an increasing sequence $(\tau(t))$ such that $P(A_{\tau(t+1)}|A_{\tau(t)}) < 1 - \varepsilon$ which implies $P(\mathbf{x} = 0) = \prod_{t=0}^{\infty} P(A_{\tau(t+1)}|A_{\tau(t)}) P(A_{\tau(0)}) \leq \lim_{t\to\infty} (1-\varepsilon)^t P(A_{t(0)}) = 0$. ∎

LEMMA 4. *Suppose there is an* $\varepsilon > 0$ *for which Condition* (U) *is satisfied and* $P(\varepsilon < \theta_t < \frac{1}{\varepsilon},\ t \geq 0) = 1$. *Then* $P(\lim_{t\to\infty} \theta_t v_t^+(a_t^*) = 0) = 1$.

Proof. We will suppose the lemma is false and show that this implies a violation of Condition (U).

Our first step is to find an interval $[\phi, \phi + \delta]$ and a time τ such that $[\phi, \phi + \delta]$ contains $\theta_\tau v_\tau^+(a_\tau^*)$ with positive probability and, in the event that $\theta_\tau v_\tau^+(a_\tau^*) \in [\phi, \phi + \delta]$, it is also true that $\theta_t v_t^+(a_t^*) \in [\phi, \phi + \delta]$ and c_t^* is near $h(\phi/\theta_t)$ with probability close to 1 for all $t \geq \tau$. Theorem 1 and Lemma 3 imply a random variable e_∞ such that $P(\lim_{t\to\infty} \theta_t v_t^+(a_t^*) = e_\infty) = 1$. If $P(e_\infty = 0) < 1$, then there is a $\psi > 0$ such that for any $\delta > 0$, $P(e_\infty \in [\psi, \psi + \delta]) > 0$. Let $\eta \equiv \varepsilon^3(1-\beta)/2$. Then, since h is uniformly continuous on any positive interval bounded away from zero, we may choose ϕ and δ, $0 < \phi < \psi < \phi + \delta$, so that (i) $P(e_\infty \in [\phi, \phi + \delta]) > 0$, (ii) $P(e_\infty = \phi) = P(e_\infty = \phi + \delta) = 0$, and (iii) $|h(\phi/\theta_t) - h((\phi + \delta)/\theta_t)| < \eta$ for $\varepsilon < \theta_t < \frac{1}{\varepsilon}$. Define $B \equiv \{e_\infty \in [\phi, \phi + \delta]\}$, and for $\tau \geq 0$, define $A_\tau \equiv \{\theta_\tau v_\tau^+(a_\tau^*) \in [\phi, \phi + \delta]\}$ and $B_\tau \equiv \{|c_t^* - h(\phi/\theta_t)| < \eta,\ \theta_t v_t^+(a_t^*) \in [\phi, \phi + \delta],\ t \geq \tau\}$. Then $\lim_{\tau\to\infty} P(A_\tau) = P(B) > 0$. Also, Lemma 1 implies $P(c_t^* - h(v_t^+(a_t^*)) = 0,\ t \geq 0) = 1$. Therefore, $P(\varepsilon < \theta_t < \frac{1}{\varepsilon}) = 1$ and the continuity of h imply $P(\lim_{t\to\infty}(c_t^* - h(\varepsilon_\infty/\theta_t)) = 0|B) = 1$. Then, since the monotonicity of h implies $P(h(\phi/\theta_t) \geq h(e_\infty/\theta_t) \geq h(\phi + \delta)|B) = 1$, we have $\lim_{\tau\to\infty} P(B_\tau) = P(B)$. Consequently, we may choose $\tau < \infty$ such that $P(B_\tau) > (1-\varepsilon)P(A_\tau) > 0$.

We show next that if the marginal utility of consumption c_j is always equated to ϕ/θ_j, then the variation in the present value of the limiting level of assets is less than $\frac{\varepsilon}{2}$ on B_τ. The monotonicity of h and the construction of B_τ imply $P(c_\tau^* < h(\phi\varepsilon) + \eta,\ t \geq \tau|B_\tau) = 1$. Therefore, letting $\alpha_2 \equiv h(\phi\varepsilon) + \eta$, Lemma 2 implies an $\alpha_1 < \infty$ such that $P(0 \leq a_t^* - c_t^* \leq \alpha_1,\ t \geq \tau|B_\tau) = 1$. Then, since $P(\varepsilon < \theta_t < \frac{1}{\varepsilon},\ t \geq 0) = 1$ implies $P(R_{\tau j}^{-1} \equiv \theta_\tau \beta^{j-\tau}/\theta_j \leq \varepsilon^{-2}\beta^{j-\tau},\ j \geq \tau) = 1$, it follows that $P(\lim_{t\to\infty}(a_t^* - c_t^*)R_{\tau t}^{-1} = 0|B_\tau) = 1$. Therefore,

$$
\begin{aligned}
&(a_t^* - c_t^*) R_{\tau t}^{-1} \\
&\qquad = a_\tau^* - c_\tau^* + \sum_{j=\tau+1}^{t} (x_j - c_j^*) R_{\tau j}^{-1} \\
&\qquad = a_\tau^* - x_\tau + \sum_{j=\tau}^{t} \left(x_j - h\left(\frac{\phi}{\theta_j}\right)\right) R_{\tau j}^{-1} + \sum_{j=\tau}^{t} \left(h\left(\frac{\phi}{\theta_j}\right) - c_j^* \right) R_{\tau j}^{-1}
\end{aligned}
$$

implies

$$P\left(a_\tau^* - x_\tau + \sum_{j=\tau}^{\infty}\left(x_j - h\left(\frac{\phi}{\theta_j}\right)\right)R_{\tau j}^{-1} + \sum_{j=\tau}^{\infty}\left(h\left(\frac{\phi}{\theta_j}\right) - c_j^*\right)R_{\tau j}^{-1} = 0 \,\middle|\, B_\tau\right) = 1. \tag{6}$$

But from $P(R_{\tau j}^{-1} \equiv \theta_\tau \beta^{j-\tau}/\theta_j \leq \varepsilon^{-2}\beta^{j-\tau},\ j \geq \tau) = 1$ and the definition of B_τ it follows that

$$P\left(\left|\sum_{j=\tau}^{\infty}\left(h\left(\frac{\phi}{\theta_j}\right) - c_j^*\right)R_{\tau j}^{-1}\right| < \frac{\eta}{(1-\beta)\varepsilon^2} \equiv \frac{\varepsilon}{2} \,\middle|\, B_\tau\right) = 1. \tag{7}$$

Combining (6) and (7) then yields

$$P\left(\left|a_\tau^* - x_\tau + \sum_{j=\tau}^{\infty}\left(x_j - h\left(\frac{\phi}{\theta_j}\right)\right)R_{\tau j}^{-1}\right| < \frac{\varepsilon}{2} \,\middle|\, B_\tau\right) = 1. \tag{8}$$

We now use these results to show that Condition (U) must be violated. Let $\alpha = x_\tau - a_\tau^* - \frac{\varepsilon}{2}$. Since $B_\tau \subset A_\tau$ and $P(B_\tau) > (1-\varepsilon)P(A_\tau)$, it follows from (8) that $P(\alpha < |\sum_{j=\tau}^{\infty}(x_j - h(\phi/\theta_j))R_{\tau j}^{-1}| < \alpha + \varepsilon | A_\tau) > 1 - \varepsilon$. But then A_τ measurable z^τ implies that the set of z^τ such that $P(\alpha < |\sum_{j=\tau}^{\infty}(x_j - h(\phi/\theta_j))R_{\tau j}^{-1}| < \alpha + \varepsilon | z^\tau) > 1 - \varepsilon$ has positive probability, which violates Condition (U). ∎

We use this lemma again in Section 5. For the moment, however, it serves as the basis for the main result of this paper.

THEOREM 4. *Suppose there is an* $\varepsilon > 0$ *such that Condition* (U) *is satisfied. If* $P(\varepsilon < \theta_t < \frac{1}{\varepsilon},\ t \geq 0) = 1$, *then* $P(\lim_{t\to\infty} c_t^* = \infty) = 1$.

Proof. If $P(\varepsilon < \theta_t < \frac{1}{\varepsilon},\ t \geq 0) = 1$, then Lemma 4 implies $P(\lim_{t\to\infty} v_t^+(a_t^*) = 0) = 1$. The conclusion then follows from Lemma 1. ∎

With sufficient uncertainty in the income and interest rate sequences, consumption will grow without bound even if the long run rate of interest is equal to the discount rate. A case in which this equality holds has been considered by Bewley (1980c, 1983) in his treatment of the optimum quantity of money. There is an asset with a fixed nominal return factor equal to β^{-1}. The real return is $r_t = \beta^{-1}(q_{t-1}/q_t)$, where the price (q_t) of the consumption good in terms of the asset is uniformly bounded away from zero and infinity. Consequently, $\theta_t = q_0/q_t$ satisfies the conditions of Theorem 4. We conclude that if the uncertainty condition is satisfied, c_τ^* converges to infinity.

For the case where the interest rate is constant, Theorem 4 implies the following corollary.

COROLLARY 2. *Suppose* $r_t = \gamma$ *for all* $t \geq 0$ *and suppose* $\beta\gamma = 1$. *If Condition* (Uγ) *is satisfied*, *then* $P(\lim_{t \to \infty} c_t^* = \infty) = 1$.

If income is bounded above, then Condition (Uγ) in Corollary 2 can be replaced by the condition that the conditional variance of discounted future income is uniformly bounded away from zero; i.e., there is a $\psi > 0$ such that $\mathrm{Var}(\sum_{j=t}^{\infty} x_j \gamma^{t-j} | z^t) \geq \psi$ for all $z^t, t \geq 0$. However, if the income stream is stochastic, but the conditions of Corollary 2 are not satisfied, there are examples where the limiting level of consumption is finite with some positive probability.

When contrasted with the outcome in the case of certainty, Corollary 2 is perhaps a surprisingly strong result. Unfortunately, the line of argument used in the proof does not provide a very convincing economic explanation. Clearly the strict concavity of the utility function must play a role. (The result does not hold if, for instance, u is a linear function over a sufficiently large domain and (x_t) is bounded.) But to simply attribute the result to risk aversion on the grounds that uncertain future returns will cause risk-averse consumers to save more, given any initial asset level, is not a completely satisfactory explanation either. In fact, it is a bit misleading. First, that argument only explains why expected accumulated assets would tend to be larger in the limit. It does not really explain why consumption should grow without bound. Second, over any finite time horizon, the argument is not even necessarily correct.

Suppose, for example, that $x_t = x_{t-1}\varepsilon_t$ for $t = 1, \dots, T$, where the (ε_t) are identically and independently distributed with $E[\varepsilon_t] = 1$ and a compact support in $\mathbb{R}$. Suppose, also, that the consumer's utility function is quadratic over a sufficiently large domain. Then if the consumer has a T-period planning horizon, it can be shown that his optimal consumption program is to set $c_t = x_t$ for all t. In particular, mean-preserving spreads of future income leave current consumption unaffected. Moreover, the expected value of consumption in any period t is just equal to period 0 income. So there is no tendency at all for consumption to rise over time.

Why then does the result change when we consider the limiting value of consumption in an infinite horizon problem? Although we have developed other arguments to establish our results, any explanation that we have been able to devise ultimately appeals in an essential way to the martingale convergence theorems.

4.4. *Limit Theorems When* (r_t) *Is Stationary*

In this subsection, we assume that the sequence of information states is generated by a stationary process and establish some restrictions on the distribution of the single period interest rate that imply the conditions of Theorem 2. Since the stationarity restriction will be placed on the entire distribution of histories, we will be explicit about conditioning on z^0.[7]

THEOREM 5. *Suppose* $P(\mathbf{x} = 0) = 0$. *If* $(r_1, r_2, \dots)$ *is stationary and ergodic and* $E[\log \beta r_1] > 0$, *then* $P(\lim_{t\to\infty} c_t^* = \infty | z^0) = 1$ *with probability* 1.

Proof. Let $\psi = E[\log \beta r_1]$. The Ergodic Theorem implies that with probability 1, $\lim_{t\to\infty} \frac{1}{t}\sum_{j=1}^{t} \log \beta r_j = \psi$ (Doob, 1953, p. 465). Therefore, for almost all $\mathbf{z}$ there is a $\tau(\mathbf{z})$ such that $\sum_{j=1}^{t} \log \beta r_j > \frac{\psi t}{2}$ for $t > \tau(\mathbf{z})$, or equivalently, $\theta_t > e^{\psi t/2}$. Therefore, $P(\lim_{t\to\infty} \theta_t = \infty) = 1$, which implies that $\{z^0 \colon P(\lim_{t\to\infty} \theta_t = \infty | z^0) = 1\}$ has probability 1. The desired result then follows from Theorem 2. ∎

For the case where $E[\log(\beta r_1)] = 0$, we need some additional regularity conditions. For $s \le t$, define $\sigma(r_s, \dots, r_t)$ as the σ-field generated by $r_s, \dots, r_t$, and define $\sigma(r_t, r_{t+1}, \dots)$ as the sigma field generated by $r_t, r_{t+1}, \dots$. Consider a nonnegative function ϕ of the positive integers. The sequence $(r_1, r_2, \dots)$ is *ϕ-mixing* if for each $t, j \ge 1$, $A_1 \in \sigma(r_1, \dots, r_t)$ and $A_2 \in \sigma(r_{t+j}, r_{t+j+1}, \dots)$ together imply that $|P(A_1 \cap A_2) - P(A_1)P(A_2)| \le \phi(j)P(A_1)$.

Condition (R). (i) $(r_1, r_2, \dots)$ is a stationary, ϕ-mixing stochastic process with $\sum_{j=1}^{\infty} (\phi(j))^{1/2} < \infty$. (ii) $E[(\log \beta r_1)^2] < \infty$ and

$$\sigma^2 \equiv E\left[(\log \beta r_1)^2\right] + 2\sum_{j=2}^{\infty} E\left[(\log \beta r_1)(\log \beta r_j)\right] \neq 0.$$

The first part of Condition (R) ensures that the dependence between r_t and r_{t+j} dies out sufficiently fast as $j \to \infty$. Suppose part (i) is satisfied and $E[\log \beta r_1] = 0$. Then $\sum_{j=2}^{\infty} E[(\log \beta r_1)(\log \beta r_j)]$ converges absolutely and $\sigma^2 = \lim_{T\to\infty} E[\frac{1}{T}(\sum_{j=1}^{T} \log \beta r_j)^2]$ (Billingsley, 1968, Lemmata 1 and 3, pp. 170, 172). In this case, part (ii) may be satisfied as well if there is sufficient variability in $\beta^t R_{0T}$. However, if $P(\beta r_1 = 1) = 1$, part (ii) is not satisfied, and we know from Theorem 4 that the behavior of (c_t^*) may depend upon

[7] Theorem A.1 establishes the existence of an optimal program only for a fixed z^t. Because we did not restrict z_t to be drawn from a compact set, we are able to show that $c_t^*(z^t)$ is a measurable function of z^t only for a fixed tail (e.g., a given z^0). Without additional restrictions we are unable to prove that c_t^* is a measurable function if all components of z^t are allowed to vary.

whether or not (x_t) is stochastic. Nevertheless, our next theorem states that if Condition (R) is satisfied and $E[\log \beta r_1] = 0$, then $\sup_t c_t^* = \infty$, without a requirement that (x_t) be stochastic.

THEOREM 6. *Suppose* $P(\mathbf{x} = 0) = 0$ *and* $(r_1, r_2, \dots)$ *satisfies Condition* (R). *Then* $E[\log \beta r_1] = 0$ *implies* $P(\sup_{t \geq 0} c_t^* = \infty | z^0) = 1$ *with probability* 1.

Proof. Define $S_t \equiv \sum_{j=1}^{t} \log \beta r_j$ $(= \log \theta_t)$ and define $Y_t(\delta) \equiv S_{\lfloor \delta t \rfloor}(\sigma^2 t)^{-1/2}$, where $0 \leq \delta \leq 1$ and $\lfloor \delta t \rfloor$ is the greatest integer less than or equal to δt. Then the functional central limit theorem implies that the distribution of the random function Y_t converges weakly to Weiner measure (Billingsley, 1968, Theorem 20.1), from which it follows that

$$\lim_{t \to \infty} P\left((\sigma^2 t)^{-1/2} \max_{j \leq t} S_j \leq \alpha \right) = 2(2\pi)^{-1/2} \int_0^{\alpha} e^{-u^2/2}\, du \tag{9}$$

for $\alpha \geq 0$ (Billingsley, 1968, p. 138, and Eq. (10.18), p. 72).

Define $\psi \equiv P(\sup_{t \geq 1} S_t < \infty)$. We shall assume that $\psi > 0$ and obtain a contradiction. Choose $\varepsilon > 0$ so that $2(2\pi)^{-1/2} \int_0^{\varepsilon} \exp(-u^2/2)\, du < \frac{\psi}{2}$ and, for $t \geq 1$, define $A_t \equiv \{(\sigma^2 t)^{-1/2} \max_{j \leq t} S_j \leq \varepsilon\}$. Then $\lim_{t \to \infty} P(A_t) < \frac{\psi}{2}$. For $t \geq 1$, define $B_t \equiv \{(\sigma^2 \tau)^{-1/2} \max_{j \leq \tau} S_j \leq \varepsilon$ for all $\tau \geq t\}$. Then there is a B such that $\{\sup_{t \geq 1} S_t < \infty\} \subset B$ and $B_t \uparrow B$. Therefore, using Eq. (9), $B_t \subset A_t$ implies $P(A_t) \geq P(B_t) \to P(B) \geq \psi$, which contradicts $\lim_{t \to \infty} P(A_t) < \frac{\psi}{2}$. We conclude that $P(\sup_{t \geq 1} S_t < \infty) = 0$, or equivalently, $P(\sup_{t \geq 1} \theta_t = \infty) = 1$. The desired conclusion then follows from Theorem 1 and Lemma 1. ∎

5. THE BUDGET CONSTRAINT

In this section, we address two questions. First, given an arbitrary stochastic income sequence, under what conditions should we expect the borrowing constraint to be binding at some information state? Second, how does a change in the borrowing constraint (or equivalently, a change in the income sequence) affect the pattern of borrowing?

5.1. *When Is Budget Constraint Sometimes Binding*?

As noted in Section 2, if the discounted value of the income sequence is finite conditional on any information state, we can represent any intertemporal budget constraint as a sequence of one period borrowing constraints. It may well turn out that none of these constraints are actually binding at the optimum, and yet the consumer is still constrained to choose a consumption program whose present value does not exceed the present value of the income stream. Our next theorem gives conditions under which this is the case.

THEOREM 7. *If* $E[r_{t+1}u^+(x_{t+1})|z^t] = \infty$ *and* $a_t^* > 0$, *then* $c_t^* < a_t^*$.

Proof. Suppose $c_t^* = a_t^*$. Then $P(c_{t+1}^* \leq x_{t+1}|z^t) = 1$. Lemma 1 then implies

$$\infty > u^-(c_t^*) \geq v_t^+(a_t^*) \geq \beta E\left[r_{t+1}v_{t+1}^+(a_{t+1}^*)|z^t\right]$$
$$\geq \beta E\left[r_{t+1}u^+(c_{t+1}^*)|z^t\right] \geq \beta E\left[r_{t+1}u^+(x_{t+1})|z^t\right] = \infty,$$

a contradiction. ∎

Since $a_t^* \geq x_t$, the following corollary follows immediately.

COROLLARY 3. *If* $E[r_{t+1}u^+(x_{t+1})|z^t] = \infty$ *and* $x_t > 0$, *then* $c_t^* < a_t^*$.

Theorem 7 and its corollary say that whenever the expected increment in disposable income in the following period is sufficiently small so that the expected marginal utility from consuming out of that increment would be infinite, the consumer chooses to consume less than his current wealth in the current period in order to pass some of his wealth to the next period. If we allow the expected marginal utility of future income to be finite, however, the borrowing constraint may well be binding, at least occasionally. This will obviously be the case if income received in each period is growing at a sufficiently high rate over time so that the consumer wants to transfer future income to present consumption. But if the income stream is suitably stochastic, a much weaker set of conditions guarantees that the budget constraint is sometimes binding.

Our next theorem may be summarized as follows. Suppose the single period interest rate never exceeds the discount rate and the marginal utility of consuming from current income alone is bounded above. Then if the long run rate of interest is less than the discount rate or if the income sequence is suitably stochastic, there is a positive probability that the borrowing constraint is binding in at least one period.

THEOREM 8. *Suppose* $P(\beta r_t \leq 1,\ t \geq 1) = 1$ *and* $P(u^+(x_t) < \alpha,\ t \geq 0) = 1$ *for some* $\alpha \in \mathbb{R}$. *If either* (i) $P(\lim_{t\to\infty}\theta_t = 0) = 1$ *or* (ii) *there is an* $\varepsilon > 0$ *for which Condition* (U) *is satisfied and* $P(\theta_t > \varepsilon,\ t \geq 0) = 1$, *then* $P(c_t^* < a_t^*,\ t \geq 0) < 1$.

Proof. We establish the theorem by contradiction. Suppose $P(0 \leq c_t^* < a_t^*,\ t \geq 0) = 1$. Then, by induction, Lemma 1 implies $v_0^-(a_0^*) \leq E[\beta r_1 v_1^-(a_1^*)] \leq E[\theta_t v_t^-(a_t^*)]$ for all $t \geq 1$. We will establish that

$\lim_{t\to\infty} E[\theta_t v_t^-(a_t^*)] = 0$ and therefore, $v_0^-(a_0^*) = 0$, violating the monotonicity and concavity of v_0.

To show that $\lim_{t\to\infty} E[\theta_t v_t^-(a_t^*)] = 0$, we first show $P(v_t^-(a_t^*) < \alpha_1) = 1$ for some $\alpha_1 \in \mathbb{R}$, and then show $P(\lim_{t\to\infty} \theta_t v_t^-(a_t^*) = 0) = 1$. Since $P(\beta r_t \leq 1,\ t \geq 0) = 1$ implies $P(0 \leq \theta_{t+1} \leq \theta_t \leq 1,\ t \geq 1) = 1$, the desired result then follows from the Dominated Convergence Theorem.

To establish $P(v_t^-(a_t^*) < \alpha_1) = 1$, note first that $P(u^+(x_t) < \alpha,\ t \geq 0) = 1$ implies either $u^+(0) < \alpha$ or $P(x_t > \phi,\ t \geq 0) = 1$ for some $\phi > 0$. If $u^+(0) < \alpha$, then Lemma A.3 and the strict concavity of v imply $P(v_t^-(a_t^*) < v_t^+(0) = u^+(0) < \alpha) = 1$, in which case we let $\alpha_1 \equiv \alpha$. If $P(x_t > \phi,\ t \geq 0) = 1$, then $P(a_t^* > \phi,\ t \geq 0) = 1$, and so Lemma 1 implies $P(v_t^-(a_t^*) \leq u^-(\phi),\ t \geq 0) = 1$, in which case we let $\alpha_1 \equiv u^-(\phi)$.

To establish $P(\lim_{t\to\infty} \theta_t v_t^-(a_t^*) = 0) = 1$, we consider Conditions (i) and (ii) separately. If Condition (i) holds, then the desired property follows immediately. If Condition (ii) holds, then Lemmata 1 and 4 imply

$$P\left(\lim_{t\to\infty} u_t^+(c_t^*) \leq \lim_{t\to\infty} v_t^+(a_t^*) \leq \frac{1}{\varepsilon}\lim_{t\to\infty} \theta_t v_t^+(a_t^*) = 0\right) = 1,$$

and so $P(\lim_{t\to\infty} c_t^* = \infty) = 1$. Another application of Lemma 1 then yields

$$P\left(\lim_{t\to\infty} \theta_t v_t^-(a_t^*) \leq \lim_{t\to\infty} \theta_t u^-(c_t^*) \leq \lim_{t\to\infty} u^-(c_t^*) = 0\right) = 1.$$

∎

If the long run rate of interest is less than the discount rate, the conclusion of Theorem 8 is not particularly surprising. In this case, the marginal return to consuming a fixed level of consumption goes to zero with time. Consequently, if the marginal utility to consuming current income is bounded, then the borrowing constraint must eventually be binding. The theorem is less intuitive for the case where the long run rate of interest is equal to the discount rate ($\beta\gamma = 1$ for a fixed interest factor γ). In this case Theorem 4 implies that consumption grows without bound. Evidently, along almost every path, the asset level first falls to its lower bound at least once before converging to infinity. However, Theorem 4 implies that this happens only a finite number of times.

5.2. *Comparative Dynamics*

Suppose the original sequence of borrowing constraints is replaced with another sequence which allows at least as much borrowing in any information state. The following theorem states that at any information state the optimal accumulated wealth under the new sequence of borrowing constraints is no higher than the optimal accumulated wealth under the original sequence of borrowing constraints.

THEOREM 9. *Let* $\mathbf{a}^0$ *and* $\mathbf{a}^1$ *be the sequence of accumulated assets associated with the solutions* $\mathbf{c}^0$ *and* $\mathbf{c}^1$ *corresponding to constraints* $\mathbf{k}^0$ *and* $\mathbf{k}^1$, *respectively. Then* $P(k_t^0 \geq k_t^1,\ t \geq 0) = 1$ *implies* $P(a_t^0 \geq a_t^1,\ t \geq 0) = 1$.

Proof. Let S_0 and S_1 be the metric spaces defined in the Appendix corresponding to $\mathbf{k}^0$ and $\mathbf{k}^1$, respectively, and let T_0 and T_1 be the corresponding T operators. Note that $S_0 \subset S_1$ so that T_1 is defined on S_0. Let v^0 and v^1 be the corresponding fixed points of T_0 and T_1. We shall show that $v^{0+}(a, z^t) > v^{1-}(a + \varepsilon, z^t)$ for all $(a, z^t) \in S_0$ and $\varepsilon > 0$.

Let f^1 represent v^1 restricted to S_0 and let $f^{n+1} \equiv T_0^n f^1$, where T_0^n is the operator T_0 applied n times. For $(a, z^t) \in S_0$, define

$$C_1(a, z^t) \equiv \arg \max_{0 \leq c \leq a - k_t^1(z^t)} \left\{ u(c) + \beta E\left[v^1\left((a-c)r_{t+1} + x_{t+1}, z^{t+1}\right) | z^t \right] \right\},$$

and for $n \geq 2$ define

$$C_n(a, z^t) \equiv \arg \max_{0 \leq c \leq a - k_t^0(z^t)} \left\{ u(c) + \beta E\left[f^{n-1}\left((a-c)r_{t+1} + x_{t+1}, z^{t+1}\right) | z^t \right] \right\}.$$

Fix $(a, z^t) \in S_0$ and $\varepsilon > 0$. Let $c_1 \equiv C_1(a + \varepsilon, z^t)$ and $c_2 \equiv C_2(a, z^t)$. Then either $c_1 > c_2 \geq 0$ or $a + \varepsilon - c_1 > a - c_2$. Therefore, using the standard envelope arguments exploited in Lemma 1, the strict concavity of u and v implies that either

$$v^{1-}(a + \varepsilon, z^t) \leq u^-(c_1) < u^+(c_2) \leq f^{2+}(a, z^t) \tag{10}$$

or

$$\begin{aligned} v^{1-}(a + \varepsilon, z^t) &\leq \beta E\left[r_{t+1} v^{1-}\left((a + \varepsilon - c_1)r_{t+1} + x_{t+1}, z^{t+1}\right) | z^t \right] \\ &< \beta E\left[r_{t+1} v^{1+}\left((a - c_2)r_{t+1} + x_{t+1}, z^{t+1}\right) | z^t \right] \leq f^{2+}(a, z^t). \end{aligned}$$

In either case, we have shown that $v^{1-}(a', z^t) < f^{2+}(a, z^t)$ for any $(a, z^t), (a', z^t) \in S_0$, $a' > a$.

Now suppose $f^{n+}(a, z^t) > f^{(n-1)-}(a', z^t)$ for any $(a, z^t), (a', z^t) \in S_0$ with $a' > a$. Fix $(a, z^t) \in S_0$ and $\varepsilon > 0$ and define $c_n \equiv C_n(a + \varepsilon, z^t)$ and $c_{n+1} \equiv C_{n+1}(a, z^t)$. Then either $c_n > c_{n+1} \geq 0$ or $a + \varepsilon - c_n > a - c_{n+1}$.

Then the envelope argument implies either

$$f^{n-}(a+\varepsilon, z^t) \leq u^-(c_n) < u^+(c_{n+1}) \leq f^{(n+1)+}(a, z^t) \tag{11}$$

or

$$\begin{aligned} f^{n-}(a+\varepsilon, z^t) &\leq \beta E\left[r_{t+1} f^{(n-1)-}\left((a+\varepsilon-c_n)r_{t+1}+x_{t+1}, z^{t+1}\right)|z^t\right] \\ &< \beta E\left[r_{t+1} f^{n+}\left((a-c_{n+1})r_{t+1}+x_{t+1}, z^{t+1}\right)|z^t\right] \\ &\leq f^{(n+1)+}(a, z^t). \end{aligned} \tag{12}$$

In either case, we have shown that $f^{n-}(a', z^t) < f^{(n+1)+}(a, z^t)$ for any $(a, z^t), (a', z^t) \in S_0, a' > a$.

Now let (ε_n) be a strictly decreasing sequence such that $\varepsilon_1 = \varepsilon$ and $\lim_{n\to\infty} \varepsilon_n = \frac{\varepsilon}{2}$. Then for $n \geq 2$, we have just shown that $v^{1-}(a+\varepsilon, z^t) < f^{1-}(a+\varepsilon_2, z^t) < \cdots < f^{(n-1)-}(a+\varepsilon_n, z^t) < f^{n-}(a+\frac{\varepsilon}{2}, z^t)$ for all $(a, z^t) \in S_0$. Since T_0 is a contraction mapping and $T_0 v^0 = v^0$, it follows that $f^n(a, z^t) \to v^0(a, z^t)$ as $n \to \infty$. Therefore, Lemma A.4 implies $v^{0+}(a, z^t) > v^{0-}(a+\frac{\varepsilon}{2}, z^t) \geq \limsup_n f^{n-}(a+\frac{\varepsilon}{2}, z^t) \geq v^{1-}(a+\varepsilon, z^t)$ for any $(a, z^t) \in S_0$ and $\varepsilon > 0$.

To show that $\mathbf{a}^1 \leq \mathbf{a}^0$, we again argue by induction. Fix any realization of $\mathbf{z} = (\ldots, z_{-1}, z_0, z_1, \ldots)$. By definition, $a_0^0(z^0) = a_0^1(z^0)$. Suppose $a_t^0(z^t) - a_t^1(z^t) \geq 0$. We will show that $a_{t+1}^0(z^{t+1}) - a_{t+1}^1(z^{t+1}) \geq 0$. Note first, that for any $w \in \mathbb{R}^N$,

$$\begin{aligned} &a_{t+1}^0(z^t, w) - a_{t+1}^1(z^t, w) \\ &\qquad = r_{t+1}(z^t, w)\left(\left(a_t^0(z^t) - a_t^1(z^t)\right) - \left(c_t^0(z^t) - c_t^1(z^t)\right)\right). \end{aligned} \tag{13}$$

Therefore, $a_{t+1}^0(z^{t+1}) < a_{t+1}^1(z^{t+1})$ implies $c_t^0(z^t) > c_t^1(z^t)$ and $a_{t+1}^0(z^t, w) < a_{t+1}^1(z^t, w)$ for all $w \in \mathbb{R}^N$. Lemma 1 then implies

$$\begin{aligned} &\beta E\left[r_{t+1}(z^t, w) v^{1-}\left(a_{t+1}^1(z^t, w), (z^t, w)\right)|z^t\right] \\ &\qquad \geq u^+\left(c_t^1(z^t)\right) > u^-\left(c_t^0(z^t)\right) \\ &\qquad \geq \beta E\left[r_{t+1}(z^t, w) v^{0+}\left(a_{t+1}^0(z^t, w), (z^t, w)\right)|z^t\right]. \end{aligned} \tag{14}$$

But we have just shown above that $a_{t+1}^0(z^t, w) < a_{t+1}^1(z^t, w)$ implies $v^{0+}(a_{t+1}^0(z^t, w), (z^t, w)) > v^{1-}(a_{t+1}^1(z^t, w), (z^t, w))$. This contradiction completes the proof. ∎

It should be clear from our discussion in Section 2 that for any change in the set of feasible consumption programs in response to a change in the borrowing constraint, there is an equivalent change in the income sequence that generates the same change in the set of feasible consumption programs. Consequently, we may reinterpret Theorem 9 as a statement about how the optimal consumption program changes in response to certain kinds of changes in the income sequence. Our next corollary states that if the income sequence changes so that at any state z^t the discounted value of income up to time t has not decreased, then the discounted value of the consumption up to time τ also does not decrease for any state z^τ.

COROLLARY 4. *Let* $\mathbf{c}^0$ *and* $\mathbf{c}^1$ *be the optimal consumption programs associated with borrowing sequence* $\mathbf{k}$ *and income sequences* $\mathbf{x}^0$ *and* $\mathbf{x}^1$, *respectively. Then*

$$P\left(\sum_{j=0}^{t}\left(x_j^1 - x_j^0\right)R_{0j}^{-1} \geq 0,\ t \geq 0\right) = 1$$

$$\textit{implies } P\left(\sum_{j=0}^{t}\left(c_j^1 - c_j^0\right)R_{0j}^{-1} \geq 0,\ t \geq 0\right) = 1.$$

Proof. Let $\mathbf{k}^0 \equiv \mathbf{k}$, $\mathbf{x} \equiv \mathbf{x}^0$, and define $\mathbf{k}^1 = (k_{-1}, k_0, k_1, \dots)$ by $k_{-1}^1 \equiv 0$ and $k_t^1 \equiv \sum_{j=0}^{t}(x_j^0 - x_j^1)R_{jt} + k_t^0$ for $t \geq 0$. By assumption, $\mathbf{x}^1$ and $\mathbf{k}$ jointly satisfy (1). Therefore, $x_t - k_t^1 + r_t k_{t-1}^1 = x_t^0 - \sum_{j=0}^{t}(x_j^0 - x_j^1)R_{jt} - k_t^0 + \sum_{j=0}^{t-1}(x_j^0 - x_j^1)R_{jt} + r_t k_{t-1}^0 = x_t^1 - k_t + r_t k_{t-1}$ implies that $\mathbf{x}$ and $\mathbf{k}^1$ do as well. Furthermore, the definitions of $\mathbf{c}^0$ and $\mathbf{c}^1$ imply that they are also the optimal consumption programs associated with the (common) income sequence, $\mathbf{x}$, and respective borrowing constraints, $\mathbf{k}^0$ and $\mathbf{k}^1$. To see this, note that for any consumption program $\mathbf{c}$, we have $\sum_{j=0}^{t}(x_j^1 - c_j)R_{jt} - k_t = \sum_{j=0}^{t}(x_j - c_j)R_{jt} - k_t^1$. Therefore, a program $\mathbf{c}$ is feasible for $\mathbf{x}^1$ and $\mathbf{k}$ if and only if it is feasible for $\mathbf{x}$ and $\mathbf{k}^1$. By assumption $\mathbf{c}^1$ is optimal for $\mathbf{x}^1$ and $\mathbf{k}$. Therefore, it is also optimal for $\mathbf{x}$ and $\mathbf{k}^1$.

Now let $\mathbf{a}^0$ and $\mathbf{a}^1$ denote the asset sequences associated with consumption programs $\mathbf{c}^0$ and $\mathbf{c}^1$ for income sequence $\mathbf{x}$. Then, for $i = 0, 1$, $a_0^i = x_0$ and $a_t^i = \sum_{j=0}^{t-1}(x_j - c_j^i)R_{jt} + x_t$ for $t \geq 1$. The hypothesis of the corollary and the construction of $\mathbf{k}^1$ imply $\mathbf{k}^1 \leq \mathbf{k}^0$. Therefore, Theorem 9 implies $0 \leq a_t^0 - a_t^1 = \sum_{j=0}^{t-1}(c_j^1 - c_j^0)R_{jt} = R_{0t}\sum_{j=0}^{t-1}(c_j^1 - c_j^0)R_{0j}^{-1}$ for all $t \geq 1$. ∎

6. MANY ASSETS

In this section, we illustrate how the argument behind Theorem 2 can be extended to the case where there are several risky assets. We show that to apply the martingale convergence theorem, it is not necessary to actually solve the portfolio problem of the consumer. So long as the consumer is able to choose a sequence of "marginal" interest rates which, in the long run, exceed the discount rate, consumption must converge to infinity.

Suppose the consumer must choose a portfolio of J risky assets in each period $t \geq 0$. If the investment in period t in the ith asset is b_i (measured in terms of period t consumption), then the payoff at $t+1$ is $b_i g_{i,t+1}$, where $g_{i,t+1}$ is a nonnegative, continuous function from $Z^{t+1}(z^0)$ to $\mathbb{R}_+$ for $i = 1, \ldots, J$, and $t \geq 0$. If $b \equiv (b_1, \ldots, b_J)$ is the portfolio chosen in period t, then the multi-asset analogue of the budget constraint is a constraint on short sales which we represent by the requirement that $b \in K_t(z^t)$. We assume that K_t is a continuous correspondence from $Z^t(z^0)$ to $\mathbb{R}^J$ and that $K_t(z^t)$ is a nonempty, closed, convex subset of $\mathbb{R}^J$ which is bounded from below; i.e., there is a function m: $Z^t(z^0) \to \mathbb{R}$ such that $K_t(z^t) \subset \{(\alpha_1, \ldots, \alpha_J) \in \mathbb{R}^J\colon \alpha_j \geq m(z^t)\}$. We also assume that $-K_t(z^t)$ is comprehensive; i.e., if $\phi_1 \in K_t(z^t)$ and $\phi_2 \in \mathbb{R}^J_+$, then $\phi_1 + \phi_2 \in K_t(z^t)$. This guarantees that it is always feasible for the consumer to increase his holding of any asset.

For any $\alpha, \beta \in \mathbb{R}^J$, let $\alpha\beta \equiv \sum_{j=1}^J \alpha_j \beta_j$. Let λ denote the J-vector of ones, and define $k_t(z^t) \equiv \inf\{b\lambda\colon b \in K_t(z^t)\}$. Interpret k_t as the minimum amount of wealth measured in terms of period t consumption that the individual may hold at the end of period t (after receiving period t income and spending period t consumption). Let $g_t \equiv (g_{1t}, \ldots, g_{Jt})$. We require that the current bounds on short sales be consistent with the borrowing constraints the individual will face in the future: $P(bg_{t+1} + x_{t+1} \geq k_{t+1} | z^t) = 1$ for all $b \in K_t(z^t)$ and all z^t.

For any state z^0, a *consumption-portfolio* program, $(\mathbf{c}, \mathbf{b})$, is a sequence of a pair of functions, c_t: $Z^t(z^0) \to \mathbb{R}_+$ and b_t: $Z^t(z^0) \to \mathbb{R}^J$, $t \geq 0$. For any $z^t \in Z^t(z^0)$ and any $a \geq k_t(z^t)$, let $\overline{K}_t(a, z^t) \equiv \{(c, b) \in \mathbb{R}_+ \times K_t(z^t)\colon 0 \leq c \leq a - b\lambda\}$ denote the set of feasible consumption-portfolio pairs for the consumer, and define

$$v(a, z^t) \equiv \max_{\mathbf{c}, \mathbf{b}} E\left[\sum_{j=t}^{\infty} u(c_j) \beta^{j-t} | z^t\right]$$

$$\text{subject to } P\Big(\big(c_j, b_j\big) \in \overline{K}_j\big(a_j, z^j\big),\ j \geq t | z^t\Big) = 1,$$

where $a_t \equiv a$ and $a_j \equiv b_{j-1} g_j + x_j$ for $j > t$. Theorem A.1 implies that this problem has a solution and that $v(a, z^t)$ is strictly concave in $a \geq k_t$. Therefore, right- and left-hand derivatives are well defined for $a > k_t$. We let $v^+(k_t, z^t) \equiv \lim_{a \downarrow k_t} v^+(a, z^t)$.

As in Section 2, let $v_t(a) \equiv v(a, z^t)$, and let $(\mathbf{c}^*, \mathbf{b}^*)$ denote the solution to the consumer's problem starting in state z^0 with wealth $a = x_0$. We shall simplify the martingale arguments by assuming that $x_0 > k_0$. The following conditions are then implied by an optimal program. They are established by the same standard arguments used to establish Lemma 1.

LEMMA 5. (i) $v_t^+(a_t^*) \geq \beta E[g_{k,t+1} v_{t+1}^+(a_{t+1}^*)|z^t]$ *for* $k = 1, \ldots, J$, *and* (ii) $v_t^+(a_t^*) \geq u^+(c_t^*)$.

To obtain the analogues of Theorems 1 and 2 from Lemma 5, we consider an arbitrary marginal portfolio, i.e., a portfolio which can be added to the existing portfolio. Given our assumptions on K_t, the only portfolios we might not be able to consider are those involving short sales. For this reason, we restrict attention to those portfolios without short sales. We say that $(b_0, b_1, \ldots)$ is a *positive portfolio program* if each b_t is a measurable function from $Z^t(z^0)$ into $\mathbb{R}_+^J - \{0\}$. For any positive portfolio program, we define the sequence of return factors by $r_{t+1}(z^{t+1}) \equiv b_t(z^t) g_{t+1}(z^{t+1}) / b_t(z^t)\lambda$ for $t \geq 0$.

The next theorem provides a condition that, if satisfied by the return sequence generated by at least one positive portfolio, implies that optimal consumption converges to infinity. Recall that $R_{0t} \equiv \prod_{j=1}^t r_j$ and $\theta_t \equiv \beta^t R_{0t}$.

THEOREM 10. *Suppose that* $(r_1, r_2, \ldots)$ *is the return sequence for some positive portfolio program*. *Then*, (i) $P(\lim_{t\to\infty} \theta_t = \infty) = 1$ *implies* $P(\lim_{t\to\infty} c_t^* = \infty) = 1$, *and* (ii) $P(\sup_{t\geq 0} \theta_t = \infty) = 1$ *implies* $P(\sup_{t\geq 0} c_t^* = \infty) = 1$.

Proof. (i) Define $d_0 \equiv 1$ and $d_t \equiv \theta_t v_t^+(a_t^*)/v_0^+(a_0^*)$. Lemma 5 implies that $v_t^+(a_t^*) \geq \beta E[r_{t+1} v_{t+1}^+(a_{t+1}^*)|z^t]$ from which it follows that d_t is a nonnegative supermartingale. Therefore, there is a random variable d_∞ with $E[d_\infty] \leq 1$ and $P(\lim_{t\to\infty} d_t = d_\infty) = 1$ (Doob, 1953, p. 324). So if $\theta_t \to \infty$, it follows that $v_t^+(a_t^*) \to 0$, and therefore, from Lemma 5 that $u^+(c_t^*) \to 0$, from which we conclude that $P(\lim_{t\to\infty} c_t^* = \infty) = 1$.

(ii) If $\theta_t \to \infty$ along a subsequence, then, along that subsequence, $v_t^+(a_t^*) \to 0$ which implies $u^+(c_t^*) \to 0$ and therefore, $P(\sup_t c_t^* = \infty) = 1$. ∎

We emphasize that there need be no relation between the optimal portfolio and the positive portfolio program that generates the sequence of interest rates used in Theorem 10.

APPENDIX

We establish the existence of an optimal solution for the problem presented in Section 6 where the consumer must choose an optimal portfolio and consumption program in each period. Recall that $z_t \in \mathbb{R}^n$

and for any $z^0 = (\dots, z_{-1}, z_0)$, $Z^t(z_0) \equiv \{z^t\colon z^t = (z^0, z_1, \dots, z^t)\}$. Fix z^0. Let (S, d) denote the metric space defined by $S \equiv \{(a, z^t)\colon a \geq k_t(z^t), z^t \in Z^t(z^0),\ t = 0, 1, \dots\}$, and

$$d\big((a, z^t), (a', z^j)\big) = \begin{cases} 1 & \text{if } j \neq t \\ \max\{|a - a'|, \rho(z_1, z_1'), \dots, \rho(z_t, z_t')\} & \text{if } j = t, \end{cases}$$

where ρ is the metric on $\mathbb{R}^n$. For any $f\colon S \to \mathbb{R}$, let $\|f\| = \sup_{s \in S}|f(s)|$, and let $D(S)$ denote the set of continuous functions $f\colon S \to \mathbb{R}$ such that $0 \leq f \leq \frac{M}{1-\beta}$ and $f(a, z^t)$ is nondecreasing and concave in a. Note that $D(S)$ is a complete metric space under the metric $\|f - g\|$.

Recall that $\overline{K}_t(a, z^t) \equiv \{(c, b) \in \mathbb{R}_+ \times K_t(z^t)\colon 0 \leq c \leq a - b\lambda\}$, where λ is the n-vector of ones, denotes the set of feasible consumption and portfolio choices for wealth level a in state z^t. Define the operator T on $D(S)$ by

$$Tf(a, z^t) \equiv \sup_{(c, b) \in \overline{K}_t(a, z^t)} \big\{u(c) + \beta E\big[f\big(bg_{t+1} + x_{t+1}, z^{t+1}\big)|z^t\big]\big\}.$$

LEMMA A.1. *T is a contraction mapping $D(S)$ into $D(S)$.*

Proof. We show first that $f \in D(S)$ implies $Tf \in D(S)$. Suppose $f \in D(S)$. It is immediate that $0 \leq Tf \leq \frac{M}{1-\beta}$ and that Tf is monotone nondecreasing.

We show next that Tf is continuous. Since $d(s, s') < 1$ implies $t = t'$, we suppress the dependence on t and let z denote z^t. For $w \in \mathbb{R}^n$, define $m(b, z, w) \equiv f(bg_{t+1}(z, w) + x_{t+1}(z, w), (z, w))$ and let $h(b, z) \equiv \int m(b, z, w) p(dw|z)$. Then $Tf(a, z) = \sup_{(c, b) \in \overline{K}_t(a, z)}\{u(c) + \beta h(b, z)\}$. Since u is continuous and $\overline{K}_t$ is continuous and compact-valued, it follows that Tf is continuous if h is continuous (Hildenbrand, 1974, Corollary, p. 30). To show that h is continuous, note that

$$\begin{aligned} |h(b, z) - h(b', z')| \leq{}& \left|\int m(b, z, w)\big(p(dw|z) - p(dw|z')\big)\right| \\ &+ \int |m(b, z, w) - m(b', z', w)|\,p(dw|z'). \end{aligned} \tag{15}$$

Define $\bar{d}((b, z), (b', z')) \equiv \max\{\rho_1(b, b'), \rho_2(z, z')\}$, where ρ_1 and ρ_2 are Euclidean metrics. For fixed (b, z), the Feller property implies that for any $\varepsilon > 0$, there is a $\delta_1 > 0$ such that $|\int m(b, z, w)(p(dw|z) - p(dw|z'))| < \varepsilon$ for $\bar{d}((b, z), (b', z')) < \delta_1$. Since $Z^t(z^0)$ is a finite-dimensional Euclidean space, the Feller property also implies that there is a compact set $A \subset \mathbb{R}^n$ and $\delta_2 > 0$ such that $\bar{d}((b,z),(b',z')) < \delta_2$ implies $\int_{\mathbb{R}^n - A} p(dw|z') < \frac{(1-\beta)\varepsilon}{M}$. Since f is uniformly continuous on compact sets, we may choose $\delta_3 > 0$ so that $\bar{d}((b,z),(b',z')) < \delta_3$ implies $|m(b,z,w) - m(b',z',w)| < \varepsilon$ for $w \in A$. Therefore, if $\bar{d}((b,z),(b',z')) < \min\{\delta_2, \delta_3\}$, we have

$$\begin{aligned} &\int |m(b,z,w) - m(b',z',w)| p(dw|z') \\ &\quad < \varepsilon\left(\frac{M}{1-\beta}\frac{1-\beta}{M}\right) + \varepsilon = 2\varepsilon. \end{aligned} \tag{16}$$

Combining these results, we have $\bar{d}((b,z),(b',z')) < \min\{\delta_1, \delta_2, \delta_3\}$ implies $|h(b,z) - h(b',z')| < 3\varepsilon$.

All that remains is to show that $Tf(a,z^t)$ is concave in a. But this follows since u is concave and $h(b,z^t)$ is concave in b. This establishes that $Tf \in D(S)$.

To establish that T is a contraction mapping, we may use Blackwell's Theorem 5 to show that given $f, g \in D(S)$, $\|Tf - Tg\| \leq \beta\|f - g\|$. ∎

LEMMA A.2. *There is a unique $v \in D(S)$ such that $Tv = v$.*

Proof. Given that T is a contraction and $D(S)$ is a complete metric space, the lemma follows immediately from the contraction mapping fixed point theorem. ∎

Define $h_v(b,z^t) \equiv \int v(bg_{t+1}(z^t,w) + x_{t+1}(z^t,w),(z^t,w))p(dw|z^t)$. Since u and h_v are continuous functions and $\bar{K}_t$ is a continuous, compact-valued correspondence, it follows that we can select measurable functions, C and B, such that for all z^t and all $a \geq k(z^t)$, $(C(a,z^t), B(a,z^t)) \in \bar{K}_t(a,z^t)$ and

$$v(a,z^t) \equiv u\big(C(a,z^t)\big) + \beta E\big[v\big(B(a,z^t)g_{t+1} + x_{t+1}, z^{t+1}\big)|z^t\big]$$

for all $(a,z^t) \in S$. (See Hildenbrand, 1974, Corollary, p. 30, Proposition 1, p. 22, and Lemma 1, p. 55.) Furthermore, C is unique since u is strictly concave, h_v is concave in b, and $\bar{K}_t$ is convex-valued.

For any z^t and any $a \geq k_t(z^t)$, define the random variables $\mathbf{a}^* = (a_t^*, a_{t+1}^*, \dots)$, $\mathbf{c}^* = (c_t^*, c_{t+1}^*, \dots)$, and $\mathbf{b}^* = (b^*{}_t, b_{t+1}^*, \dots)$ recursively as follows: $a_t^* \equiv a$, and for $j \geq t$, $c_j^* \equiv C(a_j^*, z^j)$, $b_j^* \equiv B(a_j^*, z^j)$, and $a_j^* \equiv b_{j-1}^* g_j(z^j) + x_j(z^j)$. The next theorem establishes that $(\mathbf{c}^*, \mathbf{b}^*)$ is an optimal consumption-portfolio program for the consumer, and $\mathbf{a}^*$ is the corresponding sequence of wealth levels when starting with wealth level a and state z^t.

THEOREM A.1. (i) $(\mathbf{c}^*, \mathbf{b}^*)$ *maximizes* $E[\sum_{j=t}^{\infty} u(c_j)\beta^{j-t}|z^t]$ *subject to* $P((c_j, b_j) \in \overline{K}_j(a_j, z^j),\ j \geq t|z^t) = 1$, *where* $a_t \equiv a$ *and* $a_j \equiv b_{j-1}^* g_j(z^j) + x_j(z^j)$ *for* $j > t$. (ii) *For* $a \geq k_t(z^t)$, $v(a, z^t) = E[\sum_{j=t}^{\infty} u(c_j^*)\beta^{j-t}|z^t]$, *which is a strictly concave function of* a.

Proof. It follows from Lemma A.2 and Strauch (1966, Theorem 5.1b) that $v(a, z^t) = E[\sum_{j=t}^{\infty} u(c_t^*)\beta^{j-t}|z^t]$. Condition (i) then follows from Blackwell's Theorem 6. The strict concavity of v follows from the strict concavity of u. ∎

The following results are used in Sections 4 and 5. As in those sections, Lemma A.3 is restricted to the case where there is only one asset and income is defined so that $x_t \geq 0$ and $k_t = 0$ for all $t \geq 0$. As in the text, we let $v_t(a) \equiv v(a, z^t)$.

LEMMA A.3. $P(\beta r_t \leq 1,\ t \geq 1) = 1$ *implies* $P(v_t^+(0) = u^+(0),\ t \geq 0) = 1$.

Proof. Without loss of generality it is sufficient to prove that $v_0^+(0) = u^+(0)$. Note first that if $u^+(0) = \infty$, the lemma follows immediately from Lemma 1 of the text. So we may suppose that $u^+(0) < \infty$. Since Lemma 1 implies $v_0^+(0) \geq u^+(0)$, we need to show that $v_0^+(0) \leq u^+(0)$. Let $\mathbf{c}^0$ and $\mathbf{c}^\varepsilon$ denote the optimal consumption programs for $a_0 = 0$ and $a_0 = \varepsilon > 0$, respectively, and let $\mathbf{a}^0$ and $\mathbf{a}^\varepsilon$ denote the corresponding sequence of wealth levels.

We show first that $P(a_t^0 \leq a_t^\varepsilon,\ t \geq 0) = 1$. Suppose not, and let t be the smallest j such that $P(a_j^0 > a_j^\varepsilon) > 0$. For any $z_t \in \mathbb{R}^n$, we have

$$\begin{aligned} &a_t^\varepsilon\left(z^{t-1}, z_t\right) - a_t^0\left(z^{t-1}, z_t\right) \\ &\quad = r_t\left(z^{t-1}, z_t\right)\left(\left(a_{t-1}^\varepsilon(z^{t-1}) - a_{t-1}^0(z^{t-1})\right) - \left(c_{t-1}^\varepsilon(z^{t-1}) - c_{t-1}^0(z^{t-1})\right)\right). \end{aligned} \tag{17}$$

Therefore, there is a nonnull set of z^{t-1} such that $a_{t-1}^0 \leq a_{t-1}^\varepsilon$, $c_{t-1}^0 < a_{t-1}^0$, and $P(a_t^0 > a_t^\varepsilon|z^{t-1}) = 1$. From this, Lemma 1 and the strict concavity of v_t imply

$$\begin{aligned} v_{t-1}^-\left(a_{t-1}^0\right) &\leq \beta E\left[r_t v_t^-\left(a_t^0\right)|z^{t-1}\right] \\ &< \beta E\left[r_t v_t^+\left(a_t^\varepsilon\right)|z^{t-1}\right] \leq v_{t-1}^+\left(a_{t-1}^\varepsilon\right), \end{aligned} \tag{18}$$

which violates the concavity of v_{t-1}.

By definition, $(a^{\varepsilon}_{t+1} - a^{0}_{t+1})R^{-1}_{0,t+1} = (a^{\varepsilon}_0 - a^0_0) - \sum_{j=0}^{t}(c^{\varepsilon}_j - c^0_j)R^{-1}_{0j}$ for any $t \geq 0$. Therefore, $P(a^0_t \leq a^{\varepsilon}_t,\ t \geq 0) = 1$ implies

$$E\left[\sum_{j=0}^{t}\left(c^{\varepsilon}_j - c^0_j\right)R^{-1}_{0j}\right] = \left(a^{\varepsilon}_0 - a^0_0\right) - E\left[\left(a^{\varepsilon}_{t+1} - a^0_{t+1}\right)R^{-1}_{0,t+1}\right] \leq \varepsilon. \tag{19}$$

Furthermore, using Lemma 1 and the strict concavity of u and v_t, it also follows from $P(a^0_t \leq a^{\varepsilon}_t,\ t \geq 0) = 1$ that $P(c^0_t \leq c^{\varepsilon}_t,\ t \geq 0) = 1$. Then, since $P(\beta r_t \leq 1,\ t \geq 1) = 1$ implies $P(R^{-1}_{0t} \geq \beta^t,\ t \geq 0) = 1$, it follows from the concavity of u that

$$\begin{aligned} v_0(\varepsilon) - v_0(0) &= E\left[\sum_{t=0}^{\infty}\left(u(c^{\varepsilon}_t) - u(c^0_t)\right)\beta^t\right] \\ &\leq E\left[\sum_{t=0}^{\infty}u^{+}\left(c^0_t\right)\left(c^{\varepsilon}_j - c^0_j\right)\beta^t\right] \\ &\leq E\left[\sum_{t=0}^{\infty}u^{+}\left(c^0_t\right)\left(c^{\varepsilon}_j - c^0_j\right)R^{-1}_{0t}\right] \leq \varepsilon u^{+}(0). \end{aligned} \tag{20}$$

Letting ε go to zero proves the result. ∎

The arguments behind Lemma A.3 exploit only the strict concavity of u. As long as v is well defined, the conclusions still follow.

LEMMA A.4. *Let D be an open, convex subset of $\mathbb{R}$ and let $(f_j\colon D \to \mathbb{R},\ j \geq 1)$ be a sequence of concave functions such that $f_j(x) \to f_0(x)$ for all $x \in D$. Then $f^{+}_0(x) \leq \liminf_{j\to\infty} f^{+}_j(x)$ and $f^{-}_0(x) \geq \limsup_{j\to\infty} f^{-}_j(x)$.*

Proof. Since f_j is concave, $f_j(y) - f_j(x) \leq f^{+}_j(x)(y-x)$ for $x, y \in D$. Then $f_j \to f_0$ implies that f_0 is concave and $f_0(y) - f_0(x) \leq \liminf_{j\to\infty} f^{+}_j(x)(y-x)$, which implies $f^{+}_0(x) \leq \liminf_j f^{+}_j(x)$. The proof of the second half of the lemma is similar. ∎

REFERENCES

Bewley, T. (1977). "The Permanent Income Hypothesis: A Theoretical Formulation," *Journal of Economic Theory* **16**, 252–292.

Bewley, T. (1980a). "The Permanent Income Hypothesis and Long-run Economic Stability," *Journal of Economic Theory* **22**, 377–394.

Bewley, T. (1980b). "The Permanent Income Hypothesis and Short-run Price Stability," *Journal of Economic Theory* **23**, 323–333.

Bewley, T. (1980c). "The Optimum Quantity of Money," in *Models of Monetary Economics* (J. Karenken and N. Wallace, Eds.), Minneapolis: Federal Reserve Bank.

Bewley, T. (1983). "A Difficulty with the Optimum Quantity of Money," *Econometrica* **51**, 1485–1504.

Billingsley, P. (1968). *Convergence of Probability Measures*. New York: Wiley.

Blackwell, D. (1965). "Discounted Dynamic Programming," *Annals of Mathematical Statistics* **36**, 226–235.

Doob, J. L. (1953). *Stochastic Processes*. New York: Wiley.

Friedman, M. (1957). *A Theory of the Consumption Function*. New York: National Bureau of Economic Research.

Hildenbrand, W. (1974). *Core and Equilibria of a Large Economy*. Princeton: Princeton Univ. Press.

Meyer, P. A. (1966). *Probability and Potentials*. Waltham, MA: Blaisdell.

Maitra, A. (1968). "Discounted Dynamic Programming on Compact Metric Spaces," *Sankhya Series A* **30**, 211–216.

Schechtman, J. (1976). "An Income Fluctuation Problem," *Journal of Economic Theory* **12**, 218–241.

Schechtman, J., and Escudero, V. L. S. (1977). "Some Results on 'An Income Fluctuation Problem'," *Journal of Economic Theory* **16**, 151–166.

Sotomayor, M. (1984). "On Income Fluctuations and Capital Gains," *Journal of Economic Theory* **32**, 14–35.

Strauch, R. E. (1966). "Negative Dynamic Programming," *Annals of Mathematical Statistics* **37**, 871–890.

Yaari, M. E. (1976). "A Law of Large Numbers in the Theory of Consumer's Choice Under Uncertainty," *Journal of Economic Theory* **12**, 202–217.