

Recent advances for Ramanujan type supercongruences

Sarah Chisholm, Alyson Deines, and Holly Swisher

ABSTRACT. In 1914, Ramanujan listed 17 infinite series representations of $1/\pi$ of the form

$$\sum_{k=0}^{\infty} A(k)x^k = \frac{\delta}{\pi},$$

which were later used by J. Borwein and P. Borwein and D. Chudnovsky and G. Chudnovsky to find approximations for π . Several of these formulas relate hypergeometric series to values of the gamma function. In 1997, van Hamme developed a p -adic analogue of these series called Ramanujan type supercongruences and conjectured 13 formulas relating truncated sums of hypergeometric series to values of the p -adic gamma function, three of which he proved. Since then, a handful more have been proved. In this survey, we discuss various methods to prove these supercongruences, including recent geometric interpretations.

1. Introduction

In a rather mysterious way Ramanujan [27] stated a number of representations for $1/\pi$, including

$$(1) \quad \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{k!^3} (6k+1) \frac{1}{4^k} = \frac{4}{\pi},$$

where $(a)_k$ denotes the rising factorial $(a)_k = a(a+1) \cdots (a+k-1)$. Another example due to Bauer [6] is

$$(2) \quad \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{k!^3} (4k+1)(-1)^k = \frac{2}{\pi}.$$

Amazingly, the sums in (1) and (2) are of rational numbers, resulting in a transcendental number!

Ramanujan's formulas gained popularity in the 1980's when they were discovered to provide efficient means for calculating digits of π . In 1987, J. & P. Borwein [9] proved all 17 of Ramanujan's identities, while D. & G. Chudnovsky [12] derived additional series for $1/\pi$. Digits of π were calculated in both papers resulting in a new world record (at the time) of 2,260,331,336 digits, by the Chudnovskys.

Interesting p -adic analogues of Ramanujan's formulas for $1/\pi$ were developed by van Hamme [31]. In particular, he conjectured the following congruences, which correspond to (1) and (2), respectively. For primes $p > 2$,

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{(\frac{1}{2})_k^3}{k!^3} (6k+1) \frac{1}{4^k} \equiv \left(\frac{-1}{p}\right) p \pmod{p^4}$$

and

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{(\frac{1}{2})_k^3}{k!^3} (4k+1)(-1)^k \equiv \left(\frac{-1}{p}\right) p \pmod{p^3},$$

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where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol. Congruences of this form are called Ramanujan type supercongruences. The term supercongruence originated with Coster [13]. By supercongruence, we refer to the fact that the congruence holds modulo a power of p larger than expected by general theories which use formal group laws.

All told, van Hamme conjectured 13 Ramanujan type supercongruences. Among these conjectures, he provided proofs for three of them. Since then, there have been several other proofs of a number of van Hamme's conjectures. In particular, in the work of McCarthy and Osburn [24], Mortenson [25], Zudilin [34], Kilbourn [22], and Long [23] each set of authors proved one or more of van Hamme's conjectures. Several conjectures remain open.

Ramanujan's formulas, and their associated supercongruences, are connected to certain hypergeometric series. For r a nonnegative integer and $\alpha_i, \beta_i \in \mathbb{C}$, the hypergeometric series ${}_rF_r$ is defined by

$${}_rF_r \left[\begin{matrix} \alpha_1 & \dots & \alpha_{r+1} \\ \beta_1 & \dots & \beta_r \end{matrix} ; x \right] = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \dots (\alpha_{r+1})_k}{(\beta_1)_k \dots (\beta_r)_k} \frac{x^k}{k!},$$

which converges for $|x| < 1$, and for $|x| = 1$ with appropriate conditions on α_i, β_i . One can immediately see a connection with the above equations (1)–(2), since, for example

$${}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 \end{matrix} ; 1 \right] = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^3}{k!^3}.$$

The remainder of this paper is organized as follows. In Section 2, we discuss the methods of the Borweins and Chudnovskys which provide systematic ways to arrive at formulas for Ramanujan's representations for $1/\pi$. In Section 3, we review van Hamme's p -adic analogues to Ramanujan's formulas and state his conjectures. In Section 4 we give a number of methods and recent proofs of van Hamme's conjectures. Lastly, in Section 6 we present recent work of Long, Nebe, and the authors [11], a geometric approach to Ramanujan supercongruences via $K3$ surfaces.

2. Ramanujan type formulas for $1/\pi$

Ramanujan's expansions of $1/\pi$ were even known prior to him by mathematicians including Bauer. Systematic ways to obtain such formulas have been studied by two pairs of brothers, the Borweins and the Chudnovskys. Perhaps rather surprisingly, the formulas are obtained using features of elliptic curves. In particular, one can use either the classical Legendre relation between the periods and quasi-periods or the Wronskian of the Picard-Fuchs equation associated to the curves.

Here we outline a method using the Legendre relation. The motivated reader is encouraged to consult the standard reference of Silverman [29] for further details on this section. In addition, Baruah et al. [5] and Zudilin [33] provide lovely surveys on formulas for $1/\pi$ and $1/\pi^2$.

2.1. Legendre relation. Let E/\mathbb{C} be an elliptic curve and α and β be closed paths on the group of complex points on E , $E(\mathbb{C})$, which give a basis for the homology group $H_1(E, \mathbb{Z})$. Then

$$\omega_1 = \int_{\alpha} \frac{dx}{y},$$

and

$$\omega_2 = \int_{\beta} \frac{dx}{y},$$

are \mathbb{R} -linearly independent and are called the *periods* of E . Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be the corresponding lattice for E with $\Im(\omega_2/\omega_1) > 0$. Recall that the Weierstrass \wp -function, which is an elliptic function relative to a lattice $\Lambda \subset \mathbb{C}$, is given by

$$\wp(z : \Lambda) = \frac{1}{z^2} + \sum_{\omega \in \Lambda, \omega \neq 0} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right).$$

In addition, the Weierstrass ζ -function is described as the equation

$$\zeta(z : \Lambda) = \frac{1}{z} + \sum_{\omega \in \Lambda, \omega \neq 0} \left(\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).$$

Furthermore, the Weierstrass σ -function relative to Λ is defined as

$$\sigma(z) = \sigma(z; \Lambda) = z \prod_{\omega \in \Lambda, \omega \neq 0} \left(1 - \frac{z}{\omega}\right) \exp\left(\left(\frac{z}{\omega}\right) + \frac{1}{2}\left(\frac{z}{\omega}\right)^2\right).$$

As per usual, when the lattice Λ is understood by context, we suppress it from the notation.

For a fixed lattice Λ , note the following relations between the Weierstrass \wp , ζ and σ functions,

$$\frac{d}{dz} \log \sigma(z) = \zeta(z) \quad \text{and} \quad \frac{d}{dz} \zeta(z) = -\wp(z).$$

Also, for each vector $\omega \in \Lambda$ there are constants $a, b, \eta(\omega) \in \mathbb{C}$, depending on ω such that for all $z \in \mathbb{C}$,

$$\sigma(z + \omega) = e^{az+b} \sigma(z) \quad \text{and} \quad \zeta(z + \omega) = \zeta(z) + \eta(\omega).$$

The numbers $\eta_1 := \eta(\omega_1)$, and $\eta_2 := \eta(\omega_2)$ are called the *quasi-periods* of E .

Integrating $\zeta(z)$ along a fundamental parallelogram of Λ gives the *Legendre relation* on periods and quasi-periods:

$$\eta_1 \omega_2 - \eta_2 \omega_1 = 2\pi i.$$

The complete elliptic integral of first and second kind are defined as

$$(3) \quad K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}} = \frac{\pi}{2} {}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix}; k^2 \right]$$

and

$$(4) \quad E(k) = \int_0^1 \frac{\sqrt{1-k^2 t^2} dt}{\sqrt{(1-t^2)}} = \frac{\pi}{2} {}_2F_1 \left[\begin{matrix} -\frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix}; k^2 \right],$$

respectively, providing the connection between Ramanujan supercongruences and elliptic curves. Complementary¹ integrals E', K' are defined as

$$K'(k) = K(\sqrt{1-k^2})$$

and

$$E'(k) = E(\sqrt{1-k^2}).$$

Together, the functions $K(k)$ and $K'(k)$, span the solution space of a degree-2 hypergeometric ordinary differential equation (ODE). Similarly $E(k)$ and $E'(k) - K'(k)$ span the solution space of another degree-2 hypergeometric ODE. Moreover,

$$\frac{dE}{dk} = \frac{E - K}{k}$$

and

$$\frac{dK}{dk} = \frac{E - (1-k^2)K}{k(1-k^2)}.$$

The Legendre relation can be interpreted as the following. For any $0 < k < 1$,

$$(5) \quad E(k)K'(k) + E'(k)K(k) - K(k)K'(k) = \frac{\pi}{2}.$$

If $x = k^2$, then

$$F(x) = \frac{2}{\pi} K(x) = {}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix}; x \right]$$

and

$$F'(x) = F(1-x) = {}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix}; 1-x \right]$$

¹Please note that this notation is standard in the literature, and does not indicate a derivative.

both satisfy the same order-2 differential equation

$$\left(\frac{d^2}{dx^2} + \frac{1-2x}{x(1-x)} \frac{d}{dx} - \frac{1}{4x(1-x)} \right) u(x) = 0.$$

The Wronskian of $F(x)$ is

$$\begin{aligned} (6) \quad F(x) \frac{dF(1-x)}{dx} - \frac{dF(x)}{dx} F(1-x) &= -(F(x)F'_x(x) + F_x(x)F'(x)) \\ &= \exp \left(- \int \frac{1-2x}{x(1-x)} dx \right) \\ &= -\frac{c}{x(1-x)}, \end{aligned}$$

where $c = \frac{1}{2}F(\frac{1}{2})F_x(\frac{1}{2}) = \frac{1}{\pi}$. It is straight forward to compute that

$$\begin{aligned} {}_2F_1 \left[\begin{matrix} -\frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} ; x^2 \right] &= \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^2}{k!^2} \frac{-\frac{1}{2}}{\frac{1}{2} + k - 1} x^{2k} \\ &= - \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^2}{k!^2} \frac{1}{2k-1} x^{2k} \\ &= x^2 \frac{d}{dx} \left[x^{-1} {}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} ; x^2 \right] \right]. \end{aligned}$$

What is implied here is that the Legendre relation (5) is equivalent to the Wronskian described above.

We note the following fact, which follows from hypergeometric transformation formulas [11].

LEMMA 2.1. *For $x \in \mathbb{R}$ with $|x| < 1$,*

$${}_2F_1 \left[\begin{matrix} 1-a & a \\ 1 \end{matrix} ; \frac{1-\sqrt{1-x}}{2} \right] = {}_2F_1 \left[\begin{matrix} \frac{1-a}{2} & \frac{a}{2} \\ 1 \end{matrix} ; x \right].$$

Letting $a = 1/2$ yields

$${}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} ; \frac{1-\sqrt{1-x}}{2} \right] = {}_2F_1 \left[\begin{matrix} \frac{1}{4} & \frac{1}{4} \\ 1 \end{matrix} ; x \right].$$

Clausen's formula, states that

$$(7) \quad {}_2F_1 \left[\begin{matrix} a & b \\ a+b+\frac{1}{2} \end{matrix} ; x \right]^2 = {}_3F_2 \left[\begin{matrix} 2a & 2b & a+b \\ a+b+\frac{1}{2} & 2a+2b \end{matrix} ; x \right].$$

Letting $a = 1/4$, $b = 1/4$, we see that

$${}_2F_1 \left[\begin{matrix} \frac{1}{4} & \frac{1}{4} \\ 1 \end{matrix} ; x \right]^2 = {}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 \end{matrix} ; x \right].$$

Combining these results give us a connection between the corresponding ${}_3F_2$ and ${}_2F_1$ hypergeometric series above, which leads to Ramanujan type formulas for $1/\pi$, of the form

$$(8) \quad \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{k!^3} (ak+1) \lambda^n = \frac{\delta}{\pi},$$

where δ and a are algebraic numbers [9, 12]. In particular, $a \in \mathbb{Q}(\lambda)$ with $|\lambda| < 1$, for λ parametrizing a family of elliptic curves with complex multiplication (CM)².

²When the endomorphism ring of an elliptic curve is larger than the rational integers, it is said to have complex multiplication.

2.2. Geometric connection. To see the connection with geometry, we first consider the Legendre family of elliptic curves, parametrized by λ

$$E_\lambda : y^2 = x(x-1)(x-\lambda),$$

and define

$$\omega_\lambda = \frac{dx}{y} \text{ and } \eta_\lambda = x \frac{dx}{y}.$$

Then $E(k)$ defined above is a differential of the second kind, and is a combination of ω_λ and η_λ , which corresponds to two different ways of writing the Legendre family of elliptic curves.

The elliptic integrals (3) and (4) are also related to one of the Jacobi theta functions,

$$(9) \quad K(k) = \frac{\pi}{2} \theta_3^2(q(k)),$$

where

$$q(k) = \exp\left(-\pi \frac{K'(k)}{K(k)}\right).$$

The period lattice of the corresponding elliptic curve is generated by $4K(k)$ and $2iK'(k)$. Assume that the elliptic curve has CM; this amounts to K'/K being a quadratic number. In a special case when $K' = K\sqrt{r}$ with $r \in \mathbb{Q}$, the following hold due to the Legendre relation (5)

$$(10) \quad E = \frac{\pi}{4\sqrt{r}K} + \left(1 - \frac{\alpha(r)}{\sqrt{r}}\right) K,$$

$$(11) \quad E' = \frac{\pi}{4K} + \alpha(r)K,$$

where $\alpha(r) = E'/K - \pi/4K^2$ is the so-called *singular value function* and takes algebraic values when $r \in \mathbb{Q}^+$.

Thus, when $K'/K = \sqrt{r}$, for $r \in \mathbb{Q}^+$, the Legendre relation (5) can be written in simply the terms of E and K . This approach leads to formulas for $1/\pi$ related to hypergeometric series.

To see the connection to the hypergeometric series ${}_3F_2$, we consider the $K3$ surface X_λ described by the equation

$$X_\lambda : z^2 = x(x+1)y(y+1)(x+\lambda y).$$

When $\lambda \neq -1$, this manifold is related to the one-parameter family of elliptic curves of the form

$$E_\lambda^* : y^2 = (x-1)\left(x^2 - \frac{1}{1+\lambda}\right)$$

via the so-called Shioda-Inose structure [3, 23]. In particular, for the implication in terms of arithmetic, see the paper of Ahlgren, Ono, and Penniston [3].

For $d \in \{2, 3, 4, 6\}$ let $\tilde{E}_d(t)$ denote the following families of elliptic curves parameterized by t .

$$\tilde{E}_2(t) : y^2 = x(x-1)(x-t),$$

$$\tilde{E}_3(t) : y^2 + xy + \frac{t}{27}y = x^3,$$

$$\tilde{E}_4(t) : y^2 = x(x^2 + x + \frac{t}{4}),$$

$$\tilde{E}_6(t) : y^2 + xy = x^3 - \frac{t}{432}.$$

For t such that $\tilde{E}_d(t)$ has CM, let $\lambda_d = -4t(t-1)$, and write $E_d(\lambda_d) = \tilde{E}_d(\frac{1-\sqrt{1-\lambda_d}}{2})$. Both the Borweins and Chudnovskys established the following theorem, which reiterates the identity (8) in more detail.

THEOREM 2.2. *Let $d \in \{2, 3, 4, 6\}$. For λ_d such that $\mathbb{Q}(\lambda_d)$ is totally real, and for any embedding $|\lambda_d| < 1$, there exist algebraic numbers $\delta, a \in \mathbb{Q}(\lambda_d)$ yielding the following Ramanujan type formula for $1/\pi$,*

$$(12) \quad \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (\frac{1}{d})_k (\frac{d-1}{d})_k}{(k!)^3} (\lambda_d)^k (ak+1) = \frac{\delta}{\pi}.$$

3. Ramanujan supercongruences of van Hamme

Connecting partial sums of the related hypergeometric series to values of the p -adic gamma function, van Hamme constructed analogues to 13 of Ramanujan's formulas. For instance, one of Ramanujan's formulas is given by

$$\sum_{k=0}^{\infty} (4k+1)(-1)^k \frac{\left(\frac{1}{2}\right)_k^5}{k!^5} = \frac{2}{\Gamma\left(\frac{3}{4}\right)^4},$$

where $\Gamma(x)$ is the standard Gamma function (the value on the right hand side can be expressed in terms of $1/\pi$). By considering the p -adic gamma function $\Gamma_p(x)$, van Hamme observed numerically that

$$\sum_{k=0}^{\frac{p-1}{2}} (4k+1)(-1)^k \frac{\left(\frac{1}{2}\right)_k^5}{k!^5} = \begin{cases} \frac{-p}{\Gamma_p\left(\frac{3}{4}\right)^4} & \text{if } p \equiv 1 \pmod{4} \\ 0 & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Baffled, van Hamme stated that he had no concrete explanation for his observations, including the factor of $-p/2$ which distinguishes the right hand sides of the two equations. However, he proved 3 of the 13 congruences, (C.2), (H.2), and (I.2) in the notation below. For completeness, we list all 13 of van Hamme's conjectures, together with their Ramanujan series counterpart. Here the notation $S\left(\frac{p-1}{n}\right)$, simply means take the left hand side of the corresponding Ramanujan series and truncate at $(p-1)/n$.

Ramanujan Series	Conjectures of van Hamme
(A.1) $\sum_{k=0}^{\infty} (4k+1)(-1)^k \frac{\left(\frac{1}{2}\right)_k^5}{k!^5} = \frac{2}{\Gamma\left(\frac{3}{4}\right)^4}$	(A.2) $S\left(\frac{p-1}{2}\right) \equiv \begin{cases} \frac{-p}{\Gamma_p\left(\frac{3}{4}\right)^4} \pmod{p^3}, & \text{if } p \equiv 1 \pmod{4} \\ 0 \pmod{p^3}, & \text{if } p \equiv 3 \pmod{4} \end{cases}$
(B.1) $\sum_{k=0}^{\infty} (4k+1)(-1)^k \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} = \frac{2}{\pi} = \frac{2}{\Gamma\left(\frac{1}{2}\right)^2}$	(B.2) $S\left(\frac{p-1}{2}\right) \equiv \frac{-p}{\Gamma_p\left(\frac{1}{2}\right)^2} \pmod{p^3}, \quad p \neq 2$
(C.1) $\sum_{k=0}^{\infty} (4k+1) \frac{\left(\frac{1}{2}\right)_k^4}{k!^4} = \infty$	(C.2) $S\left(\frac{p-1}{2}\right) \equiv p \pmod{p^3}, \quad p \neq 2$
(D.1) $\sum_{k=0}^{\infty} (6k+1) \frac{\left(\frac{1}{3}\right)_k^6}{k!^6} = 1.01226\dots$	(D.2) $S\left(\frac{p-1}{3}\right) \equiv -p\Gamma_p\left(\frac{1}{3}\right)^9 \pmod{p^4}, \quad \text{if } p \equiv 1 \pmod{6}$
(E.1) $\sum_{k=0}^{\infty} (6k+1)(-1)^k \frac{\left(\frac{1}{3}\right)_k^3}{k!^3} = \frac{3\sqrt{3}}{2\pi} = \frac{3}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}$	(E.2) $S\left(\frac{p-1}{3}\right) \equiv p \pmod{p^3}$
(F.1) $\sum_{k=0}^{\infty} (8k+1)(-1)^k \frac{\left(\frac{1}{4}\right)_k^3}{k!^3} = \frac{2\sqrt{2}}{\pi} = \frac{4}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}$	(F.2) $S\left(\frac{p-1}{4}\right) \equiv \frac{-p}{\Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right)} \pmod{p^3}, \quad \text{if } p \equiv 1 \pmod{4}$
(G.1) $\sum_{k=0}^{\infty} (8k+1) \frac{\left(\frac{1}{2}\right)_k^4}{k!^4} = \frac{2\sqrt{2}}{\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)^2}$	(G.2) $S\left(\frac{p-1}{4}\right) \equiv p \frac{\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{4}\right)}{\Gamma_p\left(\frac{3}{4}\right)} \pmod{p^3}, \quad \text{if } p \equiv 1 \pmod{4}$
(H.1) $\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} = \frac{\pi}{\Gamma\left(\frac{3}{4}\right)^4}$	(H.2) $S\left(\frac{p-1}{2}\right) \equiv \begin{cases} -\Gamma_p\left(\frac{1}{4}\right)^4 \pmod{p^2}, & \text{if } p \equiv 1 \pmod{4} \\ 0 \pmod{p^2}, & \text{if } p \equiv 3 \pmod{4} \end{cases}$
(I.1) $\sum_{k=0}^{\infty} \frac{1}{k+1} \frac{\left(\frac{1}{2}\right)_k^2}{k!^2} = \frac{4}{\pi} = \frac{4}{\Gamma\left(\frac{1}{2}\right)^2}$	(I.2) $S\left(\frac{p-1}{2}\right) \equiv 2p^2 \pmod{p^3}, \quad p \neq 2$
(J.1) $\sum_{k=0}^{\infty} \frac{6k+1}{4^k} (-1)^{2k} \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} = \frac{4}{\pi} = \frac{4}{\Gamma\left(\frac{1}{2}\right)^2}$	(J.2) $S\left(\frac{p-1}{2}\right) \equiv \frac{-p}{\Gamma_p\left(\frac{1}{2}\right)^2} \pmod{p^4}, \quad p \neq 2, 3$
(K.1) $\sum_{k=0}^{\infty} \frac{42k+5}{64^k} (-1)^{2k} \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} = \frac{16}{\pi} = \frac{16}{\Gamma\left(\frac{1}{2}\right)^2}$	(K.2) $S\left(\frac{p-1}{2}\right) \equiv \frac{-5p}{\Gamma_p\left(\frac{1}{2}\right)^2} \pmod{p^4}, \quad p \neq 2$
(L.1) $\sum_{k=0}^{\infty} \frac{6k+1}{8^k} (-1)^k \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} = \frac{2\sqrt{2}}{\pi} = \frac{4}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}$	(L.2) $S\left(\frac{p-1}{2}\right) \equiv \frac{-p}{\Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right)} \pmod{p^3}, \quad p \neq 2$
(M.1) $\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^4}{k!^4} : \text{unknown}$	(M.2) $S\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^3}, \quad p \neq 2$

4. Proofs of van Hamme's supercongruences

Three supercongruences (C.2), (H.2), and (I.2) were proved by van Hamme using various methods. For instance, to prove (C.2), he used a sequence of orthogonal polynomials $p_k(x)$ which satisfy a certain

recurrence relation that enabled him to deduce that

$$p_{2k}(-1/2) = \frac{\left(\frac{1}{2}\right)_k^2}{k!^2},$$

$$p_{2k+1}(-1/2) = 0.$$

Further analysis of these polynomials yields (C.2).

4.1. Apéry numbers. Let $A(n)$ denote the *Apéry numbers* defined by

$$A(n) = \sum_{j=0}^n \binom{n+j}{j}^2 \binom{n}{j}^2,$$

which were used to prove the irrationality of the values of the Riemann zeta function $\zeta(2)$ and $\zeta(3)$. In addition, define $a(n)$ to be the n th Fourier coefficient of the modular form

$$(13) \quad \eta(2z)^4 \eta(4z)^4 = \sum_{n=1}^{\infty} a(n) q^n,$$

where $q = e^{2\pi iz}$, and $\eta(z)$ is the usual Dedekind eta-function.

Beukers [7, 8] proved the congruence

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p},$$

for primes $p > 2$. Furthermore, defining the sum

$$B(n) = \sum_{j=0}^n \binom{n+j}{j}^2 \binom{n}{j}^2,$$

he proved that for primes $p > 2$,

$$(14) \quad B\left(\frac{p-1}{2}\right) \equiv b(p) \pmod{p},$$

where $b(n)$ are similarly defined as coefficients of the modular form $\eta(4z)^6 = \sum_{n=1}^{\infty} b(n) q^n$. Furthermore, he conjectured that the following statements hold modulo p^2

$$(15) \quad A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}$$

$$(16) \quad B\left(\frac{p-1}{2}\right) \equiv b(p) \pmod{p^2}.$$

Interestingly, equation (16) is actually van Hamme's (H.2) in disguise, which he later showed in his work with Stienstra [31].

4.2. Supercongruences arising from Calabi-Yau threefolds. Rodriguez-Villegas uses Calabi-Yau threefolds over finite fields to numerically discover Beukers-like supercongruences modulo p^3 [28]. For instance, for odd primes p ,

$$(17) \quad a(p) \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\left(\frac{1}{2}\right)_k^4}{k!^4} \pmod{p^3}.$$

Notice that equation (17), incredibly, is van Hamme's (M.2).

More specifically, for odd primes p , the coefficients $a(p)$ are connected to a Calabi-Yau threefold defined by

$$(18) \quad x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + w + \frac{1}{w} = 0,$$

by the relation

$$(19) \quad a(p) = p^3 - 2p^2 - 7 - N(p),$$

where $N(p)$ counts the number of solutions to the manifold (18) over \mathbb{F}_p . The relation (19) was proved by Ahlgren and Ono [2], and they used it to represent $a(p)$ in terms of Gaussian hypergeometric series. Additionally, they proved conjecture (15) of Beukers [1].

Kilbourn proved observation (17), making use of the fact that the Calabi-Yau threefold (18) is modular [22]. His method involves writing $a(p)$ in terms of Gauss sums, and then using the Gross-Koblitz formula [16] to relate the Gauss sums to the p -adic gamma function.

5. Hypergeometric methods for van Hamme's conjectures

Three of van Hamme's conjectures, (A.2), (B.2) and (J.2), have been proved using a variety of techniques involving hypergeometric series. We include details below.

5.1. Proof of Conjecture (A.2). One approach to proving conjecture (A.2) is due to McCarthy and Osburn [24]. Their proof also involves Gaussian hypergeometric series, which we now introduce. If A, B are characters of \mathbb{F}_p , the normalized Jacobi sum is given as

$$\left(\frac{A}{B}\right) = \frac{B(-1)}{p} \sum_{x \in \mathbb{F}_p} A(x) \overline{B}(1-x).$$

We extend multiplicative characters χ of \mathbb{F}_p^* to \mathbb{F}_p by defining $\chi(0) = 0$. For characters A_0, \dots, A_n and B_1, \dots, B_n of \mathbb{F}_p^* , the Gaussian hypergeometric series is defined for $x \in \mathbb{F}_p$ by

$${}_{n+1}F_n \left[\begin{matrix} A_0 & A_1 & \cdots & A_n \\ B_1 & \cdots & 0 \end{matrix} ; x \right]_p = \frac{p}{p-1} \sum_{\chi} \binom{A_0 \chi}{\chi} \binom{A_1 \chi}{B_1 \chi} \cdots \binom{A_n \chi}{B_n \chi} \chi(x).$$

When each of the A_i are Legendre symbols modulo p , and all of the B_j are trivial characters, we simply write ${}_{n+1}F_n(x)$, following suit of McCarthy and Osburn.

The Gross-Koblitz formula provides a wonderful connection between Gauss sums and the p -adic Gamma function. To state this formula, we first observe that we may view a character χ for \mathbb{F}_p^* as taking values in \mathbb{Z}_p^* . Let $\pi \in \mathbb{C}_p$ be a fixed root of $x^{p-1} + p = 0$, and let ζ_p be the unique p th root of unity in \mathbb{C}_p for which

$$\zeta_p \equiv 1 + \pi \pmod{\pi^2}.$$

We then define the Gauss sum for a character $\chi : \mathbb{F}_p \rightarrow \mathbb{C}_p$ by

$$g(\chi) = \sum_{x=0}^{p-1} \chi(x) \zeta_p^x.$$

If ω is the Teichmüller character, a primitive character defined by the property that $\omega(x) \equiv x \pmod{p}$ for $x = 0, \dots, p-1$, then the Gross-Koblitz formula states that for $0 \leq j \leq p-2$,

$$(20) \quad cg(\bar{\omega}^j) = \pi^j \Gamma_p \left(\frac{j}{p-1} \right).$$

In the proof of (A.2) by McCarthy and Osburn, they use the following theorem of Osburn and Schneider [26], which states that for an odd prime p , an integer $n \geq 2$, and an element $\lambda \in \mathbb{F}_p$,

$$-p^n {}_{n+1}F_n(\lambda) \equiv (-1)^{n+1} \left(\frac{-1}{p} \right)^{n+1} [p^2 X(p, \lambda, n) + pY(p, \lambda, n) + Z(p, \lambda, n)] \pmod{p^3}.$$

Here, $\left(\frac{\cdot}{p} \right)$ is the Legendre symbol modulo p , and $X(p, \lambda, n)$, $Y(p, \lambda, n)$, and $Z(p, \lambda, n)$ are truncated sums involving differences of generalized harmonic sums,

$$H_n(i) = \sum_{j=1}^n \frac{1}{j^i}.$$

For explicit formulas see the work of McCarthy and Osburn [24].

After careful analysis of the $X(p, \lambda, n)$ and $Y(p, \lambda, n)$ terms, they reduced the problem to showing that

$$\sum_{k=1}^{\frac{p-1}{2}} (4k+1)(-1)^k \frac{\left(\frac{1}{2}\right)_k^5}{k!^5} \equiv \left(\frac{-1}{p}\right) pZ(p, 1, 2) \pmod{p^3}.$$

From here, they used the following special case of Whipple's ${}_7F_6$ hypergeometric transformation

$$(21) \quad {}_6F_5 \left[\begin{matrix} a & 1 + \frac{a}{2} & c & d & e & f \\ & \frac{a}{2} & 1+a-c & 1+a-d & 1+a-e & 1+a-f \end{matrix} ; -1 \right] = \frac{\Gamma(1+a-e)\Gamma(1+a-f)}{\Gamma(1+a)\Gamma(1+a-e-f)} \cdot {}_3F_2 \left[\begin{matrix} 1+a-c-d & e & f \\ & 1+a-c & 1+a-d \end{matrix} ; 1 \right]$$

which enabled them to prove the result.

5.2. Proofs of Conjecture (B.2). Conjecture (B.2) of van Hamme has been proved in two different ways. Mortenson [25] observed that the supercongruence (A.2) shared the right hand side with a supercongruence of Beukers. Namely, for odd primes p

$$\sum_{k=0}^{\infty} (4k+1)(-1)^k \frac{\left(\frac{1}{2}\right)_k^5}{k!^5} \equiv p \sum_{k=0}^{\frac{p-1}{2}} \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} \pmod{p^3}.$$

Armed with this idea, Mortenson proved (B.2) from van Hamme's list. More specifically,

$$\sum_{k=0}^{\frac{p-1}{2}} (4k+1)(-1)^k \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} \equiv p \left(\frac{-1}{p}\right) \pmod{p^3}.$$

To do so, he first demonstrated a technical lemma to evaluate a quotient of gamma functions. Then, following McCarthy and Osburn, he utilized the special case of Whipple's transformation (21).

Zudilin proved conjecture (B.2) by using a method known as the WZ method [34]. This method, designed by Wilf and Zeilberger [32], is a powerful tool for proving identities for hypergeometric series. In fact, equipped with the WZ method, Zudilin was able to prove a p -adic analog for a formula for $1/\pi^2$ discovered by Guillera [17]. More specifically, Guillera's series

$$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^3 \left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{k!^5} (120k^2 + 34k + 3) \frac{1}{2^{4k}} = \frac{32}{\pi^2},$$

has the p -adic analog for odd primes p

$$\sum_{k=0}^{p-1} \frac{\left(\frac{1}{2}\right)_k^3 \left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{k!^5} (120k^2 + 34k + 3) \frac{1}{2^{4k}} \equiv 3p^2 \pmod{p^5}.$$

5.3. Proof of Conjecture (J.2). The approach by Long [23] is more general for conjecture (J.2). Here, motivated by the techniques of McCarthy and Osburn, Mortenson, and Zudilin, she utilized particular hypergeometric series identities and evaluations in order to prove the following more general theorem.

THEOREM 5.1. *Let $p > 3$ be prime and r a positive integer. Then*

$$\sum_{k=0}^{\frac{p^r-1}{2}} (4k+1) \frac{\left(\frac{1}{2}\right)_k^4}{k!^4} \equiv p^r \pmod{p^{3+r}}.$$

In addition, she proved conjecture (J.2) of van Hamme,

$$\sum_{k=0}^{\frac{p-1}{2}} (6k+1) \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} 4^{-k} \equiv (-1)^{\frac{p-1}{2}} p \pmod{p^4}.$$

Her method differs with the use of particularly nice hypergeometric evaluation identities. For instance, the “strange” valuation of Gosper [15],

$${}_5F_4 \left[\begin{matrix} 2a & 2b & 1-2b & 1+\frac{2a}{3} \\ a+b-1 & a+b+\frac{1}{2} & \frac{2a}{3} & 1+2a+2n \end{matrix} ; \frac{1}{4} \right] = \frac{(a+\frac{1}{2})_n(a+1)_n}{(a+b+\frac{1}{2})_n(a-b+1)_n}.$$

6. Ramanujan supercongruences arising from $K3$ surfaces

We now turn our attention to recent work of Long, Nebe, and the authors on Ramanujan supercongruences arising from $K3$ surfaces. First, we describe some work of Atkin and Swinnerton-Dyer on noncongruence subgroups which will be involved in our discussion.

Atkin and Swinnerton-Dyer [4] studied the arithmetic properties of coefficients of modular forms of noncongruence subgroups of $SL_2(\mathbb{Z})$, i.e. those finite index subgroup of $SL_2(\mathbb{Z})$ not containing any principal congruence subgroups. Due to the lack of efficient Hecke theory, these coefficients are not as desirable as the coefficients of Hecke eigenforms; for example, the discriminant function, to name one, describes some of the coefficients. Despite this difficulty, Atkin and Swinnerton-Dyer discovered interesting p -adic analogues of classical Hecke recursions. To illustrate, we state their result in a special case. Let

$$E : y^2 = x^3 - Bx - C$$

be a nonsingular elliptic curve over \mathbb{Z}_p such that E has good reduction modulo p . Let ξ be any local uniformizer of E at infinity over \mathbb{Z}_p that is a formal power series of $\frac{-x}{y}$ with coefficients in \mathbb{Z}_p . Then the holomorphic differential

$$(22) \quad -\frac{dx}{2y} = (1 + \sum_{n \geq 1} a(n)\xi^n)d\xi$$

has coefficients in \mathbb{Z}_p .

Then, taking $A = p + 1 - \#(E/\mathbb{F}_p)$, and the coefficient $a(n)$ from the holomorphic differential (22), the following congruence holds for all $n \geq 1$,

$$a(np) - Aa(n) + pa(n/p) \equiv 0 \pmod{p^{1+\text{ord}_p n}}.$$

Atkin and Swinnerton-Dyer’s results inspired the later work of Cartier [10] and Katz [20]. Here we shall use Katz’s approach. In the spirit of their results, a sequence $\{a(n)\}$ is said to satisfy a *weight k Atkin and Swinnerton-Dyer (ASD) congruence* at a fixed prime p if there are p -adic integers A_1, \dots, A_{s-1} such that for all $n \geq 1$,

$$a(np) + A_1a(n) + A_2a(n/p) + \dots + A_s a(n/p^{s-1}) \equiv 0 \pmod{p^{(k-1)(\text{ord}_p n+1)}}.$$

We require that the congruence is written in the way which is compatible with the congruences satisfied by the logarithms of 1-dimensional commutative formal groups [30, Appendix A.8]. The convention that $a(n/p^e) = 0$ if $p^e \nmid n$ is taken in the preceding congruence. Moreover, the characteristic polynomial

$$T^s + A_1T^{s-1} + \dots + \dots + A_s$$

customarily has important arithmetic meaning. In the case of the above mentioned elliptic curves, when E has a model defined over \mathbb{Q} , then

$$T^2 - AT + p$$

is the Hecke polynomial of a weight 2 Hecke eigenform. This is the result of the Taniyama-Shimura-Weil conjecture, since proved by Wiles, Taylor-Wiles et al. A common technique in proving weight 2 ASD congruences is the theory of formal groups [19]. As a result, any ASD congruence of weight greater than 2 is also referred to as a supercongruence.

6.1. Results of Chisholm, Deines, Long, Nebe, and Swisher. Alternatively, for $d \in \{2, 3, 4, 6\}$ we give a geometric proof of Ramanujan type supercongruences related to hypergeometric series of the form

$${}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{d} & 1 - \frac{1}{d} \\ 1 & 1 \end{matrix} ; \lambda \right]$$

truncated modulo p^2 . We outline the specific case for $d = 2$ and note that the results for the other cases follow similarly.

Consider the $K3$ surface X_λ described by the equation

$$X_\lambda : z^2 = x(x+1)y(y+1)(x+\lambda y).$$

This manifold is a natural 2-dimensional analogue of the one-parameter family of elliptic curves of the form

$$E_\lambda : y^2 = (x-1) \left(x^2 - \frac{1}{1+\lambda} \right).$$

Each elliptic curve E_λ has a model over the field $F_\lambda = \mathbb{Q}(j(E_\lambda))$, where the j -invariant of E_λ is given by

$$j(E_\lambda) = 64 \frac{(\lambda+4)^3}{\lambda^2}.$$

In fact, E_λ is isomorphic to $E_2(\lambda)$ as defined in Section 2.

Let $\mathbb{Q}(\tau)$ denote an imaginary quadratic number field. When $\lambda \in \overline{\mathbb{Q}}$ is such that E_λ admits complex multiplication by an order of $\mathbb{Q}(\tau)$, following Ramanujan's idea, there exist numbers a and λ in $K_\lambda = \mathbb{Q}(\tau, j(\tau), \lambda)$, and an algebraic number δ , giving an instance of the Ramanujan type formula [9, 12]

$$(23) \quad \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{k!^3} (ak+1) \lambda^k = \frac{\delta}{\pi}.$$

Now we will state the main result in full generality for $d \in \{2, 3, 4, 6\}$. Take $E_d(\lambda_d)$ as in Section 2.2.

THEOREM 6.1. *Let $\lambda_d \in \overline{\mathbb{Q}}$ such that $\mathbb{Q}(\lambda_d)$ is totally real, the elliptic curve $E_d(\lambda_d)$ has complex multiplication, and $|\lambda_d| < 1$ for an embedding of λ_d to \mathbb{C} . For each prime p that is unramified in $\mathbb{Q}(\sqrt{1-\lambda_d})$ and coprime to the discriminant of $E_d(\lambda_d)$ such that a, λ_d can be embedded in \mathbb{Z}_p (and we fix such embeddings), then*

$$(24) \quad \sum_{k=0}^{p-1} \frac{(\frac{1}{2})_k (\frac{1}{d})_k (\frac{d-1}{d})_k}{k!^3} (\lambda_d)^k (ak+1) \equiv \text{sgn} \cdot \left(\frac{1-\lambda_d}{p} \right) \cdot p \pmod{p^2},$$

where $\left(\frac{1-\lambda_d}{p} \right)$ is the Legendre symbol, and $\text{sgn} = \pm 1$, equaling 1 if and only if $E_d(\lambda_d)$ is ordinary modulo p .

The proof involves ASD congruences, discussed above, which was inspired by results due to Katz. Also required are elliptic curves with complex multiplication with a quartic twist. As well, we view the curve with a model over the ring of Witt vectors, and consider its de Rham cohomology. The theory of modular forms plays an integral role as well as essential identities due to Clausen.

6.2. Ramanujan type supercongruences for λ in \mathbb{Q} . In this case the coefficients of the Ramanujan formula (23) are obtained from the Hilbert class field $K_\lambda = \mathbb{Q}$. In fact, the congruence

$$(25) \quad \sum_{k=0}^{\frac{p-1}{2}} \frac{(\frac{1}{2})_k^3}{k!^3} (4k+1) (-1)^k \equiv (-1)^{\frac{p-1}{2}} p \pmod{p^3},$$

is an example of such a formula with $a = 4$, $\lambda = -1$, taken modulo p^3 . Corresponding to this supercongruence is the conjectural ASD congruence of the form

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{(\frac{1}{2})_k^3}{k!^3} (-1)^k \equiv \alpha_p \pmod{p^3}.$$

The congruence pertains to ordinary primes, i.e. $p \equiv 1, 3 \pmod{8}$ and α_p is the p -adic unit root of

$$H_p(X, \lambda) = X^2 - a_p X + \psi_\lambda(p)p^2.$$

The quadratic function $H_p(X, \lambda)$ is the p th Hecke polynomial of a weight 3 Hecke cuspidal eigenform $g(z) = \sum b(n)q^n$. Explicitly, the first few coefficients of $H_p(X, \lambda)$ are

$$a_5 = a_{13} = a_{23} = 0, \quad a_{17} = 2, \quad a_{19} = 34, \quad a_{67} = -62, \quad a_{73} = -142,$$

which can be identified from the L -function of X_λ .

For the corresponding elliptic curve at $\lambda = -1$, E_{-1} , whenever the curve has *ordinary* reduction, for example at the primes $p = 17, 19, 67$, the ASD congruence becomes

$$(26) \quad \sum_{k=0}^{\frac{p-1}{2}} \frac{(\frac{1}{2})_k^3}{k!^3} (-1)^k \equiv b(p) \pmod{p^3}.$$

When E_{-1} is *supersingular* at p , for example at the primes $p = 3, 13, 23$, then the ASD congruence is given by

$$(27) \quad \sum_{k=0}^{\frac{p-1}{2}} \frac{(\frac{1}{2})_k^3}{k!^3} (-1)^k \equiv b(p) = 0 \pmod{p^2}.$$

It is also possible to write the coefficient $b(p)$ in terms of an explicit value of a p -adic Gamma function as related to Gauss or Jacobi sums, using the Gross-Koblitz formula (20). These congruences can be proved modulo p , by counting points on the $K3$ surface X_{-1} . Also, congruence (26) can be proved modulo p^2 , using Coster and van-Hamme's result [14]. The keen reader is encourage to consult the work of Kibelbek et al. for further details [21]. Numerical results similar to those in equations (26) and (27) are observed when $\lambda = 1/4, -1/8, 1/64$.

6.3. Ramanujan type supercongruences for λ not in \mathbb{Q} . For larger fields K_λ , less has been formulated, and here we provide some numerical evidence computed using **Sage**. Specific values for a, λ and δ are considered below. We claim that the infinite sums, when reduced modulo p^2 , take the particular form stated in their respective conjecture. We note that in private communication with Long, Zudilin conjectured that Theorem 6.1 should in fact hold modulo p^3 for each λ for which E_λ has complex multiplication. For further conjectured examples, see also the work of Guillera [18].

Example 1. [9, Exercise 5]

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{k!^3} \left[\frac{12}{3-\sqrt{3}} k + 1 \right] (3-\sqrt{3})(2-\sqrt{3})((2-\sqrt{3})^4)^k = \frac{\sqrt{2}}{3^{1/4}} \cdot \frac{1}{\pi}.$$

Here $a = 12/(3-\sqrt{3})$, $\lambda = (2-\sqrt{3})^4$, and the Hilbert class field is $K_\lambda = \mathbb{Q}(\sqrt{3}, \sqrt{-1})$. This sum in condensed form is

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{k!^3} [(3-\sqrt{3}) + 12k] (2-\sqrt{3})^{4k+1} = \frac{\sqrt{2}}{3^{1/4}\pi}.$$

Truncating the sum modulo p^2 we make the following claim.

CONJECTURE 6.2. *For any prime $p \equiv \pm 1 \pmod{12}$,*

$$\sum_{k=0}^{p-1} \frac{(\frac{1}{2})_k^3}{k!^3} [(3-\sqrt{3}) + 12k] (2-\sqrt{3})^{4k+1} \equiv \left(\frac{-2\sqrt{3}}{p} \right) (3-\sqrt{3})(2-\sqrt{3})p \pmod{p^2}.$$

This congruence holds for primes at least up to 10,000.

Example 2. [9, (5.5.15)]

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{k!^3} \left[\frac{4\sqrt{5}}{\sqrt{5}-1} k + 1 \right] \frac{(\sqrt{5}-1)}{2} ((\sqrt{5}-2)^2)^k = \sqrt{\sqrt{5}+2} \cdot \frac{1}{\pi}.$$

Here $a = 4\sqrt{5}/(\sqrt{5} - 1)$, $\lambda = (\sqrt{5} - 2)^2$, and the Hilbert class field is $K_\lambda = \mathbb{Q}(\sqrt{5}, \sqrt{-1})$. To see how this follows from equation (5.5.15) [9], let $N = 5$, $k_5 = \frac{\sqrt{\sqrt{5}-1}-\sqrt{3-\sqrt{5}}}{2}$, $k_5^2 = \frac{1}{2} - \sqrt{\sqrt{5}-2}$, $\alpha(5) = \frac{\sqrt{5}-\sqrt{2\sqrt{5}-2}}{2}$, $G_5^{-12} = \sqrt{5} - 2$, and

$$a_n(5) = (\alpha(5) - \sqrt{5}k_5^2) + n\sqrt{5}(k_5'^2 - k_5^2) = \sqrt{\sqrt{5}-2} \left[(\sqrt{5}-1)/2 + 2n\sqrt{5} \right].$$

The condensed form is

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{k!^3} [(\sqrt{5}-1)/2 + 2k\sqrt{5}] (\sqrt{5}-2)^{2k} = \frac{\sqrt{\sqrt{5}+2}}{\pi}.$$

We conjecture the following.

CONJECTURE 6.3. *For any prime $p \equiv \pm 1 \pmod{10}$,*

$$\sum_{k=0}^{p-1} \frac{(\frac{1}{2})_k^3}{k!^3} [(\sqrt{5}-1)/2 + 2k\sqrt{5}] (\sqrt{5}-2)^{2k} \equiv \left(\frac{-(\sqrt{5}-2)}{p} \right) \frac{(\sqrt{5}-1)}{2} p \pmod{p^2}.$$

This congruence holds for primes at least up to 10,000.

Example 3. [9, Exercise 6]

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{k!^3} \left[\frac{28}{7-2\sqrt{6}}k + 1 \right] (7-2\sqrt{6})(\sqrt{3}-\sqrt{2})^2(-(\sqrt{3}-\sqrt{2})^8)^k = \frac{2}{3} \cdot \frac{1}{\pi}.$$

Here $a = 28/(7-2\sqrt{6})$, $\lambda = -(\sqrt{3}-\sqrt{2})^8$, and the Hilbert class field is $K_\lambda = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. In condensed form, this is

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{k!^3} (-1)^k [(7-2\sqrt{6}) + 28k] (\sqrt{3}-\sqrt{2})^{8k+2} = \frac{2}{3\pi}.$$

Modulo p^3 we claim the following.

CONJECTURE 6.4. *For any prime $p \equiv \pm 1 \pmod{24}$,*

$$\sum_{k=0}^{p-1} \frac{(\frac{1}{2})_k^3}{k!^3} (-1)^k [(7-2\sqrt{6}) + 28k] (\sqrt{3}-\sqrt{2})^{8k+2} \equiv \left(\frac{-1}{p} \right) (7-2\sqrt{6})(\sqrt{3}-\sqrt{2})^2 p \pmod{p^3}.$$

This congruence holds for primes at least up to 10,000. It is our desire that the numerous techniques described in this paper will prove fruitful in further investigation of these conjectures.

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DEPT. OF MATHEMATICS AND STATISTICS, UNIVERSITY OF CALGARY, CALGARY, ALBERTA, CANADA T2N 1N4
E-mail address: `chisholm@math.ucalgary.ca`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA 98195, USA
E-mail address: `adeines@math.washington.edu`

DEPARTMENT OF MATHEMATICS, OREGON STATE UNIVERSITY, CORVALLIS, OR, 97331, USA
E-mail address: `swisherh@math.oregonstate.edu`