# QUIVER FLAG VARIETIES AND MIRROR SYMMETRY 

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# Quiver flag varieties and mirror symmetry 


#### Abstract

Quiver flag zero loci are subvarieties of quiver flag varieties cut out by sections of representation theoretic vector bundles. Grassmannians are an example of quiver flag varieties. The Abelian/non-Abelian correspondence is a conjecture relating the Gromov-Witten invariants of a non-Abelian GIT quotient to the same invariants of an Abelian GIT quotient. In the first chapter, we show how the conjecture in the case of Grassmannians arises from Givental's loop space mirror heuristics. We then prove the Abelian/non-Abelian Correspondence for quiver flag zero loci: this allows us to compute their genus zero Gromov-Witten invariants. We determine the ample cone of a quiver flag variety. In joint work with Tom Coates and Alexander Kasprzyk, we use these results to find all four-dimensional Fano manifolds that occur as quiver flag zero loci in ambient spaces of dimension up to 8, and compute their quantum periods. In this way we find at least 141 new four-dimensional Fano manifolds. In the last chapter, we describe a conjectural method for finding mirrors to these fourfolds, and implement this in several examples.


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## 0

## Introduction

Quiver flag varieties are a generalization of type A flag varieties that were introduced by Craw [18] based on work of King [35]. They are fine moduli spaces for stable representations of the associated quiver (see 2.1.3). Like flag varieties and toric complete intersections, quiver flag varieties can be constructed as GIT quotients of a vector space (see 2.1.1). Unlike toric varieties, the quotienting group for a quiver flag variety is in general non-Abelian; this increases the complexity of their structure considerably: specifically, it places them largely outside of the range of known mirror symmetry constructions.

The Abelian/non-Abelian Correspondence of Ciocan-Fontanine-Kim-Sabbah relates the Gromov-Witten theory of a non-Abelian GIT quotient to that of an Abelian GIT quotient. In Chapter 1, we show how this relation can be seen for the Grassmannian just from considering the loop space of Givental. This calculation isn't rigorous, but can be seen as motivation for the rest of the thesis. In Chapter 2, the main focus is to prove the Abelian/non-Abelian correspondence for quiver flag varieties.

The two perspectives on quiver flag varieties - as fine moduli spaces and as GIT quotients - give two different ways to consider them as ambient spaces. From the moduli space perspective, smooth projective varieties with collections of vector bundles together with appropriate maps between them come with natural maps into the quiver flag variety. From the GIT perspective, one is led to consider subvarieties which occur as zero loci of sections of representation theoretic vector bundles. If the ambient GIT quotient is a toric variety, these subvarieties are toric complete
intersections; if the ambient space is a quiver flag variety, we call these subvarieties quiver flag zero loci. While in this thesis we emphasize the GIT quotient perspective, the moduli space perspective should be kept in mind as further evidence of the fact that quiver flag varieties are natural ambient spaces. All smooth Fano varieties of dimension less than or equal to three can be constructed as either toric complete intersections or quiver flag zero loci. These constructions of the Fano threefolds were given in [13]: see Theorem A. 1 there as well as the explicit constructions in each case. While there is an example in dimension 66 of a Fano variety which is neither a toric complete intersection nor a quiver flag zero locus (see [13]), one might nevertheless hope that most four-dimensional smooth Fano variety are either toric complete intersections or quiver flag zero loci. The classification of four dimensional Fano varieties is open.

Chapter 2 studies quiver flag varieties with a view towards understanding them as ambient spaces of Fano fourfolds. Specifically, [16] classified smooth four dimensional Fano toric complete intersections with codimension at most four in the ambient space. This heavily computational search relied on understanding the geometry and quantum cohomology of toric varieties from their combinatorial structure. The guiding motivation of the chapter is to establish comparable results for quiver flag varieties to enable the same search to be carried out in this context. For example, we determine the ample cone of a quiver flag variety from the path space of the associated quiver: in this way, we are able to efficiently determine a sufficient condition for whether a quiver flag zero locus is Fano.

The main result of this thesis is the proof of the Abelian/non-Abelian Correspondence of Ciocan-Fontanine-Kim-Sabbah for Fano quiver flag zero loci. This allows us to compute their genus zero Gromov-Witten invariants*. From the perspective of the search for four dimensional Fano quiver flag zero loci, the importance of this result is that it allows us to compute the quantum period. The quantum period (a generating function built out of certain genus 0 Gromov-Witten invariants) is the invariant that we use to distinguish deformation families of Fano fourfolds: if two quiver flag zero loci have different period sequences, they are not deformation equivalent.

In Chapter 3, which reports on work which is joint with Tom Coates and Alexander Kasprzyk, we use the structure theory developed in Chapter 2 to find fourdimensional Fano manifolds that occur as quiver flag zero loci in ambient spaces of dimension up to 8 , and compute their quantum periods. 141 of these quantum periods were previously unknown. Thus we find at least 141 new four-dimensional

[^0]Fano manifolds. The quantum periods, and quiver flag zero loci that give rise to them, are recorded in Appendix A. Figure 1 overleaf shows the distribution of degree and Euler number for the four-dimensional quiver flag zero loci that we found, and for four-dimensional Fano toric complete intersections. Our primary motivation is as follows. There has been much recent interest in the possibility of classifying Fano manifolds using Mirror Symmetry. It is conjectured that, under Mirror Symmetry, $n$-dimensional Fano manifolds should correspond to certain very special Laurent polynomials in $n$ variables [12]. This conjecture has been established in dimensions up to three [13], where the classification of Fano manifolds is known [32, 41]. Little is known about the classification of four-dimensional Fano manifolds, but there is strong evidence that the conjecture holds for four-dimensional toric complete intersections [16]. The results of Chapter 3 will provide a first step towards establishing the conjectures for these four dimensional Fano quiver flag zero loci.

In the final chapter of the thesis, Chapter 4, we discuss future directions of this work. Specifically, we discuss toric degenerations of quiver flag varieties, and their role in finding mirrors of Fano quiver flag zero loci. For a certain family of quivers, we provide a systematic (and still conjectural) method of finding Laurent polynomial mirrors of quiver flag zero loci which are subvarieties of these quivers.


Figure 1: Degrees and Euler numbers for four-dimensional Fano quiver flag zero loci and toric complete intersections; cf. [16, Figure 5]. Quiver flag zero loci that are not toric complete intersections are highlighted in red.

## 1

## Mirror heuristics

In [23], the authors Galkin and Iritani recover the Laurent polynomial mirror of projective space using Givental's equivariant loop space heuristics. In this chapter, we find the analogue for Grassmannians. We show how the Abelian/non-Abelian correspondence for Grassmannians arises from the same considerations. That is, we use a heuristic argument to produce the mirror oscillatory integral to the Grassmannian, and show that it takes the form predicted by Hori-Vafa and the Abelian/non-Abelian correspondence.

### 1.1 Quantum cohomology and the quantum differential equations

We first briefly review quantum cohomology and the quantum differential equations. Let $X$ be a smooth Fano variety. The quantum cohomology ring is defined by giving a deformation of the usual cup product of $H^{*}(X)$ for every $t \in H^{*}(X)$. The structural constants defining the new product are given by Gromov-Witten invariants.

A nodal curve $C$ is a projective, connected curve with singularities that are at most nodes, that is, of the local form $x y=0$. An $n$-pointed nodal curve is pair $(C, \epsilon)$ where $C$ is a nodal curve, and $\epsilon$ is a set $\left\{p_{1}, \ldots, p_{n}\right\}$ of $n$ non-singular points on $C$. The moduli space of stable maps $\bar{M}_{g, n}(X, \beta)$ parametrizes stable maps $f: C \rightarrow X$ up to isomorphism. Here $C$ is a possibly nodal curve of arithmetic genus $g$, with $n$ marked points, and $f_{*}([C])=\beta$. In general, $\bar{M}_{g, n}(X, \beta)$ may have components of different dimensions; however, it is possible to define a virtual fundamental class of
the expected dimension:

$$
(\operatorname{dim}(X)-3)(g-1)+\int_{\beta} c_{1}(X)+n
$$

There are natural maps $e v_{i}: \bar{M}_{g, n}(X, \beta) \rightarrow X$, where $e v_{i}([f: C \rightarrow X])=f\left(p_{i}\right)$. Let $\alpha_{1}, \ldots, \alpha_{n} \in H^{*}(X)$.

Definition 1.1.1. A Gromov-Witten invariant of $X$ is

$$
\int_{\left[\bar{M}_{0, n}(X, \beta)\right]^{v i r t}} e v_{1}^{*}\left(\alpha_{1}\right) \cup \cdots \cup e v_{n}^{*}\left(\alpha_{n}\right)
$$

for some $n \in \mathbb{Z}_{>0}, \alpha_{i} \in H^{*}(X)$ and $\beta \in H_{2}(X)$.

Let $\left\{T_{i}\right\}$ be a homogenous basis of $H^{*}(X, \mathbb{C})$ and $\left\{T^{i}\right\}$ a dual basis. Let $t \in$ $H^{2}(X, \mathbb{C})$. The small quantum product is defined by

$$
\left\langle T^{a} o_{t} T^{b}, T^{c}\right\rangle:=\sum_{d \in H_{2}(X)} e^{\int_{d} t} \int_{\left[M_{0,3}(X, d)\right]^{v i r t}} e v_{1}^{*}\left(T^{a}\right) e v_{2}^{*}\left(T^{b}\right) e v_{3}^{*}\left(T^{c}\right) .
$$

If $T_{1}, \ldots, T_{r}$ are a basis of $H^{2}(X, \mathbb{C})$, and $t_{i}$ a parameter for $T_{i}$, define $q_{i}:=e^{t_{i}}$. For $d=\sum_{i=1}^{r} d_{i} T_{i}$, write $q^{d}=q_{1}^{d_{1}} \cdots q_{r}^{d_{r}}$. We can re-write the above as

$$
\left\langle T^{a} \circ T^{b}, T^{c}\right\rangle:=\sum_{d \in H_{2}(X)} q^{d} \int_{\left[M_{0,3}(X, d)\right]^{v i r t}} e v_{1}^{*}\left(T^{a}\right) e v_{2}^{*}\left(T^{b}\right) e v_{3}^{*}\left(T^{c}\right),
$$

This gives a product on $T^{*}\left(H^{*}(X)\right)=H^{*}(X) \times H^{*}(X)$ at every point in $H^{2} X$. Associated to this product structure is the quantum differential equations. As $H^{*}(X) \times H^{*}(X)$ is trivial over $H^{*}(X)$, there is a natural flat connection $d$ given by the parameters on the base.

$$
\nabla_{i}^{z}=\nabla_{\frac{\partial}{\partial t_{i}}} s=z d_{\frac{\partial}{\partial t_{i}}} s+T_{i} \circ .
$$

In fact, this is a flat connection (see, for example, [5]). The quantum differential equations are the differential equations satisfied by the sections.

This connection also gives quantum cohomology $H^{*}(X) \otimes \mathbb{C}[z]\left[\left[q_{1}, \ldots, q_{r}\right]\right]$ the structure of a D module. Here we follow [31]. Let $D$ be a Heisenberg algebra:

$$
D:=\mathbb{C}[z]\left[\left[q_{1}, \ldots, q_{r}\right]\right]\left[p_{1}, \ldots, p_{r}\right],
$$

with $\operatorname{deg}\left(q_{i}\right)=\int_{P D\left(t_{i}\right)} c_{1}\left(T_{X}\right)$, and $\operatorname{deg}\left(p_{i}\right)=\operatorname{deg}(z)=2$. The commutation relations
are given by

$$
\left[p_{a}, q_{b}\right]=z \delta_{b}^{a} q_{b},\left[p_{a}, p_{b}\right]=\left[q_{a}, q_{b}\right]=0,\left[p_{a}, f\right]=z \frac{\partial}{\partial q_{a}} f, f \in \mathbb{C}\left[\left[q_{1}, \ldots, q_{r}\right]\right] .
$$

Then $D$ operates on the quantum cohomology ring $H^{*}(X) \otimes \mathbb{C}[z]\left[\left[q_{1}, \ldots, q_{r}\right]\right]$ by $q_{a} \mapsto q_{a}$ and $p_{a} \mapsto \nabla_{a}^{z}$.

In [26], Givental conjectured that equivariant Floer cohomology (not rigorously defined) should have the structure of a D module, and that this D module should be isomorphic to the quantum D module.

Let $L X$ be the loop space of $X$ : that is, the space of free contractible loops in $X$. The symplectic form $\omega$ on $X$ induces a symplectic form on $L X$ : vector fields on $X$ can be identified with vector fields on $X$ over a loop $\gamma$; given two such vector fields $w, v$, the symplectic form is given by

$$
\oint \omega(w(\gamma(t), v(\gamma(t))) d t .
$$

Reparametrization of loops gives an action of $S^{1}$ on $L X$ which preserves the symplectic form; however, the Hamiltonian associated to it is multivalued: it assigns to a loop the symplectic area of a disc contracting the loop. To make this better defined, let $\tilde{L X}$ be a covering space of $L X$. Givental then discusses the $S^{1}$ equivariant Floer cohomology of $\tilde{L X}$ : by definition this is the cohomology of the critical set of the Hamiltonian. That is, it is the cohomology of the fixed points of the $S^{1}$ action, which are constant loops. We therefore get a copy of $X$ at each level of the covering space. Assuming that $X$ is simply connected, we can identify $\pi_{2}(X)$ with $H_{2}(X, \mathbb{Z})$. Note that the deck transformation group of $L X \rightarrow L X$ is $\pi_{1}(L X)=\pi_{2}(X)$. Let $q_{1}, \ldots, q_{r}$ be a basis of the lattice. As an additive object, the cohomology is then identified with

$$
H^{*}\left(X, \mathbb{C}[z]\left[q_{1}^{ \pm}, \ldots, q_{r}^{ \pm}\right]\right)
$$

Here $z$ is the equivariant parameter, and the $q_{i}$ can be understood as determining the level in $\tilde{L X}$. Givental shows that the Heisenberg algebra $D$ above acts on this cohomology as follows. Given a basis Poincaré dual to the chosen one for $H_{2}(X, \mathbb{Z})$, let $\omega_{1}, \ldots, \omega_{r}$ be the associated $S^{1}$ equivariant symplectic forms on $\tilde{L X}$. Then we obtain $H_{1}, \ldots, H_{r}$, the Hamiltonians for the $S^{1}$ action with respect to each symplectic form. Define the action of $p_{i}$ in $D$ on the Floer cohomology by mapping

$$
p_{i} \mapsto \omega_{i}+z H_{i} .
$$

One can check that this gives the Floer cohomology the structure of a D module.

Givental conjectured that these two D modules - the one associated to quantum cohomology and the one associated to Floer cohomology - are isomorphic. Suppose there exists $c \in \mathcal{M}:=H^{*}\left(X, \mathbb{C}[z]\left[q_{1}^{ \pm}, \ldots, q_{r}^{ \pm}\right]\right)$such that the cohomology is generated by $c$ as D module (otherwise, we can simply consider the sub- D module generated by $c$ ). This gives an identification of the equivariant Floer cohomology with $D / I_{c}$, where $I_{c}$ is the ideal of operators which annihilate $c$. Givental's conjecture implies that $c$ is a solution to the quantum differential equations.
$H^{*}(X, \mathbb{C})$ can be mapped into $\mathcal{M}$ : given a Poincaré dual cycle $\gamma$, consider the cycle in the critical set at each level in $\tilde{L X}$. Then take the downwards gradient flow with respect to the Hamiltonian (a infinite dimensional version of the unstable manifold): the corresponding cycle is the desired element in $\mathcal{M}$. If $H^{*}(X)$ is generated in degree 2 (which is not the case for the Grassmannian), then the Poincaré dual of the image of the fundamental cycle gives the proposed $c$. It consists of the boundary values of all holomorphic discs in $X$. Givental did a formal computation to find a solution, which is the $I$ function (later proved to be a solution to the quantum differential equations using different methods). Below, we do a formal calculation to show that one obtains the oscillatory integrals that satisfy the quantum differential equations of the Grassmannian given the Abelian/non-Abelian correspondence. In the second chapter of the thesis, we prove the Abelian/non-Abelian correspondence for quiver flag varieties rigorously.

### 1.2 The algebraic loop space for GIT quotients

Following Givental and [23], we use an algebraic analogue of the loop space. Let $V$ be a $\mathbb{C}$ vector space, equipped with a $G$ action for a group $G$ that is a product of $G l\left(r_{i}\right)$, so that $G$ acts linearly. Choosing coordinates on $V$, we have the standard symplectic form $\omega$. By the Kempf-Ness theorem, the GIT quotient (after choosing a stability condition) $X:=V / / G$ is diffeomorphic to the symplectic quotient $\mu^{-1}(u) / K$, where $K$ is the maximal compact subgroup of $G$ such that $K_{\mathbb{C}}=G$, and $\mu: V \rightarrow \mathfrak{k}^{*}$ is a moment map for the action.

Example 1.2.1 (The Grassmannian). Let $V=\operatorname{Mat}(r \times N ; \mathbb{C})$ where $G=G l(r)$ acts by multiplication on the left. $K:=U(r)$ is the unitary group. $\mathfrak{k}$ is the skew Hermitian matrices, and $\mathfrak{k}^{*}$ is identified with Hermitian matrices via the pairing $\left\langle h_{1}, h_{2}\right\rangle:=i \operatorname{tr}\left(h_{1} h_{2}\right) \in \mathbb{R}$. One can check that this action is Hamiltonian with moment map $\mu(A):=\pi A A^{*}$ where the symplectic form is

$$
\omega:=\sum_{i=1}^{N} \sum_{j=1}^{r} \frac{\sqrt{-1}}{2} d a_{i, j} \wedge d \overline{a_{i, j}} .
$$

Example 1.2.2 (Quiver flag varieties). Let $(Q, \mathbf{r})$ be the data giving a quiver flag variety (see 2.1.1). Let $K:=\prod_{i=1}^{\rho} U\left(r_{i}\right)$, and

$$
V:=\bigoplus_{a \in Q_{1}} \operatorname{Hom}_{\mathbb{C}}\left(\mathbb{C}^{r_{s(a)}}, \mathbb{C}^{r_{t(a)}}\right)
$$

acting by change of basis. Write coordinates on $V$ as $\left[\left[a_{i, j}^{(a)}\right]_{1 \leq i \leq r_{s(a)}, 1 \leq j \leq r_{t(a)}}\right]_{a \in Q_{1}}$. Let $\omega$ be the standard symplectic form. Then this action is Hamiltonian with moment map

$$
\mu\left(\left(A_{a}\right)_{a \in Q_{1}}\right)=\left(\pi \sum_{a \in Q_{1}, t(a)=i} A_{a} A_{a}^{*}-\pi \sum_{a \in Q_{1}, s(a)=i} A_{a}^{*} A_{a}\right)_{i=1}^{\rho} .
$$

Let $V\left[\zeta, \zeta^{-1}\right]:=\oplus_{n=-\infty}^{\infty} V \zeta^{n}$ be the infinite dimensional vector space over $\mathbb{C}$ identified with replacing the scalar entries of a vector in $V$ with Laurent polynomials in $\zeta$. This induces an action of $G$ (and $K$ ) on $V\left[\zeta^{ \pm 1}\right]$. If $b_{i}$ were coordinates on $V$, then we can write coordinates on $V\left[\zeta^{ \pm 1}\right]$ as $b_{i}^{(n)}, n \in \mathbb{Z}$ and define

$$
\omega_{\infty}:=\sum_{n \in \mathbb{Z}} \sum_{i=1}^{\operatorname{dim}(V)} b_{i}^{(n)} \wedge \overline{b_{i}^{(n)}} .
$$

We can similarly define

$$
\mu_{\infty}: V\left[\zeta^{ \pm 1}\right] \rightarrow \mathfrak{k}^{*} ; \mu_{\infty}\left(\left(v_{n}\right)_{n \in \mathbb{Z}}\right) \mapsto \sum_{n \in \mathbb{Z}} \mu\left(v_{n}\right) .
$$

Because the $K$ action was defined just by extending linearly, it follows that $\mu_{\infty}$ is a moment map for the $K$ action on $V\left[\zeta^{ \pm 1}\right]$. The polynomial loop space of $X$ is defined to be

$$
L_{\text {poly }}(X):=\mu_{\infty}^{-1}(u) / K
$$

Let $\omega_{u}$ be the induced symplectic form.
$L_{\text {poly }}(X)$ should be considered as the algebraic analogue of the covering of the infinite loop space. An element of $L_{\text {poly }}(X)$ defines a loop in $X$ by varying $\zeta \in S^{1}$. $S^{1}$ acts on $L_{\text {poly }}(X)$ by re-parametrizing the loops; that is $\zeta \mapsto \lambda \zeta$ for $\lambda \in S^{1}$. Let $H_{u}$ be the Hamiltonian for this action; it is the restriction and quotient to $L_{p o l y}(X)$ of

$$
\pi \sum_{n \in \mathbb{Z}} \sum_{i} n\left|b_{i}^{(n)}\right|^{2}
$$

on $V\left[\zeta^{ \pm 1}\right]$.
The analogue of the deck transformation of the covering space is an action for each element of $\pi_{2}(X) \cong H_{2}(X) \cong \chi(G)^{\vee}$ (assuming $X$ is simply connected and a Mori dream space). Given a co-character $\chi: \mathbb{C}^{*} \rightarrow G$, we get an action on of $\mathbb{C}^{*}$ on $V$,
which for some choice of basis and $\beta_{i}$ is given by, for $\zeta \in \mathbb{C}^{*}$ :

$$
\zeta \cdot\left[b_{i}^{(n)}\right] \mapsto\left[\zeta^{\beta_{i}} b_{i}^{(n)}\right] .
$$

Hence we can interpret this as a deck transformation on $V\left[\zeta^{ \pm 1}\right]$ in the obvious way.
Example 1.2.3. Suppose $X:=\mathbb{P}^{n} \times \mathbb{P}^{m}$, so that $G:=\mathbb{C}^{*} \times \mathbb{C}^{*}$ acts on $V=\mathbb{C}^{m+1} \times \mathbb{C}^{n+1}$ by scaling each factor. An element of $V\left[\zeta^{ \pm 1}\right]$ is $(\underline{f}, \underline{g})$ where $\underline{f}$ is an $n+1$ tuple of Laurent polynomials in $\zeta$, and $\underline{g}$ is an $m+1$ tuple. $\operatorname{Hom}\left(\mathbb{C}^{*}, G\right) \cong \mathbb{Z}^{2}$, so taking $(a, b) \in \mathbb{Z}^{2}$, the corresponding deck transformation is

$$
(\underline{f}, \underline{g}) \mapsto\left(\zeta^{a} \underline{f}, \zeta^{b} \underline{g}\right) .
$$

Now we are ready to start considering the integral indicated by Givental, which we have discussed in the first section. For each co-character $\chi$, there is copy of $X$ denoted $X_{\chi}$ at the level $\chi$ in $L_{\text {poly }}(X)$. That is, $X_{\chi}$ is the equivalence classes of elements of $\chi(V) . X_{0}$ is just the image of the constant polynomial $V$ in $V\left[\zeta^{ \pm 1}\right]$ in $L_{\text {poly }}(X)$. The image of the fundamental class of $X$ under the map $H^{*}(X) \rightarrow$ $\mathcal{M}=H^{*}\left(X, \mathbb{C}[z]\left[q_{1}^{ \pm}, \ldots, q_{r}^{ \pm}\right]\right)$is given by the class Poincaré dual to the closure of the stable manifold of $X_{0}$. The class of the closure of the stable manifold, denoted $\Delta$, is the image of $V[\zeta] \cap \mu_{\infty}^{-1}(u)$ under the quotient $\mu_{\infty}^{-1}(u) \rightarrow L_{\text {poly }}(X) . \Delta$ defines a class in $\mathcal{M}:=H^{*}\left(X, \mathbb{C}[z]\left[q_{1}^{ \pm}, \ldots, q_{r}^{ \pm}\right]\right)$.

### 1.3 Mirror heuristics for Grassmannians

For Grassmannians, we can be much more explicit. From now on, assume we are in the case of the Grassmannian, using the notation of Example 1.2.1: $V=\operatorname{Mat}(r \times$ $N, \mathbb{C})$ with coordinates $a_{i j}$. The coordinates on $V\left[\zeta^{ \pm}\right]$are given by $a_{i j}^{(n)}$. The moment map is

$$
\mu(A)=\pi \sum_{n} A^{(n)}\left(A^{(n)}\right)^{*}
$$

if $A^{(n)}=\left[a_{i j}^{(n)}\right]_{1 \leq i \leq r, 1 \leq j \leq N}$. In these coordinates, $H$ is given by

$$
\pi \sum_{n \in \mathbb{Z}} \sum_{i, j} n\left|a_{i j}^{(n)}\right|^{2} .
$$

Givental conjectures that $\Delta$ satisfies the quantum differential equations of $X$. As he suggests, one can instead take the Fourier transform: consider the integral

$$
\begin{equation*}
\int_{\Delta} e^{\omega_{u} / z-H_{u}} . \tag{1.1}
\end{equation*}
$$

For shorthand, we refer to this integral as the mirror integral for the rest of this chapter.

Following the case of projective space (in [23]), we would like to write this as an integral over $V[\zeta]$ instead.

Suppose $\phi$ is a principal one form for the principal $K$ bundle $\pi: \mu^{-1}(u) \rightarrow L_{\text {poly }}(X)$. By definition, $\phi$ is a $\mathfrak{k}$-valued one form which is $K$ equivariant. Choose a basis $f_{1}, \ldots, f_{k}$ of $\mathfrak{k}^{*}$. We can define $k$ scalar-valued one forms $\phi_{1}, \ldots, \phi_{k}$ via $\phi_{i}:=\left\langle\phi, f_{i}\right\rangle$. As the restriction of $\phi_{1} \wedge \cdots \wedge \phi_{k}$ to a fiber is the volume form,

$$
\int_{\Delta} e^{\omega_{u} / z-H_{u}}=\int_{\mu^{-1}(u) \cap V[\zeta]} e^{\omega / z-H} \phi_{1} \wedge \cdots \wedge \phi_{k} .
$$

We then change coordinates $a_{i j}^{(n)} \mapsto \sqrt{z} a_{i j}^{(n)}$ (having chosen $\phi$ such that $\phi_{i}$ are invariant under this change of coordinates, as in [23]), and obtain

$$
\int_{\mu^{-1}(u / z) \cap V[\zeta]} e^{\omega-z H} \phi_{1} \wedge \cdots \wedge \phi_{k}
$$

Note that $\mu^{*}\left(\delta_{u / z} f_{1} \wedge \cdots \wedge f_{n}\right)=\delta(\mu-u / z) d \mu_{1} \wedge \cdots \wedge d \mu_{k}$ where $\delta$ is the Dirac delta function and the $d \mu_{i}$ are defined as follows. The one form $d \mu$ is $\mathfrak{k}^{*}$-valued. Let $e_{1}, \ldots, e_{k}$ be a dual basis to the $f_{1}, \ldots, f_{k}$ : define $d \mu_{i}=\left\langle d \mu, e_{i}\right\rangle$. Now note that the mirror integral can be written as

$$
\int_{V[\zeta]} e^{\omega-z H} \delta(\mu-u / z) \phi_{1} \wedge \cdots \wedge \phi_{k} \wedge d \mu_{1} \wedge \cdots \wedge d \mu_{k} .
$$

Suppose that $\phi$ is scaled so that the top degree term of $e^{\omega} \wedge \phi_{1} \wedge \cdots \wedge \phi_{k} \wedge d \mu_{1} \wedge \cdots \wedge d \mu_{k}$ is

$$
d \mathrm{vol}=\bigwedge_{i=1}^{r} \bigwedge_{j=1}^{N} \bigwedge_{n \in \mathbb{Z}} \frac{\sqrt{-1}}{2} d a_{i j}^{(n)} \wedge d \overline{a_{i j}^{(n)}} .
$$

Then, in analogue to the finite dimensional situation, we write the mirror integral as

$$
\int_{V[\zeta]} \delta(\mu-u / z) e^{-z H} d \mathrm{vol} .
$$

As $\mu$ is vector (or rather, matrix) valued, we use the matrix $\delta$ function defined by [50]: for $C$ an $r \times r$ Hermitian matrix,

$$
\begin{gathered}
\delta(C):=\frac{1}{2^{r} \pi^{r^{2}}} \int_{\mathfrak{k}^{*}} e^{i T r\left(X T^{t}\right)}[d T], \\
d T:=\prod_{j=1}^{r} d t_{j j} \prod_{1 \leq i<j \leq r} d \operatorname{Re}\left(t_{i j}\right) d \operatorname{Im}\left(t_{i j}\right) .
\end{gathered}
$$

Here $t_{i j}$ are the usual coordinates on Hermitian matrices. We take the transpose (which amounts to changing variables) just to ease notation. This is equivalent to taking a product of scalar delta functions, one for each coordinate.

Using the definitions of $H$ and $\mu$, the mirror integral becomes

$$
\frac{1}{2^{r} \pi^{r^{2}}} \int_{\mathfrak{k}^{*}}[d T] \prod_{\substack{1 \leq i \leq r, r \\ 1 \leq j \leq r}}\left(e^{-i t_{i j} u_{i j} / z}\right) \int_{V[\zeta]} d \operatorname{vol} \prod_{n=0}^{\infty}\left(\prod_{1 \leq i, j \leq r} \prod_{k=1}^{N}\left(e^{i \pi a_{i k}^{(n)} a_{j k}^{(n)}} \overline{i d j}^{\substack{1 \leq}} \prod_{\substack{1 \leq i \leq r, 1 \leq j \leq N}} e^{-\pi n z\left|a_{i j}^{(n)}\right|^{2}}\right) .\right.
$$

We can compute the integral over $V[\zeta]$ as it is an (infinite) product of Gaussian integrals. Recall that

$$
\begin{equation*}
\int_{-\infty}^{\infty} e^{-a x^{2}+b x+c} d x=\sqrt{\frac{\pi}{a}} e^{\frac{b^{2}}{4 a}+c} . \tag{1.2}
\end{equation*}
$$

It's easiest to do this by fixing $n$ and $l$ and considering the integral over $a_{i l}^{(n)}$ for all $i$. Recall that $i$ and $l$ index the rows and columns of elements of $V=\operatorname{Mat}(r \times N ; \mathbb{C})$. In the proof of the following proposition, we start at the bottom row, $i=r$, and go up, and show that that the following recursive definition gives the integral:

$$
\begin{gathered}
A_{r}^{(n)}(i)=i t_{i r}, C_{r}^{(n)}=n z-i t_{r r}, i<r \\
A_{k}^{(n)}(i):=i t_{i k}+\sum_{j=k+1}^{r}\left(A_{j}^{(n)}(k)\right)^{t} A_{j}^{(n)}(i) / C_{j}^{(n)}, i<k \\
C_{k}^{(n)}:=n z-i t_{k k}-\sum_{j=k+1}^{r} A_{j}^{(n)}(k)\left(A_{j}^{(n)}(k)\right)^{t} / C_{j}^{(n)} .
\end{gathered}
$$

Here taking the transpose means $t_{i j} \mapsto t_{j i}$ in the formula.
Proposition 1.3.1. The mirror integral is

$$
\frac{1}{2^{r} \pi^{r^{2}}} \int_{\mathfrak{k}^{*}}[d T] \prod_{1 \leq i, j \leq r}\left(e^{-i t_{i j} u_{i j} / z}\right) \prod_{n=0}^{\infty} \prod_{k=1}^{r} \frac{1}{\left(C_{k}^{(n)}\right)^{N}}
$$

Proof. From now on, we suppress the ( $n$ ) notation as $n$ is fixed.
We compute the integral step by step, starting by integrating out the $a_{r l} \bar{a}_{r l}$ variables. We claim that after integrating up to $k$ (so from $r, \ldots, k+1$ ), the term involving $a_{k l}$ is

$$
\begin{equation*}
e^{\pi\left(\sum_{i=1}^{k-1} A_{k}(i) a_{i l}\right) \overline{a_{k l}}+\pi\left(\sum_{i=1}^{k-1} A_{k}^{t}(i) \overline{a_{i l}}\right) a_{k l}-\pi C_{k}^{(n)}\left|a_{k k}\right|^{2}} . \tag{1.3}
\end{equation*}
$$

As the $k=r$ step (the induction step) is straightforward, assume the statement is true for $j>k$. After changing coordinates to the real and imaginary parts of $a_{j l}$, we
can use (1.2) twice to see that at the $j^{\text {th }}$ step we get a contribution of

$$
\frac{1}{C_{j}} e^{\pi\left(\sum_{i=1}^{j-1} A_{j}(i) a_{i l}\right)\left(\sum_{i=1}^{j-1} A_{j}^{t}(i) \overline{a_{i l}}\right) / C_{j}} .
$$

Note that here we use the identity $-(a-b)^{2}+(a+b)^{2}=4 a b$.
Now we gather all the terms involving $a_{k l}$, including the contributions from the original integrand and each of the $k-1$ integrations previously - and it is precisely as given in (1.3).

To prove the proposition, note that for $k=1$, the final step, we are computing

$$
\int_{\mathbb{C}} e^{-\pi C_{1}^{(n)}\left|a_{11}\right|^{2}} \prod_{i=2}^{r} \frac{1}{C_{i}^{(n)}} d a_{1 l} \wedge d \overline{a_{1 l}},
$$

which yields the proposition.
We can write this in a much nicer form. To do this we use the following lemma. Let $A$ be an $n \times n$ matrix. If $S, T \subset\{1, \ldots, n\}, \# S=\# T$, then $A_{S T}$ denotes the determinant of the minor of $A$ obtained by removing rows $S$ and columns $T$. If $S=\{i\}, T=\{j\}$ we denote it $A_{i j}$.

Lemma 1.3.2. Let $A$ be an $n \times n$ matrix, $1<i<j<n$.

$$
B:=\left[\begin{array}{ll}
A_{i i} & A_{i j} \\
A_{j i} & A_{j j}
\end{array}\right]
$$

Then if $A_{\{i, j\}\{i, j\}} \neq 0$,

$$
\operatorname{det}(A)=\operatorname{det}(B) / A_{\{i, j\}\{i, j\}} .
$$

Proof. The base case $n=2$ is obvious. Suppose it is true for $n-1$. It suffices to prove the case $\{i, j\}=\{1,2\}$. By induction, we can write

$$
A_{12}=\operatorname{det}\left(\left[\begin{array}{ll}
A_{\{1,2\}\{1,2\}} & A_{\{1,2\}\{2,3\}} \\
A_{\{1,3\}\{1,2\}} & A_{\{1,3\}\{2,3\}}
\end{array}\right]\right) / A_{\{1,2,3\}\{1,2,3\}} .
$$

We can similarly expand the other entries in $B$. Taking the determinant of $B$, one gets

$$
\begin{array}{r}
\left(A_{\{1,2\}\{2,3\}}\left(-A_{\{1,3\}\{1,3\}} A_{\{2,3\}\{1,2\}}+A_{\{1,3\}\{1,2\}} A_{\{2,3\}\{1,3\}}\right)\right. \\
+A_{\{1,2\}\{1,3\}}\left(A_{\{1,3\}\{2,3\}} A_{\{2,3\}\{1,2\}}-A_{\{1,3\}\{1,2\}} A_{\{2,3\}\{2,3\}}\right) \\
\left.+A_{\{1,2\}\{1,2\}}\left(-A_{\{1,3\}\{2,3\}} A_{\{2,3\}\{1,3\}}+A_{\{1,3\}\{1,3\}} A_{\{2,3\}\{2,3\}}\right)\right) / A_{\{1,2,3\}\{1,2,3\}} .
\end{array}
$$

Each term can be re-written using the induction step. For example,

$$
-A_{\{1,3\}\{1,3\}} A_{\{2,3\}\{1,2\}}+A_{\{1,3\}\{1,2\}} A_{\{2,3\}\{1,3\}} / A_{\{1,2,3\}\{1,2,3\}}=A_{31} .
$$

Then we get

$$
\frac{1}{A_{\{1,2,3\}\{1,2,3\}}}\left(A_{\{1,2\}\{2,3\}} A_{31}-A_{\{1,2\}\{1,3\}} A_{32}+A_{\{1,2\}\{1,2\}} A_{33}\right) .
$$

Expand the $n-2 \times n-2$ minors using Laplace's formula, going across the third row. The first term in each expansion looks like, for $i=1,2,3$,

$$
a_{3 i} A_{\{1,2,3\}\{1,2,3\}} A_{3 i} .
$$

Cancelling the $A_{\{1,2,3\}\{1,2,3\}}$, the sum of these first terms is:

$$
\operatorname{det}(A)-\sum_{i=4}^{n}(-1)^{i+1} a_{3 i} A_{3 i} .
$$

The rest of the expansion of Laplace's formula contributes

$$
\frac{1}{A_{\{1,2,3\}\{1,2,3\}}} \sum_{i=4}^{n}(-1)^{i+1} a_{3 i}\left(A_{\{1,2,3\}\{i, 2,3\}} A_{31}-A_{\{1,2,3\}\{1, i, 3\}} A_{32}+A_{\{1,2,3\}\{1,2, i\}} A_{33}\right) .
$$

So it suffices to show that

$$
A_{\{1,2,3\}\{i, 2,3\}} A_{31}-A_{\{1,2,3\}\{1, i, 3\}} A_{32}+A_{\{1,2,3\}\{1,2, i\}} A_{33}-A_{\{1,2,3\}\{1,2,3\}} A_{3 i}=0
$$

In fact, this is one of the quadratic Plucker relations cutting out the complete flag variety (see [40, pp. 277]). Let $\tau=\{1,4, \ldots, n\}-\{i\}$ and $\sigma=\{2, \ldots, n\}$. Let $\pi$ be a permutation of $\{1, \ldots, n\}$. Define

$$
\begin{gathered}
\pi(\tau)=\{\pi(1), 4, \ldots, \hat{i}, \ldots, n\}, \\
\pi(\sigma)=\{\pi(2), \ldots, \pi(n)\}
\end{gathered}
$$

Denote $p_{S, T}$ as the Plücker coordinate given by taking the determinant of the matrix with rows taken from $S$ and and columns from $T$. Then the relation is equivalent to

$$
\sum_{\pi \in S_{n}} \operatorname{sign}(\pi) p_{\{4, \ldots, n\}, \pi(\tau)} p_{\{1,2,4, \ldots, n\}, \pi(\sigma)}=0 .
$$

This is multi-linear and alternating in the $n$ columns of the $n-1 \times n$ matrix formed by removing the third row of $A$. As these columns form at most an $n-1$ dimensional space, the relation is identically zero.

Proposition 1.3.3. Let $I$ be the $r \times r$ identity matrix. Let $E^{(n)}:=n z I-\left[i t_{i j}\right]$. Then

$$
\prod_{k=1}^{r} C_{k}^{(n)}=\operatorname{det}\left(E^{(n)}\right)
$$

Proof. We suppress $n$ from the notation. Denote the entries of $E$ by $e_{i j}$. We prove by induction on $k$ (starting at $r$ ) that for $i<k$ :

$$
\begin{aligned}
& A_{k}(i)=\frac{-1}{\prod_{j=k+1}^{r} C_{j}} E_{\{1, \ldots, i-1, i+1, \ldots, k\}\{1, \ldots, k-1\}}, \\
& A_{k}^{t}(i)=\frac{-1}{\prod_{j=k+1}^{r} C_{j}} E_{\{1, \ldots, k-1\}\{1, \ldots, i-1, i+1, \ldots, k\}} .
\end{aligned}
$$

We use this to extend the definition of $A_{k}(i)$ to $i=k$, and prove that

$$
C_{k}=-A_{k}(k):=\frac{1}{\prod_{j=k+1}^{r} C_{j}} E_{\{1, \ldots, k-1\}\{1, \ldots, k-1\}} .
$$

If $k=r$ this is obvious. Suppose it is true for $j>k$ for some $k \geq 1$. Note that

$$
\begin{gathered}
A_{k}(i):=-e_{i k}+\sum_{j=k+1}^{r}\left(A_{j}(k)\right)^{t} A_{j}(i) / C_{j} \\
C_{k}
\end{gathered}=e_{k k}-\sum_{j=k+1}^{r} A_{j}(k)\left(A_{j}(k)\right)^{t} / C_{j} .
$$

So the relation between $C_{k}, A_{k}(k)$ is clear. Now by induction:

$$
\begin{align*}
A_{k}(i)= & \frac{1}{\prod_{j=k+1}^{r} C_{j}^{j-k}}(\underbrace{-e_{i k} \prod_{j=k+1}^{r} C_{j}^{j-k}}_{\text {term A }} \\
& +\underbrace{\left.\sum_{j=k+1}^{r}\left(E_{\{1, \ldots, j-1\}\{1, \ldots, \hat{k}, \ldots, j\}} E_{\{1, \ldots, \hat{i}, \ldots, j\}\{1, \ldots, j-1\}}\right) \frac{\prod_{s=k+1}^{r} C_{s}^{(s-k)}}{C_{j} \prod_{s=j+1}^{r} C_{s}^{2}}\right)}_{\text {term B }} . \tag{1.4}
\end{align*}
$$

Note that $\prod_{s=t}^{r} C_{s}=E_{\{1, \ldots, t-1\}\{1, \ldots, t-1\}}, t>k$. So in particular

$$
\begin{gathered}
\prod_{j=k+1}^{r} C_{j}^{j-k}=\prod_{i=k+1}^{r} E_{\{1, \ldots, i-1\}}\{1, \ldots, i-1\}, \\
\frac{\prod_{s=k+1}^{r} C_{s}^{(s-k)}}{C_{j} \prod_{s=j+1}^{r} C_{s}^{2}}=\prod_{s=k+1}^{j-1} E_{\{1, \ldots, s-1\}\{1, \ldots, s-1\}} \prod_{s=j+2}^{r} E_{\{1, \ldots, s-1\}\{1, \ldots, s-1\}} .
\end{gathered}
$$

Consider the sum of term A and the $j=r$ contribution of term B in (1.4). Together,
they simplify to

$$
\begin{array}{r}
\prod_{i=k+1}^{r-1} E_{\{1, \ldots, i-1\}\{1, \ldots, i-1\}}\left(e_{r k} e_{i r}-e_{i k} e_{r r}\right)  \tag{1.5}\\
=-\left(\prod_{i=k+1}^{r-2} E_{\{1, \ldots, i-1\}\{1, \ldots, i-1\}}\right) E_{\{1, \ldots, \hat{i}, \ldots, r-1\}\{1, \ldots, \hat{k}, \ldots, r-1\}} E_{\{1, \ldots, r-1\}\{1, \ldots, r-1\}} .
\end{array}
$$

Now consider the sum of (1.5) and the $j=r-1$ term in term B in (1.4): this simplifies to

$$
\begin{array}{r}
\left(\prod_{i=k+1}^{r-2} E_{\{1, \ldots, i-1\}\{1, \ldots, i-1\}}\right)\left(E_{\{1, \ldots, r-2\}\{1, \ldots, \hat{k}, \ldots, r-1\}} E_{\{1, \ldots, \hat{i}, \ldots, r-1\}\{1, \ldots, r-2\}}\right. \\
\left.-E_{\{1, \ldots, \hat{i}, \ldots, r-1\}\{1, \ldots, \hat{k}, \ldots, r-1\}} E_{\{1, \ldots, r-2\}\{1, \ldots, r-2\}}\right)
\end{array}
$$

The right hand factor is a determinant of the form found in the lemma for the $3 \times 3$ matrix obtained from $E$ by removing rows $\{1, \ldots, \hat{i}, \ldots, r-2\}$ and columns $\{1, \ldots, \hat{k}, \ldots, r-2\}$. Applying the lemma we obtain:

$$
-\left(\prod_{i=k+1}^{r-3} E_{\{1, \ldots, i-1\}\{1, \ldots, i-1\}}\right) E_{\{1, \ldots, \hat{i}, \ldots, r-2\}\{1, \ldots, \hat{k}, \ldots, r-2\}} E_{\{1, \ldots, r-3\}\{1, \ldots, r-3\}} E_{\{1, \ldots, r-1\}\{1, \ldots, r-1\}}
$$

We now see that we will be able to repeat this process until we have simplified to a single term (inside the brackets of (1.4)):

$$
-\prod_{s=k+2}^{r} E_{\{1, \ldots, s-1\}\{1, \ldots, s-1\}} E_{\{1, \ldots, \hat{i}, \ldots, k\},\{1, \ldots, k-1\}}
$$

When we consider the factor, we arrive at the induction statement. The statement for the 'transpose' follows from the invariance of the determinant under transpose. This proves the proposition, as it implies that

$$
C_{1}=\frac{1}{C_{2} \ldots C_{r}} \operatorname{det} E^{(n)}
$$

Therefore the mirror integral is

$$
\frac{1}{2^{r} \pi^{r^{2}}} \int_{\mathfrak{k}^{\star}}[d T]\left(e^{-i \operatorname{Tr}\left(u T^{t}\right) / z}\right) \prod_{n=0}^{\infty} \frac{1}{\operatorname{det}\left(E^{(n)}\right)^{N}}
$$

We can change variables again to remove the transpose. The infinite product looks like a matrix version of the Zeta function. Before we can use zeta function regularization, however, we have to use the Harish-Chandra formula, which allows us to
integrate over just diagonal Hermitian matrices.
Let $a_{1}, \ldots, a_{n}$ be the entries of a diagonal matrix $A$. Then the Vandermonde determinant of $A$ is

$$
V(A)=\prod_{i<j}\left(a_{i}-a_{j}\right)
$$

Theorem 1.3.4 (The Harish-Chandra formula). Let $\Phi$ be a conjugation invariant function of Hermitian matrices, let $U$ be an $r \times r$ Hermitian matrix with eigenvalues $u_{1}, \ldots, u_{n}$. Let $U^{\prime}$ be the diagonal matrix conjugate to $U$. Then

$$
\int_{\mathbb{k}^{*}} \Phi(T) e^{-i t r(T U)}[d T]=(-2 \pi i)^{r(r-1) / 2} V\left(U^{\prime}\right)^{-1} \int_{\mathbb{R}^{r}} \Phi(D) e^{-i t r(D Y)} V(D) d D
$$

where $D$ is a real diagonal matrix.

Applying this to the integral we have obtained (taking $\left.U=\operatorname{diag}\left(u_{1}, \ldots, u_{r}\right)\right)$ :

$$
\frac{(-2 \pi i)^{r(r-1) / 2}}{2^{r} \pi^{r^{2}} V(U)} \int_{\mathbb{R}^{r}}[d D] \prod_{i=1}^{r}\left(e^{-i d_{i} u_{i} / z}\right) \prod_{j=1}^{r} \prod_{n=0}^{\infty} \frac{1}{\left(-i d_{j}+n z\right)^{N}} V(D) .
$$

### 1.4 Zeta function Regularization

The zeta function is

$$
\zeta(s, a):=\sum_{n=1}^{\infty}(n+a)^{-s} .
$$

It converges for $s$ such that $\mathbb{R} e(s)>1$, but can be extended meromorphically to the whole plane by analytic continuation. This gives a way of making sense of infinite sums and products where they may not converge. In particular, we can apply this to products of the form

$$
\prod_{n=1}^{\infty}(a n+b)=\exp \left(\sum_{n=1}^{\infty} \log (a n+b)\right) .
$$

Let $f(s)=a^{-s} \zeta(s, b / a)=\sum_{n=1}^{\infty}(a n+b)^{-s}$. Then

$$
f^{\prime}(s)=\sum_{n=1}^{\infty} \log (a n+b)(a n+b)^{-s},
$$

and hence

$$
\prod_{n=1}^{\infty}(a n+b)=\exp \left(f^{\prime}(0)\right)
$$

On the other hand, as $\zeta(0, b / a)=-1 / 2-b / a$ and

$$
\frac{\partial}{\partial s} \zeta(0, b / a)=1 / 2 \log (2 \pi)+\log (\Gamma(b / a+1))
$$

we also see that:

$$
\begin{array}{r}
f^{\prime}(0)=\left.\left((-\log (a)) a^{-s} \zeta(s, b / a)+a^{-s} \frac{\partial}{\partial s} \zeta(s, b / a)\right)\right|_{s=0}= \\
\quad((-1 / 2-b / a) \log (a)+\log (\Gamma(b / a+1)+1 / 2 \log (2 \pi) .
\end{array}
$$

So we conclude that

$$
\prod_{n=1}^{\infty}(a n+b) \sim a^{-1 / 2-b / a} \Gamma(b / a+1)(2 \pi)^{1 / 2} .
$$

We can apply this to remove the infinite sum in the mirror integral, as

$$
\prod_{j=1}^{r} \prod_{n=0}^{\infty} \frac{1}{\left(-i d_{j}+n z\right)^{N}} \sim \prod_{j=1}^{r}\left((2 \pi z)^{-1 / 2} z^{-i d_{j} / z} \Gamma\left(-i d_{j} / z\right)\right)^{N} .
$$

Together with a change of coordinates $d_{j} \rightarrow z d_{j}$, the mirror integral is

$$
\left.\frac{(-2 \pi i)^{r(r-1) / 2}}{2^{r} \pi^{r^{2}}(2 \pi z)^{N r / 2} V(U)} z^{N} \int_{\mathbb{R}^{r}} \prod_{j=1}^{r}\left(\mathrm{~d} d_{j} e^{-i\left(u_{j}+N \log z\right) d_{j}}\right) \Gamma\left(-i d_{j}\right)^{N}\right) V(D) .
$$

The integral representation for the Gamma function is

$$
\Gamma(w)=\int_{0}^{\infty} x^{w-1} e^{-w} d x
$$

Placing this in the mirror integral, we obtain

$$
\begin{aligned}
& \frac{(-2 \pi i)^{r(r-1) / 2}}{2^{r} \pi^{r^{2}}(2 \pi z)^{N r / 2} V(U)} z^{N} \\
& \int_{\mathbb{R}^{r}} \prod_{j=1}^{r}\left(\mathrm{~d} d_{j} \int_{[0, \infty)^{N}} \frac{\mathrm{~d} x_{j 1}}{x_{j 1}} \cdots \frac{\mathrm{~d} x_{j N}}{x_{j N}} e^{-i\left(u_{j}+N \log z+\sum_{i=1}^{N} \log \left(x_{j i}\right)\right) d_{j}} e^{\sum_{i=1}^{N} x_{j i}}\right) V(D) .
\end{aligned}
$$

This is the mirror for $\left(\mathbb{P}^{N-1}\right)^{r}$ with an extra factor of $V(D)$. It matches precisely with the mirror from Hori-Vafa.

There doesn't seem to be a way to get a Laurent polynomial mirror from this integral, unlike in the case of projective space. In Chapter 4, we suggest different methods for finding mirrors of quiver flag varieties and their subvarieties.

## 2

## Four dimensional Fano quiver flag Zero

In this chapter, which is based on work which appears in [33], we discuss quiver flag varieties and certain of their subvarieties, which we call quiver flag zero loci. We give a different construction of quiver flag varieties as subvarieties of products of Grassmannians, and use this to prove the Abelian/non-Abelian correspondence for quiver flag zero loci.

### 2.1 Quiver flag varieties

Quiver flag varieties are generalizations of Grassmannians and type A flag varieties ([18]). Like flag varieties, they are GIT quotients and fine moduli spaces. We begin by recalling Craw's construction. A quiver flag variety $M(Q, \mathbf{r})$ is determined by a quiver $Q$ and a dimension vector $\mathbf{r}$. The quiver $Q$ is assumed to be finite and acyclic, with a unique source. Let $Q_{0}=\{0,1, \ldots, \rho\}$ denote the set of vertices of $Q$ and let $Q_{1}$ denote the set of arrows. Without loss of generality, after reordering the vertices if necessary, we may assume that $0 \in Q_{0}$ is the unique source and that the number $n_{i j}$ of arrows from vertex $i$ to vertex $j$ is zero unless $i<j$. Write $s, t: Q_{1} \rightarrow Q_{0}$ for the source and target maps, so that an arrow $a \in Q_{1}$ goes from $s(a)$ to $t(a)$. The dimension vector $\mathbf{r}=\left(r_{0}, \ldots, r_{\rho}\right)$ lies in $\mathbb{N}^{\rho+1}$, and we insist that $r_{0}=1 . M(Q, \mathbf{r})$ is defined to be the moduli space of $\theta$-stable representations of the quiver $Q$ with dimension vector $\mathbf{r}$. Here $\theta$ is a fixed stability condition defined below, determined by the dimension vector.

### 2.1.1 Quiver flag varieties as Git quotients.

Consider the vector space

$$
\operatorname{Rep}(Q, \mathbf{r})=\bigoplus_{a \in Q_{1}} \operatorname{Hom}\left(\mathbb{C}^{r_{s(a)}}, \mathbb{C}^{r_{t(a)}}\right)
$$

and the action of $\mathrm{GL}(\mathbf{r}):=\prod_{i=0}^{\rho} \mathrm{GL}\left(r_{i}\right)$ on $\operatorname{Rep}(Q, \mathbf{r})$ by change of basis. The diagonal copy of GL(1) in GL(r) acts trivially, but the quotient $G:=\mathrm{GL}(\mathbf{r}) / \mathrm{GL}(1)$ acts effectively; since $r_{0}=1$, we may identify $G$ with $\prod_{i=1}^{\rho} \mathrm{GL}\left(r_{i}\right)$. The quiver flag variety $M(Q, \mathbf{r})$ is the $\operatorname{GIT}$ quotient $\operatorname{Rep}(Q, \mathbf{r}) \|_{\theta} G$, where the stability condition $\theta$ is the character of $G$ given by

$$
\theta(g)=\prod_{i=1}^{\rho} \operatorname{det}\left(g_{i}\right), \quad g=\left(g_{1}, \ldots, g_{\rho}\right) \in \prod_{i=1}^{\rho} \mathrm{GL}\left(r_{i}\right)
$$

For the stability condition $\theta$, all semistable points are stable. To identify the $\theta$-stable points in $\operatorname{Rep}(Q, \mathbf{r})$, set $s_{i}=\sum_{a \in Q_{1}, t(a)=i} r_{s(a)}$ and write

$$
\operatorname{Rep}(Q, \mathbf{r})=\bigoplus_{i=1}^{\rho} \operatorname{Hom}\left(\mathbb{C}^{s_{i}}, \mathbb{C}^{r_{i}}\right)
$$

Then $w=\left(w_{i}\right)_{i=1}^{\rho}$ is $\theta$-stable if and only if $w_{i}$ is surjective for all $i$.
Example 2.1.1. Consider the quiver $Q$ given by

so that $\rho=1, n_{01}=n$, and the dimension vector $\mathbf{r}=(1, r)$. Then $\operatorname{Rep}(Q, \mathbf{r})=$ $\operatorname{Hom}\left(\mathbb{C}^{n}, \mathbb{C}^{r}\right)$, and the $\theta$-stable points are surjections $\mathbb{C}^{n} \rightarrow \mathbb{C}^{r}$. The group $G$ acts by change of basis, and therefore $M(Q, \mathbf{r})=\operatorname{Gr}(n, r)$, the Grassmannian of $r$ dimensional quotients of $\mathbb{C}^{n}$. More generally, the quiver

gives the flag of quotients $\operatorname{Fl}(n, a, b, \ldots, c)$.

Quiver flag varieties are non-Abelian GIT quotients unless the dimension vector $\mathbf{r}=(1,1, \ldots, 1)$. In this case $G \cong \prod_{i=1}^{\rho} \mathrm{GL}_{1}(\mathbb{C})$ is Abelian, and $M(Q ; \mathbf{r})$ is a toric variety. We call such $M(Q, \mathbf{r})$ toric quiver flag varieties. Not all toric varieties are toric quiver flag varieties.

### 2.1.2 Quiver flag varieties as ambient spaces: Quiver flag zero loci

As mentioned in the introduction, GIT quotients have a special class of subvarieties, sometimes called representation theoretic subvarieties. In this subsection, we discuss these subvarieties in the specific case of quiver flag varieties.

We have expressed the quiver flag variety $M(Q, \mathbf{r})$ as the geometric quotient by $G$ of the stable locus $\operatorname{Rep}(Q, \mathbf{r})^{s} \subset \operatorname{Rep}(Q, \mathbf{r})$. A representation $E$ of $G$, therefore, defines a vector bundle $E_{G} \rightarrow M(Q, \mathbf{r})$ with fiber $E$; here $E_{G}=E \times{ }_{G} \operatorname{Rep}(Q, \mathbf{r})^{s}$. In the next chapter, we will study subvarieties of quiver flag varieties cut out by regular sections of such bundles. If $E_{G}$ is globally generated, a generic section cuts out a smooth subvariety. We refer to such subvarieties as quiver flag zero loci, and such bundles as representation theoretic bundles. As mentioned above, quiver flag varieties can also be considered natural ambient spaces via their moduli space construction ([18], [19]).

The representation theory of $G=\prod_{i=1}^{\rho} \mathrm{GL}\left(r_{i}\right)$ is well-understood, and we can use this to write down the bundles $E_{G}$ explicitly. Irreducible polynomial representations of $\mathrm{GL}(r)$ are indexed by partitions (or Young diagrams) of length at most $r$. The irreducible representation corresponding to a partition $\alpha$ is the Schur power $S^{\alpha} \mathbb{C}^{r}$ of the standard representation of $\mathrm{GL}(r)$ [22, chapter 8]. For example, if $\alpha$ is the partition $(k)$ then $S^{\alpha} \mathbb{C}^{r}=\operatorname{Sym}^{k} \mathbb{C}^{r}$, the $k$ th symmetric power, and if $\alpha$ is the partition $(1,1, \ldots, 1)$ of length $k$ then $S^{\alpha} \mathbb{C}^{r}=\Lambda^{k} \mathbb{C}^{r}$, the $k$ th exterior power. Irreducible polynomial representations of $G$ are therefore indexed by tuples ( $\alpha_{1}, \ldots, \alpha_{\rho}$ ) of partitions, where $\alpha_{i}$ has length at most $r_{i}$. The tautological bundles on a quiver flag variety are representation theoretic: if $E=\mathbb{C}^{r_{i}}$ is the standard representation of the $i^{\text {th }}$ factor of $G$, then $W_{i}=E_{G}$. More generally, the representation indexed by $\left(\alpha_{1}, \ldots, \alpha_{\rho}\right)$ is $\otimes_{i=1}^{\rho} S^{\alpha_{i}} \mathbb{C}^{r_{i}}$, and the corresponding vector bundle on $M(Q, \mathbf{r})$ is $\otimes_{i=1}^{\rho} S^{\alpha_{i}} W_{i}$.
Example 2.1.2. Consider the vector bundle $\operatorname{Sym}^{2} W_{1}$ on $\operatorname{Gr}(8,3)$. By duality which sends a quotient $\mathbb{C}^{8} \rightarrow V \rightarrow 0$ to a subspace $0 \rightarrow V^{*} \rightarrow\left(\mathbb{C}^{8}\right)^{*}$ - this is equivalent to considering the vector bundle $\operatorname{Sym}^{2} S_{1}^{*}$ on the Grassmannian of 3-dimensional subspaces of $\left(\mathbb{C}^{8}\right)^{*}$, where $S_{1}$ is the tautological sub-bundle. A generic symmetric 2form $\omega$ on $\left(\mathbb{C}^{8}\right)^{*}$ determines a regular section of $\mathrm{Sym}^{2} S_{1}^{*}$, which vanishes at a point $V^{*}$ if and only if the restriction of $\omega$ to $V^{*}$ is identically zero. So the associated quiver flag zero locus is the orthogonal Grassmannian $\operatorname{OGr}(3,8)$.

### 2.1.3 Quiver flag varieties as moduli spaces.

To give a morphism to $M(Q, \mathbf{r})$ from a scheme $B$ is the same as to give:

- globally generated vector bundles $W_{i} \rightarrow B, i \in Q_{0}$, of rank $r_{i}$ such that $W_{0}=$ $\mathcal{O}_{B}$; and
- morphisms $W_{s(a)} \rightarrow W_{t(a)}, a \in Q_{1}$ satisfying the $\theta$-stability condition
up to isomorphism. Thus $M(Q, \mathbf{r})$ carries universal bundles $W_{i}, i \in Q_{0}$. It is also a Mori Dream Space (see Proposition 3.1 in [18]). The GIT description gives an isomorphism between the Picard group of $M(Q, \mathbf{r})$ and the character group $\chi(G) \cong$ $\mathbb{Z}^{\rho}$ of $G$. When tensored with $\mathbb{Q}$, the fact that this is a Mori Dream space (see Lemma 4.2 in [30]) implies that this isomorphism induces an isomorphism of wall and chamber structures given by the Mori structure (on the effective cone) and the GIT structure (on $\chi(G) \otimes \mathbb{Q}$ ); in particular, the GIT chamber containing $\theta$ is the ample cone of $M(Q, \mathbf{r})$. The Picard group is generated by the determinant line bundles $\operatorname{det}\left(W_{i}\right), i \in Q_{0}$.


### 2.1.4 Quiver flag varieties as towers of Grassmannian bundles.

Generalizing Example 2.1.1, all quiver flag varieties are towers of Grassmannian bundles [18, Theorem 3.3]. For $0 \leq i \leq \rho$, let $Q(i)$ be the subquiver of $Q$ obtained by removing the vertices $j \in Q_{0}, j>i$, and all arrows attached to them. Let $\mathbf{r}(i)=\left(1, r_{1}, \ldots, r_{i}\right)$, and write $Y_{i}=M(Q(i), \mathbf{r}(i))$. Denote the universal bundle $W_{j} \rightarrow Y_{i}$ by $W_{j}^{(i)}$. Then there are maps

$$
M(Q, \mathbf{r})=Y_{\rho} \rightarrow Y_{\rho-1} \rightarrow \cdots \rightarrow Y_{1} \rightarrow Y_{0}=\operatorname{Spec} \mathbb{C},
$$

induced by isomorphisms $Y_{i} \cong \operatorname{Gr}\left(\mathcal{F}_{i}, r_{i}\right)$, where $\mathcal{F}_{i}$ is the locally free sheaf

$$
\mathcal{F}_{i}=\bigoplus_{a \in Q_{1}, t(a)=i} W_{s(a)}^{(i-1)}
$$

of rank $s_{i}$ on $Y_{i-1}$. This makes clear that $M(Q, \mathbf{r})$ is a smooth projective variety of dimension $\sum_{i=1}^{\rho} r_{i}\left(s_{i}-r_{i}\right)$, and that $W_{i}$ is the pullback to $Y_{\rho}$ of the tautological quotient bundle over $\operatorname{Gr}\left(\mathcal{F}_{i}, r_{i}\right)$. Thus $W_{i}$ is globally generated, and $\operatorname{det}\left(W_{i}\right)$ is nef. Furthermore the anti-canonical line bundle of $M(Q, \mathbf{r})$ is

$$
\begin{equation*}
\bigotimes_{a \in Q_{1}} \operatorname{det}\left(W_{t(a)}\right)^{r_{s(a)}} \otimes \operatorname{det}\left(W_{s(a)}\right)^{-r_{t(a)}} . \tag{2.1}
\end{equation*}
$$

In particular, $M(Q, \mathbf{r})$ is Fano if $s_{i}>s_{i}^{\prime}:=\sum_{a \in Q_{1}, s(a)=i} r_{t(a)}$. This condition is not if and only if.

### 2.1.5 The Euler SEQUENCE

Quiver flag varieties, like both Grassmannians and toric varieties, have an Euler sequence.

Proposition 2.1.3. Let $X=M(Q, \mathbf{r})$ be a quiver flag variety, and for a $\in Q_{1}$, denote $W_{a}:=W_{s(a)}^{*} \otimes W_{t(a)}$. There is a short exact sequence

$$
0 \rightarrow \bigoplus_{i=1}^{\rho} W_{i}^{*} \otimes W_{i} \rightarrow \bigoplus_{a \in Q_{1}} W_{a} \rightarrow T_{X} \rightarrow 0
$$

Proof. We proceed by induction on the Picard rank $\rho$ of $X$. If $\rho=1$ then this is the usual Euler sequence for the Grassmannian. Suppose that the proposition holds for quiver flag varieties of Picard rank $\rho-1$, for $\rho>1$. Then the fibration $\pi: \operatorname{Gr}\left(\pi^{*} \mathcal{F}_{\rho}, r_{\rho}\right) \rightarrow Y_{\rho-1}$ from §2.1.4 above gives a short exact sequence

$$
0 \rightarrow W_{\rho}^{*} \otimes W_{\rho} \rightarrow \pi^{*} \mathcal{F}_{\rho}^{*} \otimes W_{\rho} \rightarrow S^{*} \otimes W_{\rho} \rightarrow 0
$$

where $S$ is the kernel of the projection $\pi^{*} \mathcal{F}_{\rho} \rightarrow W_{\rho}$. Note that

$$
\pi^{*} \mathcal{F}_{\rho}^{*} \otimes W_{\rho}=\bigoplus_{a \in Q_{1}, t(a)=\rho} W_{a}
$$

Pulling back the short exact sequence from the induction hypothesis and summing with the above, we obtain

$$
0 \rightarrow \bigoplus_{i=1}^{\rho} W_{i}^{*} \otimes W_{i} \rightarrow \bigoplus_{a \in Q_{1}} W_{a} \rightarrow \pi^{*} T_{Y_{\rho-1}} \oplus S^{*} \otimes W_{\rho} \rightarrow 0
$$

This shows the proposition.

If $X$ is a quiver flag zero locus cut out of the quiver flag variety $M(Q, \mathbf{r})$ by a regular section of the representation theoretic vector bundle $E$ then there is a short exact sequence

$$
\begin{equation*}
\left.0 \rightarrow T_{X} \rightarrow T_{M(Q, \mathbf{r})}\right|_{X} \rightarrow E \rightarrow 0 \tag{2.2}
\end{equation*}
$$

Thus $T_{X}$ is the K-theoretic difference of representation theoretic vector bundles.

### 2.2 Quiver flag varieties as subvarieties

There are three well-known constructions of flag varieties: as GIT quotients, as homogeneous spaces, and as subvarieties of products of Grassmannians. Craw's
construction gives quiver flag varieties as GIT quotients. General quiver flag varieties are not homogeneous spaces, so the second construction does not generalize. In this section we generalize the third construction of flag varieties, exhibiting quiver flag varieties as subvarieties of products of Grassmannians. It is this description that will allow us to prove the Abelian/non-Abelian correspondence for quiver flag varieties.

Proposition 2.2.1. Let $M_{Q}:=M(Q, \mathbf{r})$ be a quiver flag variety with $\rho>1$. Then $M_{Q}$ is cut out of $Y=\prod_{i=1}^{\rho} \operatorname{Gr}\left(H^{0}\left(M_{Q}, W_{i}\right), r_{i}\right)$ by a tautological section of

$$
E=\bigoplus_{a \in Q_{1}, s(a) \neq 0} S_{s(a)}^{*} \otimes Q_{t(a)}
$$

where $S_{i}$ and $Q_{i}$ are the pullbacks to $Y$ of the tautological sub-bundle and quotient bundle on the $i^{\text {th }}$ factor of $Y$.

Proof. As vector spaces, there is an isomorphism $H^{0}\left(M_{Q}, W_{i}\right) \cong e_{0} \mathbb{C} Q e_{i}$, where $\mathbb{C} Q$ is the path algebra over $\mathbb{C}$ of $Q$ (Corollary 3.5, [18]). This isomorphism identifies a basis of global sections of $W_{i}$ from the set of paths from vertex 0 to $i$ in the quiver. Let $e_{a} \in \mathbb{C} Q$ be the element associated to the arrow $a \in Q_{1}$. Thus

$$
H^{0}\left(M_{Q}, W_{i}\right)=\bigoplus_{a \in Q_{1}, t(a)=i, s(a) \neq 0} H^{0}\left(M_{Q}, W_{s(a)}\right) \oplus \bigoplus_{a \in Q_{1}, s(a)=0, t(a)=i} \mathbb{C} e_{a} .
$$

Let $F_{i}=\oplus_{t(a)=i} Q_{s(a)}$. Combining the tautological surjective morphisms

$$
H^{0}\left(M_{Q}, W_{s(a)}\right) \otimes \mathcal{O}_{Y}=H^{0}\left(Y, Q_{s(a)}\right) \otimes \mathcal{O}_{Y} \rightarrow Q_{s(a)}
$$

gives the exact sequence

$$
0 \rightarrow \underset{t(a)=i, s(a) \neq 0}{ } S_{s(a)} \rightarrow H^{0}\left(M_{Q}, W_{i}\right) \otimes \mathcal{O}_{Y} \rightarrow F_{i} \rightarrow 0
$$

Thus

$$
\left(H^{0}\left(M_{Q}, W_{i}\right)^{*} \otimes \mathcal{O}_{Y}\right) / F_{i}^{*} \cong \bigoplus_{t(a)=i, s(a) \neq 0} S_{s(a)}^{*}
$$

and it follows that $E=\oplus_{i=2}^{\rho} \operatorname{Hom}\left(Q_{i}^{*},\left(H^{0}\left(M_{Q}, W_{i}\right)^{*} \otimes \mathcal{O}_{Y}\right) / F_{i}^{*}\right)$.
Consider the section $s$ of $E$ given by the compositions

$$
Q_{i}^{*} \rightarrow H^{0}\left(M_{Q}, W_{i}\right)^{*} \otimes \mathcal{O}_{Y} \rightarrow\left(H^{0}\left(M_{Q}, W_{i}\right)^{*} \otimes \mathcal{O}_{Y}\right) / F_{i}^{*} .
$$

The section $s$ vanishes at quotients $\left(V_{1}, \ldots, V_{\rho}\right)$ if and only if $V_{i}^{*} \subset \oplus_{t(a)=i} V_{s(a)}^{*}$; dually, the zero locus is where there is a surjection $F_{i} \rightarrow Q_{i}$ for each $i$. We now
identify $Z(s)$ with $M(Q, \mathbf{r})$. Since the $W_{i}$ are globally generated, there is a unique map

$$
f: M_{Q} \rightarrow Y=\prod_{i=1}^{\rho} \operatorname{Gr}\left(H^{0}\left(M_{Q}, W_{i}\right), r_{i}\right)
$$

such that $Q_{i}$ on $Y$ pulls back to $W_{i}$ on $M(Q, \mathbf{r})$. In particular, $f^{*}\left(F_{i}\right)$ is the pullback to $M_{Q}$ of the bundle $\pi^{*} \mathcal{F}_{i}$ from $\S 2.1 .4$ (here $\pi$ is the projection $Y_{\rho} \rightarrow Y_{\rho-1}$ ). By construction of $M_{Q}$ there are surjections

$$
\pi^{*} \mathcal{F}_{i}=f^{*}\left(\oplus_{a \in Q_{1}, t(a)=i} Q_{s(a)}\right) \rightarrow W_{i} \rightarrow 0,
$$

so $f\left(M_{Q}\right) \subset Z(s)$.
Any variety $X$ with vector bundles $V_{i}$ of rank $r_{i}$ for $i=1, \ldots, \rho$ and maps $H^{0}\left(M_{Q}, W_{i}\right) \rightarrow$ $V_{i} \rightarrow 0$ that factor as

$$
H^{0}\left(M_{Q}, W_{i}\right) \rightarrow \bigoplus_{t(a)=i} V_{s(a)} \rightarrow V_{i}
$$

has a unique map to $M(Q, \mathbf{r})$ as the $V_{i}$ form a flat family of $\theta$-stable representations of $Q$ of dimension $\mathbf{r}$. The $\left(\left.Q_{i}\right|_{Z(s)}\right)_{i=1}^{\rho}$ on $Z(s)$ give precisely such a set of vector bundles. The surjections $\left.H^{0}\left(M_{Q}, W_{i}\right) \rightarrow Q_{i}\right|_{Z(s)} \rightarrow 0$ follow from the fact that these are restrictions of the tautological bundles on a product of Grassmannians. That these maps factor as required is precisely the condition that $s$ vanishes.

Let $g: Z(s) \rightarrow M_{Q}$ be the induced map. By the universal property of $M(Q, \mathbf{r})$, the composition $g \circ f: M_{Q} \rightarrow Z(s) \rightarrow M_{Q}$ must be the identity. The composition $f \circ g: Z(s) \rightarrow M(Q, \mathbf{r}) \rightarrow Y$ must be the inclusion $Z(s) \rightarrow Y$ by the universal property of $Y$. Therefore $Z(s)$ and $M(Q, \mathbf{r})$ are canonically isomorphic.

Suppose that $X$ is a quiver flag zero locus cut out of $M(Q, \mathbf{r})$ by a regular section of a representation theoretic vector bundle $E_{G}$ determined by a representation $E$. The product of Grassmannians $Y=\prod_{i=1}^{\rho} \operatorname{Gr}\left(H^{0}\left(W_{i}\right), r_{i}\right)$ is a GIT quotient $V^{s s} / G$ for the same group $G$ (one can see this by constructing $Y$ as a quiver flag variety). Therefore $E$ also determines a vector bundle $E_{G}^{\prime}$ on $Y$ :

$$
E_{G}^{\prime}:=E \times V^{s s} / G \rightarrow Y .
$$

We see that $X$ is deformation equivalent to the zero locus of a generic section of the vector bundle

$$
\begin{equation*}
F:=E_{G}^{\prime} \oplus \bigoplus_{a \in Q_{1}, s(a) \neq 0} S_{s(a)}^{*} \otimes Q_{t(a)} \tag{2.3}
\end{equation*}
$$

Although $Y$ is a quiver flag variety, this is not generally an additional model of $X$ as a quiver flag zero locus, as the summand $S_{s(a)}^{*} \otimes Q_{t(a)}$ in $F$ does not in general
come from a representation of $G$. We refer to the summands of $F$ of this form as arrow bundles.

Remark 2.2.2. Suppose $\alpha$ is a non-negative Schur partition. Then [47] shows that $S^{\alpha}\left(Q_{i}\right)$ is globally generated on $Y$ (using the notation as above). This implies that $S^{\alpha}\left(W_{i}\right)$ is globally generated on $M(Q, \mathbf{r})$.

### 2.3 Equivalences of quiver flag zero loci

The representation of a given variety $X$ as a quiver flag zero locus, if it exists, is far from unique. In this section we describe various methods of passing between different representations of the same quiver flag zero locus. This is important in practice, because our systematic search for four-dimensional quiver flag zero loci described in the Appendix finds a given variety in many different representations. Furthermore, geometric invariants of a quiver flag zero locus $X$ can be much easier to compute in some representations than in others. The observations in this section allow us to compute invariants of four-dimensional Fano quiver flag zero loci using only a few representations, where the computation is relatively cheap, rather than doing the same computation many times and using representations where the computation is expensive (see 3.4 for more details). However, the results of this section will be only used in the Appendix: the rest of the chapter is independent of this section.

### 2.3.1 Dualising

As we saw in the previous section, a quiver flag zero locus $X$ given by $(M(Q, \mathbf{r}), E)$ can be thought of as a zero locus in a product of Grassmannians $Y$. Unlike general quiver flag varieties, Grassmannians come in canonically isomorphic dual pairs:


The isomorphism interchanges the tautological quotient bundle $Q$ with $S^{*}$, where $S$ is the tautological sub-bundle. One can then dualize some or none of the Grassmannian factors in $Y$, to get different models of $X$. Depending on the representations in $E$, after dualizing, $E$ may still be a representation theoretic vector bundle, or the direct sum of a representation theoretic vector bundle with bundles of the form $S_{i}^{*} \otimes W_{j}$. If this is the case, one can then undo the product representation process to obtain another model $\left(M\left(Q^{\prime}, \mathbf{r}^{\prime}\right), E_{G}^{\prime}\right)$ of $X$.

Example 2.3.1. Consider $X$ given by the quiver

and bundle $\wedge^{2} W_{2}$; here and below the vertex numbering is indicated in blue. Then writing it as a product:

with bundle $\wedge^{2} W_{2} \oplus S_{1}^{*} \otimes W_{2}$ (as in equation (2.3)) and dualizing the first factor, we get

with bundle $\wedge^{2} W_{2} \oplus W_{1} \otimes W_{2}$, which is a quiver flag zero locus.

### 2.3.2 Removing arrows

Example 2.3.2. Note that $\operatorname{Gr}(n, r)$ is the quiver flag zero locus given by $(\operatorname{Gr}(n+$ $\left.1, r), W_{1}\right)$. This is because the space of sections of $W_{1}$ is $\mathbb{C}^{n+1}$, where the image of the section corresponding to $v \in \mathbb{C}^{n+1}$ at the point $\phi: \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{r}$ in $\operatorname{Gr}(n+1, r)$ is $\phi(v)$. This section vanishes precisely when $v \in \operatorname{ker} \phi$, so we can consider its zero locus to be $\operatorname{Gr}\left(\mathbb{C}^{n+1} /\langle v\rangle, r\right) \cong \operatorname{Gr}(n, r)$. The restriction of $W_{1}$ to $\operatorname{Gr}(n, r)$ is its tautological quotient bundle, and the restriction of $S$ is the direct sum of the tautological subbundle on $\operatorname{Gr}(n, r)$ with $\mathcal{O}_{\operatorname{Gr}(n, r)}$.

This example generalises. Let $M(Q, \mathbf{r})$ be a quiver flag variety. A choice of arrow $i \rightarrow j$ in $Q$ determines a canonical section of $W_{i}^{*} \otimes W_{j}$, and the zero locus of this section is $M\left(Q^{\prime}, \mathbf{r}\right)$, where $Q^{\prime}$ is the quiver obtained from $Q$ by removing one arrow from $i \rightarrow j$.

Example 2.3.3. Similarly, $\operatorname{Gr}(n, r)$ is the zero locus of a section of $S^{*}$, the dual of the tautological sub-bundle, on $\operatorname{Gr}(n+1, r+1)$. The exact sequence $0 \rightarrow W_{1}^{*} \rightarrow$ $\left(\mathbb{C}^{n+1}\right)^{*} \rightarrow S^{*} \rightarrow 0$ shows that a global section of $S^{*}$ is given by a linear map $\psi$ : $\mathbb{C}^{n+1} \rightarrow \mathbb{C}$. The image of the section corresponding to $\psi$ at the point $s \in S$ is $\psi(s)$, where we evaluate $\psi$ on s via the tautological inclusion $S \rightarrow \mathbb{C}^{n+1}$. Splitting
$\mathbb{C}^{n+1}=\mathbb{C}^{n} \oplus \mathbb{C}$ and choosing $\psi$ to be projection to the second factor shows that $\psi$ vanishes precisely when $S \subset \mathbb{C}^{n}$, that is, precisely along $\operatorname{Gr}(n, r)$. The restriction of $S$ to $\operatorname{Gr}(n, r)$ is its tautological sub-bundle, and the restriction of $W_{1}$ is the direct sum of its tautological quotient bundle and $\mathcal{O}_{\operatorname{Gr}(n, r)}$.

### 2.3.3 Grafting

Let $Q$ be a quiver. We say that $Q$ is graftable at $i \in Q_{0}$ if:

- $r_{i}=1$ and $0<i<\rho$;
- if we remove all of the arrows out of $i$ we get a disconnected quiver.

Call the quiver with all arrows out of $i$ removed $Q^{i}$. If $i$ is graftable, we call the grafting set of $i$

$$
\left\{j \in Q_{0} \mid 0 \text { and } j \text { are in different components of } Q^{i}\right\} .
$$

Example 2.3.4. In the quiver below, vertex 1 is not graftable.


If we removed the arrow from vertex 0 to vertex 2, then vertex 1 would be graftable and the grafting set would be $\{2\}$.

Proposition 2.3.5. Let $M(Q, \mathbf{r})$ be a quiver flag variety and let $i$ be a vertex of $Q$ that is graftable. Let $J$ be its grafting set. Let $Q^{\prime}$ be the quiver obtained from $Q$ by replacing each arrow $i \rightarrow j$, where $j \in J$, by an arrow $0 \rightarrow j$. Then

$$
M(Q, \mathbf{r})=M\left(Q^{\prime}, \mathbf{r}\right) .
$$

Proof. Define $V_{j}:=W_{i}^{*} \otimes W_{j}$ for $j \in J$, and $V_{j}:=W_{j}$ otherwise.
Note that by construction of $J$, for $j \in J$, there is a surjective morphism

$$
W_{i}^{\oplus d_{i j}} \rightarrow W_{j} \rightarrow 0 .
$$

Here $d_{i j}$ is the number of paths $i \rightarrow j$. Tensoring this sequence with $W_{i}^{*}$ shows that $V_{j}$ is globally generated.

Now we show that the $V_{j}, j \in\{0, \ldots, \rho\}$ are a $\theta$-stable representation of $Q^{\prime}$. It suffices
to check that there are surjective morphisms

$$
\bigoplus_{a \in Q_{1}^{\prime}, t(a)=j} V_{s(a)} \rightarrow V_{j} .
$$

If $j \notin J$, this is just the same surjection given by the fact that the $W_{i}$ are a $\theta$-stable representation of $Q$. If $j \in J$, one must, as above, tensor the sequence from $Q$ with $W_{i}^{*}$. The $V_{j}$ then give a map $M(Q, \mathbf{r}) \rightarrow M\left(Q^{\prime}, \mathbf{r}\right)$. Reversing this procedure shows that this is a canonical isomorphism.

Example 2.3.6. Consider the quiver flag zero locus $X$ given by the quiver in (a) below, with bundle

$$
W_{1} \otimes W_{3} \oplus W_{1}^{\oplus 2} \oplus \operatorname{det} W_{1}
$$

Notice we have chosen a different labelling of the vertices for convenience. Writing $X$ inside a product of Grassmannians gives $W_{1} \otimes W_{3} \oplus W_{1}^{\oplus 2} \oplus \operatorname{det} W_{1}$ on the quiver in (b), with arrow bundle $S_{2}^{*} \otimes W_{1}$. Removing the two copies of $W_{1}$ using Example 2.3.2 gives

$$
W_{1} \otimes W_{3} \oplus \operatorname{det} W_{1}
$$

on the quiver in (c), with arrow bundle $S_{2}^{*} \otimes W_{1}$. We now apply Example 2.3.3 to remove $\operatorname{det} W_{1}=\operatorname{det} S_{1}^{*}=S_{1}^{*}$. As mentioned in Example 2.3.3, $W_{1}$ on (c) becomes $W_{1} \oplus \mathcal{O}$ after removing $S_{1}^{*}$. The arrow bundle therefore becomes

$$
S_{2}^{*} \otimes\left(W_{1} \oplus \mathcal{O}\right)=S_{2}^{*} \oplus S_{2}^{*} \otimes W_{1}
$$

Similarly, $W_{1} \otimes W_{3}$ becomes $W_{3} \oplus W_{1} \otimes W_{3}$. We can remove the new $S_{2}^{*}$ and $W_{3}$ summands (reducing the $\operatorname{Gr}(8,6)$ factor to $\operatorname{Gr}(7,5)$ and the $\operatorname{Gr}(8,2)$ factor to $\operatorname{Gr}(7,2)$ respectively). Thus, we see that $X$ is given by $W_{1} \otimes W_{3}$ on the quiver in (d), with arrow bundle $S_{2}^{*} \otimes W_{1}$. Dualising at vertices 1 and 2 now gives the quiver in (e), with arrow bundle $S_{1}^{*} \otimes W_{2} \oplus S_{1}^{*} \otimes W_{3}$. Finally, undoing the product representation of \$2.2 exhibits $X$ as the quiver flag variety for the quiver in (f).
(a)

(b)

(c)



### 2.4 The ample cone

We now discuss how to compute the ample cone of a quiver flag variety. This is essential if one wants to search systematically for quiver flag zero loci that are Fano. In [18], Craw gives a conjecture that would in particular solve this problem, by relating a quiver flag variety $M(Q, \mathbf{r})$ to a toric quiver flag variety. We give a counterexample to this conjecture, and determine the ample cone of $M(Q, \mathbf{r})$ in terms of the combinatorics of the quiver: this is Theorem 2.4.14 below. Our method also involves a toric quiver flag variety: the Abelianization of $M(Q, \mathbf{r})$.

### 2.4.1 The multi-graded Plücker embedding

Given a quiver flag variety $M(Q, \mathbf{r})$, Craw ( $\S 5$ of [18], Example 2.9 in [19]) defines a multi-graded analogue of the Plücker embedding:

$$
p: M(Q, \mathbf{r}) \hookrightarrow M\left(Q^{\prime}, \mathbf{1}\right) \quad \text { with } \mathbf{1}=(1, \ldots, 1)
$$

Here $Q^{\prime}$ is the quiver with the same vertices as $Q$ but with the number of arrows $i \rightarrow j, i<j$ given by

$$
\operatorname{dim}\left(\operatorname{Hom}\left(\operatorname{det}\left(W_{i}\right), \operatorname{det}\left(W_{j}\right)\right) / S_{i, j}\right)
$$

where $S_{i, j}$ is spanned by maps which factor through maps to $\operatorname{det}\left(W_{k}\right)$ with $i<k<j$. This induces an isomorphism $p^{*}: \operatorname{Pic}\left(M\left(Q^{\prime}, \mathbf{1}\right)\right) \otimes \mathbb{R} \rightarrow \operatorname{Pic}(M(Q, \mathbf{r})) \otimes \mathbb{R}$ that sends $\operatorname{det}\left(W_{i}^{\prime}\right) \mapsto \operatorname{det}\left(W_{i}\right)$. In [18], it is conjectured that this induces a surjection of Cox rings $\operatorname{Cox}\left(M\left(Q^{\prime}, \mathbf{1}\right)\right) \rightarrow \operatorname{Cox}(M(Q, \mathbf{r}))$. This would give information about the Mori wall and chamber structure of $M(Q, \mathbf{r})$. In particular, by the proof of Theorem 2.8 of [37], a surjection of Cox rings together with an isomorphism of Picard groups (which we have here) implies an isomorphism of effective cones.

We provide a counterexample to the conjecture. To do this, we exploit the fact that quiver flag varieties are Mori Dream Spaces, and so the Mori wall and chamber structure on $\operatorname{NE}^{1}(M(Q, \mathbf{r})) \subset \operatorname{Pic}(M(Q, \mathbf{r}))$ coincides with the GIT wall and chamber structure. This gives GIT characterizations for effective divisors, ample divisors,
nef divisors, and the walls.
Theorem 2.4.1. [21] Let $X$ be a Mori Dream Space obtained as a GIT quotient in which $G$ acts on $V=\mathbb{C}^{N}$ with stability condition $\tau \in \chi(G)=\operatorname{Hom}\left(G, \mathbb{C}^{*}\right)$. Identifying $\operatorname{Pic}(X) \cong \chi(G)$, we have that:

- $v \in \chi(G)$ is ample if $V^{s}(v)=V^{s s}(v)=V^{s}(\tau)$.
- $v$ is on a wall if $V^{s s}(v) \neq V^{s}(v)$.
- $v \in \operatorname{NE}^{1}(X)$ if $V^{s s} \neq \varnothing$.

When combined with King's characterisation [35] of the stable and semistable points for the GIT problem defining $M(Q, \mathbf{r})$, this determines the ample cone of any given quiver flag variety. In Theorem 2.4.14 below we make this effective, characterising the ample cone in terms of the combinatorics of $Q$. We can also use 2.4.1 to see a counterexample to Conjecture 6.4 in [18].

Example 2.4.2. Consider the quiver $Q$ and dimension vector $\mathbf{r}$ as in (a). The target $M\left(Q^{\prime}, \mathbf{1}\right)$ of the multi-graded Plücker embedding has the quiver $Q^{\prime}$ shown in (b).
(a)

(b)


One can see this by noting that $\operatorname{Hom}\left(\operatorname{det}\left(W_{2}\right), \operatorname{det}\left(W_{1}\right)\right)=0$, and that after taking $\wedge^{3}\left(\right.$ respectively $\left.\wedge^{2}\right)$ the surjection $\mathcal{O}^{\oplus 5} \rightarrow W_{1} \rightarrow 0$ (respectively $\mathcal{O}^{\oplus 10} \rightarrow W_{2} \rightarrow 0$ ) becomes

$$
\mathcal{O}^{\oplus 10} \rightarrow W_{1} \rightarrow 0\left(\text { respectively } \mathcal{O}^{\oplus 45} \rightarrow W_{2} \rightarrow 0\right) .
$$

In this case, $M\left(Q^{\prime}, \mathbf{1}\right)$ is a product of projective spaces and so the effective cone coincides with the nef cone, which is just the closure of the positive orthant. The ample cone of $M(Q, \mathbf{r})$ is indeed the positive orthant, as we will see later. However, here we will find an effective character not in the nef cone. We will use King's characterisation (Definition 1.1 of [35]) of semi-stable points with respect to a character $\chi$ of $\prod_{i=0}^{\rho} G l\left(r_{i}\right)$ : a representation $R=\left(R_{i}\right)_{i \in Q_{0}}$ is semi-stable with respect to $\chi=\left(\chi_{i}\right)_{i=0}^{\rho}$ if and only if

- $\sum_{i=0}^{\rho} \chi_{i} \operatorname{dim}_{\mathbb{C}}\left(R_{i}\right)=0$; and
- for any subrepresentation $R^{\prime}$ of $R, \sum_{i=0}^{\rho} \chi_{i} \operatorname{dim}_{\mathbb{C}}\left(R_{i}^{\prime}\right) \geq 0$.

Consider the character $\chi=(-1,3)$ of $G$, which we lift to a character of $\prod_{i=0}^{\rho} G l\left(r_{i}\right)$ by taking $\chi=(-3,-1,3)$. We will show that there exists a representation $R=$
$\left(R_{0}, R_{1}, R_{2}\right)$ which is semi-stable with respect to $\chi$. The maps in the representation are given by a triple $(A, B, C) \in \operatorname{Mat}(3 \times 5) \times \operatorname{Mat}(2 \times 3) \times \operatorname{Mat}(2 \times 3)$. Suppose that

$$
\text { A has full rank, } \quad B=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right], \quad C=\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right],
$$

and that $R^{\prime}$ is a subrepresentation with dimensions $a, b$, $c$. We want to show that $-3 a-b+3 c \geq 0$. If $a=1$ then $b=3$, as otherwise the image of $A$ is not contained in $R_{1}^{\prime}$. Similarly, this implies that $c=2$. So suppose that $a=0$. The maps $B$ and $C$ have no common kernel, so $b>0$ implies $c>0$, and $-b+3 c \geq 0$ as $b \leq 3$. Therefore $R$ is a semi-stable point for $\chi$, and as quiver flag varieties are Mori Dream Spaces, $\chi$ is in the effective cone.

Therefore, there cannot exist a Mori embedding of $M(Q, r)$ into $M\left(Q^{\prime}, \mathbf{1}\right)$ because it would induce an isomorphism of effective cones.

### 2.4.2 Abelianization

We consider now the toric quiver flag variety associated to a given quiver flag variety $M(Q, \mathbf{r})$ which arises from the corresponding Abelian quotient. Let $T \subset G$ be the diagonal maximal torus. Then the action of $G$ on $\operatorname{Rep}(Q, \mathbf{r})$ induces an action of $T$ on $\operatorname{Rep}(Q, \mathbf{r})$, and the inclusion $i: \chi(G) \hookrightarrow \chi(T)$ allows us to interpret the special character $\theta$ as a stability condition for the action of $T$ on $\operatorname{Rep}(Q, \mathbf{r})$. The Abelian quotient is then $\operatorname{Rep}(Q, \mathbf{r}) \|_{i(\theta)} T$. Let us see that $\operatorname{Rep}(Q, \mathbf{r}) \|_{\theta} T$ is a toric quiver flag variety. Let $\lambda=\left(\lambda_{1}, \ldots, \lambda_{\rho}\right)$ denote an element of $T=\prod_{i=1}^{\rho}\left(\mathbb{C}^{*}\right)^{r_{i}}$, where $\lambda_{j}=\left(\lambda_{j 1}, \ldots, \lambda_{j r_{j}}\right)$. Let $\left(w_{a}\right)_{a \in Q_{1}} \in \operatorname{Rep}(Q, \mathbf{r})$. Here $w_{a}$ is an $r_{t(a)} \times r_{s(a)}$ matrix. The action of $\lambda$ on $\left(w_{a}\right)_{a \in Q_{1}}$ is defined by

$$
w_{a}(i, j) \mapsto \lambda_{s(a) i}^{-1} w_{a}(i, j) \lambda_{t(a) j} .
$$

Hence this is the same as the group action on the quiver $Q^{\text {ab }}$ with vertices

$$
Q_{0}^{\mathrm{ab}}=\left\{v_{i j}: 0 \leq i \leq \rho, 1 \leq j \leq r_{i}\right\}
$$

and the number of arrows between $v_{i j}$ and $v_{k l}$ is the number of arrows in the original quiver between vertices $i$ and $k$. Clearly $i(\theta) \in \chi(T)$ is the character prescribed by §2.1.1. Hence

$$
\operatorname{Rep}(Q, \mathbf{r}) \|_{\theta} T=M\left(Q^{\mathrm{ab}}, \mathbf{1}\right) .
$$

We call $Q^{\mathrm{ab}}$ the Abelianized quiver.

Example 2.4.3. Let $Q$ be the quiver


Then $Q^{\mathrm{ab}}$ is


Martin [39] has studied the relationship between the cohomology of Abelian and non-Abelian quotients. We state his result specialized to quiver flag varieties, then extend this to a comparison of the ample cones. To simplify notation, denote $M_{Q}=$ $M(Q, \mathbf{r}), M_{Q^{\text {ab }}}=M\left(Q^{\mathrm{ab}},(1, \ldots, 1)\right)$ and $V=\operatorname{Rep}(Q, \mathbf{r})=\operatorname{Rep}\left(Q^{\mathrm{ab}},(1, \ldots, 1)\right)$. For $v \in \chi(G)$, let $V_{v}^{s}(T)$ denote the $T$-stable points of $V$ and $V_{v}^{s}(G)$ denote the $G$ stable points, dropping the subscript if it is clear from context. It is easy to see that $V^{s}(G) \subset V^{s}(T)$. The Weyl group $W$ of $(G, T)$ is $\prod_{i=1}^{\rho} S_{r_{i}}$, where $S_{r_{i}}$ is the symmetric group on $r_{i}$ letters. Let $\pi: V^{s}(G) / T \rightarrow V^{s}(G) / G$ be the projection. The Weyl group acts on the cohomology of $M\left(Q^{\text {ab }}, \mathbf{1}\right)$, and also on the Picard group, by permuting the $W_{v_{i 1}}, \ldots, W_{v_{i_{i}}}$. It is well-known (see e.g. Atiyah-Bott [4]) that

$$
\pi^{*}: H^{*}\left(V^{s}(G) / T\right)^{W} \cong H^{*}\left(M_{Q}\right)
$$

Theorem 2.4.4. [39] There is a graded surjective ring homomorphism

$$
\phi: H^{*}\left(M_{Q^{\mathrm{ab}}}, \mathbb{C}\right)^{W} \rightarrow H^{*}\left(V^{s}(G) / T, \mathbb{C}\right) \xrightarrow{\pi^{*}} H^{*}\left(M_{Q}, \mathbb{C}\right)
$$

where the first map is given by the restriction $V^{s}(T) / T \rightarrow V^{s}(G) / T$. The kernel is the annihilator of $e=\prod_{i=1}^{\rho} \prod_{1 \leq j, k \leq r_{i}} c_{1}\left(W_{v_{i j}}^{*} \otimes W_{v_{i k}}\right)$.

Remark 2.4.5. This means that any class $\sigma \in H^{*}\left(M_{Q}\right)$ can be lifted (non-uniquely) to a class $\tilde{\sigma} \in H^{*}\left(M_{Q^{\mathrm{ab}}}\right)$. Moreover, e $\cap \tilde{\sigma}$ is uniquely determined by $\sigma$.

Corollary 2.4.6. Let $E$ be a representation of $G$ defining representation theoretic bundles $E_{G} \rightarrow M_{Q}$ and $E_{T} \rightarrow M_{Q^{\text {ab }}}$. Then $\phi\left(c_{i}\left(E_{T}\right)\right)=c_{i}\left(E_{G}\right)$.

Proof. Recall that

$$
\begin{aligned}
& E_{G}=\left(V^{s}(G) \times E\right) / G \rightarrow M_{Q}, \\
& E_{T}=\left(V^{s}(T) \times E\right) / T \rightarrow M_{Q^{\mathrm{ab}}} .
\end{aligned}
$$

Define

$$
E_{G}^{\prime}=\left(V^{s}(G) \times E\right) / T \rightarrow V^{s}(G) / T .
$$

Let $f$ be the inclusion $V^{s}(G) / T \rightarrow V^{s}(T) / T$. Clearly $f^{*}\left(E_{T}\right)=E_{G}^{\prime}$ as $E_{G}^{\prime}$ is just the restriction of $E_{T}$. Considering the square

we see that $\pi^{*}\left(E_{G}\right)=E_{G}^{\prime}$. Then we have that $f^{*}\left(E_{T}\right)=\pi^{*}\left(E_{G}\right)$, and so in particular $f^{*}\left(c_{i}\left(E_{T}\right)\right)=\pi^{*}\left(c_{i}\left(E_{G}\right)\right)$. The result now follows from Martin's theorem (Theorem 2.4.4).

Remark 2.4.7. Note that $E_{T}$ always splits as a direct sum of line bundles on $M\left(Q^{a b},(1, \ldots, 1)\right)$, as any representation of $T$ splits into rank one representations. In particular, this means that if $\left(Q, E_{G}\right)$ defines a quiver flag zero locus, $\left(Q^{a b}, E_{T}\right)$ which is also a toric complete intersection.

The corollary shows that in degree 2, the inverse of Martin's map is

$$
i: c_{1}\left(W_{i}\right) \mapsto \sum_{j=1}^{r_{i}} c_{1}\left(W_{v_{i j}}\right)
$$

In particular, using (2.1), we have that $i\left(\omega_{M_{Q}}\right)=\omega_{M_{Q^{\text {ab }}}}$, where $\omega_{X}$ is the canonical bundle of $X$.

Proposition 2.4.8. Let $\operatorname{Amp}(Q), \operatorname{Amp}\left(Q^{\mathrm{ab}}\right)$ denote the ample cones of $M_{Q}$ and $M_{Q^{\text {ab }}}$ respectively. Then

$$
i(\operatorname{Amp}(Q))=\operatorname{Amp}\left(Q^{\mathrm{ab}}\right)^{W} .
$$

Proof. Let $\alpha$ be a character for $G$, denoting its image under $i: \chi(G) \hookrightarrow \chi(T)$ as $\alpha$ as well. The image of $i$ is $W$-invariant, and in fact $i(\chi(G))=\chi(T)^{W}$ (this reflects that $W$-invariant lifts of divisors are unique).

Note that $V_{\alpha}^{s s}(G) \subset V_{\alpha}^{s s}(T)$. To see this, suppose $v \in V$ is semi-stable for $\alpha$ as a character of $G$. Let $\lambda: \mathbb{C}^{*} \rightarrow T$ be a one-parameter subgroup of $T$ such that
$\lim _{t \rightarrow 0} \lambda(t) \cdot v$ exists. By inclusion, $\lambda$ is a one-parameter subgroup of $G$, and so $\langle\alpha, \lambda\rangle \geq 0$ by semi-stability of $v$. Hence $v \in V_{\alpha}^{s s}(T)$. It follows that, if $\alpha \in \operatorname{NE}^{1}\left(M_{Q}\right)$, then $V_{\alpha}^{s s}(G) \neq \varnothing$, so $V_{\alpha}^{s s}(T) \neq \varnothing$, and hence by Theorem 2.4.1 $\alpha \in \mathrm{NE}^{1}\left(M_{Q^{\mathrm{ab}}}\right)^{W}$.

Ciocan-Fontanine-Kim-Sabbah use duality to construct a projection [10]

$$
p: \operatorname{NE}_{1}\left(M_{Q^{\mathrm{ab}}}\right) \rightarrow \mathrm{NE}_{1}\left(M_{Q}\right) .
$$

Suppose that $\alpha \in \operatorname{Amp}(Q)$. Then for any $C \in \mathrm{NE}_{1}\left(M_{Q^{\mathrm{ab}}}\right), i(\alpha) \cdot C=\alpha \cdot p(C)>0$. So $i(\alpha) \in \operatorname{Amp}\left(Q^{\mathrm{ab}}\right)^{W}$.

Let $\operatorname{Wall}(G) \subset \operatorname{Pic}\left(M_{Q}\right)$ denote the union of all GIT walls given by the $G$ action, and similarly for $\operatorname{Wall}(T)$. Recall that $\alpha \in \operatorname{Wall}(G)$ if and only if it has a non-empty strictly semi-stable locus. Suppose $\alpha \in \operatorname{Wall}(G)$, with $v$ in the strictly semi-stable locus. That is, there exists a non-trivial $\lambda: \mathbb{C}^{*} \rightarrow G$ such that $\lim _{t \rightarrow 0} \lambda(t) \cdot v$ exists and $\langle\alpha, \lambda\rangle=0$. Now we don't necessarily have $\operatorname{Im}(\lambda) \subset T$, but the image is in some maximal torus, and hence there exists $g \in G$ such that $\operatorname{Im}(\lambda) \subset g^{-1} T g$. Consider $\lambda^{\prime}=g \lambda g^{-1}$. Then $\lambda^{\prime}\left(\mathbb{C}^{*}\right) \subset T$. Since $g \cdot v$ is in the orbit of $v$ under $G$, it is semi-stable with respect to $G$, and hence with respect to $T$. In fact, it is strictly semi-stable with respect to $T$, since $\lim _{t \rightarrow 0} \lambda^{\prime}(t) g \cdot v=\lim _{t \rightarrow 0} g \lambda(t) \cdot v$ exists, and $\left\langle\alpha, \lambda^{\prime}\right\rangle=\langle\alpha, \lambda\rangle=0$. So as a character of $T, \alpha$ has a non-empty strictly semi-stable locus, and we have shown that

$$
i(\operatorname{Wall}(G)) \subset \operatorname{Wall}(T)^{W}
$$

This means that the boundary of $i(\operatorname{Amp}(Q))$ has empty intersection with $\operatorname{Amp}\left(Q^{\mathrm{ab}}\right)^{W}$. Since both are full dimensional cones in the $W$ invariant subspace, the inclusion $i(\operatorname{Amp}(Q)) \subset \operatorname{Amp}\left(Q^{\mathrm{ab}}\right)^{W}$ is in fact an equality.

Remark 2.4.9. Note that the proof of this proposition provides a stronger result. The inclusion of walls in the effective chamber of $Q$ into the walls of the effective chamber of $Q^{a b}$ implies that the wall-and-chamber decomposition of $\mathrm{NE}_{1}\left(M_{Q}\right)$ is just the one restricted from $\mathrm{NE}_{1}\left(M_{Q^{a b}}\right)$. Notice, however, that it can happen that $V_{\theta}^{s s}(G)=\varnothing$, but $V_{\theta}^{s s}(T) \neq \varnothing$. So the Weyl invariant part of the effective cone of the Abelianized quiver may have chambers that do not show up in the effective cone of the non-Abelian quotient (as flag varieties demonstrate).

Example 2.4.10. Consider again the example


The Abelianization of this quiver is


Walls are generated by collections of divisors that generate cones of codimension 1 . We then intersect them with the Weyl invariant subspace, generated by $(1,1,1,0,0)$ and $(0,0,0,1,1)$. In this subspace, the walls are generated by

$$
(1,1,1,0,0), \quad(0,0,0,1,1), \quad(-2,-2,-2,3,3) .
$$

Combined with Example 2.4.2, this determines the wall-and-chamber structure of the effective cone of $M(Q, \mathbf{r})$. That is, it has three walls, each generated by one of $v_{1}:=(1,0), v_{2}:=(-2,3)$, and $v_{3}=(0,1)$. There are two maximal cones generated by $\left(v_{1}, v_{3}\right)$ and $\left(v_{2}, v_{3}\right)$ respectively.

### 2.4.3 The toric case

As a prelude to determining the ample cone of a general quiver flag variety, we first consider the toric case. Recall that a smooth projective toric variety (or orbifold) can be obtained as a GIT quotient of $\mathbb{C}^{N}$ by a $\rho$-dimensional torus.

Definition 2.4.11. The GIT data for a toric variety is an $\rho$-dimensional torus $K$ with cocharacter lattice $L=\operatorname{Hom}\left(\mathbb{C}^{*}, K\right)$, and $m$ characters $D_{1}, \ldots, D_{m} \in L^{\vee}$, together with a stability condition $w \in L^{\vee} \otimes \mathbb{R}$.

These linear data give a toric variety (or Deligne-Mumford stack) as the quotient of an open subset $U_{w} \subset \mathbb{C}^{m}$ by $K$, where $K$ acts on $\mathbb{C}^{m}$ via the map $K \rightarrow\left(\mathbb{C}^{*}\right)^{m}$ defined by the $D_{i} . U_{w}$ is defined as

$$
\left\{\left(z_{1}, \ldots, z_{m}\right) \in \mathbb{C}^{m} \mid w \in \operatorname{Cone}\left(D_{i}: z_{i} \neq 0\right)\right\}
$$

that is, its elements can have zeroes at $z_{i}, i \in I$, only if $w$ is in the cone generated by $D_{i}, i \notin I$. Assume that all cones given by subsets of the divisors that contain $w$ are full dimensional, as is the case for toric quiver flag varieties. Then the ample cone is the intersection of all of these.

In [20], the GIT data for a toric quiver flag variety is detailed; we present it slightly differently. The torus is $K=\left(\mathbb{C}^{*}\right)^{\rho}$. Let $e_{1}, \ldots, e_{\rho}$ be standard basis of $L^{\vee}=\mathbb{Z}^{\rho}$ and
set $e_{0}:=0$. Then each $a \in Q_{1}$ gives a weight $D_{a}=-e_{s(a)}+e_{t(a)}$. The stability condition is $\mathbf{1}=(1,1, \ldots, 1)$. Identify $L^{\vee} \cong \operatorname{Pic} M(Q, \mathbf{1})$. Then $D_{a}=W_{a}:=W_{s(a)}^{*} \otimes W_{t(a)}$.
A minimal generating set for a full dimensional cone for a toric quiver flag variety is given by $\rho$ linearly independent $D_{a_{i}}, a_{i} \in Q_{1}$. Therefore for each vertex $i$ with $1 \leq i \leq \rho$, we need an arrow $a_{i}$ with either $s(a)=i$ or $t(a)=i$, and these arrows should be distinct. For the positive span of these divisors to contain 1 requires that $D_{a_{i}}$ has $t\left(a_{i}\right)=i$. Fix such a set $S=\left\{a_{1}, \ldots, a_{\rho}\right\}$, and denote the corresponding cone by $C_{S}$. As mentioned, the ample cone is the intersection of such cones $C_{S}$. The set $S$ determines a path from 0 to $i$ for each $i$, given by concatenating (backwards) $a_{i}$ with $a_{s\left(a_{i}\right)}$ and so on; let us write $f_{i j}=1$ if $a_{j}$ is in the path from 0 to $i$, and 0 otherwise. Then

$$
e_{i}=\sum_{j=1}^{\rho} f_{i j} D_{a_{j}} .
$$

This gives us a straightforward way to compute the cone $C_{S}$. Let $B_{S}$ be the matrix with columns given by the $D_{a_{i}}$, and let $A_{S}=B_{S}^{-1}$. The columns of $A_{S}$ are given by the aforementioned paths: the $j$ th column of $A_{S}$ is $\sum_{i=1}^{\rho} f_{i j} e_{i}$. If $c \in \operatorname{Amp}(Q)$, then $A_{S} c \in A_{S}(\operatorname{Amp}(Q)) \subset A_{S}\left(C_{S}\right)$. Since $A_{S} D_{a_{i}}=e_{i}$, this means that $A_{S} c$ is in the positive orthant.

Proposition 2.4.12. Let $M(Q, 1)$ be a toric quiver flag variety. Let $c \in \operatorname{Amp}(Q)$, $c=\left(c_{1}, \ldots, c_{\rho}\right)$, be an ample class, and suppose that vertex $i$ of the quiver $Q$ satisfies the following condition: for all $j \in Q_{0}$ such that $j>i$, there is a path from 0 to $j$ not passing through $i$. Then $c_{i}>0$.

Proof. Choose a collection $S$ of arrows $a_{j} \in Q_{1}$ such that the span of the associated divisors $D_{a_{j}}$ contains the stability condition 1 , and such that the associated path from 0 to $j$ for any $j>i$ does not pass through $i$. Then the $(i, i)$ entry of $A_{S}$ is 1 and all other entries of the $i^{\text {th }}$ row are zero. As $A_{S} c$ is in the positive orthant, $c_{i}>0$.

### 2.4.4 The ample cone of a quiver flag variety

First, note the following corollary of the previous section.
Corollary 2.4.13. Let $M(Q, \mathbf{r})$ be a quiver flag variety, not necessarily toric. If $c=\left(c_{1}, \ldots, c_{\rho}\right) \in \operatorname{Amp}(Q)$ and $r_{j}>1$, then $c_{j}>0$.

Proof. Consider the Abelianized quiver. For any vertex $v \in Q_{0}^{\text {ab }}$, we can always choose a path from the origin to $v$ that does not pass through $v_{j 1}$ : if there is an arrow between $v_{j 1}$ and $v$, then there is an arrow between $v_{j 2}$ and $v$, so any path
through $v_{j 1}$ can be rerouted through $v_{j 2}$. Then we obtain that the $j 1$ entry of $i(c)$ is positive - but this is just $c_{j}$.

Let $M(Q, \mathbf{r})$ be a quiver flag variety and $Q^{a b}$ be the associated Abelianized quiver. Here paths are defined to be directed paths consisting of at least one arrow. A path passes through a vertex $i$ if either the source or the target of one of the arrows in the path is $i$. For each $i \in\{1, \ldots, \rho\}$, define

$$
T_{i}:=\left\{j \in Q_{0} \mid \text { all paths from the source to } v_{j 1} \text { pass through } v_{i 1} \text { in } Q^{a b}\right\} .
$$

Note that $i \in T_{i}$, as every path from 0 to $v_{i 1}$ passes through $v_{i 1}$ by definition. There are no paths from the source to the source, which is therefore not in $T_{i}$ for any $i$. If $r_{i}>1$ then $T_{i}=\{i\}$.

Theorem 2.4.14. The nef cone of $M(Q, \mathbf{r})$ is given by the following inequalities. Suppose that $a=\left(a_{1}, \ldots, a_{\rho}\right) \in \operatorname{Pic}\left(M_{Q}\right)$. Then $a$ is nef if and only if

$$
\begin{equation*}
\sum_{j \in T_{i}} r_{j} a_{j} \geq 0 \quad i=1,2, \ldots, \rho . \tag{2.4}
\end{equation*}
$$

Proof. We have already shown that the Weyl invariant part of the nef cone of $M_{Q^{a b}}:=$ $M\left(Q^{a b}, \mathbf{1}\right)$ is the image of the nef cone of $M_{Q}:=M(Q, \mathbf{r})$ under the natural map $i$ : $\operatorname{Pic}\left(M_{Q}\right) \rightarrow \operatorname{Pic}\left(M_{Q^{a b}}\right)$. Label the vertices of $Q^{a b}$ as $v_{i j}, i \in\{0, \ldots, \rho\}, j \in\left\{1, \ldots, r_{i}\right\}$, and write coordinates on $\operatorname{Pic}\left(M_{Q^{a b}}\right)$ as $\left(b_{i j}\right)$ (with respect to the basis given by the tautological line bundles). The inequalities defining the ample cone of $M_{Q^{a b}}$ are given by a choice of arrow $A_{i j} \in Q_{1}^{a b}, t\left(A_{i j}\right)=v_{i j}$ for each $v_{i j}$. This determines a path $P_{i j}$ from $0 \rightarrow v_{i j}$ for each vertex $v_{i j}$. For each $v_{i j}$ the associated inequality is:

$$
\begin{equation*}
\sum_{v_{i j} \in P_{k l}} b_{k l} \geq 0 \tag{2.5}
\end{equation*}
$$

Suppose that $a$ is nef. We want to show that $a$ satisfies the inequalities (2.4). We do this by finding a collection of arrows such that the inequality (2.5) applied to $i(a)$ is just the inequality (2.4).

It suffices to do this for $i$ such that $r_{i}=1$ (as we have already shown that the inequalities are the same in the $r_{i}>1$ case). Choose a set of arrows such that the associated paths avoid $v_{i 1}$ if possible: in other words, if $v_{i 1} \in P_{k l}$, then assume $k \in T_{i}$. Notice that if $v_{i 1} \in P_{k l_{1}}$, then $v_{i 1} \in P_{k l_{2}}$. By assumption $i(a)$ satisfies the $i^{t h}$ inequality associated to this collection of arrows, that is:

$$
\sum_{k \in T_{i}} r_{k} a_{k}=\sum_{v_{i 1} \in P_{k l}} a_{k} \geq 0
$$

Therefore, if $C$ is the cone defined by (2.4), we have shown that $\operatorname{Nef}\left(M_{Q}\right) \subset C$.
Suppose now that $a \in C$ and take a choice of arrows $A_{k l}$. Write $i(a)=\left(a_{i j}\right)$. We prove that the inequalities (2.5) are satisfied starting at $v_{\rho r_{\rho}}$. For $\rho$, the inequality is $a_{\rho r_{\rho}} \geq 0$, which is certainly satisfied. Suppose the $(i j+1),(i j+2), \ldots,\left(\rho r_{\rho}\right)$ inequalities are satisfied. The inequality we want to establish for $(i j)$ is

$$
\sum_{v_{i 1} \in P_{k l}} a_{k l}=a_{i j}+\sum_{k \in T_{i}-\{i\}} \sum_{l=1}^{r_{l}} a_{k l}+\Gamma=a_{i}+\sum_{k \in T_{i}-\{i\}} r_{k} a_{k}+\Gamma \geq 0,
$$

where

$$
\Gamma=\sum_{s\left(A_{k l}\right)=v_{i j}, k \notin T_{i}}\left(a_{k l}+\sum_{v_{k l} \in P_{s t}} a_{s t}\right) .
$$

This uses the fact that for $k \in T_{i}, v_{i 1} \in P_{k l}$ for all $l$, and that if $k \notin T_{i}$, and $v_{k l} \in P_{s t}$, we also have that $s \notin T_{i}$.

As $a \in C$ it suffices to show that $\Gamma \geq 0$. By the induction hypothesis $a_{k l}+\sum_{v_{k l} \in P_{s t}} a_{s t} \geq$ 0 , and therefore $\Gamma \geq 0$. This shows that $i(a)$ satisfies (2.5).

Example 2.4.15. The quiver flag variety given by the quiver

has ample cone defined by the inequalities

$$
c_{1}+2 c_{2} \geq 0, c_{2} \geq 0
$$

Notice that vertex 1 in the previous example can be grafted (by 2.3.5) and that the resulting quiver flag variety has ample cone given by the positive orthant. This holds more generally.

Corollary 2.4.16. Let $Q$ be a quiver which has no graftable vertices. Then the nef cone is the positive orthant.

Proof. The corollary followings from noting that for $i \in\{1, \ldots, \rho\}, r_{i}>0$ we have that $T_{i} \neq\{i\}$ if and only if $T_{i}$ is graftable.

### 2.4.5 Nef line bundles are globally generated

We conclude this section by proving that nef line bundles on quiver flag varieties are globally generated. Craw [18] has shown that the nef line bundles $\operatorname{det}\left(W_{i}\right)$ on
$M(Q, \mathbf{r})$ are globally generated; they span a top-dimensional cone contained in the nef cone (and thus all line bundles in this cone are globally generated). Nef line bundles on toric varieties are known to be globally generated. This result for quiver flag varieties will be important for us because in order to use the Abelian/nonAbelian Correspondence to compute the quantum periods of quiver flag zero loci, we need to know that the bundles involved are convex. Convexity is a difficult condition to understand geometrically, but it is implied by global generation.

To prove the proposition, we will need the following lemma about the structure of the $T_{i}$. The set $\left\{T_{i}: i \in\{1, \ldots, \rho\}\right\}$ has a partial order given by

$$
T_{i} \leq T_{j} \Leftrightarrow T_{i} \subset T_{j}
$$

(this order, rather than the opposite one, is chosen as it has the property that $T_{i} \leq T_{j}$ implies $i \leq j$ ).

Lemma 2.4.17. For all $j$, the set $\left\{T_{i} \leq T_{j}\right\}$ is a chain.

Proof. Observe that if $i \in T_{j} \cap T_{k}$ for $j<k$, then $T_{k} \subset T_{j}$ : if all paths from 0 to $i 1$ pass through both $j 1$ and $k 1$, then all paths from 0 to $k 1$ must pass through $j 1$. So $k \in T_{j}$ and hence $T_{k} \subset T_{j}$. Therefore, if $T_{j} \leq T_{i}$ and $T_{k} \leq T_{i}$ for $j<k$, then $i \in T_{j} \cap T_{k}$, and so $T_{j} \leq T_{k}$. Hence $\left\{T_{k} \mid T_{k} \leq T_{j}\right\}$ is totally-ordered for all $j$.

Proposition 2.4.18. Let $L$ be a nef line bundle on $M(Q, r)$. Then $L$ is globally generated.

Proof. Let $M:=\left\{T_{i} \mid T_{i}\right.$ is minimal $\}$. By the lemma, $\{1, \ldots, \rho\}=\bigsqcup_{T_{i} \in M} T_{i}$. Suppose $L$ is given by the character $\left(a_{1}, \ldots, a_{\rho}\right)$. Write $L$ as $L=\otimes_{T_{i} \in M} L_{T_{i}}$, where each $L_{T_{i}}$ comes from a character $\left(b_{1}, \ldots, b_{\rho}\right) \in \chi(G)$ satisfying $b_{j}=0$ if $j \notin T_{i}$.
$L$ is nef if and only if all the $L_{T_{i}}, T_{i} \in M$ are nef. To see this, note that for each $j$ the inequality

$$
\sum_{k \in T_{j}} r_{k} a_{k} \geq 0
$$

involves terms from a minimal $T_{i}$ if and only if $j \in T_{i}$, in which case it involves only terms from $T_{i}$. It therefore suffices to show the statement of the proposition for each $L_{j}$. Therefore suppose that $\left\{j \mid a_{j} \neq 0\right\} \subset T_{i}$ for $T_{i}$ minimal. If $r_{i}>1$, then $T_{i}=\{i\}$, so $L=\operatorname{det}\left(W_{i}\right)^{\otimes a_{i}}$ which is globally generated. So we further assume that $r_{i}=1$. For $k \in T_{i}, k>i$, define $h^{\prime}(k)$ such that $T_{h^{\prime}(k)}$ is the maximal element such that $T_{i} \leq T_{h^{\prime}(k)}<T_{k}$. This is well-defined because the set $\left\{T_{j} \mid T_{j}<T_{k}\right\}$ is a chain.

A section of $L$ is a $G$-equivariant section of the trivial line bundle on $\operatorname{Rep}(Q, \mathbf{r})$, where the action of $G$ on the line bundle is given by the character $\Pi \chi_{i}^{a_{i}}$. A point
of $\operatorname{Rep}(Q, \mathbf{r})$ is given by $\left(\phi_{a}\right)_{a \in Q_{1}}, \phi_{a}: \mathbb{C}^{r_{s(a)}} \rightarrow \mathbb{C}^{r_{t(a)}}$, where $G$ acts by change of basis. A choice of path $i \rightarrow j$ on the quiver gives an equivariant map $\operatorname{Rep}(Q, \mathbf{r}) \rightarrow$ $\operatorname{Hom}\left(\mathbb{C}^{r_{i}}, \mathbb{C}^{r_{j}}\right)$ where $G$ acts on the image by $g \cdot \phi=g_{j} \phi g_{i}^{-1}$. If $r_{i}=r_{j}=1$, such maps can be composed.

For $j \in T_{i}$, define $f_{j}$ as follows:

- If $j=i$, let $f_{i}$ be any homogeneous polynomial of degree $d_{i}=\sum_{k \in T_{i}} r_{k} a_{k} \geq 0$ in the maps given by paths $0 \rightarrow i$. Therefore $f_{i}$ is a section of the line bundle given by the character $\chi_{i}^{d_{i}}$.
- If $j>i, r_{j}=1$, let $f_{j}$ be any homogeneous polynomial of degree $d_{j}=\sum_{k \in T_{j}} r_{k} a_{k} \geq$ 0 in the maps given by paths $h^{\prime}(j) \rightarrow j$. Note that $r_{h^{\prime}(j)}=1$ as by construction $j, h^{\prime}(j) \in T_{h^{\prime}(j)}$. So $f_{j}$ defines a section of the line bundle given by character $\chi_{h(j)}^{-d_{j}} \chi_{j}^{d_{j}}$.
- If $j>i, r_{j}>1$, let $f_{j}$ be a homogeneous polynomial of degree $a_{k} \geq 0$ in the minors of the matrix whose columns are given by the paths $h^{\prime}(j) \rightarrow j$. That is, $f_{j}$ is a section of the line bundle given by character $\chi_{h^{\prime}(j)}^{-r_{j} a_{j}} \chi_{j}^{a_{j}}$.

For any $x \in \operatorname{Rep}(Q, \mathbf{r})$ which is semi-stable, and for any $j \in T_{i}$, there exists an $f_{j}$ as above with $f_{j}(x) \neq 0$, because $j \in T_{h^{\prime}(j)}$. Fixing $x$, choose such $f_{j}$. Define $\sigma:=\prod_{j \in T_{i}} f_{j}: \operatorname{Rep}(Q, \mathbf{r}) \rightarrow \mathbb{C}$. Then $\sigma$ defines a section of the line bundle given by character

$$
\prod_{j \in T_{i}} \chi_{j}^{b_{j}}=\chi_{i}^{d_{i}} . \prod_{j \in T_{i}, j \neq i, r_{j}=1} \chi_{h^{\prime}(j)}^{-d_{j}} \chi_{j}^{d_{j}} \cdot \prod_{j \in T_{i}, j \neq i, r_{j}>1} \chi_{h^{\prime}(j)}^{-r_{j} a_{j}} \chi_{j}^{a_{j}} .
$$

We need to check that $b_{l}=a_{l}$ for all $l$. This is obvious for $l \in T_{i}$ with $r_{l}>1$. For $r_{l}=1$,

$$
b_{l}=\sum_{j \in T_{l}} r_{j} a_{j}-\sum_{k \in T_{l}-\{l\}, h^{\prime}(k)=l} \sum_{j \in T_{k}} r_{j} a_{j} .
$$

This simplifies to $a_{l}$ because $T_{l}-\{l\}=\sqcup_{j \in T_{l}, h^{\prime}(j)=l} T_{j}$. Therefore $\sigma$ gives a well-defined non-vanishing section of $L$, so $L$ is globally generated.

### 2.5 The Abelian/non-Abelian Correspondence

The main theoretical result of this thesis, Theorem 2.5.4 below, proves the Abelian/nonAbelian Correspondence with bundles [10, Conjecture 6.1.1] for quiver flag zero loci. This determines all genus-zero Gromov-Witten invariants, and hence the quantum cohomology, of quiver flag varieties, as well as all genus-zero Gromov-Witten invariants of quiver flag zero loci involving cohomology classes that come from the ambient
space. In particular, it determines the quantum period (see Definition 2.5.1) of a quiver flag varieties or quiver flag zero locus $X$ with $c_{1}\left(T_{X}\right) \geq 0$.

### 2.5.1 A brief Review of Gromov-Witten theory

We give a very brief review of Gromov-Witten theory, mainly to fix notation, See $[13,10]$ for more details and references. Let $Y$ be a smooth projective variety. Given $n \in \mathbb{Z}_{\geq 0}$ and $\beta \in H_{2}(Y)$, let $M_{0, n}(Y, \beta)$ be the moduli space of genus zero stable maps to $Y$ of class $\beta$, and with $n$ marked points [36]. While this space may be highly singular and have components of different dimensions, it has a virtual fundamental class $\left[M_{0, n}(Y, \beta)\right]^{\text {virt }}$ of the expected dimension $[8,38]$. There are natural evaluation maps $e v_{i}: M_{0, n}(Y, \beta) \rightarrow Y$ taking the class of a stable map $f: C \rightarrow Y$ to $f\left(x_{i}\right)$, where $x_{i} \in C$ is the $i^{\text {th }}$ marked point. There is also a line bundle $L_{i} \rightarrow M_{0, n}(Y, \beta)$ whose fiber at $f: C \rightarrow Y$ is the cotangent space to $C$ at $x_{i}$. The first Chern class of this line bundle is denoted $\psi_{i}$. Define:

$$
\begin{equation*}
\left\langle\tau_{a_{1}}\left(\alpha_{1}\right), \ldots, \tau_{a_{n}}\left(\alpha_{n}\right)\right\rangle_{n, \beta}=\int_{\left[M_{0, n}(Y, \beta)\right]^{v i r t}} \prod_{i=1}^{n} e v_{i}^{*}\left(\alpha_{i}\right) \psi_{i}^{a_{i}} \tag{2.6}
\end{equation*}
$$

where the integral on the right-hand side denotes cap product with the virtual fundamental class. If $a_{i}=0$ for all $i$, this is called a (genus zero) Gromov-Witten invariant and the $\tau$ notation is omitted; otherwise it is called a descendant invariant. It is deformation invariant.

We consider a generating function for descendant invariants called the J-function. Write $q^{\beta}$ for the element of $\mathbb{Q}\left[H_{2}(Y)\right]$ representing $\beta \in H_{2}(Y)$. Write $N(Y)$ for the Novikov ring of $Y$ :

$$
N(Y)=\left\{\begin{array}{l|l}
\sum_{\beta \in \mathrm{NE}_{1}(Y)} c_{\beta} q^{\beta} & \begin{array}{l}
c_{\beta} \in \mathbb{C}, \text { for each } d \geq 0 \text { there are only finitely } \\
\text { many } \beta \text { such that } \omega \cdot \beta \leq d \text { and } c_{\beta} \neq 0
\end{array}
\end{array}\right\}
$$

Here $\omega$ is the Kähler class on $Y$. The J-function assigns an element of $H^{*}(Y) \otimes$ $N(Y)\left[\left[z^{-1}\right]\right]$ to every element of $H^{*}(Y)$, as follows. Let $\phi_{1}, \ldots, \phi_{N}$ be a homogeneous basis of $H^{*}(Y)$, and let $\phi^{1}, \ldots, \phi^{N}$ be the Poincaré dual basis. Then the J -function is given by

$$
\begin{equation*}
J_{X}(\tau, z):=1+\tau z^{-1}+z^{-1} \sum_{i}\left\langle\left\langle\phi_{i} /(z-\psi)\right\rangle\right\rangle \phi^{i} . \tag{2.7}
\end{equation*}
$$

Here 1 is the unit class in $H^{0}(Y), \tau \in H^{*}(Y)$, and

$$
\begin{equation*}
\left\langle\left\langle\phi_{i} /(z-\psi)\right\rangle\right\rangle=\sum_{\beta \in \mathrm{NE}_{1}(Y)} q^{\beta} \sum_{n=0}^{\infty} \sum_{a=0}^{\infty} \frac{1}{n!z^{a+1}}\left\langle\tau_{a}\left(\phi_{i}\right), \tau, \ldots, \tau\right\rangle_{n+1, \beta} . \tag{2.8}
\end{equation*}
$$

The small J-function is the restriction of the J-function to $H^{0}(Y) \oplus H^{2}(Y)$; closed forms for the small J-function of toric complete intersections and toric varieties are known [25].

Definition 2.5.1. The quantum period $G_{Y}(t)$ is the component of $J(0)$ along $1 \in$ $H^{\bullet}(Y)$ after the substitutions $z \mapsto 1, q^{\beta} \mapsto t^{\left\langle-K_{Y}, \beta\right\rangle}$. This is a power series in $t$.

The quantum period satisfies an important differential equation called the quantum differential equation.

A vector bundle $E \rightarrow Y$ is defined to be convex if for every genus 0 stable map $f: C \rightarrow Y, H^{1}\left(C, f^{*} E\right)=0$. Globally generated vector bundles are convex. Let $X \subset Y$ be the zero locus of a generic section of a convex vector bundle $E \rightarrow Y$ and let e denote the total Chern class, which evaluates on a vector bundle $F$ of rank $r$ as

$$
\begin{equation*}
\mathbf{e}(F)=\lambda^{r}+\lambda^{r-1} c_{1}(F)+\cdots+\lambda c_{r-1}(F)+c_{r}(F) . \tag{2.9}
\end{equation*}
$$

The notation here indicates that one can consider $\mathbf{e}(F)$ as the $\mathbb{C}^{*}$-equivariant Euler class of $F$, with respect to the canonical action of $\mathbb{C}^{*}$ on $F$ which is trivial on the base of $F$ and scales all fibers. In this interpretation, $\lambda \in H_{\mathbb{C}^{*}}^{*}(p t)$ is the equivariant parameter. The twisted J-function $J_{\mathbf{e}, E}$ is defined exactly as the J-function (2.7), but replacing the virtual fundamental class which occurs there (via equations (2.8) and (2.6)) by $\left[M_{0, n}(Y, \beta)\right]^{v i r t} \cap \mathbf{e}\left(E_{0, n, \beta}\right)$, where $E_{0, n, \beta}$ is $\pi_{*}\left(e v_{n+1}^{*}(E)\right)$, $\pi: M_{0, n+1}(Y, \beta) \rightarrow M_{0, n}(Y, \beta)$ is the universal curve, and $e v_{n+1}: M_{0, n+1}(Y, \beta) \rightarrow Y$ is the evaluation map. $E_{0, n, \beta}$ is a vector bundle over $M_{0, n}(Y, \beta)$, because $E$ is convex. Functoriality for the virtual fundamental class [34] implies that

$$
\left.j^{*} J_{\mathbf{e}, E}(\tau, z)\right|_{\lambda=0}=J_{X}\left(j^{*} \tau, z\right)
$$

where $j: X \rightarrow Y$ is the embedding [11, Theorem 1.1]. Thus one can compute the quantum period of $X$ from the twisted J-function. We will use this to compute the quantum period of Fano fourfolds which are quiver flag zero loci.

The Abelian/non-Abelian correspondence is a conjecture [10] relating the J-functions (and more broadly, the quantum cohomology Frobenius manifolds) of GIT quotients $V / / G$ and $V / / T$, where $T \subset G$ is the maximal torus. It also extends to considering zero loci of representation theoretic bundles, by relating the associated twisted Jfunctions. As the Abelianization $V / / T$ of a quiver flag variety $V / / G$ is a toric quiver
flag variety, the Abelian/non-Abelian correspondence conjectures a closed form for the J-functions of Fano quiver flag zero loci. Ciocan-Fontanine-Kim-Sabbah proved the Abelian/non-Abelian correspondence (with bundles) when $V / / G$ is a flag manifold [10]. We will build on this to prove the conjectures when $V / / G$ is a quiver flag variety.

### 2.5.2 The I-Function

We give the J-function in the way usual in the literature: first, by defining a cohomology-valued hypergeometric function called the I-function (which should be understood as a mirror object, but we omit this perspective here), then relating the J-function to the I-function. We follow the construction given by [10] in our special case. Let $X$ be a quiver flag zero locus given by $\left(Q, E_{G}\right)$ (where we assume $E_{G}$ is globally generate), and write $M_{Q}=M(Q, \mathbf{r})$ for the ambient quiver flag variety. Let $\left(Q^{\text {ab }}, E_{T}\right)$ be the associated Abelianized quiver and bundle, $M_{Q^{\mathrm{ab}}}=M\left(Q^{\mathrm{ab}},(1, \ldots, 1)\right)$. Assume, moreover, that $E_{T}$ splits into nef line bundles; this implies that $E_{T}$ is convex. To define the I-function, we need to relate the Novikov rings of $M_{Q}$ and $M_{Q^{\mathrm{ab}}}$. Let $\operatorname{Pic} Q$ (respectively $\operatorname{Pic} Q^{\text {ab }}$ ) denote the Picard group of $M_{Q}$ (respectively of $M_{Q^{\mathrm{ab}}}$ ), and similarly for the cones of effective curves and effective divisors. The isomorphism $\operatorname{Pic} Q \rightarrow\left(\operatorname{Pic} Q^{\text {ab }}\right)^{W}$ gives a projection $p: \mathrm{NE}_{1}\left(M_{Q^{\text {ab }}}\right) \rightarrow \mathrm{NE}_{1}\left(M_{Q}\right)$. In the bases dual to $\operatorname{det}\left(W_{1}\right), \ldots, \operatorname{det}\left(W_{\rho}\right)$ of Pic $M_{Q}$ and $W_{i j}, 1 \leq i \leq \rho, 1 \leq j \leq r_{i}$ of Pic $M_{Q^{\mathrm{ab}}}$, this is

$$
p:\left(d_{1,1}, \ldots, d_{1, r_{1}}, d_{2,1}, \ldots, d_{\rho, r_{\rho}}\right) \mapsto\left(\sum_{i=1}^{r_{1}} d_{1 i}, \ldots, \sum_{i=1}^{r_{\rho}} d_{\rho i}\right) .
$$

For $\beta=\left(d_{1}, \ldots, d_{\rho}\right)$, define

$$
\epsilon(\beta)=\sum_{i=1}^{\rho} d_{i}\left(r_{i}-1\right) .
$$

Then, following [10, equation 3.2.1], the induced map of Novikov rings $N\left(M_{Q^{\text {ab }}}\right) \rightarrow$ $N\left(M_{Q}\right)$ sends

$$
q^{\tilde{\beta}} \mapsto(-1)^{\epsilon(\beta)} q^{\beta}
$$

where $\beta=p(\tilde{\beta})$. We write $\tilde{\beta} \rightarrow \beta$ if and only if $\tilde{\beta} \in \mathrm{NE}_{1}\left(M_{Q^{\mathrm{ab}}}\right)$ and $p(\tilde{\beta})=\beta$.
For a representation theoretic bundle $E_{G}$ of rank $r$ on $M_{Q}$, let $D_{1}, \ldots, D_{r} \in \operatorname{Pic}\left(Q^{a b}\right)$ be the divisors on $M_{Q^{\text {ab }}}$ giving the split bundle $E_{T}$. Given $\tilde{d} \in \mathrm{NE}_{1}\left(M_{Q^{\text {ab }}}\right)$ define

$$
I_{E_{G}}(\tilde{d})=\frac{\prod_{i=1}^{r} \prod_{m \leq 0}\left(D_{i}+m z\right)}{\prod_{i=1}^{r} \prod_{m \leq\left\langle\tilde{d}, D_{i}\right\rangle}\left(D_{i}+m z\right)}
$$

Notice that all but finitely many factors cancel here. If $E$ is K-theoretically a representation theoretic bundle, in the sense that there exists $A_{G}, B_{G}$ such that

$$
0 \rightarrow A_{G} \rightarrow B_{G} \rightarrow E \rightarrow 0
$$

is an exact sequence, we define

$$
\begin{equation*}
I_{E}(\tilde{d})=\frac{I_{B_{G}(\tilde{d})}}{I_{A_{G}(\tilde{d})}} \tag{2.10}
\end{equation*}
$$

Example 2.5.2. The Euler sequence from Proposition 2.1.3 shows that for the tangent bundle $T_{M_{Q}}$

$$
I_{T_{M_{Q}}}(\tilde{d})=\frac{\prod_{a \in Q_{1}^{a b}} \prod_{m \leq 0}\left(D_{a}+m z\right)}{\prod_{a \in Q_{1}^{a b}} \prod_{m \leq\left\langle\tilde{d}, D_{a}\right\rangle}\left(D_{a}+m z\right)} \frac{\prod_{i=1}^{\rho} \prod_{j \neq k} \prod_{m \leq\left\langle\tilde{d}, D_{i j}-D_{i k}\right\rangle}\left(D_{i j}-D_{i k}+m z\right)}{\prod_{i=1}^{\rho} \prod_{j \neq k} \prod_{m \leq 0}\left(D_{i j}-D_{i k}+m z\right)} .
$$

Here $D_{i j}$ is the divisor corresponding to the tautological bundle $W_{i j}$ for vertex $i j$, and $D_{a}:=-D_{s(a)}+D_{t(a)}$ is the divisor on $M_{Q^{a b}}$ corresponding to the arrow $a \in Q_{1}^{a b}$.

Example 2.5.3. If $X$ is a quiver flag zero locus in $M_{Q}$ defined by the bundle $E_{G}$, then the adjunction formula (see equation (2.2)) implies that

$$
I_{T_{X}}(\tilde{d})=I_{T_{M_{Q}}}(\tilde{d}) / I_{E_{G}}(\tilde{d}) .
$$

Define the I-function of $X \subset M_{Q}$ to be

$$
I_{X, M_{Q}}(z)=\sum_{d \in \mathrm{NE}_{1}\left(M_{Q}\right)} \sum_{\tilde{d} \rightarrow d}(-1)^{\epsilon(d)} q^{d} I_{T_{X}}(\tilde{d}) .
$$

Note that $I_{T_{X}}(\tilde{d})$ is homogenous of degree $\left(i\left(K_{X}\right), \tilde{d}\right)$, so defining the grading of $q^{d}$ to be $\left(-K_{X}, d\right), I_{X, M_{Q}}(z)$ is homogeneous of degree 0 . If $X$ is Fano, we can write $I_{T_{X}}(\tilde{d})$ as

$$
\begin{equation*}
z^{\left(\omega_{X}, \tilde{d}\right)}\left(b_{0}+b_{1} z^{-1}+b_{2} z^{-2}+\cdots\right), b_{i} \in H^{2 i}(X) \tag{2.11}
\end{equation*}
$$

Since $I_{X, M_{Q}}$ is invariant under the action of the Weyl group on the $D_{i j}$, by viewing these as Chern roots of the tautological bundles $W_{i}$ we can express it as a function in the Chern classes of the $W_{i}$. We can therefore regard the I-function as an element

[^1]of $H \cdot\left(M_{Q}, \mathbb{C}\right) \otimes N\left(M_{Q}\right) \otimes \mathbb{C}\left[\left[z^{-1}\right]\right]$. If $X$ is Fano,
\[

$$
\begin{equation*}
I_{X, M_{Q}}(z)=1+z^{-1} C+O\left(z^{-2}\right) \tag{2.12}
\end{equation*}
$$

\]

where $O\left(z^{-2}\right)$ denotes terms of the form $\alpha z^{k}$ with $k \leq-2$ and $C \in H^{0}\left(M_{Q}, \mathbb{C}\right) \otimes$ $N\left(M_{Q}\right)$; furthermore (by (2.11)) $C$ vanishes if the Fano index of $X$ is greater than 1.

Theorem 2.5.4. Let $X$ be a Fano quiver flag zero locus given by $\left(Q, E_{G}\right)$, and let $j: X \rightarrow M_{Q}$ be the embedding of $X$ into the ambient quiver flag variety. Then

$$
J_{X}(0, z)=e^{-c \mid z} j^{*} I_{X, M_{Q}}(z)
$$

where $c=j^{*} C$.
Remark 2.5.5. Via the Divisor Equation and the String Equation [46, §1.2], Theorem 2.5.4 determines $J_{X}(\tau, z)$ for $\tau \in H^{0}(X) \oplus H^{2}(X)$.

### 2.5.3 Proof of Theorem 2.5.4

Givental has defined [27,15] a Lagrangian cone $\mathcal{L}_{X}$ in the symplectic vector space $H_{X}:=H^{*}(X, \mathbb{C}) \otimes N(X) \otimes \mathbb{C}\left(\left(z^{-1}\right)\right)$ that encodes all genus-zero Gromov-Witten invariants of $X$. Note that $J_{X}(\tau, z) \in H_{X}$ for all $\tau$. The J-function has the property that $(-z) J_{X}(\tau,-z)$ is the unique element of $\mathcal{L}_{X}$ of the form

$$
-z+\tau+O\left(z^{-1}\right)
$$

(see $[15, \S 9]$ ) and this, together with the expression (2.12) for the I-function and the String Equation

$$
J_{X}(\tau+c, z)=e^{c / z} J_{X}(\tau, z)
$$

shows that Theorem 2.5.4 follows immediately from Theorem 2.5.6 below. Theorem 2.5.6 is stronger: it does not require the hypothesis that the quiver flag zero locus $X$ be Fano.

Theorem 2.5.6. Let $X$ be a quiver flag zero locus given by $\left(Q, E_{G}\right)$, and let $j: X \rightarrow$ $M_{Q}$ be the embedding of $X$ into the ambient quiver flag variety. Then $(-z) j^{*} I_{X, M_{Q}}(-z) \in$ $\mathcal{L}_{X}$.

Proof. Let $Y=\prod_{i=1}^{\rho} \operatorname{Gr}\left(H^{0}\left(W_{i}\right), r_{i}\right)$. Denote by $Y^{a b}=\prod_{i=1}^{\rho} \mathbb{P}\left(H^{0}\left(W_{i}\right)\right)^{\times r_{i}}$ the Abelianization of $Y$. In $\S 2.2$ we constructed a vector bundle $V$ on $Y$ such that $M_{Q}$ is cut
out of $Y$ by a regular section of $V$ :

$$
V=\bigoplus_{i=2}^{\rho} Q_{i} \otimes H^{0}\left(W_{i}\right)^{*} / F_{i}^{*}
$$

where $F_{i}=\oplus_{t(a)=i} Q_{s(a)}$. $V$ is globally generated and hence convex. It is not representation theoretic, but it is K-theoretically: the sequence

$$
0 \rightarrow F_{i}^{*} \otimes Q_{i} \rightarrow H^{0}\left(W_{i}\right)^{*} \otimes Q_{i} \rightarrow H^{0}\left(W_{i}\right)^{*} \otimes Q_{i} / F_{i}^{*} \rightarrow 0
$$

is exact. Let $i: M_{Q} \rightarrow Y$ denote the inclusion.
Both $Y$ and $M_{Q}$ are GIT quotients by the same group; we can therefore canonically identify a representation theoretic vector bundle $E_{G}^{\prime}$ on $Y$ such that $\left.E_{G}^{\prime}\right|_{M_{Q}}$ is $E_{G}$. Our quiver flag zero locus $X$ is cut out of $Y$ by a regular section of $V^{\prime}=V \oplus E_{G}^{\prime}$. Note that

$$
I_{T_{M_{Q}}}(\tilde{d}) / I_{V}(\tilde{d})=I_{T_{Y}}(\tilde{d}) / I_{V^{\prime}}(\tilde{d})
$$

The I-function $I_{X, M_{Q}}$ defined by considering $X$ as a quiver flag zero locus in $M_{Q}$ with the bundle $E_{G}$ then coincides with the pullback $i^{*} I_{X, Y}$ of the I-function defined by considering $X$ as a quiver flag zero locus in $Y$ with the bundle $V^{\prime}$. It therefore suffices to prove that

$$
(-z)(i \circ j)^{*} I_{X, Y}(-z) \in \mathcal{L}_{X}
$$

We consider a $\mathbb{C}^{*}$-equivariant counterpart of the I-function, defined as follows. $\lambda$ is the equivariant parameter given by the action on the bundle which is trivial on the base, as in (2.9). For a representation theoretic bundle $W_{G}$ on $Y$, let $D_{1}, \ldots, D_{r}$ be the divisors on $Y^{a b}$ giving the split bundle $W_{T}$, and for $\tilde{d} \in \mathrm{NE}_{1}\left(Y^{\mathrm{ab}}\right)$ set

$$
I_{W_{G}}^{\mathrm{C}^{*}}(\tilde{d})=\frac{\prod_{i=1}^{r} \prod_{m \leq 0}\left(\lambda+D_{i}+m z\right)}{\prod_{i=1}^{r} \prod_{m \leq\left\langle\tilde{d}, D_{i}\right\rangle}\left(\lambda+D_{i}+m z\right)}
$$

We extend this definition to bundles on $Y$ - such as $V^{\prime}$ - that are only K-theoretically representation theoretic in the same way as (2.10). Let $\tilde{s}_{i}:=\operatorname{dim} H^{0}\left(W_{i}\right)$. Recalling that

$$
I_{T_{Y}}(\tilde{d})=\frac{\prod_{i=1}^{\rho} \prod_{j \neq k} \Pi_{m \leq\left\langle\tilde{d}, D_{i j}-D_{i k}\right\rangle}\left(D_{i j}-D_{i k}+m z\right)}{\prod_{i=1}^{\rho} \prod_{j \neq k} \prod_{m \leq 0}\left(D_{i j}-D_{i k}+m z\right)} \frac{\prod_{i=1}^{\rho} \prod_{j=1}^{r_{i}} \Pi_{m \leq 0}\left(D_{i j}+m z\right)^{\tilde{s}_{i}}}{\prod_{i=1}^{\rho} \prod_{j=1}^{r_{i}} \prod_{m \leq\left\langle\tilde{d}, D_{i j}\right\rangle}\left(D_{i j}+m z\right)^{\tilde{s}_{i}}},
$$

we define

$$
I_{X, Y}^{\mathbb{C}^{*}}(z)=\sum_{d \in \mathrm{NE}_{1}(Y)} \sum_{\tilde{d} \rightarrow d}(-1)^{\epsilon(d)} q^{d} I_{T_{Y}}(\tilde{d}) / I_{V^{\prime}}^{\mathbb{C}^{*}}(\tilde{d})
$$

The I-function $I_{X, Y}$ can be obtained by setting $\lambda=0$ in $I_{X, Y}^{\mathbb{C}^{*}}$. In view of [11, Theorem 1.1], it therefore suffices to prove that

$$
(-z) I_{X, Y}^{\mathbb{C}^{*}}(-z) \in \mathcal{L}_{\mathrm{e}, V^{\prime}}
$$

where $\mathcal{L}_{\mathbf{e}, V^{\prime}}$ is the Givental cone for the Gromov-Witten theory of $Y$ twisted by the total Chern class e and the bundle $V^{\prime}$.

If $V^{\prime}$ were a representation theoretic bundle, this would follow immediately from the work of Ciocan-Fontanine-Kim-Sabbah: see the proof of Theorem 6.1.2 in [10]. In fact $V^{\prime}$ is only K -theoretically representation theoretic, but their argument can be adjusted almost without change to this situation. Suppose that $A_{G}$ and $B_{G}$ are representation theoretic vector bundles, and that

$$
0 \rightarrow A_{G} \rightarrow B_{G} \rightarrow V \rightarrow 0
$$

is exact. Then we can also consider an exact sequence

$$
0 \rightarrow A_{T} \rightarrow B_{T} \rightarrow F \rightarrow 0
$$

on the Abelianization, and define $V_{T}:=F$. Using the notation of the proof of $[10$, Theorem 6.1.2], the point is that

$$
\Delta(V) \Delta\left(A_{G}\right)=\Delta\left(B_{G}\right)
$$

Here, $\Delta(V)$ is the twisting operator that appears in the Quantum Lefschetz theorem [15]. We can then follow the same argument for

$$
\Delta\left(B_{G}\right) / \Delta\left(A_{G}\right)
$$

After Abelianizing, we obtain $\Delta\left(B_{T}\right) / \Delta\left(A_{T}\right)=\Delta(F)$, and conclude that

$$
(-z) I_{X, Y}^{\mathbb{C}^{*}}(-z) \in \mathcal{L}_{\mathrm{e}, V^{\prime}}
$$

as claimed. This completes the proof.

## 3

## The search for Fano fourfolds

In this chapter, we describe the computer search for four dimensional Fano quiver flag zero loci with codimension at most four. Code to perform this and similar analyses, using the computational algebra system Magma [9], is available at the repository [1]. A database of Fano quiver flag varieties, which was produced as part of the calculation, is available at the repository [2]. This work is joint with T. Coates and A. Kasprzyk, and also appears in the appendices of the paper [33]. The tables with the results of our computations can be found in Appendix A.

### 3.1 Classifying quiver flag varieties

The first step is to find all Fano quiver flag varieties of dimension at most 8. A nonnegative integer matrix $A=\left[a_{i, j}\right]_{0 \leq i, j \leq \rho}$ and a dimension vector $\mathbf{r} \in \mathbb{Z}_{>0}^{\rho+1}$ determine a vertex-labelled directed multi-graph: the $\rho+1$ vertices are labelled by the $r_{i}$, and the adjacency matrix for the graph is $A$. Here, by adjacency matrix of a directed multi-graph, we mean the $\rho+1 \times \rho+1$ square matrix $A=\left[a_{i j}\right]$ such that $a_{i j}$ is the number of (directed) edges from $i$ to $j$. Such a graph, if it is acyclic with a unique source, and the label of the source is 1 , also determines a quiver flag variety. Two $(A, \mathbf{r})$ pairs can determine the same graph and hence the same quiver flag varieties.

Definition 3.1.1. A pair $(A, \mathbf{r})$ determining a quiver flag variety is in normal form if $\mathbf{r}$ is increasing and, under all permutations of the $\rho+1$ indices that preserve $\mathbf{r}$, the columns of $A$ are lex minimal.

Two pairs in normal form determine the same quiver flag variety (and hence the
same graph) if and only if they are equal.
Recall that quiver flag varieties are towers of Grassmannians (see §2.1.4), and that the $i$ th step in the tower is given by the relative Grassmannian $\operatorname{Gr}\left(\mathcal{F}_{i}, r_{i}\right)$, where $\mathcal{F}_{i}$ is a vector bundle of rank $s_{i}$. Using this construction it is easy to see that if $s_{i}=r_{i}$ then this quiver flag variety is equivalent to the quiver flag variety $\tilde{Q}$ with vertex $i$ removed, and one arrow $k \rightarrow j$ for every path of the form $k \rightarrow i \rightarrow j$. Therefore we can assume that $s_{i}>r_{i}$, and hence that every vertex contributes strictly positively to the dimension of the quiver flag variety. With this constraint, there are only finitely many quiver flag varieties with dimension at most 8 , and each such has at most 9 vertices.

The algorithm to build all quiver flag varieties with dimension at most 8 is as follows. Start with the set $S$ of all Grassmannians of dimension at most 8. Given an element of $S$ of dimension less than 8 , add one extra labelled vertex and extra arrows into this vertex, in all possible ways such that the dimension of the resulting quiver flag variety is at most 8. Put these in normal form and include them in $S$. Repeat until there are no remaining elements of $S$ of dimension less than 8 .

In this way we obtain all quiver flag varieties of dimension at most 8 . We then compute the ample cone and anti-canonical bundle for each, and discard any which are not Fano. We find 223044 Fano quiver flag varieties of dimension at most 8; 223017 of dimension $4 \leq d \leq 8$. Of these 50617 (respectively 50612) are non-toric quiver flag varieties. Note that in many cases we find the same variety multiple times: for example, in the table below, there are two quiver flag varieties of dimension 2 and Picard rank 1 given by $\mathbb{P}^{2}=\operatorname{Gr}(3,1)$ and $\mathbb{P}^{2}=\operatorname{Gr}(3,2)$.

|  | $\rho$ |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 1 |  |  |  |  |  |  |  |
| 2 | 2 | 3 |  |  |  |  |  |  |
| 3 | 2 | 8 | 11 |  |  |  |  |  |
| 4 | 3 | 17 | 44 | 48 |  |  |  |  |
| 5 | 2 | 27 | 118 | 262 | 231 |  |  |  |
| 6 | 4 | 41 | 264 | 903 | 1647 | 1202 |  |  |
| 7 | 2 | 54 | 498 | 2484 | 7005 | 10618 | 6541 |  |
| 8 | 4 | 74 | 872 | 5852 | 23268 | 54478 | 69574 | 36880 |

Table 3.1: The number of Fano quiver flag varieties by dimension $d$ and Picard rank $\rho$

Remark 3.1.2. In our codebase we define new Magma intrinsics:

```
QuiverFlagVariety(A,r),
```

which creates a quiver flag variety from an adjacency matrix $A$ and dimension vector $\mathbf{r}$, and

```
QuiverFlagVarietyFano(id),
```

which creates a Fano quiver flag variety, in normal form, from its ID [1, 2]. We assign IDs to Fano quiver flag varieties of dimension at most 8, in the range $\{1 \ldots 223044\}$, by placing them in normal form and then ordering them first by dimension, then by Picard rank, then lexicographically by dimension vector, then lexicographically by the columns of the adjacency matrix. We also define Magma intrinsics NefCone(Q), MoriCone(Q), PicardLattice(Q), and CanonicalClass(Q) that compute the nef cone, Mori cone, Picard lattice and canonical class of a quiver flag variety $Q$, and an intrinsic PeriodSequence $(Q, 1)$ that computes the first $l+1$ terms of the Taylor expansion of the regularised quantum period of $Q$. See $\S 3.5$ for more details.

### 3.2 The class of vector bundles that we consider

We consider all bundles $E$ on a given quiver flag variety that:

- are direct sums of bundles of the form

$$
\begin{equation*}
L \otimes S^{\alpha_{1}}\left(W_{1}\right) \otimes \cdots \otimes S^{\alpha_{\rho}}\left(W_{\rho}\right) \tag{3.1}
\end{equation*}
$$

where each $S^{\alpha_{i}}$ is a non-negative Schur power and $L$ is a nef line bundle; and

- have rank $c$, where $c$ is four less than the dimension of the ambient quiver flag variety.

Remark 2.2.2 shows that non-negative Schur powers $S^{\alpha}\left(W_{i}\right)$ are globally generated, and Proposition 2.4.18 shows that nef line bundles are globally generated. Since the tensor product of globally generated vector bundles is globally generated, the first condition ensures that $E$ is globally generated. In particular, therefore, the zero locus $X$ of a generic section of $E$ is smooth. The second condition ensures that the zero locus $X$, if non-empty, is a fourfold. Global generation also implies that the bundle $E$ is convex, which allows us to compute the quantum period of $X$ as described in §2.5.1.

Consider a summand as in (3.1). We can represent the partition $\alpha_{i}$ as a length $r_{i}$ decreasing sequence of non-negative integers, and write $L=\otimes_{j=1}^{\rho}\left(\operatorname{det} W_{j}\right)^{a_{j}}$ where $a_{j}$ may be negative. Therefore each such summand is determined by a length $\rho$ sequence of generalised partitions: the partition (with possibly negative entries) corresponding to index $i$ is $\alpha_{i}+\left(a_{i}, \ldots, a_{i}\right)$.

Remark 3.2.1. In our codebase we define a new Magma intrinsic

```
QuiverFlagBundle(Q,[A1,...,Ak])
```

which creates a bundle of the above form, on the quiver flag variety $Q$, from a sequence of generalised partitions $(A 1, \ldots, A k)$.

We also define an intrinsic FirstChernClass(E) that computes the first Chern class of such a bundle $E$; intrinsics Degree(E) and EulerNumber(E) that compute the degree and Euler number* of the zero locus $X$ of a generic section of $E$.

Finally, we define intrinsics HilbertCoefficients(E, l) and PeriodSequence(E, 1) that compute the first $l+1$ terms of, respectively, the Hilbert series of $X$ and the Taylor expansion of the regularised quantum period of X. See §3.5.

### 3.3 Classifying quiver flag bundles

In this step, we describe the algorithm for determining all bundles on a given quiver flag variety that determine a smooth four-dimensional Fano quiver flag zero locus. A vector bundle as above is determined by a tuple $(A, \mathbf{r}, P)$, where $A$ is an adjacency matrix, $\mathbf{r}$ is a dimension vector, and $P=\left(P_{1}, \ldots, P_{k}\right)$ is a sequence where each $P_{i}$ is a length- $\rho$ sequence of generalised partitions such that the $j$ th partition in each $P_{i}$ is of length $r_{j}$. Note that we regard the summands (3.1) in our vector bundles as unordered; also, as discussed above, different pairs $(A, \mathbf{r})$ can determine the same quiver flag variety. We therefore say that a tuple $(A, \mathbf{r}, P)$ is in normal form if the pair $(A, \mathbf{r})$ is in normal form, $P$ is in lex order, and under all permutations of the vertices preserving these conditions, the sequence $P$ is lex minimal; we work throughout with tuples in normal form.

Given a Fano quiver flag variety $M(Q, \mathbf{r})$ of dimension $4+c, c \leq 4$, with anti-canonical class $-K_{Q}$ and nef cone $\operatorname{Nef}(Q)$, we search for all bundles $E$ such that

- $E$ is a direct sum of bundles of the form (3.1);

[^2]- $\operatorname{rank}(E)=c$;
- $-K_{Q}-c_{1}(E) \in \operatorname{Amp}(Q)$.

The last condition ensures that the associated quiver flag zero locus $X$, if non-empty, is Fano. We proceed as follows. We first find all possible summands that can occur; that is, all irreducible vector bundles $E$ of the form (3.1) such that $\operatorname{rank}(E) \leq c$ and $-K_{Q}-c_{1}(E) \in \operatorname{Amp}(Q)$. Let $\operatorname{Irr}(Q)$ be the set of all such bundles. Write $\operatorname{Irr}(Q)=\operatorname{Irr}(Q)_{1} \sqcup \operatorname{Irr}(Q)_{2}$, where $\operatorname{Irr}(Q)_{1}$ contains vector bundles of rank strictly larger than 1 , and $\operatorname{Irr}(Q)_{2}$ contains only line bundles. We then search for two vector bundles $E_{1}, E_{2}$ such that $E_{i}$ is a direct sum of bundles from $\operatorname{Irr}(Q)_{i}$ and that $E=E_{1} \oplus E_{2}$ satisfies the conditions above.

For each $x \in \operatorname{Nef}(Q)$ such that $-K_{Q}-x$ is ample, we find all possible ways to write $x$ as

$$
\begin{equation*}
x=\sum_{i=1}^{l} a_{i} \tag{3.2}
\end{equation*}
$$

where the $a_{i}$ are (possibly repeated) elements of a Hilbert basis for $\operatorname{Nef}(Q)$. There are only finitely many decompositions (3.2); finding them efficiently is a knapsacktype problem that has already been solved [16]. For each $\tilde{c} \leq c$ and each partition of the $a_{i}$ into at most $c / 2$ groups $S_{1}, \ldots, S_{s}$, we find all possible choices of $F_{1}, \ldots, F_{s} \in$ $\operatorname{Irr}_{1}$ such that

$$
c_{1}\left(F_{i}\right)=\sum_{j \in S_{i}} a_{j} \quad \operatorname{rank}\left(F_{1}\right)+\cdots+\operatorname{rank}\left(F_{s}\right)=\tilde{c}
$$

Set $E_{1}=F_{1} \oplus \cdots \oplus F_{s}$. Then for each $y \in \operatorname{Nef}(Q)$ such that $-K_{Q}-x-y$ is ample, we again find all ways of writing

$$
y=\sum_{j=1}^{m} b_{j}
$$

as a sum of Hilbert basis elements. Each partition of the $b_{j}$ into $c-\tilde{c}$ groups gives a choice of nef line bundles $L_{1}, \ldots, L_{c-\tilde{c}} \in \operatorname{Irr}_{2}(Q)$, and we set $E_{2}=\oplus L_{j}$.

Remark 3.3.1. Treating the higher rank summands $\operatorname{Irr}_{2}$ and line bundles $\operatorname{Irr}_{1}$ separately here is not logically necessary, but it makes a huge practical difference to the speed of the search.

### 3.4 Classifying quiver flag zero loci

For each of the Fano quiver flag varieties $Q$ of dimension between 4 and 8, found in $\S 3.1$, we use the algorithm described in $\S 3.3$ to find all bundles on $Q$ of the form
described in $\S 3.2$. This produces 10788446 bundles. Each such bundle $E$ determines a quiver flag zero locus $X$ that is either empty or a smooth Fano fourfold. We discard any varieties that are empty or disconnected (first doing a cheap check by Remark 3.4.1 and then computing the degree), and for the remainder compute the first fifteen terms of the Taylor expansion of the regularised quantum period of $X$, using Theorem 2.5.4. For many of the quiver flag zero loci that we find, this computation is extremely expensive (the main factor is the Picard rank of the abelianised quiver, as this determines the size of the cohomology ring of the abelianised quiver flag variety where the computations are done; Gröbner basis calculations that allow the computation of products in this ring become more expensive as the size of the ring grows). In practice, therefore, it is essential to use the equivalences described in §2.3 to replace such quiver flag zero loci by equivalent and more tractable models. The number of equivalence classes is far smaller than the number of quiver flag zero loci that we found, and so this replaces roughly 10 million calculations, many of which are hard, by around half a million calculations, almost all of which are easy. In this way we find 749 period sequences. We record these period sequences, together with the construction, Euler number, and degree for a representative quiver flag zero locus, in Appendix A below. 141 of the period sequences that we find are new. Thus we find at least 141 new four-dimensional Fano manifolds ${ }^{\dagger}$.

Remark 3.4.1. A computationally cheap sufficient condition for a quiver flag zero locus to be empty arises as follows. If $W$ is the tautological quotient bundle on $G r(n, r)$, where $2 r-1>n$, then a generic global section of $\wedge^{2} W$ or $\operatorname{Sym}^{2} W$ has an empty zero locus. Thus if $i$ is a vertex in a quiver $Q$ such that all arrows into $i$ are from the source, and $2 r_{i}-1>n_{0 i}=s_{i}$, then there are no global sections of $\wedge^{2} W_{i}$ or Sym $^{2} W_{i}$ with non-empty zero locus: to see this, apply Proposition 2.2.1 to $Q$.

### 3.5 Cohomological computations for Quiver flag zero loci

In this section we describe how we compute the degree, Euler characteristic, Hilbert coefficients, and Taylor expansion of the regularised quantum period for quiver flag varieties and quiver flag zero loci. This relies on Martin's integration formula [39] and Theorem 2.5.4.

Let $V$ be a smooth projective variety with an action of $G$ on $V$, let $T$ be a maximal torus in $G$, and consider the GIT quotients $V / / G$ and $V / / T$ determined by a character of $G$. Let $\pi: V^{s s}(G) / T \rightarrow V / / G$ be the projection and $i: V^{s s}(G) / T \rightarrow V^{s s}(T) / T=$

[^3]$V / / T$ be the inclusion. Let $W$ be the Weyl group, and $e=\prod_{\lambda \in \operatorname{Roots}(G)} c_{1}\left(L_{\lambda}\right)$, where $L_{\lambda}$ is the line bundle on $V / / T$ associated to the character $\lambda$.

Theorem 3.5.1 (Martin's Integration Formula, [39]). For any $a \in H^{*}(V / / G, \mathbb{C})$ and any $\tilde{a} \in H^{*}(V / / T, \mathbb{C})$ satisfying $\pi^{*}(a)=i^{*}(\tilde{a})$

$$
\int_{V / / G} a=\frac{1}{|W|} \int_{V / / T} \tilde{a} \cup e .
$$

If $a \in H^{*}(V / / G, \mathbb{C})$ and $\tilde{a} \in H^{*}(V / / T, \mathbb{C})$ satisfy $\pi^{*}(a)=i^{*}(\tilde{a})$ then we say that $\tilde{a}$ is a lift of $a$.

In our case the Abelianization $V / / T$ is a smooth toric variety, and the cohomology rings of such varieties, being Stanley-Reisner rings, are easy to work with computationally [9, 48]. For example, we can use this to compute the number of components $h^{0}\left(X, \mathcal{O}_{X}\right)$ of a Fano quiver flag zero locus $X$. By Kodaira vanishing, $h^{0}\left(X, \mathcal{O}_{X}\right)=\chi(X)$, and applying the Hirzebruch-Riemann-Roch theorem gives

$$
\begin{equation*}
\chi\left(\mathcal{O}_{X}\right)=\int_{X} \operatorname{ch}\left(\mathcal{O}_{X}\right) \cup T d\left(T_{X}\right)=\int_{X} T d\left(T_{X}\right) . \tag{3.3}
\end{equation*}
$$

We need to find a lift of the Todd class of $T_{X}$. Writing $T_{X}$ as a K-theoretic quotient of representation theoretic bundles via the Euler sequence, as in the proof of Theorem 2.5.4, gives the lift that we seek; we then use Martin's formula to reduce the integral (3.3) to an integral in the cohomology ring of the Abelianization. The same approach allows us to compute the first two terms $\chi\left(X,-K_{X}\right), \chi\left(X,-2 K_{X}\right)$ of the Hilbert series of $X$ as well as the degree and Euler characteristic of $X$. To compute the first few Taylor coefficients of the quantum period of $X$, we combine this approach with the explicit formula in Theorem 2.5.4.

## 4

## Future directions: Laurent polynomial mirrors for Fano quiver flag zero loci

In the final chapter of this thesis, we discuss mirrors for the quiver flag zero loci found in Appendix A. This is work in progress. First, we briefly describe the conjectures relating Fano varieties and Laurent polynomials, and explain how toric degenerations are expected to play a role. Secondly, using new coordinates on quiver flag varieties, we provide a recipe for finding toric degenerations of quiver flag varieties modelled on the Gonciulea-Lakshmibai toric degeneration [28] of a flag variety. In the third section, we introduce ladder diagrams, a combinatorial picture used to describe the toric degeneration of [28], and generalise them to Y shaped quiver flag varieties. We give a new interpretation of ladder diagrams as quivers. Finally, even beyond the complete intersection case, we use these constructions to find mirrors of some of the new Fano fourfolds which are subvarieties of Y shaped quiver flag varieties. We also present two degenerations of quiver flag varieties beyond this context. Many of the proofs of this section are still work in progress, and so the main focus is on the examples.

### 4.1 Mirror symmetry for Fano varieties

Conjecturally (see, for example [3] and [24]), $n$ dimensional Fano varieties up to deformation should correspond to certain Laurent polynomials in $n$ variables up to a type of equivalence called mutation.

Definition 4.1.1. Let $f$ be a Laurent polynomial in $\mathbb{C}\left[x_{1}^{ \pm}, \ldots, x_{n}^{ \pm}\right]$. The classical period of $f$, denoted $\pi_{t}(f)$ is

$$
\frac{1}{(2 \pi i)^{n}} \int_{\left(S^{1}\right)^{n}} \frac{1}{1-t f} \frac{d x_{1}}{x_{1}} \wedge \cdots \wedge \frac{d x_{n}}{x_{n}} .
$$

Repeated applications of the residue theorem allows one to re-write $\pi_{f}(t)$ as $\sum_{i=0}^{\infty} a_{i} t^{i}$, where $a_{i}$ is the constant term in the expansion of $f^{i}$.

Let $X$ be a smooth Fano variety. A Laurent polynomial $f \in \mathbb{C}\left[x_{1}^{ \pm}, \ldots, x_{n}^{ \pm}\right]$is mirror to $X$ if the quantum period $X$ is equal to $\pi_{f}(t)$, the classical period of $f$.

The class of Laurent polynomials mirror to Fano varieties is conjectured to be rigid maximally mutable Laurent polynomials. To define this class, we first need to define the notion of a mutation. We follow [3]. Mutations are compositions of two types of operations - GL $(n, \mathbb{Z})$ equivalences and certain birational transformations - on a Laurent polynomial $f$. For the first, let $A=\left[a_{i j}\right] \in \operatorname{GL}(n ; \mathbb{Z})$. $A$ defines a $\operatorname{GL}(n, \mathbb{Z})$ equivalence $\phi:\left(\mathbb{C}^{*}\right)^{n} \rightarrow\left(\mathbb{C}^{*}\right)^{n}$ via

$$
\left(x_{1}, \ldots, x_{n}\right) \mapsto\left(\prod_{i=1}^{n} x_{i}^{a_{1 i}}, \ldots, \prod_{i=1}^{n} x_{i}^{a_{n i}}\right)
$$

This defines a new Laurent polynomial $\phi^{*}(f)$. For the second type of map, write

$$
f=\sum_{i} C_{i}\left(x_{1}, \ldots, x_{n-1}\right) x_{n}^{i}
$$

and suppose that $C_{i}$ is divisible by $h^{-i}$ for a fixed Laurent polynomial $h\left(x_{1}, \ldots, x_{n-1}\right)$. This defines a birational transformation $\phi:\left(\mathbb{C}^{*}\right)^{n} \rightarrow\left(\mathbb{C}^{*}\right)^{n}$ via

$$
\left(x_{1}, \ldots, x_{n}\right) \mapsto\left(x_{1}, \ldots, x_{n-1}, h x_{n}\right) .
$$

We obtain a new Laurent polynomial $g$ by pullback where

$$
g=\sum_{i} h^{i} C_{i}\left(x_{1}, \ldots, x_{n-1}\right) x_{n}^{i} .
$$

A one-step mutation is defined to be the composition of a $\operatorname{GL}(n, \mathbb{Z})$ equivalence, a birational transformation of this type, and another $\operatorname{GL}(n, \mathbb{Z})$ equivalence. A mutation is a composition of one-step mutations.

A polytope $P^{\prime}$ is defined to be a mutation of a polytope $P$ if there exists $f, f^{\prime}$ such that $P^{\prime}$ is the Newton polytope of $f^{\prime}, P$ is the Newton polytope of $f$, and $f^{\prime}$ is a mutation of $f$. A mutation of $P$ is said to be compatible with a Laurent polynomial
$f$ if the mutation of $P$ is induced from a mutation of $f$.
Definition 4.1.2. Let $f$ be a Laurent polynomial with Newton polytope P. $f$ is rigid maximally mutable if there is a set of mutations $S$ on $P$ such that $f$ is compatible with all mutations in $S$ and up to scaling, $f$ is the only Laurent polynomial compatible with all mutations in $S$.

A limited amount is known about rigid maximally mutable Laurent polynomials. In dimension 4 , a polytope may support two different such polynomials.

For Fano toric complete intersections in Fano toric varieties subject to some extra, technical conditions (the ability to find a nef partition), there is a well understood method of producing a Laurent polynomial mirror, called the Przyjalkowski method. The paper [17] explains this in detail. However, it isn't known whether this always produces a rigid maximally mutable Laurent polynomial (but there are no known counterexamples). One can formally follow the same method when the toric variety and the toric complete intersection are singular: below, we show examples when this method does not produce a rigid maximally mutable Laurent polynomial. The correct mirror, in this case, is the rigid maximally mutable Laurent polynomial with the same Newton polytope. There is code written by Kasprzyk to find maximally mutable Laurent polynomials.

To understand mirrors for Fano varieties which are not toric complete intersections, toric degenerations are one of the main tools used.

Definition 4.1.3. Let $X$ be a smooth variety. A flat family $\pi: \mathfrak{X} \rightarrow U \subset \mathbb{C}$ is a toric degeneration of $X$ if the generic fiber is $X$ and the special fiber $X_{0}$ of $\pi$ is a toric variety. We also require that the family is $\mathbb{Q}$-Gorenstein.

A toric degeneration is called a small toric degeneration if in addition, $\mathfrak{X}$ is irreducible, $X_{0}$ has at worst Gorenstein terminal singularities, and for any $t$, the map

$$
\operatorname{Pic}(\mathfrak{X} / U) \rightarrow \operatorname{Pic}\left(X_{t}\right)
$$

is an isomorphism.
Batyrev conjectured (see [6]) that if $X$ is a smooth Fano variety with small toric degeneration to some $X_{0}$, then the formal mirror to $X_{0}$ (produced as described above) should be a mirror to $X$. Given a toric complete intersection $X$ such that the Przyjalkowski method produces a mirror $f$, [29] and [17] produce a toric degeneration of $X$ to the toric variety with fan the spanning fan of the Newton polytope of $f$.

Below, we will describe a certain toric degeneration of flag varieties, which was first constructed by [28]. This is a small toric degeneration. If $X$ is a Fano complete
intersection in a flag variety, the degeneration of the ambient space degenerates $X$ to complete intersection in the degenerate toric variety. Applying the Przyjalkowski method to this toric complete intersection, one can find a Laurent polynomial mirror for $X$. We will discuss a generalisation of this degeneration for certain quiver flag varieties, and use it to produce conjectural mirrors to their quiver flag zero loci, going beyond the complete intersection case.

### 4.2 SAGBI basis degenerations of quiver flag varieties

### 4.2.1 A DEGENERATION of A FLAG VARIETY

In this section, we follow Miller and Sturmfels in [40] to present the toric degeneration of the flag variety from [28]. Consider the flag variety $\mathrm{Fl}\left(n ; r_{1}, \ldots, r_{\rho}\right)$. Let $k_{i}:=n-r_{i}$. The Cox ring of the flag variety is generated by the top aligned minors of size $k_{1}, \ldots, k_{\rho}$ of the matrix

$$
\left[\begin{array}{ccc}
x_{11} & \cdots & x_{1 n} \\
\vdots & & \vdots \\
x_{k_{\rho} 1} & \cdots & x_{k_{\rho} n}
\end{array}\right] .
$$

One can also think of these minors as sections of the determinants of the duals of tautological sub-bundles on $\operatorname{Fl}\left(n ; r_{1}, \ldots, r_{\rho}\right)$, and so they define the Plücker map $\mathrm{Fl}\left(n ; r_{1}, \ldots, r_{\rho}\right) \rightarrow \prod_{i=1}^{\rho} \mathbb{P}^{\left(k_{i}\right)-1}$. The variables can be thought of as coordinates on $V$, where $V$ is the vector space such that $V / / G=\operatorname{Fl}\left(n ; r_{1}, \ldots, r_{\rho}\right)$.

Consider the subalgebra $A \subset \mathbb{C}\left[x_{i j}: 1 \leq i \leq r_{1}, 1 \leq j \leq n\right]$ generated by these minors. In [40], they show that the basis given by the minors is a SAGBI basis for $A$ under the monomial order given by the lex ordering on the $x_{i j}$. That is, for any $f$ in the algebra, the initial term of $f$ is a monomial in the initial terms of the basis. A SAGBI basis defines a flat degeneration of the flag variety in $\prod_{i=1}^{\rho} \mathbb{P}^{\left(k_{i}^{n}\right)-1}$ to the toric subvariety defined by the monomials which are the initial terms of the basis elements.

Example 4.2.1. Consider the flag variety $\operatorname{Gr}(4,2)$. Then the matrix above is

$$
\left[\begin{array}{llll}
x_{11} & x_{12} & x_{13} & x_{14} \\
x_{21} & x_{22} & x_{23} & x_{24}
\end{array}\right] .
$$

The algebra is generated by the six $2 \times 2$ minors of this matrix. Their initial terms are

$$
x_{11} x_{22}, x_{11} x_{23}, x_{11} x_{24}, x_{12} x_{23}, x_{12} x_{24}, x_{13} x_{24} .
$$

The toric degeneration of $\operatorname{Gr}(4,2)$ is the closure of the map $\left(\mathbb{C}^{*}\right)^{8} \rightarrow \mathbb{P}^{5}$ defined by these monomials.

### 4.2.2 Coordinates on quiver flag varieties

The first step towards generalising this construction to quiver flag varieties is to choose appropriate coordinates (or equivalently, appropriate line bundles). One option would be to consider the Cox ring of the quiver flag variety. In section 4.5.2, we do this in a particular example. However, generators of the Cox ring of a quiver flag variety are not known in general, so in practice this isn't helpful. As an alternative, we propose specific line bundles coming from the subvariety construction of quiver flag varieties 2.2.1.

Let $M(Q, \mathbf{r})$ be quiver flag variety. Use 2.2 .1 to write $M(Q, \mathbf{r})$ as a subvariety of

$$
Y:=\prod_{i=1}^{\rho} \operatorname{Gr}\left(\tilde{s_{i}}, r_{i}\right), \tilde{s_{i}}=\operatorname{dim} H^{0}\left(W_{i}\right) .
$$

The line bundles required are $\left.\operatorname{det}\left(S_{i}^{*}\right)\right|_{M(Q, \mathbf{r})} i=1, \ldots, \rho ;\left.\operatorname{det}\left(S_{i}^{*}\right)\right|_{M(Q, \mathbf{r})}$ has a basis of sections given by the maximal minors of a $\tilde{s_{i}}-r_{i} \times \tilde{s_{i}}$ matrix. Before writing down the general construction, we do an example

Example 4.2.2. Consider the quiver flag variety $M_{Q}$ given by


This quiver flag variety can be seen as a subvariety of $Y=\operatorname{Gr}(4,2) \times \operatorname{Gr}(5,1) \times$ $\operatorname{Gr}(8,1)$. A point of this space given by a triple $\left(V_{1} \subset \mathbb{C}^{4}, V_{2} \subset \mathbb{C}^{8}, V_{3} \subset \mathbb{C}^{5}\right)$, of dimension 2,7 , and 4 respectively, is in the subvariety $M(Q, \mathbf{r})$ if $\{0\} \oplus V_{1} \subset V_{3}, V_{1} \oplus$ $V_{1} \subset V_{2}$. A basis of sections of the $\left.\operatorname{det}\left(S_{i}^{*}\right)\right|_{M_{Q}}$ are given by the minors of the 3 matrices below. The entries of these matrices should be seen as coordinates on $V=\operatorname{Mat}(2 \times 4) \times \operatorname{Mat}(7 \times 8) \times \operatorname{Mat}(4 \times 5)$ (so that $Y$ is a GIT quotient of $V)$; the form of the matrices comes from the conditions on the $V_{i}$ cutting out $M_{Q}$. The first set of minors are the size 2 minors of

$$
\left[\begin{array}{llll}
x_{11} & x_{12} & x_{13} & x_{14} \\
x_{21} & x_{22} & x_{23} & x_{24}
\end{array}\right]
$$

The second set of minors are the $7 \times 7$ minors of

$$
\left[\begin{array}{cccccccc}
z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & z_{16} & z_{17} & z_{18} \\
z_{21} & z_{22} & z_{23} & z_{24} & z_{25} & z_{26} & z_{27} & z_{28} \\
z_{31} & z_{32} & z_{33} & z_{34} & z_{35} & z_{36} & z_{37} & z_{38} \\
x_{11} & x_{12} & x_{13} & x_{14} & 0 & 0 & 0 & 0 \\
x_{21} & x_{22} & x_{23} & x_{24} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x_{11} & x_{12} & x_{13} & x_{14} \\
0 & 0 & 0 & 0 & x_{21} & x_{22} & x_{23} & x_{24}
\end{array}\right]
$$

and the third set of minors are the $4 \times 4$ minors of

$$
\left[\begin{array}{ccccc}
y_{11} & y_{12} & y_{13} & y_{14} & y_{15} \\
y_{21} & y_{22} & y_{23} & y_{24} & y_{25} \\
0 & x_{11} & x_{12} & x_{13} & x_{14} \\
0 & x_{21} & x_{22} & x_{23} & x_{24}
\end{array}\right]
$$

The general construction proceeds exactly as in the example. That is, a point in $Y$ can be given as $\rho$ vector subspaces $\left(V_{i} \subset \mathbb{C}^{\tilde{s_{i}}}\right)_{i=1}^{\rho}$; this point lies in $M(Q, \mathbf{r})$ iff

$$
\bigoplus_{a \in Q_{1}, s(a) \neq 0, t(a)=i} V_{s(a)} \oplus \mathbb{C}^{n_{0 i}} \subset V_{i} .
$$

This is by Proposition 2.2.1. Recall that $n_{0 i}$ is the number of arrows from 0 to $i$. $Y$ is the GIT quotient of $\prod_{i=1}^{\rho} \operatorname{Mat}\left(\left(\tilde{s}_{i}-r_{i}\right) \times \tilde{s}_{i}\right)$ by $\prod_{i=1}^{\rho} \mathrm{GL}\left(\tilde{s}_{i}-r_{i}\right)$ with stability condition in the positive orthant. From the above, we can see that $M(Q, \mathbf{r})$ is the GIT quotient of the $V^{\prime} \subset V$ intersected with the stable locus. We write down coordinates on $V^{\prime}$ (via the entries of the $\left(M_{i}\right) \in V^{\prime}$ ) using Proposition 2.2.1.

For each $i$, by definition, $M_{i}$ is $\left(\tilde{s}_{i}-r_{i}\right) \times \tilde{s}_{i}$ matrix. Each $a \in Q_{1}, t(a)=i$ corresponds to $r_{s(a)}$ columns; this gives a partition of the columns of $M_{i}$. We can also partition the rows where there is a subset for each $a \in Q_{1}, t(a)=i, s(a) \neq 0$ of size $\tilde{s}_{s(a)}-r_{s(a)}$, and one remaining subset of size $s_{i}-r_{i}$ as

$$
\tilde{s}_{i}-\sum_{a \in Q_{1}, t(a)=i, s(a) \neq 0}\left(\tilde{s}_{s(a)}-r_{s(a)}\right)-r_{i}=s_{i}-r_{i} .
$$

One can see this by considering the conditions which define $M(Q, \mathbf{r})$ in $Y$.
Then for this matrix to describe an element of $V^{\prime}$, the submatrix of $M_{i}$ corresponding to the rows given for $a \in Q_{1}, t(a)=i, s(a) \neq 0$ is 0 except for the sub-submatrix of corresponding to the columns determined by $a$ : this sub-submatrix is $M_{s(a)}$.

In this way we get coordinates on $Y^{\prime}$, and we see that the sections of $\left.\operatorname{det}\left(S_{i}^{*}\right)\right|_{M(Q, \mathbf{r})}$ are spanned by the minors of $M_{i}$. They define an embedding $M(Q, \mathbf{r}) \rightarrow \prod_{i=1}^{\rho} \mathbb{P}^{\left(\mathbb{s}_{i}\right)-1}$. If for some monomial order they are a SAGBI basis, we would obtain a toric degeneration of $M(Q, \mathbf{r})$.

### 4.2.3 Toric degenerations of Y-Shaped quivers

There is a class of quiver for which I conjecture that the minors defined in the previous section form a SAGBI basis (the details of the proof are still work in progress). We call these $Y$ shaped quivers, and they are characterised as follows. If $\{0, \ldots, \rho\}$ is a labelling of the vertices such that $n_{i j}=0$ if $i \geq j$, then vertex 1 can have at most 2 arrows out of it, and all other non-source vertices can have at most one arrow out. For any vertex $i$, there is at most one arrow $a$ with $t(a)=i$ and $s(a) \neq 0$. We can assume that for all $j>1$, there is a path $1 \rightarrow j$, as otherwise the associated quiver flag variety is a product of two Y-shaped quiver flag varieties. The quiver in Example 4.2.2 is not a Y-shaped quiver because of the double arrow.

Example 4.2.3. The following is an example of a $Y$-shaped quiver:


By definition, there are at most 2 arrows out of vertex 1 in a Y-shaped quiver. If there are two, call them $a_{1}$ and $a_{2}$, and define a partition $S_{1} \sqcup S_{2}=\{1, \ldots, \rho\}$ by $i \in S_{2}$ if $i=1$ or the path from $1 \rightarrow i$ contains $a_{2}$. Essentially, we just subdivide the two branches of the quiver.

Let $Q$ be a Y-shaped quiver. Let $\left(A_{i}\right)_{i=1}^{\rho}$ be the $\rho$ matrices with coordinate entries defined by the previous subsection for a Y-shaped quiver $Q$. For each $i$, there are $\left(s_{i}-r_{i}\right)\left(\tilde{s_{i}}\right)$ new variables appearing in $A_{i}$ as the entries of $s_{i}-r_{i}$ rows. We use the partition $S_{1} \sqcup S_{2}$ to define an order on these coordinates: variables introduced in $A_{i}$ take priority over variables introduced in $A_{j}$ if $i \in S_{2}$ and $j \in S_{1}$.

More formally, denote the variables $x_{j, k}^{(i)}, 1 \leq j \leq s_{i}-r_{i}, 1 \leq k \leq \tilde{s}_{i}$. We now define an order on the $x_{j k}^{(i)}$ for all $i$. For a given $i$, we define

$$
x_{11}^{(i)}>x_{12}^{(i)}>\cdots>x_{\left(s_{i}-r_{i}\right) \tilde{s}_{i}}^{(i)} .
$$

Secondly, if $i_{1} \in S_{1}, i_{2} \in S_{2}$ then $x_{j k}^{\left(i_{1}\right)}<x_{l m}^{\left(i_{2}\right)}$. If $i_{1}, i_{2} \in S_{2}$, and $i_{2}>i_{1}$, then $x_{j k}^{\left(i_{1}\right)}<x_{j k}^{\left(i_{2}\right)}$. If $i_{1}, i_{2} \in S_{1}$, and $i_{2}>i_{1}$, then $x_{j k}^{\left(i_{1}\right)}>x_{j k}^{\left(i_{2}\right)}$.

Conjecture 4.2.4. The $k_{i} \times k_{i}$ minors of the $A_{i}$ for all $i$ form a SAGBI basis under the above defined order.

To prove this, it should be possible to simply follow [40], which proves the statement for flag varieties (which are an example of a $Y$ shaped quiver).

### 4.3 LADDER DIAGRAMS FOR CERTAIN DEGENERATIONS

In [7], the authors Batyrev, Ciocan-Fontanine, Kim, and van Straten use ladder diagrams to give a concrete description of the toric variety to which the flag variety degenerates. In this section, we give a new description of the degenerate toric variety by considering the ladder diagram as a quiver. We then generalise this construction to the degenerations of the Y-shaped quiver described in the previous section. We then explain the importance of this description in finding mirrors to quiver flag zero loci.

For a general definition of a ladder diagram of a flag variety, see Definition 2.1.1 in [7]. It can also be described as follows: the ladder diagram of $\operatorname{Gr}(n, r)$ is an $n-r \times r$ grid of unit squares such that the bottom left corner is at $(0,0)$. Let $O$ denote this vertex. For example, the ladder diagram of $\operatorname{Gr}(5,2)$ is

where $O$ is marked. The ladder diagram of $\operatorname{Fl}\left(n, r_{1}, \ldots, r_{\rho}\right)$ is the union of the ladder diagrams of $\operatorname{Gr}\left(n, r_{i}\right)$ for all $i$ : for example, the ladder diagram of $\operatorname{Fl}(5,3,2,1)$ is


The authors in [7] associate to the ladder diagram another graph, and then describe the polytope of the degeneration of the flag variety given above by paths in this graph. Instead, we associate to the ladder diagram a quiver.

The first step is to add more vertices to the ladder diagram. For $\operatorname{Gr}(n, r)$, add vertices at $(i, j)$ for $1 \leq j<r, 1 \leq i<n-r$ and at $(n-r, r)$. So for $\operatorname{Gr}(5,2)$, the new
diagram is


For a flag variety, the new diagram is again the union of the diagrams for each $\operatorname{Gr}\left(n, r_{i}\right)$, with an extra vertex at $\left(n_{i-1}-r_{i-1}, r_{i}\right)$ for each $i>1$. So the ladder diagram for $\mathrm{Fl}(5,3,2,1)$ is


To make this a quiver, we consider paths between vertices where one is allowed to travel up and to the right only. We call such a path primitive if it doesn't pass through any vertices other than its source and target. We define the ladder quiver to be the quiver where the vertices are the vertices in the ladder diagram, and the number of arrows between two vertices is the number of primitive paths in the ladder diagram between them. This quiver is denoted $Q=L\left(n, r_{1}, \ldots, r_{\rho}\right)$. This defines a quiver flag variety $M(Q, \mathbf{1})$ which is a GIT quotient $V / /_{\theta} T$. We define the toric variety $X\left(n, r_{1}, \ldots, r_{\rho}\right)$ to be the GIT quotient $V / /_{\kappa} T$ where $\kappa$ is the canonical stability condition.

For example, the quiver with dimension vector associated to $\operatorname{Gr}(5,2)$ is


Theorem 4.3.1. The degenerate toric variety described by [28] and in the previous section is $X\left(n, r_{1}, \ldots, r_{\rho}\right)$.

Proof. Sketch. In [7] they describe the rays of the fan of the Fano toric variety of [28]. It suffices to check that the cokernel of the ray map is precisely given by the transpose of the weight matrix of the toric variety described above, as the higher dimensional cones are determined by the fact that the toric variety is Fano.

We now define ladder quivers for Y-shaped quivers. First, suppose $Q$ is a Y-shaped quiver such that there is only one arrow out of vertex 1 . An example of such a quiver is


Build the ladder diagram almost exactly as for the flag case: it is the union of the ladder diagrams of $\operatorname{Gr}\left(\tilde{s}_{i}, r_{i}\right)$ for $i=1, \ldots, \rho$; however, we truncate this diagram by insisting that the maximum height of the diagram between $x=0$ and $\tilde{s}_{i}-r_{i}$ is at most $r_{i}$. We add vertices as in the flag case: at interior points and at the intersection points $\left(\tilde{s}_{i-1}-r_{i-1}, r_{i}\right)$ for each $i>1$. The ladder diagram of the above example is then:


One notices that the corresponding toric variety has the correct dimension.
We can now describe the proposed ladder diagram of a general $Y$ shaped quiver $Q$. Assume that there are exactly two arrows out of vertex 1 . Recall that we have partitioned the non-source vertices $\{1, \ldots, \rho\}=S_{1} \sqcup S_{2}$ according to which of the two branches of the quiver the vertex is on, and we assume $1 \in S_{2}$. Consider the subquiver of $Q$ with vertices $S_{1} \cup\{0,1\}$ : this is a Y-shaped quiver for which we know how to build a ladder diagram. Take this ladder diagram, and reflect it across the line $y=-x$, and then translate it so that what was the origin is at $\left(\tilde{s}_{i}-r_{i}, r_{i}\right)$. The ladder diagram of $Q$ is the union of this ladder diagram with the ladder diagram of the second subquiver with vertices $S_{2} \cup\{0\}$.

Example 4.3.2. We draw the ladder diagram for


Set $S_{1}:=\{2\}, S_{2}:=\{1,3,4\}$. The reflected diagram of the quiver with vertices $\{0,1,2\}$
and the ladder diagram of the quiver with vertices $\{0,1,3,4\}$ are pictured below:


The ladder diagram for the entire quiver is


As in the flag case, we can define a toric variety using the ladder diagram of a Yshaped quiver, by interpreting the ladder diagram as a quiver (now without a unique source). I conjecture that this ladder diagram is indeed the toric degeneration of the quiver flag variety from the previous section. I have checked this in many examples. The details of the proof are still a work in progress, but, roughly, this should involve using certain globally generated line bundles (one for each $i \in Q_{0}$ ) to embed the associated toric variety of the ladder quiver into $\prod_{i=1}^{\rho} \mathbb{P}^{\left(\frac{\tilde{s}_{i}}{k_{i}}\right)-1}$, and then check that the image is precisely that of the SAGBI basis degeneration.

There is a natural identification between the sections of the $L_{i}$ and initial terms of the monomials of the $i^{\text {th }}$ matrix from 4.2.2. We illustrate this for the quiver flag variety in Example 4.3.2. The matrices which give the minors "defining" the embedding of the quiver flag variety in $\prod_{i=1}^{\rho} \mathbb{P}^{\left(\bar{s}_{k_{i}}^{s_{i}}\right)-1}$ are

$$
\begin{gathered}
A_{1}=\left[\begin{array}{ccccc}
x_{11}^{(1)} & x_{12}^{(1)} & x_{13}^{(1)} & x_{14}^{(1)} & x_{15}^{(1)} \\
x_{21}^{(1)} & x_{22}^{(1)} & x_{23}^{(1)} & x_{24}^{(1)} & x_{25}^{(1)}
\end{array}\right], \quad A_{2}=\left[\begin{array}{ccccc}
x_{11}^{(1)} & x_{12}^{(1)} & x_{13}^{(1)} & x_{14}^{(1)} & x_{15}^{(1)} \\
x_{21}^{(1)} & x_{22}^{(1)} & x_{23}^{(1)} & x_{24}^{(1)} & x_{25}^{(1)} \\
x_{11}^{(2)} & x_{12}^{(2)} & x_{13}^{(2)} & x_{14}^{(2)} & x_{15}^{(2)}
\end{array}\right], \\
A_{3}= \\
{\left[\begin{array}{ccccccc}
x_{11}^{(3)} & x_{12}^{(3)} & x_{13}^{(3)} & x_{14}^{(3)} & x_{15}^{(3)} & x_{16}^{(3)} & x_{17}^{(3)} \\
0 & 0 & x_{11}^{(1)} & x_{12}^{(1)} & x_{13}^{(1)} & x_{14}^{(1)} & x_{15}^{(1)} \\
0 & 0 & x_{21}^{(1)} & x_{22}^{(1)} & x_{23}^{(1)} & x_{24}^{(1)} & x_{25}^{(1)}
\end{array}\right],}
\end{gathered}
$$

$$
A_{4}=\left[\begin{array}{cccccccc}
x_{11}^{(4)} & x_{12}^{(4)} & x_{13}^{(4)} & x_{14}^{(4)} & x_{15}^{(4)} & x_{16}^{(4)} & x_{17}^{(4)} & x_{18}^{(4)} \\
x_{21}^{(4)} & x_{22}^{(4)} & x_{23}^{(4)} & x_{24}^{(4)} & x_{25}^{(4)} & x_{26}^{(4)} & x_{27}^{(4)} & x_{28}^{(4)} \\
x_{31}^{(4)} & x_{32}^{(4)} & x_{33}^{(4)} & x_{34}^{(4)} & x_{35}^{(4)} & x_{36}^{(4)} & x_{37}^{(4)} & x_{38}^{(4)} \\
x_{41}^{(4)} & x_{42}^{(4)} & x_{43}^{(4)} & x_{44}^{(4)} & x_{45}^{(4)} & x_{46}^{(4)} & x_{47}^{(4)} & x_{48}^{(4)} \\
0 & x_{11}^{(3)} & x_{12}^{(3)} & x_{13}^{(3)} & x_{14}^{(3)} & x_{15}^{(3)} & x_{16}^{(3)} & x_{17}^{(3)} \\
0 & 0 & 0 & x_{11}^{(1)} & x_{12}^{(1)} & x_{13}^{(1)} & x_{14}^{(1)} & x_{15}^{(1)} \\
0 & 0 & 0 & x_{21}^{(1)} & x_{22}^{(1)} & x_{23}^{(1)} & x_{24}^{(1)} & x_{25}^{(1)}
\end{array}\right]
$$

The identification is given by labelling the ladder diagram of the quiver as follows:


For example, sections of $L_{3}$ correspond to paths from $(0,0)$ (the blue vertex) to $(3,4)$. For each such path, we identify it with the monomial which is the product of all the variables in the path. So, for example, the path marked in red below corresponds to $x_{13}^{(3)} x_{12}^{(1)} x_{23}^{(1)}$, which is the initial term of the minor of $A_{3}$ given by the choice of columns 3, 4, 5.


To prove that this ladder diagram is the degeneration given by 4.2 .4 would require that this identification provides an isomorphism of between the cones defined by the monomials corresponding to the sections of $L_{i}$ and the monomials of the initial terms of the minors of $A_{i}$.

### 4.4 Mirrors of quiver flag zero loci

The usefulness of ladder diagrams becomes clear when we start trying to find mirrors of quiver flag zero loci. Let $L(Q)$ be a ladder diagram for some Y-shaped quiver $Q$, and let $X$ be the associated toric variety (to which, at least conjecturally, $M(Q, \mathbf{r})$ degenerates). The paths in $L(Q)$ between vertices are associated with Weil divisors in $X$, by first associating them to a monomial in the Cox ring of $X$. One might hope that a quiver flag zero locus in $Q$ degenerates to a toric complete intersection in $X$ to which we can apply the Przyjalkowski method.

We explain how to associate to a quiver flag zero locus in a Y-shaped quiver the weights and divisor data necessary for the Przyjalkowski method (not necessarily admitting a nef partition). For complete intersections in flag varieties, this was first noted in [7]. Given a flag variety $\mathrm{Fl}\left(n, r_{1}, \ldots, r_{\rho}\right)$ and $Z$ a complete intersection quiver flag zero locus, $Z$ degenerates to a complete intersection in $X\left(n, r_{1}, \ldots, r_{\rho}\right)$. The line bundle $\operatorname{det}\left(Q_{i}\right)$ on the flag variety corresponds to the rank one reflexive sheaf arising from the Weil divisor corresponding to the paths from $(0,0)$ to $\left(n_{i}-\right.$ $r_{i}, r_{i}$ ) on the ladder diagram. In [7], they prove that this path gives in fact a Cartier divisor, and we call the associated line bundle $L_{i}$. To deal with general zero loci, the ideal situation would be to find $r_{i}$ line bundles (or at least rank one reflexive sheaves) on $X\left(n, r_{1}, \ldots, r_{\rho}\right)$ whose direct sum corresponds to $Q_{i}$. If they exist, their tensor product would equal $L_{i}$. There are multiple choices of $r_{i}$ such rank one reflexive sheaves on $X\left(n, r_{1}, \ldots, r_{\rho}\right)$ whose tensor product corresponds to $L_{i}$. For example, consider the three paths in the ladder quiver of $\operatorname{Gr}(6,3)$ :


In general, though, these are not Cartier divisors. In many examples, this works to find mirrors of quiver flag zero loci, by using these $r$ Weil divisors to choose Weil divisors corresponding to Schur powers of $Q$. We illustrate the full method below in an example of a quiver flag zero locus for which a Laurent polynomial mirror previously wasn't known.

Example 4.4.1. Consider the quiver flag zero loci on $\operatorname{Gr}(8,6)$ with bundles

$$
\wedge^{5} Q \oplus \operatorname{det}(Q) \oplus \operatorname{det}(Q)
$$

The summand $\wedge^{5} Q$ is a rank 6 bundle. Suppose $L_{1}, \ldots, L_{6}$ are the 6 rank one reflexive
sheaves on $X(8,6)$ corresponding to $Q$ (there's only one choice here). By considering $\wedge^{5}\left(L_{1} \oplus \cdots \oplus L_{6}\right)$, it is clear that the six rank one reflexive sheaves corresponding to $\wedge^{5} Q$ on $X(8,6)$ are given by the following six paths on the ladder diagram


The ladder diagram determines a weight matrix for $X(8,6)$ :

$$
\left[\begin{array}{cccccccccccccccccc}
1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right] .
$$

The weights for the rank one reflexive sheaves are given by

$$
\left[\begin{array}{cccccccc}
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1
\end{array}\right] .
$$

The ladder diagrams give a way of picking a nef partition for these bundles. One can formally follow the Przyjalkowski method (see [177) to produce a Laurent polynomial. This Laurent polynomial is not rigid maximally mutable, but it is also does not have the correct period sequence. There is a unique rigid maximally mutable Laurent polynomial on the Newton polytope of this polynomial. It has the correct classical
period, up to 10 terms. The Laurent polynomial is

$$
\begin{aligned}
& x y z w+x y z+x y w+2 x y+x y / w+x z w+x z \\
& +x w+2 x+x / w+x / z+x /(z w)+y z w+y z+y w \\
& +2 y+y / w+z w+z+w+2 / w+2 / z+2 /(z w)+1 / y \\
& +1 /(y w)+1 /(y z)+1 /(y z w)+1 / x+1 /(x w)+1 /(x z)+1 /(x z w)+1 /(x y) \\
& +1 /(x y w)+1 /(x y z)+1 /(x y z w) .
\end{aligned}
$$

We can generalise the above discussion to the ladder diagrams of Y-shaped quivers, because they are built out of the ladder diagrams of Grassmannian factors.

Example 4.4.2. Consider the quiver flag zero locus given by the quiver

with bundles $W_{1} \otimes W_{2}$. This Fano variety has PID 115 (see the tables in Appendix A). The paths on the ladder diagram which give the divisors suggested by the above method is


Again, to find a mirror with the correct period sequence, one must find a rigid maximally mutable Laurent polynomial supported on the resulting Newton polytope. This is

$$
x+y w+y+z+w+1 / x+1 /(x w)+1 /(x z)+z /(x y w)+1 /(x y)+1 /(x y w)+1 /(x y z)
$$

Example 4.4.3. Consider the quiver flag zero locus given by the quiver flag variety

and bundle $W_{1} \oplus W_{2}$. It corresponds to PID 20. The toric degeneration is given by the following ladder diagram:


The mirror produced is

$$
x+y+z+w+z / y+1 /(y w)+w / x+1 /(x z) .
$$

Example 4.4.4. Consider the quiver flag zero locus with PID 232 given by the quiver flag variety

and the quiver flag bundles $\operatorname{det}\left(W_{1}\right) \oplus W_{1} \otimes W_{3}$. The ladder diagram and paths given by the prescribed method are


The mirror is

$$
\begin{array}{r}
x+y+z+w+w / y+1 / y+1 / x+1 /(x w) \\
+1 /(x z)+1 /(x z w)+2 /(x y)+1 /(x y w)+1 /(x y z)+1 /(x y z w)+1 /\left(x^{2} z w\right) \\
+1 /\left(x^{2} y w\right)+2 /\left(x^{2} y z w\right)+1 /\left(x^{2} y z w^{2}\right)+1 /\left(x^{3} y z w^{2}\right) .
\end{array}
$$

In total, of the approximately forty quiver flag zero loci I have attempted to find mirrors for (from the tables in Appendix A), I have been successful in about thirty. There are two sources of failures. The first source is when there is no nef partition supporting the choice of divisors. The next example is an example of this; in this case (but not usually), one is able to find a degeneration of the complete intersection to a toric variety; we find the Laurent polynomial associated to this toric variety and after taking the rigid maximally mutable Laurent polynomial on its polytope, find a mirror.

Example 4.4.5 (PID 104). Consider the quiver flag variety obtained from the quiver flag variety with PID 104 in the tables by grafting:


The quiver flag zero locus is given by bundles $W_{1} \otimes W_{4} \oplus W_{2} \otimes W_{4}$. The toric degeneration is given by the product of the 3 toric varieties given by the three ladder diagrams below:


Notice that there is no nef partition which will give these bundles, because there is no choice of basis of divisors in the first ladder diagram such that all chosen divisors are in the positive span. We instead construct a toric degeneration, using similar ideas to that of [29]. Suppose the fan of the toric variety is in the lattice $N_{\mathbb{R}}$. I find $v_{1}, \ldots, v_{4} \in\left(N^{\vee}\right)_{\mathbb{R}}$ such that they define binomial sections of the four rank one reflexive sheaves; the associated toric subvariety has the following Laurent polynomial mirror, with matching period sequence.
$x+y+z+w+y /(x w)+1 / x+1 /(x w)+w /(x z)+1 /(x z)+1 /(x y)+w /(x y z)+1 /(x y z)$

In other examples, I am unable to find a mirror because the degeneration has too low Picard rank. Consider the quiver flag variety which appeared in one of the factors in the previous example:

with toric degeneration given by the ladder diagram


Notice that from the quiver flag variety, we would expect to have a class group of
rank at least 3 (generated by two Weil divisors coming from $W_{2}$ and one Cartier divisor from $W_{1}$ ), but the toric degeneration (which is in this case smooth: it is $\left.\mathbb{P}_{\mathbb{P}^{2}}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1))\right)$ has rank only 2 . In the previous example, we are still able to find a mirror, because the bundles only involve $W_{2}$. However, in the case of PID 15 , where the quiver flag variety is

and the bundles are $W_{1} \otimes W_{2}$, this method fails to produce a mirror. The ladder diagram is


Writing the degeneration of $W_{2}$ to $L_{1} \oplus L_{2}$ and that of $W_{1}$ and $L_{3}$ we see that $L_{1}$ and $L_{2}$ are the same sheaf.

### 4.5 Degenerations beyond Y-shaped quivers

To find mirrors of quiver flag zero loci more generally, we need to find good degenerations of quiver flag varieties beyond $Y$ shaped quivers. In the last section of this chapter and thesis, we give two examples of such degenerations. The first example is a SAGBI basis degeneration of the sections of the $\operatorname{det}\left(S_{i}^{*}\right)$ (see 4.2.2) which cannot be represented as a ladder diagram: instead it is represented by what one might call a bound ladder diagram. Bound quivers were used in [20] to produce certain subvarieties of toric quiver flag varieties. For the second example, no monomial order on the sections of the $\operatorname{det}\left(S_{i}^{*}\right)$ that I could find produced a good degeneration of the quiver flag variety. However, I produce a degeneration using the entire Cox ring.

### 4.5.1 A DEgEneration of a quiver flag variety with a double arrow

Consider the quiver flag variety $M_{Q}$


The coordinates on $M_{Q}$ given by $\S 4.2 .2$ are the maximal minors of the matrices

$$
\left[\begin{array}{llll}
x_{11} & x_{12} & x_{13} & x_{14} \\
x_{21} & x_{22} & x_{23} & x_{24}
\end{array}\right],\left[\begin{array}{cccccccc}
x_{11} & x_{12} & x_{13} & x_{14} & 0 & 0 & 0 & 0 \\
x_{21} & x_{22} & x_{23} & x_{24} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x_{11} & x_{12} & x_{13} & x_{14} \\
0 & 0 & 0 & 0 & x_{21} & x_{22} & x_{23} & x_{24} \\
z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & z_{16} & z_{17} & z_{18} \\
z_{21} & z_{22} & z_{23} & z_{24} & z_{25} & z_{26} & z_{27} & z_{28} \\
z_{31} & z_{32} & z_{33} & z_{34} & z_{35} & z_{36} & z_{37} & z_{38}
\end{array}\right] .
$$

These minors define $M_{Q}$ as a subvariety of $\mathbb{P}^{5} \times \mathbb{P}^{6}$. Define a monomial ordering induced by

$$
z_{1 i_{1}}>x_{1 i_{2}}>z_{2 i_{3}}>x_{2 i_{4}}>z_{3 i_{5}}
$$

and the lex ordering within the rows. This defines a toric degeneration of $M_{Q}$; the associated Laurent polynomial has correct period sequence. One can also describe the degeneration as a subvariety of the ladder diagram


Notice that the ladder diagram is the ladder diagram for the quiver flag variety

$M_{Q}$ is a subvariety of this quiver flag variety cut out by a section of $S_{1}^{*} \otimes W_{2}$.


To describe the subvariety of the ladder diagram, recall that each arrow in the corresponding ladder quiver determines a variable in the Cox ring of the toric variety. We label the vertices in the ladder diagram by their Cartesian coordinates, so that the source is at $(0,0)$. We draw the relevant arrows on the diagram below, and label
them in the text for further clarity.


Label three of the paths from $(0,0)$ to $(7,1)$ with variables $x_{1}, x_{2}, x_{3}$ (these arrows are marked in yellow on the above diagram). Label the arrow from from $(0,1)$ to $(2,2)$ as $y_{1}$ (in red), the arrow from $(1,1)$ to $(2,2)$ as $y_{2}$ (in green), the arrow from $(0,1)$ to $(1,1)$ as $y_{3}$ (in blue), the arrow from $(1,1)$ to $(2,1)$ as $y_{4}$ (in orange), and the arrow from $(2,1)$ to $(2,2)$ as $y_{5}$ (in violet). Then the ideal determining the toric variety is given by the binomial relations

$$
\left(x_{1} y_{2} y_{3}-y_{1} x_{2}, x_{1} y_{3} y_{4} y_{5}-y_{1} x_{3}, x_{2} y_{3} y_{4} y_{5}-x_{3} y_{2} y_{3}\right)
$$

In other words, this identifies the right most two boxes with the uppermost two boxes.

The quiver flag zero locus $X$ given by $M_{Q}$ and the bundle $W_{2}^{\oplus 3}$ has period sequence PID 29. Pulling back the divisors indicated by choosing three distinct paths from $(0,0)$ to $(7,1)$ in the ladder diagram result in the following candidate Laurent polynomial mirror with matching period sequence (up to ten terms) to $X$ :

$$
x+y+z+w+w / z+1 /(y z)+z /(x w)+1 /(x w)+1 /(x y)+1 /(x y z) .
$$

### 4.5.2 A Cox ring degeneration

Consider the quiver flag variety $M_{Q}$ given by the quiver


None of the monomial orders I tried on the coordinates given by $\S 4.2 .2$ produced a degeneration with a correct period sequence. Instead, one can use the entire Cox ring (or rather, what I conjecture to be the entire Cox ring). From Proposition 2.4.8,
we can see that the effective cone in $\mathbb{R}^{4}$ of $M_{Q}$ is contained in the cone generated by

$$
\begin{aligned}
& e_{i}: i \in\{1, \ldots, 4\},[-1,1,1,0],[-1,1,1,0],[-1,1,0,1], \\
& {[-1,2,0,0],[-1,0,2,0],[-1,0,0,2],}
\end{aligned}
$$

as this is the Weyl invariant part of the effective cone of the abelianisation of $M_{Q}$. In fact, the last three bundles have no global sections (one can also check directly using the GIT characterisations that they are not in the effective cone). All the others do have sections. Write coordinates on $\operatorname{Rep}(Q, \mathbf{r})$ as

$$
\left[\begin{array}{llll}
x_{11} & x_{12} & x_{13} & x_{14}  \tag{4.1}\\
x_{21} & x_{22} & x_{23} & x_{24}
\end{array}\right], \quad\left[\begin{array}{ll}
a_{1} & a_{2}
\end{array}\right], \quad\left[\begin{array}{ll}
b_{1} & b_{2}
\end{array}\right], \quad\left[\begin{array}{ll}
c_{1} & c_{2}
\end{array}\right] .
$$

For example, the global sections of $\operatorname{det}\left(W_{1}\right)^{*} \otimes W_{2} \otimes W_{3}$ (corresponding to weight $[-1,1,1,0])$ are generated by

$$
\operatorname{det}\left(\left[\begin{array}{ll}
a_{1} & a_{2} \\
b_{1} & b_{2}
\end{array}\right]\right)
$$

The global sections of $W_{2}$ are given by the $1 \times 1$ minors of

$$
\left[\begin{array}{llll}
a_{1} x_{11}+a_{2} x_{21} & a_{1} x_{12}+a_{2} x_{22} & a_{1} x_{13}+a_{2} x_{23} & a_{1} x_{14}+a_{2} x_{24}
\end{array}\right] .
$$

I conjecture that the sections of all of these line bundles generate the Cox ring. If this is the case, by [30, Proposition 2.11], these sections define a map from $M_{Q}$ to the toric variety $Y$ given with weights

$$
\left[\begin{array}{ccccccccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}\right]
$$

and stability condition in the positive orthant. These weights are given by the six sections of $\operatorname{det}\left(W_{1}\right)$, the single section of each of $\operatorname{det}\left(W_{1}\right)^{*} \otimes W_{2} \otimes W_{3}, \operatorname{det}\left(W_{1}\right)^{*} \otimes W_{2} \otimes$ $W_{4}$, and $\operatorname{det}\left(W_{1}\right)^{*} \otimes W_{3} \otimes W_{4}$, and of the four sections each of $W_{2}, W_{3}, W_{4}$. Choosing a monomial order of the variables given by their appearance in (4.1) produces a degeneration of $M_{Q}$ in $Y$ to the image of the monomials

$$
\begin{aligned}
& x_{11} x_{22}, x_{11} x_{23}, x_{11} x_{24}, x_{12} x_{23}, x_{12} x_{24}, x_{13} x_{24}, \\
& a_{1} b_{2}, a_{1} c_{2}, b_{1} c_{2}, a_{1} x_{1 i}, b_{1} x_{1 i}, c_{1} x_{1 i}, i \in\{1, \ldots, 4\} .
\end{aligned}
$$

Using Mathematica one can find the binomial ideal defining the subvariety of the degeneration. This subvariety is the Fano toric variety $X$ with weights

$$
\left[\begin{array}{lllllllllllll}
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 2 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{array}\right] .
$$

Using these weights, one can follow the Przyjalkowski method to find a Laurent polynomial. This Laurent polynomial is not rigid maximally mutable. However, its Newton polytope supports a unique rigid maximally mutable Laurent polynomial: it has the same period sequence as $M_{Q}$, up to 10 terms. This is the only example I know where the Przyjalkowski method for a toric variety (not a complete intersection) fails to produce a maximally mutable Laurent polynomial.

The rank one reflexive sheaves corresponding to the weights

$$
\left[\begin{array}{l}
0 \\
1 \\
0 \\
0
\end{array}\right], \quad\left[\begin{array}{l}
0 \\
0 \\
1 \\
0
\end{array}\right], \quad\left[\begin{array}{l}
0 \\
0 \\
0 \\
1
\end{array}\right]
$$

pullback to $W_{2}, W_{3}$ and $W_{4}$ on $M_{Q}$ by construction. However, on $X, W_{2}$ and $W_{3}$ pullback to the same rank one reflexive sheaves. This is similar to the situation for PID 15 described above. As a result, this toric degeneration cannot be used to find a mirror for the quiver flag zero locus $W_{2} \oplus W_{3} \oplus W_{4}$ (PID 26), but only that of the quiver flag zero locus given by $W_{4}$.

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## A

## Regularized quantum periods for quiver flag zero loci

## A.0.3 The TABLE OF REPRESENTATIVES

As described in Chapter 3, we divided the 4-dimensional quiver flag zero loci $X$ that we found into 749 buckets, according to the first 15 terms of the Taylor expansion of the regularised quantum period of $X$. We refer to these Taylor coefficients as the period sequence. Table 1 below gives, for each of the 749 period sequence buckets, a representative quiver flag zero locus $X$ as well as the degree and Euler number of $X$. (In some cases we do not know that all the quiver flag zero loci in a bucket are isomorphic, but we checked that they all have the same degree, Euler number, and Hilbert series.) The quiver flag zero locus $X$ is represented by the adjacency matrix and dimension vector of its ambient quiver flag variety $Y=M(Q, \mathbf{r})$, together with the sequence of generalised partitions that determine a vector bundle $E \rightarrow Y$ such that $X$ is the zero locus of a generic section of $E$. The generalised partitions are written as Young diagrams, with:

- $\varnothing$ representing the empty Young diagram;
- a filled Young diagram, such as $\boldsymbol{\square}$, representing the dual to the vector bundle represented by the unfilled Young diagram $\square$.

Filled Young diagrams that occur always represent line bundles.

The entries in Table 1 give representatives of each period sequence bucket that are chosen so as to make the computation of geometric data (the period sequence etc.) straightforward*. Even though the Table is constructed by considering all four-dimensional Fano manifolds that occur as quiver flag zero loci in codimension up to four, in all four cases there is no tractable representative as a quiver flag zero locus of low codimension. In these cases the Table contains a representative as a quiver flag zero locus in higher codimension; the reader who prefers models in lower-dimensional ambient spaces should consult Table A.1.


Table A.1: Representatives for certain Period IDs in codimension at most four

## A.0.4 The table of period sequences

Table 2 records the first 8 terms of the period sequence, $\alpha_{0}, \alpha_{1}, \ldots, \alpha_{7}$, for each of the 749 period sequence buckets. It also records, where they exist, the names of known four-dimensional Fano manifolds which have the same first fifteen terms of the period sequence. Notation is as follows:

- $\mathbb{P}^{n}$ denotes $n$-dimensional complex projective space;
- $Q^{n}$ denotes a quadric hypersurface in $\mathbb{P}^{n+1}$;
- $\mathrm{FI}_{k}^{4}$ is the $k$ th four-dimensional Fano manifold of index 3 , as in [14, §5];
- $V_{k}^{4}$ is the $k$ th four-dimensional Fano manifold of index 2 and Picard rank 1, as in [14, §6.1];

[^4]- $\mathrm{MW}_{k}^{4}$ is the $k$ th four-dimensional Fano manifold of index 2 and Picard rank at least 2 , as in $[14, \S 6.2]$;
- $\mathrm{B}_{\square} \mathrm{S}_{k}^{4}$ is the $k$ th four-dimensional toric Fano manifold, as in $[14, \S 7]$;
- $\operatorname{Str}_{k}$ are the Strangeway fourfolds described in [14, §8];
- $\mathrm{CKP}_{k}$ is the $k$ th four-dimensional toric complete intersection, as in [16];
- $S_{k}^{2}$ denotes the del Pezzo surface of degree $k$;
- $V_{k}^{3}$ denotes the three-dimensional Fano manifold of Picard rank 1, Fano index 1 , and degree $k$;
- $B_{k}^{3}$ denotes the three-dimensional Fano manifold of Picard rank 1, Fano index 2 , and degree $8 k$;
- $\mathrm{MM}_{\rho-k}^{3}$ denotes the $k$ th entry in the Mori-Mukai list of three-dimensional Fano manifolds of Picard rank $\rho$ [41, 42, 43, 44, 45]. We use the ordering as in [13], which agrees with the original papers of Mori-Mukai except when $\rho=4$.

Remark A.0.1. It appears from Table 2 as if the period sequences with IDs 72 and 73 might coincide. This is not the case. The coefficients $\alpha_{8}, \alpha_{9}$, and $\alpha_{10}$ in these cases are:

| Period ID | $\alpha_{8}$ | $\alpha_{9}$ | $\alpha_{10}$ |
| :---: | :---: | :---: | :---: |
| 72 | 32830 | 212520 | 1190952 |
| 73 | 32830 | 227640 | 1190952 |

Remark A.0.2. 590 of the period sequences that we find coincide with period sequences for toric complete intersections, at least for the first 15 terms. 579 of these are realised by quiver flag zero loci that are also toric complete intersections. For the remaining 11 cases - period sequences with IDs 17, 48, 73, 144, 145, 158, 191, 204, 256, 280, and 282 - there is no model as a toric complete intersection that is also a quiver flag zero locus in codimension at most four. In four of these cases with IDs 17, 48, 144, and 256 - the toric complete intersection period sequence is realised by a smooth four-dimensional toric variety.

Table A.2: Certain 4-dimensional Fano manifolds with Fano index 1 that arise as quiver flag zero loci


Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector |  |  |  | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 11 | $\begin{array}{llll} 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | 1 |  | 1 | 1 |  | 405 | 12 |
| 12 | $\begin{array}{llll} 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}$ | 1 |  | 1 | 1 | $(\square, \square, \varnothing)$ | 384 | 13 |
| 13 | $\begin{array}{lll} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{array}$ | 1 | 1 | 1 |  | $(\square, \varnothing),(\square, \varnothing)$ | 351 | 9 |
| 14 | $\begin{array}{lll} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 5 & 0 \end{array}$ | 1 | 1 | 1 |  | $(\square, \square),(\square, \square)$ | 486 | 9 |
| 15 | $\begin{array}{lll} 0 & 5 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$ | 1 | 1 | 2 |  | $(\square, \square)$ | 433 | 9 |
| 16 | $\begin{array}{llll} 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | 1 | 1 | 1 | 1 | $(\square, \varnothing, \varnothing)$ | 401 | 13 |
| 17 | $\begin{array}{llll} 0 & 3 & 5 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}$ | 1 | 1 | 1 | 2 | $(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing),(\varnothing, \square, \square)$ | 406 | 13 |
| 18 | $\begin{array}{lllll} 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}$ | 1 | 1 | 1 | 11 | $(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \varnothing, \varnothing)$ | 322 | 18 |
| 19 | $\begin{array}{lll} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{array}$ | 1 | 1 |  |  | $(\varnothing, \square),(\square, \varnothing)$ | 378 | 10 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 20 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array} 0_{0}^{1}$ | 1112 | $(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing)$ | 358 | 13 |
| 21 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array} 0_{1}^{1}$ | 1112 | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 347 | 13 |
| 22 | $\begin{array}{llll}0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0\end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 330 | 14 |
| 23 | $\begin{array}{lll} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{array}$ | 111 | $(\square, ■),(\square, \square)$ | 297 | 13 |
| 24 | $\begin{array}{ll} 0 & 5 \\ 0 & 0 \end{array}$ | 12 |  | 405 | 6 |
| 25 | $\begin{array}{lll}0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{array}$ | 122 | $(\square, \varnothing),(\square, \varnothing)$ | 325 | 10 |
| 26 | 0 0 0 0 4 <br> 0 0 0 0 0 <br> 0 0 0 0 0 <br> 0 0 0 0 0 <br> 0 1 1 1 0 |  | $(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \varnothing, \varnothing)$ | 290 | 18 |
| 27 | $\begin{array}{lll}0 & 7 \\ 0 & 0\end{array}$ | 1 | ( $\square$ ), ( $\square$ ) | 324 | 12 |
| 28 | $\begin{array}{lll} 0 & 1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 111 | ( $\varnothing, \square$ ),(■,■) | 292 | 14 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 29 | $\begin{array}{llll}0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}$ | 112 | ( $\square, \square),(\square, \square)$ | 273 | 9 |
| 30 |  | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \square)$ | 261 | 17 |
| 31 | $\begin{array}{lll} 0 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | 111 | (■, $\square$ ),(■, ■) | 244 | 16 |
| 32 | $\begin{array}{llll}0 & 5 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}$ | 114 | $(\square, \boxminus)$ | 225 | 5 |
| 33 | $\begin{array}{ll} 0 & 6 \\ 0 & 0 \end{array}$ | 1 | (■) | 243 | 27 |
| 34 | $\begin{array}{lll} 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | 111 | $(\square, \square)$ | 211 | 29 |
| 35 | $\begin{array}{lll} 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | 1111 |  | 544 | 8 |
| 36 | $\begin{array}{llll}0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 4 & 0\end{array}$ | 111 | $(\square, ■),(\square, \square)$ | 512 | 8 |
| 37 | $\begin{array}{llll} 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}$ |  |  | 464 | 12 |
| 38 | $\begin{array}{lllll} 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{array}$ |  | $(\square, \varnothing, \square),(\square, \square, \square)$ | 431 | 11 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 39 | $\begin{array}{lll} 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | 111 | $(\varnothing, \square)$ | 480 | 8 |
| 40 | $\begin{array}{lll} 0 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | 111 | $(\square, \square)$ | 416 | 10 |
| 41 | $\begin{array}{llll} 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\square, \varnothing, \square)$ | 400 | 12 |
| 42 | $\begin{array}{llll} 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \square)$ | 383 | 13 |
| 43 | $\begin{array}{lll} 0 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\square, \varnothing)$ | 350 | 12 |
| 44 | $\begin{array}{llll} 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ |  | 480 | 12 |
| 45 | $\begin{array}{lll} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\square, \square),(\square, \varnothing)$ | 432 | 9 |
| 46 | $\begin{array}{llll} 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\square, \varnothing, \square),(\square, \varnothing, \square)$ | 496 | 12 |
| 47 | $\begin{array}{llll} 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ |  | 432 | 12 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 48 | $\begin{array}{llll} 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & \\ \end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \varnothing, \square)$ | 433 | 13 |
| 49 | $\begin{array}{llll} 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ |  | 432 | 12 |
| 50 | $\begin{array}{lllll} 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ |  | 432 | 16 |
| 51 | $\begin{array}{llll} 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ |  | 400 | 12 |
| 52 | $\begin{array}{lllll} 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}$ | $1 \begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\square, \varnothing, \varnothing, \varnothing)$ | 384 | 16 |
| 53 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\square, \square, \square \square)$ | 378 | 12 |
| 54 | $\begin{array}{llll} 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square)$ | 464 | 16 |
| 55 | $\begin{array}{llll} 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square)$ | 416 | 12 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 56 | $\begin{array}{llll} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square)$ | 384 | 13 |
| 57 | $\begin{array}{lllll} 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square, \square),(\square, \varnothing, \square, \square)$ | 384 | 16 |
| 58 | $\begin{array}{lllll} 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}$ | $1 \begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \square)$ | 357 | 17 |
| 59 | $\begin{array}{llll} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\square, \square, \varnothing)$ | 336 | 14 |
| 60 | $\begin{array}{lllll} 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \square, \square)$ | 357 | 16 |
| 61 | $\begin{array}{llll} 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 336 | 13 |
| 62 | $\begin{array}{lll} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\varnothing, \square),(\square, \varnothing)$ | 324 | 12 |
| 63 | $\begin{array}{llll} 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing)$ | 336 | 12 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 64 | $\begin{array}{llll} 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\square, \varnothing, \varnothing),(\square, \square, \square)$ | 303 | 13 |
| 65 | $\begin{array}{lll} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\square, \square),(\square, \square)$ | 270 | 9 |
| 66 | $\begin{array}{llll} 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\square, \varnothing, \square),(\square, \varnothing, \varnothing)$ | 480 | 12 |
| 67 | $\begin{array}{llll} 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ |  | 432 | 12 |
| 68 | $\begin{array}{lll} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\varnothing, \square),(\square, \square)$ | 432 | 8 |
| 69 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\varnothing, \square, \varnothing)$ | 368 | 13 |
| 70 | $\begin{array}{lll} 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\square, \square)$ | 352 | 12 |
| 71 | $\begin{array}{lllll} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ |  | 448 | 16 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 72 | $\begin{array}{lllll} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \end{array}$ |  | $(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \varnothing, \square)$ | 389 | 16 |
| 73 |  | $1 \begin{array}{lllll}1 & 1 & 2\end{array}$ | $(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \varnothing, \square, \varnothing),(\square, \varnothing, \varnothing, \square)$ | 369 | 17 |
| 74 | $\begin{array}{llll} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \square),(\varnothing, \square, \varnothing)$ | 352 | 13 |
| 75 | $\begin{array}{lll} 0 & 0 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | $1 \begin{array}{lllll}1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \square)$ | 368 | 12 |
| 76 | $\begin{array}{llll} 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | 1122 | ( $\varnothing, \square, \square)$ | 337 | 13 |
| 77 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ |  | $(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \square, \square)$ | 347 | 16 |
| 78 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ |  | $(■, \square, \varnothing, \varnothing),(■, \square, \varnothing, \square),(\square, \varnothing, \varnothing, \varnothing)$ | 331 | 17 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 79 | $\begin{array}{llll} 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \square)$ | 335 | 15 |
| 80 |  | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing)$ | 305 | 13 |
| 81 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \varnothing, \varnothing)$ | 368 | 12 |
| 82 | $\begin{array}{lll} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | ( $\varnothing, \square, \square)$ | 352 | 12 |
| 83 |  | $1{ }_{1} 111$ | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 336 | 14 |
| 84 | 0 0 0 3 2 <br> 0 0 0 0 0 <br> 0 0 0 0 0 <br> 0 0 0 0 1 <br> 0 2 2 0 0 | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square, ■),(\square, \varnothing, \varnothing, \varnothing)$ | 352 | 16 |
| 85 | $\begin{array}{lllll} 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}$ | 11111 | $(\varnothing, \varnothing, \square),(\square, \square, \square)$ | 346 | 12 |
| 86 | $\begin{array}{lllll} 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 \end{array}$ |  | $(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \square, \square)$ | 310 | 17 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 87 | $\begin{array}{llll} 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 299 | 15 |
| 88 | $\begin{array}{lllll} 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square, \varnothing),(\square, \varnothing, \varnothing, \varnothing)$ | 289 | 18 |
| 89 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing)$ | 304 | 13 |
| 90 | $\begin{array}{lllll} 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \square),(\square, \varnothing, \varnothing, \varnothing)$ | 309 | 18 |
| 91 | $\begin{array}{llll} 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing)$ | 273 | 15 |
| 92 | $\begin{array}{lllll} 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \square),(\square, \square, \varnothing, \square)$ | 299 | 17 |
| 93 | $\begin{array}{llll} 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \square, \square)$ | 282 | 14 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 94 | $\begin{array}{llll} 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\square, \varnothing, ■),(\square, \varnothing, \square)$ | 288 | 8 |
| 95 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} 0_{0}$ | $1{ }_{1} 1111$ | $(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing)$ | 282 | 18 |
| 96 | $\begin{array}{llll} 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\square, \varnothing, \square)$ | 266 | 16 |
| 97 | $\begin{array}{llll} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\square, \varnothing, ■),(\square, \square, \varnothing)$ | 249 | 17 |
| 98 | $\begin{array}{lll} 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{array}$ | 111 | $(\square, \varnothing)$ | 216 | 16 |
| 99 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}$ |  | $(\varnothing, \varnothing, \square),(\square, \varnothing, \square)$ | 480 | 16 |
| 100 | $\begin{array}{llll}0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0\end{array}$ | 11111 | $(\varnothing, \square, \square),(\square, \varnothing, \square)$ | 384 | 13 |
| 101 | $\begin{array}{lll}0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{array}$ | 122 | $(\varnothing, \square),(\square, \varnothing)$ | 352 | 9 |
| 102 | $\begin{array}{llll}0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0\end{array}$ | 112 | $(\varnothing, \boxminus),(\varnothing, \boxminus),(\square, \varnothing)$ | 320 | 10 |


| Period ID | Adjacency matrix | Dimension vector |  |  |  |  | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 103 | $\begin{array}{lllll} 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \end{array}$ | 1 | 1 | 1 | 1 | 1 | $(\varnothing, \square, \varnothing, \square),(\square, \varnothing, \varnothing, \square)$ | 362 | 17 |
| 104 | $\begin{array}{lllll} 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}$ | 1 | 1 | 1 | 1 | 2 | $(\varnothing, \square, \square, \square),(\square, \varnothing, \square, \square)$ | 305 | 18 |
| 105 | $\begin{array}{llll} 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}$ | 1 | 1 | 1 | 1 |  | $(\varnothing, \varnothing, \square),(\square, \square, \square)$ | 352 | 16 |
| 106 | $\begin{array}{llll} 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{array}$ | 1 | 1 | 1 | 1 |  | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 304 | 14 |
| 107 | $\begin{array}{lllll} 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 \end{array}$ | 1 | 1 | 1 | 1 | 1 | $(\varnothing, \square, \square, \square),(\square, \varnothing, \square, \square)$ | 304 | 17 |
| 108 | $\begin{array}{lllll} 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & 0 \end{array}$ | 1 | 1 | 1 |  | 1 | $(\varnothing, \square, \square, \varnothing),(\square, \varnothing, \varnothing, \varnothing)$ | 283 | 18 |
| 109 | $\begin{array}{lll}0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{array}$ |  | 1 |  |  |  | $(\varnothing, \boxminus),(\varnothing, \boxminus),(\varnothing, \boxminus)$ | 272 | 8 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 110 | $\begin{array}{lllll} 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, ■),(\square, \square, ■)$ | 256 | 17 |
| 111 | $\begin{array}{lll} 0 & 1 & 6 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | 111 | $(\varnothing, \square),(\varnothing, \square)$ | 320 | 0 |
| 112 | $\begin{array}{lll} 0 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 111 | ( $\varnothing, \square$ ), (■, $\square$ ) | 304 | 12 |
| 113 | $\begin{array}{lll} 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 111 | (■, $\square$ ), ( $\square, \square)$ | 272 | 15 |
| 114 |  |  | $(\varnothing, \varnothing, \square),(\varnothing, \square, \square)$ | 282 | 16 |
| 115 | $\begin{array}{lll} 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}$ | 112 | ( $\square, \square \square$ | 272 | 10 |
| 116 | $\begin{array}{lllll} 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}$ | 1122 | $(\varnothing, \boxminus, \varnothing),(\varnothing, \boxminus, \varnothing),(\square, \varnothing, \varnothing)$ | 257 | 15 |
| 117 |  | $1{ }_{1} 111$ | $(\varnothing, \square, \square),(\square, \square, \varnothing)$ | 256 | 18 |
| 118 | $\begin{array}{llll}0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0\end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing)$ | 256 | 15 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 119 | $\begin{array}{lll} 0 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | 111 | $(\varnothing, \square),(\square, \varnothing)$ | 260 | 16 |
| 120 | $\begin{array}{llll} 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\varnothing, \square, \square)$ | 229 | 19 |
| 121 | $\begin{array}{llll} 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\square, \varnothing, \square)$ | 230 | 20 |
| 122 | $\begin{array}{llll} 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \square, \varnothing)$ | 213 | 19 |
| 123 | $\begin{array}{lll} 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\square, \square),(\square, \varnothing)$ | 196 | 16 |
| 124 | $\begin{array}{lll} 0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 2\end{array}$ | $(\square, \varnothing),(\square, \square),(\square, \boxminus)$ | 240 | 13 |
| 125 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 2\end{array}$ | $(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing)$ | 211 | 15 |
| 126 | $\begin{array}{lll} 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | $1 \quad 12$ | $(\varnothing, \boxminus),(\square, \boxminus)$ | 224 | 13 |
| 127 | $\begin{array}{lll} 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\varnothing, \square \square)$ | 240 | -12 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 128 | $\begin{array}{lll} 0 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | 111 | $(\square, \square)$ | 224 | 20 |
| 129 |  | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square)$ | 202 | 24 |
| 130 | $\begin{array}{ll} 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | ( $\square, \square, \square)$ | 180 | 32 |
| 131 | $\begin{array}{lll} 0 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 111 | ( $\square, \square)$ | 163 | 31 |
| 132 | $\begin{array}{lll} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 111 | $(\square, \square),(\square, \varnothing),(\square, \varnothing),(\square, \varnothing)$ | 192 | 18 |
| 133 | $\begin{array}{llll} 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ |  | 464 | 12 |
| 134 |  | 11111 |  | 448 | 12 |
| 135 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & 2 \end{array}$ |  | $(\varnothing, \varnothing, \square, \square),(\varnothing, \square, \varnothing, \varnothing)$ | 384 | 16 |
| 136 | $\begin{array}{lll}0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}$ | 112 | $(\varnothing, \boxminus),(\square, \varnothing),(\square, \square)$ | 384 | 8 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 137 | $\begin{array}{lll} 0 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$ | 111 | $(\varnothing, \square)$ | 352 | 10 |
| 138 | $\begin{array}{ll} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array} 0_{0}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \varnothing)$ | 320 | 16 |
| 139 | $\begin{array}{llll}0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 4 & 0\end{array}$ | 111 | $(\square, \varnothing),(\square, \varnothing)$ | 320 | 12 |
| 140 | $\begin{array}{lll}0 & 0 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 4 & 0\end{array}$ | 111 | $(\varnothing, \square),(\square, \square)$ | 432 | 12 |
| 141 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} 0_{0}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 400 | 12 |
| 142 | $\begin{array}{llll}0 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0\end{array}$ | 112 | $(\varnothing, \boxminus),(\varnothing, \boxminus)$ | 352 | 10 |
| 143 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \end{array}$ |  | $(\varnothing, \varnothing, \varnothing, \square),(\square, \square, \varnothing, \square \square)$ | 400 | 16 |
| 144 | 0 3 3 2 4 <br> 0 0 0 0 0 <br> 0 0 0 1 0 <br> 0 0 0 0 0 <br> 0 0 0 0 0 | $1 \begin{array}{lllll}1 & 1 & 2\end{array}$ | $(\varnothing, \varnothing, \square, \square),(\square, \varnothing, \varnothing, \square)$ | 369 | 17 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 145 | $\begin{array}{llll} 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 2 & 2\end{array}$ | $(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing)$ | 353 | 14 |
| 146 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \varnothing, \square),(\square, \varnothing, \square, \square \square)$ | 368 | 16 |
| 147 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \square),(\varnothing, \square, \varnothing, \varnothing)$ | 367 | 15 |
| 148 | $\begin{array}{lllll} 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\square, \varnothing, \varnothing, \varnothing)$ | 351 | 15 |
| 149 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing)$ | 352 | 13 |
| 150 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \varnothing, \varnothing)$ | 352 | 16 |
| 151 | $\begin{array}{lllll} 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \varnothing, \square)$ | 331 | 17 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 152 |  | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \varnothing, \varnothing)$ | 326 | 16 |
| 153 | $\begin{array}{ll} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array} 0_{1}^{1}$ | $11_{1} 11$ | $(\square, \varnothing, \square)$ | 319 | 13 |
| 154 |  | 1112 | $(■, \square, \square),(\varnothing, \varnothing, \boxminus),(\square, \varnothing, \varnothing)$ | 320 | 12 |
| 155 | $\begin{array}{llllllll} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 2 & 0 \end{array}$ |  | $(\varnothing, \varnothing, \varnothing, \square, \square),(\varnothing, \varnothing, \varnothing, \square, \varnothing),(\square, \varnothing, \square, \varnothing, \square \mathbf{)}$ | 310 | 21 |
| 156 | $\begin{array}{llll} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} 0$ | $\begin{array}{lllllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\square, \varnothing, \square, \varnothing)$ | 303 | 17 |
| 157 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}$ |  | $(\varnothing, \varnothing, \square),(\square, \varnothing, \varnothing)$ | 304 | 16 |
| 158 | $\begin{array}{lll} 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 122 | ( $\square, \square)$ | 274 | 16 |
| 159 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 \\ & & & \\ \end{array}$ | $(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing)$ | 288 | 15 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 160 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \square)$ | 256 | 12 |
| 161 | $\begin{array}{ll} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \varnothing, \boldsymbol{m})$ | 416 | 16 |
| 162 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 \end{array}$ |  | $(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \square, ■)$ | 384 | 16 |
| 163 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}$ |  | $(\varnothing, \square, ■),(\square, \square, \boldsymbol{\square})$ | 378 | 15 |
| 164 |  |  | $(\varnothing, \varnothing, \square),(\square, \square, \boldsymbol{m})$ | 368 | 12 |
| 165 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing)$ | 336 | 12 |
| 166 | $\begin{array}{llll} 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}$ | 11811 | $(\square, \varnothing, ■),(\square, \square, ■)$ | 335 | 13 |
| 167 | $\begin{array}{llll}0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 0\end{array}$ | 1111 | $(\square, \varnothing)$ | 272 | 16 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 168 | $\begin{array}{lllll} 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 & 0 \end{array}$ |  | $(\varnothing, \varnothing, \varnothing, \square),(\square, \varnothing, \square, ■)$ | 352 | 16 |
| 169 |  |  | $(\varnothing, \square, \varnothing, ■),(\square, \varnothing, \square, ■)$ | 335 | 16 |
| 170 |  |  | $(\varnothing, \varnothing, \varnothing, \square),(\square, \square, \varnothing, \square \square)$ | 336 | 16 |
| 171 |  |  | $(\varnothing, \varnothing, \square, \varnothing),(\square, \square, \varnothing, \square \square)$ | 320 | 17 |
| 172 | $\begin{array}{llllll} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \varnothing),(\square, \varnothing, \varnothing, \varnothing)$ | 314 | 16 |
| 173 | $\begin{array}{llll} 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}$ | 1122 | $(\varnothing, \boxminus, \varnothing),(\varnothing, \boxminus, \varnothing)$ | 298 | 13 |
| 174 | 0 0 0 1 2 <br> 0 0 1 0 0 <br> 0 0 0 1 0 <br> 0 0 0 0 0 <br> 0 3 2 0 0 | $\begin{array}{lllll} 1 & 1 & 1 & 1 \end{array}$ | $(\varnothing, \square, \varnothing, ■),(\square, \varnothing, \square, ■)$ | 309 | 19 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 175 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \square, \square)$ | 288 | 15 |
| 176 | $\begin{array}{llll} 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \square, \square)$ | 304 | 14 |
| 177 | $\begin{array}{llllll} 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \end{array}$ | $\begin{array}{lllllll}1 & 1 & 1 & 1 & 1 & 1\end{array}$ | $(\square, \varnothing, \square, \varnothing, \square)$ | 325 | 20 |
| 178 | $\begin{array}{llll} 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 2\end{array}$ | $(\varnothing, \varnothing, \boxminus),(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing),(\square, \square, \varnothing)$ | 304 | 12 |
| 179 | $\begin{array}{lllll} 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \square),(\square, \varnothing, \varnothing, \varnothing)$ | 304 | 16 |
| 180 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 2 & 0 \end{array}$ | $1 \begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \varnothing),(\varnothing, \square, \square, \square \square)$ | 304 | 17 |
| 181 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \square \square)$ | 304 | 16 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 182 |  | 1112 | ( $\varnothing, \square, \square),(\square, \square, \square)$ | 273 | 15 |
| 183 |  |  | $(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \varnothing, \varnothing)$ | 293 | 16 |
| 184 | $\begin{array}{llll} 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | ( $\square, \varnothing, \square)$ | 272 | 13 |
| 185 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 & 0 \end{array}$ |  | $(\square, \varnothing, \varnothing, ■),(\square, \square, \varnothing, ■)$ | 277 | 17 |
| 186 | $\begin{array}{lllll} 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}$ | $1{ }_{1}^{1} 111$ | $(\varnothing, \varnothing, \square),(\square, \varnothing, \varnothing)$ | 252 | 16 |
| 187 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 & 0 \end{array}$ |  | $(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \varnothing, \varnothing)$ | 277 | 17 |
| 188 | $\begin{array}{llll} 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | 1112 | $(\varnothing, \varnothing, \boxminus),(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing),(\varnothing, \square, \square)$ | 272 | 12 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 189 |  |  | $(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \square, ■)$ | 262 | 19 |
| 190 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} 0_{1}^{1}$ | 1112 | $(\varnothing, \varnothing, \boxminus),(\varnothing, \varnothing, \boxminus),(\square, \varnothing, \varnothing)$ | 256 | 14 |
| 191 |  | 1112 | $(\varnothing, \varnothing, \boxminus),(\varnothing, \varnothing, \boxminus)$ | 241 | 17 |
| 192 | $\begin{array}{llllll} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & 0 \end{array}$ |  | $(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \square, \varnothing, \varnothing)$ | 256 | 18 |
| 193 | $\begin{array}{lllll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, ■),(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing),(\square, \square, \square \square)$ | 256 | 14 |
| 194 | $\begin{array}{lllll} 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{array}$ |  | $(\varnothing, \varnothing, \square, \varnothing),(\square, \square, \varnothing, ■)$ | 252 | 22 |
| 195 | $\begin{array}{llll}0 & 3 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 5 \\ 0 & 0 & 0\end{array}$ | 112 | $(\boldsymbol{m}, ~ 日),(\boldsymbol{m}, ~ \exists),(\varnothing, \square)$ | 288 | 10 |
| 196 | $\begin{array}{llll}0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0\end{array}$ | $1 \begin{array}{lllll}1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing)$ | 304 | 12 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 197 | $\begin{array}{llll} 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\square, \varnothing, \square)$ | 255 | 19 |
| 198 | $\begin{array}{lllll} 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \varnothing, \square),(\square, \varnothing, \square, \square)$ | 330 | 20 |
| 199 | $\begin{array}{lllll} 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \square),(\square, \square, \varnothing, \square)$ | 288 | 17 |
| 200 | $\begin{array}{llllll} 0 & 0 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}$ | $\begin{array}{lllllll}1 & 1 & 1 & 1 & 1 & 1\end{array}$ | $(\square, \varnothing, \varnothing, \varnothing, \varnothing)$ | 299 | 20 |
| 201 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 2\end{array}$ | $(\varnothing, \square, \square, \square),(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \varnothing, \varnothing)$ | 278 | 16 |
| 202 | $\begin{array}{llll} 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 2\end{array}$ | $(\varnothing, \varnothing, \boxminus),(\varnothing, \boxminus, \varnothing),(\square, \varnothing, \varnothing)$ | 267 | 13 |
| 203 | $\begin{array}{lllll} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \square),(\varnothing, \square, \square, \square)$ | 271 | 19 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 204 | $\begin{array}{lllll} 0 & 0 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}$ | $1 \begin{array}{llllll}1 & 1 & 1 & 1 & 2\end{array}$ | $(\varnothing, \square, \varnothing, \boxminus),(\square, \varnothing, \square, \square)$ | 257 | 19 |
| 205 | $\begin{array}{llllll} 0 & 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{array}$ | $\begin{array}{lllllll}1 & 1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \varnothing, \square),(\varnothing, \varnothing, \square, \square, \square),(\square, \square, \varnothing, \varnothing, \square)$ | 283 | 21 |
| 206 | $\begin{array}{lllll} 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & 0 \end{array}$ | $1 \begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \square),(\square, \varnothing, \square, \square)$ | 277 | 18 |
| 207 | $\begin{array}{lllll} 0 & 0 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \square),(\square, \varnothing, \varnothing, \varnothing)$ | 262 | 18 |
| 208 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 1 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \square),(\varnothing, \varnothing, \square, ■),(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \varnothing, \varnothing)$ | 257 | 17 |
| 209 | $\begin{array}{lllll} 0 & 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\square, \varnothing, \varnothing, \varnothing),(\square, \varnothing, \varnothing, \varnothing)$ | 257 | 17 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 210 | 0 0 1 2 3 <br> 0 0 0 0 0 <br> 0 0 0 1 0 <br> 0 0 0 0 0 <br> 0 2 1 0 0 | $1 \begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | ( $\varnothing, \varnothing, \square, \square),(\square, \varnothing, \square, ■)$ | 256 | 18 |
| 211 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} 0_{0}$ | 1112 | $(\varnothing, \varnothing, \boxminus),(\varnothing, \varnothing, \boxminus),(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing)$ | 241 | 15 |
| 212 | $\begin{array}{ll} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array} 0_{1}^{1}$ | 1112 | ( $\square, \varnothing, \square)$ | 235 | 15 |
| 213 |  | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \varnothing)$ | 256 | 18 |
| 214 |  | 1111 | $(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing)$ | 240 | 16 |
| 215 | 0 0 1 2 3 <br> 0 0 0 0 0 <br> 0 0 0 0 0 <br> 0 0 0 0 0 <br> 0 2 1 0 0 <br>     0 | $\begin{array}{lllllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square, \varnothing),(\square, \varnothing, \square, ■)$ | 235 | 19 |
| 216 | $\begin{array}{llll} 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\square, \square, \varnothing)$ | 219 | 19 |
| 217 | $\begin{array}{llll} 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing)$ | 225 | 16 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 218 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \end{array}$ |  | $(\varnothing, \square, \varnothing, ■),(\square, \square, \varnothing, ■)$ | 229 | 21 |
| 219 | $\begin{array}{llll}0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0\end{array}$ | 113 | $(\varnothing, \boxminus),(\varnothing, \boxminus),(\varnothing, \boxminus),(\square, \varnothing)$ | 224 | 9 |
| 220 |  |  | $(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing)$ | 195 | 18 |
| 221 | $\begin{array}{lll} 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | 111 | $(\varnothing, \square),(\square, \varnothing)$ | 288 | 16 |
| 222 | $\begin{array}{llll}0 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0\end{array}$ | 111 | ( $\varnothing, \square$ ), (■, $\square$ ) | 240 | 16 |
| 223 |  |  | $(\varnothing, \square, \varnothing),(\varnothing, \square, \boldsymbol{\square})$ | 256 | 16 |
| 224 | $\begin{array}{llll} 0 & 3 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}$ | 1122 | $(\varnothing, \square, \varnothing),(\varnothing, \boxminus, \varnothing),(\square, \varnothing, \square)$ | 251 | 14 |
| 225 | $\begin{array}{llll} 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \square, \varnothing)$ | 230 | 18 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 226 | $\begin{array}{llll} 0 & 3 & 5 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 2\end{array}$ | $(\varnothing, \square, \square),(\varnothing, \square, \square)$ | 225 | 18 |
| 227 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 2\end{array}$ | $(\varnothing, \square, \square, \varnothing),(\varnothing, \square, \square, \boxminus),(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \varnothing, \varnothing)$ | 236 | 19 |
| 228 | $\begin{array}{llll} 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 220 | 18 |
| 229 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 2 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \square),(\varnothing, \square, \varnothing, \varnothing),(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \square, \square \square)$ | 241 | 18 |
| 230 | $\begin{array}{lll} 0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | $1 \begin{array}{lll}1 & \\ \end{array}$ | $(\varnothing, \boxminus),(\square, \square),(\square, \varnothing)$ | 240 | 8 |
| 231 | $\begin{array}{llll} 0 & 3 & 5 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing),(\square, \varnothing, \boxminus)$ | 225 | 17 |
| 232 | $\begin{array}{llll} 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1 & 2\end{array}$ | $(\varnothing, \varnothing, \boxminus),(\varnothing, \square, \square)$ | 230 | 14 |
| 233 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 2 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \square, \varnothing, \varnothing)$ | 235 | 19 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 234 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 \end{array}$ |  | $(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \square, \varnothing)$ | 215 | 21 |
| 235 | $\begin{array}{llll} 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | 1122 | $(\varnothing, \square, 日),(\square, \square, \varnothing)$ | 199 | 16 |
| 236 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} 0_{0}$ | $1{ }_{1} 111$ | $(\varnothing, \square, \varnothing),(\square, \square, \square)$ | 208 | 18 |
| 237 |  |  | $(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing)$ | 200 | 24 |
| 238 | $\begin{array}{lll}0 & 5 & 4 \\ 0 & 5 & \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}$ | 112 | $(\varnothing, \boxplus),(\square, \varnothing),(\square, \square)$ | 192 | 0 |
| 239 | $\begin{array}{llll}0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0\end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\square, \varnothing, \square)$ | 266 | 20 |
| 240 |  |  | $(\varnothing, \square, \square),(\square, \varnothing, \square)$ | 224 | 19 |
| 241 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 & 0 \end{array}$ |  | $(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \square, \square, \boldsymbol{\square})$ | 235 | 20 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 242 | $\begin{array}{llll} 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}$ | 1113 | $(\varnothing, \square, \boxminus),(\square, \varnothing, \boxminus)$ | 224 | 16 |
| 243 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \square, \varnothing)$ | 203 | 20 |
| 244 |  | 1112 | $(\varnothing, \varnothing, \boxminus),(\varnothing, \square, \boxminus)$ | 203 | 17 |
| 245 |  | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 208 | 16 |
| 246 | $\begin{array}{lll} 0 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$ | 111 | ( $\square, \square)$ | 192 | 30 |
| 247 | $\begin{array}{lll}0 & 0 & \\ 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 7 & 0\end{array}$ | 111 | $(\square, \square),(\square, \varnothing),(\square, \varnothing),(\square, \square \square)$ | 208 | 16 |
| 248 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 208 | 20 |
| 249 | $\begin{array}{lll} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & 0 \end{array}$ | 111 | $(\varnothing, \square),(\square, \square),(\square, \varnothing),(\square, \varnothing)$ | 176 | 16 |
| 250 | $\begin{array}{lllll}0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0\end{array}$ | $1 \begin{array}{lllll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, ■),(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing)$ | 193 | 19 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 251 |  | 1111 | $(\square, \square, \square)$ | 176 | 33 |
| 252 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \end{array}$ | 11111 | $(\varnothing, \square, \square),(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing)$ | 177 | 21 |
| 253 | $\begin{array}{llll}0 & 5 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0\end{array}$ | 112 | $(\varnothing, \boxminus),(\varnothing, \boxminus),(\square, \square)$ | 177 | 17 |
| 254 | $\begin{array}{llll}0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0\end{array}$ | 111 | $(\square, \varnothing),(\square, \square)$ | 160 | 30 |
| 255 | $\begin{array}{lll} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{array}$ | 111 | $(\square, \varnothing),(\square, \square)$ | 144 | 23 |
| 256 | $\begin{array}{lllll} 0 & 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}$ |  | $(\varnothing, \varnothing, \varnothing, \square),(\varnothing, \varnothing, \varnothing, \square)$ | 385 | 17 |
| 257 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, ■),(\varnothing, \square, \varnothing)$ | 384 | 12 |
| 258 | $\begin{array}{lll} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{array}$ | 111 | $(\varnothing, \square),(\square, \varnothing)$ | 320 | 10 |
| 259 | $\begin{array}{llll}0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 4 & 0\end{array}$ | 1111 | $(\square, ■),(\square, \square)$ | 256 | 16 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 260 |  | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \square, \boldsymbol{m})$ | 384 | 16 |
| 261 |  |  | $(\varnothing, \varnothing, \square, \square),(\square, \varnothing, \varnothing, \varnothing)$ | 336 | 16 |
| 262 |  |  | $(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \square, \square \square)$ | 320 | 16 |
| 263 | $\begin{array}{llll} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}$ | $1 \begin{array}{lllll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \square, \square)$ | 304 | 18 |
| 264 |  | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 288 | 13 |
| 265 | $\begin{array}{llll} 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}$ | 1122 | $(\varnothing, \varnothing, \boxminus),(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing)$ | 273 | 14 |
| 266 | $\begin{array}{llll} 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 250 | 19 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 267 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square, \square \square),(\square, \varnothing, \varnothing, \varnothing)$ | 336 | 20 |
| 268 | $\begin{array}{llll} 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \varnothing, \varnothing)$ | 320 | 12 |
| 269 | $\begin{array}{llllll} 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{array}$ | $\begin{array}{lllllll}1 & 1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \varnothing, \square \square, \square),(\varnothing, \varnothing, \square, \square, \varnothing),(\square, \square, \varnothing, \varnothing, \square)$ | 320 | 20 |
| 270 | $\begin{array}{lllll} 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \square),(\varnothing, \square, \varnothing, \varnothing)$ | 304 | 16 |
| 271 | $\begin{array}{lllll} 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square, ■),(\square, \varnothing, \varnothing, \varnothing)$ | 293 | 18 |
| 272 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \varnothing),(\varnothing, \square, \square, \square \square)$ | 288 | 16 |
| 273 | $\begin{array}{lllll} 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \square, \square)$ | 282 | 18 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector |  |  |  |  | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 274 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}$ | 1 | 1 | 1 | 1 |  | $(\varnothing, \varnothing, \square),(\varnothing, \square, \varnothing)$ | 288 | 16 |
| 275 | $\begin{array}{llll} 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}$ | 1 | 1 | 1 | 1 |  | $(\varnothing, \varnothing, \square),(\varnothing, \square, \varnothing)$ | 272 | 14 |
| 276 | $\begin{array}{llll} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | 1 | 1 | 1 | 1 |  | $(\varnothing, \square, \varnothing)$ | 286 | 16 |
| 277 | $\begin{array}{llllll} 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{array}$ | 1 |  | 1 |  | 11 | $(\varnothing, \varnothing, \varnothing, \varnothing, \square),(\varnothing, \varnothing, \square, ■, \square),(\square, \square, \varnothing, \varnothing, \square)$ | 278 | 21 |
| 278 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 2 & 0 \end{array}$ | 1 | 1 | 1 |  | 1 | $(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \square, \square, \square \square)$ | 272 | 17 |
| 279 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \end{array}$ | 1 | 1 | 1 | 1 |  | $(\varnothing, \square, \varnothing),(\square, \square, \square \square)$ | 256 | 14 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 280 | $\begin{array}{lll} 0 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 122 | $(\varnothing, \boxminus),(\square, \varnothing)$ | 228 | 20 |
| 281 |  |  | $(\varnothing, \square, \square, ■),(\square, \varnothing, \varnothing, \varnothing)$ | 256 | 18 |
| 282 |  | 1112 | $(\varnothing, \square, \boxminus),(\square, \square, \varnothing),(\square, \square, \square)$ | 225 | 19 |
| 283 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}$ |  | $(\varnothing, \square, ■),(\square, \square, \square)$ | 224 | 19 |
| 284 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \varnothing, \varnothing)$ | 324 | 18 |
| 285 | $\begin{array}{llllll} 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}$ | $\begin{array}{llllllll}1 & 1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \varnothing, \varnothing)$ | 293 | 21 |
| 286 | $\begin{array}{lllll} 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 0 \end{array}$ |  | $(\varnothing, \varnothing, \square, \square),(\varnothing, \square, \square, ■)$ | 288 | 16 |



Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 294 | $\begin{array}{llll} 0 & 3 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 2 & 2\end{array}$ | $(\varnothing, \varnothing, \boxminus),(\varnothing, \boxminus, \varnothing),(\square, \square, \varnothing)$ | 241 | 14 |
| 295 | $\begin{array}{llll} 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 2\end{array}$ | $(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing)$ | 244 | 13 |
| 296 | $\begin{array}{lllll} 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \square),(\varnothing, \square, \varnothing, \varnothing)$ | 246 | 18 |
| 297 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}$ | $1 \begin{array}{lllll}1 & 1 & 1 & 1 & 2\end{array}$ | $(\varnothing, \varnothing, \varnothing, \boxminus),(\varnothing, \square, \varnothing, \varnothing),(\square, \square, \square, \varnothing)$ | 246 | 17 |
| 298 | $\begin{array}{lllll} 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square, \varnothing),(\square, \varnothing, \square, \square)$ | 240 | 19 |
| 299 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing)$ | 239 | 13 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 300 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 2 & 0 \end{array}$ | $1 \begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \varnothing, \square \square),(\varnothing, \varnothing, \square, \square),(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \square, \square \square)$ | 236 | 22 |
| 301 | $\begin{array}{llll} 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing)$ | 226 | 18 |
| 302 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 2 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, ■),(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \varnothing, \varnothing),(\square, \varnothing, \square, \square \square)$ | 230 | 18 |
| 303 | $\begin{array}{llll} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1 & 2\end{array}$ | $(\varnothing, \boldsymbol{\square}, \boxminus),(\varnothing, \boldsymbol{\square}, \boxminus),(\varnothing, \varnothing, \square)$ | 209 | 16 |
| 304 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square \square, \square \square),(\square, \varnothing, \square)$ | 256 | 8 |
| 305 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \square \square),(\varnothing, \square, \varnothing, \varnothing)$ | 246 | 18 |
| 306 | $\begin{array}{lllll} 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \varnothing),(\varnothing, \square, \square, \square)$ | 245 | 19 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 307 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\square, \varnothing, \square)$ | 238 | 22 |
| 308 |  | 11111 | $(\varnothing, \varnothing, \square),(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 220 | 16 |
| 309 | $\begin{array}{lllllll} 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \end{array}$ | $\begin{array}{llllllll}1 & 1 & 1 & 1 & 1 & 1\end{array}$ | $(\square, \square, \varnothing, \varnothing, \square)$ | 251 | 21 |
| 310 |  |  | $(\varnothing, \varnothing, \square, \square),(\square, \square, \varnothing, ■)$ | 230 | 18 |
| 311 |  |  | $(\varnothing, \varnothing, \square, \varnothing),(\square, \varnothing, \varnothing, \varnothing)$ | 250 | 22 |
| 312 | $\begin{array}{llll}0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & 0\end{array}$ | 111 | ( $\varnothing, \square$ ), ( $\square, \square),(\square, \varnothing),(\square, \square \square)$ | 224 | 16 |
| 313 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} 0_{0}$ | 11111 | $(\varnothing, \square, ■),(\varnothing, \square, \square)$ | 240 | 12 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 314 | $\begin{array}{lllll} 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \square),(\square, \varnothing, \square, \square)$ | 230 | 18 |
| 315 | $\begin{array}{llll} 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \varnothing, \varnothing)$ | 218 | 14 |
| 316 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \square, \square \square)$ | 208 | 20 |
| 317 | $\begin{array}{lllll} 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \varnothing),(\varnothing, \square, \square, \square)$ | 214 | 20 |
| 318 | $\begin{array}{lllll} 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 2 & 2\end{array}$ | $(■, \square, \varnothing, \varnothing),(\varnothing, \varnothing, \varnothing, \boxminus),(\varnothing, \varnothing, \varnothing, \boxminus),(\square, \varnothing, \varnothing, \varnothing)$ | 209 | 15 |
| 319 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \varnothing),(\square, \varnothing, \square, \square \square)$ | 209 | 20 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 320 | $\begin{array}{ll} 0 & 3 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} 0_{0}^{0} 0$ | 1112 | $(\varnothing, \square, \varnothing),(\varnothing, \square, \square),(\varnothing, \square, \boxminus)$ | 193 | 20 |
| 321 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}$ | 1112 | $(\square, \varnothing, \exists)$ | 202 | 18 |
| 322 |  | 1112 | $(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing),(\square, \square, \square)$ | 198 | 9 |
| 323 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & 2 \end{array}$ |  | $(\varnothing, \varnothing, \square, ■),(\varnothing, \square, \square, ■)$ | 202 | 22 |
| 324 |  | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\square, \varnothing, \varnothing),(\square, \square, \varnothing)$ | 178 | 20 |
| 325 | $\begin{array}{lllll} 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \varnothing)$ | 186 | 20 |
| 326 | $\begin{array}{llll} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}$ | 1122 | $(\boldsymbol{m}, \exists, \varnothing),(\boldsymbol{m}, \square, \varnothing),(\mathbf{\square}, \varnothing, \boxminus)$ | 209 | 17 |
| 327 | $\begin{array}{lllll} 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \square, \square)$ | 204 | 15 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 328 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 \end{array}$ | $1 \begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \varnothing),(\varnothing, \square, \square, \square \square)$ | 214 | 20 |
| 329 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \square, \varnothing, \varnothing)$ | 209 | 19 |
| 330 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \square),(\square, \varnothing, \varnothing),(\square, \square, \square \square)$ | 204 | 20 |
| 331 | $\begin{array}{llll} 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \square)$ | 188 | 19 |
| 332 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \square),(\square, \varnothing, \square, \square \square)$ | 194 | 19 |
| 333 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \varnothing),(\square, \square, \varnothing, \square)$ | 188 | 22 |
| 334 | $\begin{array}{llll} 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \varnothing)$ | 244 | 24 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 335 |  | 11111 | $(\varnothing, \square, \varnothing),(\square, \varnothing, \square)$ | 208 | 22 |
| 336 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \varnothing, ■),(\square, \varnothing, \varnothing),(\square, \square, \square)$ | 198 | 18 |
| 337 |  | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 173 | 23 |
| 338 |  |  | $(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \square, \square)$ | 199 | 15 |
| 339 |  |  | $(■, \square, \varnothing, \varnothing),(\square, \varnothing, \varnothing, \varnothing),(\square, \varnothing, \varnothing, \varnothing),(\square, \varnothing, \varnothing, \varnothing)$ | 199 | 13 |
| 340 |  | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \square, \varnothing, \square \square)$ | 214 | 24 |
| 341 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} 0_{0}^{1}$ | 1112 | $(\varnothing, \square, \varnothing),(\square, \varnothing, \boxminus)$ | 178 | 19 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 342 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} 0_{0}$ | 1112 | $(\varnothing, \varnothing, \boxminus),(\square, \square, \varnothing)$ | 188 | 16 |
| 343 |  | 1112 | $(\varnothing, \varnothing, \boxminus),(\varnothing, \square, \boxminus),(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing)$ | 183 | 17 |
| 344 |  | 1122 | $(\boldsymbol{m}, ~,, \square),(\varnothing, \square, \varnothing)$ | 178 | 17 |
| 345 |  |  | $(\varnothing, \square, \square),(\square, \square, ■)$ | 177 | 21 |
| 346 |  | 1122 | ( $\varnothing, \square, \square)$ | 150 | 18 |
| 347 |  |  | $(\varnothing, \square, \square),(\varnothing, \square, \varnothing)$ | 176 | 20 |
| 348 |  | 11811 | $(\square, \square, \varnothing),(\square, \square, \varnothing)$ | 166 | 22 |
| 349 | $\begin{array}{llll}0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0\end{array}$ |  | $(\varnothing, \square, \varnothing),(\square, \square, \varnothing)$ | 166 | 20 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 350 | $\begin{array}{lll} 0 & 4 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$ | 123 | $(\varnothing, \boxminus),(\varnothing, \boxminus),(\varnothing, \boxminus)$ | 177 | 18 |
| 351 | $\begin{array}{lllll} 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square, \square),(\square, \varnothing, \square, \varnothing)$ | 187 | 24 |
| 352 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 3 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing),(\square, \square, \square \square)$ | 172 | 20 |
| 353 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 3 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\varnothing, \square \square, \square \square),(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing)$ | 178 | 22 |
| 354 | $\begin{array}{lll} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\varnothing, \square),(\square, \square)$ | 176 | 16 |
| 355 | $\begin{array}{lll} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\square, \square),(\square, \square \square)$ | 160 | 32 |
| 356 | $\begin{array}{llll} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square)$ | 149 | 34 |
| 357 | $\begin{array}{llll} 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \square)$ | 162 | 19 |

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| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 358 | $\begin{array}{llll} 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}$ | 11111 | $(\square, \varnothing, \square),(\square, \square, \square)$ | 151 | 25 |
| 359 |  |  | $(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing)$ | 160 | 24 |
| 360 |  | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\square, \varnothing, \varnothing),(\square, \square, \varnothing)$ | 145 | 33 |
| 361 | $\begin{array}{llll}0 & 4 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0\end{array}$ | 112 | $(\varnothing, \boxminus),(\varnothing, \boxminus),(\square, \boxminus)$ | 129 | 31 |
| 362 | $\begin{array}{llll}0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 4 & 0\end{array}$ | 111 | $(\square, \square),(\square, \varnothing)$ | 112 | 52 |
| 363 | $\begin{array}{lll}0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 4 & 0\end{array}$ | 111 | $(\varnothing, \square),(\square, \boldsymbol{\square})$ | 384 | 16 |
| 364 | $\begin{array}{llll}0 & 3 & \\ 0 & 3 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0\end{array}$ | 112 | $(\boldsymbol{m}, ~]),(\boldsymbol{m}, ~ \square),(\boldsymbol{\square}, ~ \square),(\square, \varnothing)$ | 320 | 8 |
| 365 | $\begin{array}{lll}0 & 6 \\ 0 & 0\end{array}$ | 12 | ( $\square$ ), ( $\mathrm{\square}),(\square),(\square)$ | 224 | 12 |
| 366 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{array}$ |  | $(\varnothing, \varnothing, \square),(\square, \square, \boldsymbol{m})$ | 336 | 20 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 367 | $\begin{array}{lllll} 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \square),(\varnothing, \square, \varnothing, \square)$ | 304 | 16 |
| 368 | $\begin{array}{llll} 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \square, \square)$ | 282 | 20 |
| 369 | $\begin{array}{lllll} 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \square, \varnothing, \varnothing)$ | 304 | 20 |
| 370 | $\begin{array}{lllll} 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \square),(\square, \square, \varnothing, \square \square)$ | 294 | 25 |
| 371 | $\begin{array}{llllll} 0 & 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{array}$ | $\begin{array}{lllllll}1 & 1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \varnothing, \square, \square),(\varnothing, \varnothing, \square, \square, \square),(\varnothing, \square, \varnothing, \square, \square)$ | 288 | 20 |
| 372 | $\begin{array}{llll} 0 & 3 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 2 & 2\end{array}$ | $(\varnothing, \varnothing, \boxminus),(\varnothing, \square, \varnothing),(\varnothing, \boxminus, \varnothing),(\square, \varnothing, \varnothing)$ | 272 | 12 |
| 373 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \square \square),(\square, \varnothing, \varnothing)$ | 256 | 16 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 374 | $\begin{array}{lll} 0 & 7 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$ | $1 \quad 12$ | $(\square, \square),(\square, \square)$ | 193 | 21 |
| 375 | $\begin{array}{llll} 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \square, \square)$ | 208 | 20 |
| 376 | $\begin{array}{llll} 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\square, \varnothing, \square),(\square, \square, \varnothing)$ | 186 | 33 |
| 377 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \varnothing),(\varnothing, \square, \varnothing, \varnothing)$ | 288 | 24 |
| 378 | $\begin{array}{lllll} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \square \square),(\square, \varnothing, \square, \square)$ | 282 | 22 |
| 379 | $\begin{array}{llllll} 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{array}$ | $\begin{array}{llllllll}1 & 1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \varnothing, \varnothing, \square)$ | 272 | 20 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 380 | $\begin{array}{lllll} 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \varnothing, \square),(\varnothing, \square, \square, \square)$ | 256 | 22 |
| 381 | $\begin{array}{lllll} 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \square),(\varnothing, \square, \varnothing, \varnothing)$ | 256 | 16 |
| 382 | $\begin{array}{llllll} 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \end{array}$ | $\begin{array}{lllllll}1 & 1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \square, \varnothing),(\varnothing, \square, \varnothing, \square, \square),(\varnothing, \square, \varnothing, \varnothing, \square)$ | 246 | 21 |
| 383 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \square),(\varnothing, \square, \varnothing),(\square, \square, \square \square)$ | 240 | 16 |
| 384 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, ■),(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing)$ | 224 | 13 |
| 385 | $\begin{array}{lll} 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | 122 | $(\varnothing, \boxminus),(\varnothing, \boxminus),(\boxminus, \varnothing),(\boxminus, \varnothing)$ | 193 | 15 |
| 386 | $\begin{array}{llll} 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing)$ | 252 | 20 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 387 | $\begin{array}{lllll} 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \varnothing, \square),(\varnothing, \square, \square, \square)$ | 256 | 20 |
| 388 | $\begin{array}{llll} 0 & 0 & 0 & 5 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1 & 2\end{array}$ | $(\varnothing, \varnothing, \boxminus),(\varnothing, \varnothing, \boxminus),(\varnothing, \varnothing, \boxminus),(\square, \varnothing, \varnothing)$ | 240 | 12 |
| 389 | $\begin{array}{lll} 0 & 2 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$ | 111 | $(\varnothing, \square),(\varnothing, \square)$ | 192 | 20 |
| 390 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \square),(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \square, \varnothing, \varnothing),(\varnothing, \square, \square, \square \square)$ | 240 | 16 |
| 391 | $\begin{array}{llllll} 0 & 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{array}$ | $\begin{array}{lllllll}1 & 1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \square, \square),(\varnothing, \square, \varnothing, \square \square \square),(\square, \varnothing, \varnothing, \square, \varnothing)$ | 230 | 22 |
| 392 | $\begin{array}{lllll} 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square, \square),(\square, \varnothing, \square, \square)$ | 219 | 20 |
| 393 | $\begin{array}{lllll} 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1 \\ & & & & \end{array}$ | $(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \square, \varnothing, \varnothing)$ | 220 | 22 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 394 | $\begin{array}{lllll} 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \square),(\varnothing, \square, \square, \square)$ | 214 | 19 |
| 395 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\varnothing, \square, \square \square)$ | 208 | 14 |
| 396 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 3 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \varnothing),(\varnothing, \square, \square, \square \square),(\square, \varnothing, \varnothing, \varnothing),(\square, \varnothing, \square, \square \square)$ | 204 | 18 |
| 397 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\square, \varnothing, \varnothing),(\square, \square, \square)$ | 192 | 18 |
| 398 | $\begin{array}{lll} 0 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 2\end{array}$ | $(\varnothing, \boxminus),(\varnothing, \square)$ | 164 | 20 |
| 399 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \square, \varnothing, \square \square)$ | 230 | 22 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 400 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & x_{3} & 1 & 0 \end{array}$ | $1 \begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \varnothing, \square),(\square, \square, \square, \square \square \square)$ | 224 | 12 |
| 401 |  |  | $(\varnothing, \varnothing, \varnothing, \square),(\varnothing, \square, \varnothing, \varnothing)$ | 224 | 22 |
| 402 | $\begin{array}{ll} 0 & 0 \\ 0 & 3 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{array} 0$ | 11111 | $(\varnothing, \varnothing, \square),(\square, \square, \varnothing)$ | 204 | 20 |
| 403 |  | $1 \begin{array}{lllll}1 & 1 & 1\end{array}$ | $(■, \varnothing, \square, \varnothing),(\varnothing, \varnothing, \varnothing, \boxminus),(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \square, ■, \boxminus)$ | 204 | 18 |
| 404 | $\begin{array}{ll} 0 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} 0_{0}^{0} 0$ | 1122 | $(\boldsymbol{m}, \varnothing, \boxminus),(\boldsymbol{m}, ~ \exists, \varnothing),(\mathbf{m}, \boxminus, \varnothing)$ | 193 | 17 |
| 405 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}$ |  | $(\varnothing, \varnothing, \square),(\square, \square, \square)$ | 192 | 22 |
| 406 | $\begin{array}{lllll} 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}$ | 1122 | $(\boldsymbol{m}, ~(),, \varnothing),(\boldsymbol{m}, ~(, ~ \varnothing),(\varnothing, \square, \varnothing)$ | 183 | 16 |
| 407 | $\begin{array}{lll} 0 & 6 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}$ | $1{ }^{1} 2$ | (■, $\square$ ), (■, ${ }^{\text {, }}$ ) | 161 | 31 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 408 | $\begin{array}{llllll} 0 & 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{array}$ | $\begin{array}{lllllll}1 & 1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \varnothing, \square, \square),(\varnothing, \square, \varnothing, \square, \square),(\square, \square, \varnothing, \varnothing, \square)$ | 209 | 23 |
| 409 | $\begin{array}{llll} 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \varnothing, \square)$ | 182 | 22 |
| 410 | $\begin{array}{lllll} 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \end{array}$ | $1 \begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \square),(\square, \varnothing, \square, \varnothing)$ | 197 | 24 |
| 411 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \varnothing),(\varnothing, \square, \varnothing, \varnothing)$ | 203 | 19 |
| 412 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \square, \square, \square \square)$ | 188 | 21 |
| 413 | $\begin{array}{llll} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1 & 2\end{array}$ | $(\varnothing, \square \square, \boxminus),(\varnothing, \varnothing, \square),(\square, \square \square, \varnothing)$ | 188 | 14 |
| 414 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 3 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \square, \varnothing, \varnothing),(\varnothing, \square, \square, \square \square),(\square, \varnothing, \varnothing, \varnothing)$ | 188 | 20 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 415 |  | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \square),(\varnothing, \square, \boldsymbol{\square}),(\square, \varnothing, \varnothing)$ | 184 | 24 |
| 416 |  |  | $(\varnothing, \square, \square, \square),(\square, \varnothing, \varnothing, \varnothing)$ | 182 | 24 |
| 417 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{array}$ |  | $(\varnothing, \square, \varnothing, \varnothing),(\square, \varnothing, \square, ■)$ | 182 | 22 |
| 418 | $\begin{array}{llll}0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0\end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 176 | 20 |
| 419 |  | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\square, \varnothing, \square),(\square, \square, \varnothing)$ | 162 | 21 |
| 420 | $\begin{array}{lllll} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{array}$ | 11111 | $(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing)$ | 161 | 22 |
| 421 | $\begin{array}{lllll} 0 & 0 & 0 & 5 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | 1112 | $(\varnothing, \varnothing, \square),(\varnothing, \varnothing, \boxminus),(\varnothing, \varnothing, \boxminus),(\varnothing, \square, \varnothing)$ | 183 | 15 |
| 422 | $\begin{array}{llll}0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0\end{array}$ | 1112 | $(\varnothing, \square, \varnothing),(\varnothing, \square, \square),(\square, \varnothing, \boxminus)$ | 172 | 18 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 423 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array} 0_{0}^{0} 0$ | 1112 | $(■, \square, \square),(\varnothing, \varnothing, \boxminus),(\varnothing, \varnothing, \boxminus),(\square, \varnothing, \varnothing)$ | 167 | 19 |
| 424 | $\begin{array}{llll} 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}$ | 1112 | $(\varnothing, \varnothing, \boxminus),(\varnothing, \varnothing, \boxminus),(\square, \square, \varnothing),(\square, ■, \boxminus)$ | 161 | 20 |
| 425 | $\begin{array}{lll} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{array}$ | 111 | ( $\varnothing, \square$ ), (■,■) | 160 | 30 |
| 426 | $\begin{array}{llll}0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{array}$ | 123 | $(\square, \varnothing),(\square, \boxminus)$ | 147 | 19 |
| 427 |  |  | $(\varnothing, \varnothing, \square),(\varnothing, \square, \boldsymbol{m})$ | 192 | 4 |
| 428 |  | 1112 | $(■, \square, \square),(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing)$ | 163 | 17 |
| 429 | $\begin{array}{lllll} 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 \end{array}$ |  | $(\varnothing, \square, \square, \varnothing),(\square, \varnothing, \varnothing, \varnothing)$ | 167 | 24 |
| 430 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & \\ & & \end{array}$ | $(■, \square, \varnothing),(\square, \square, \boxminus),(\varnothing, \varnothing, \boxminus),(\varnothing, \square, \varnothing)$ | 162 | 18 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 431 | $\begin{array}{lll} 0 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}$ | 122 | $(\varnothing, \boxminus),(\varnothing, \square)$ | 148 | 20 |
| 432 |  |  | $(\varnothing, \varnothing, \square),(\varnothing, \square, ■),(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing)$ | 156 | 20 |
| 433 | $\begin{array}{lllll} 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\square, \square, \square)$ | 141 | 33 |
| 434 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \square),(\varnothing, \square, \square)$ | 161 | 24 |
| 435 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \square),(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing)$ | 151 | 22 |
| 436 | $\begin{array}{llll} 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}$ | $1 \begin{array}{lllll}1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \square)$ | 151 | 25 |
| 437 | $\begin{array}{lllll} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square \square),(\square, \varnothing, \square)$ | 148 | 22 |
| 438 | $\begin{array}{lllll} 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}$ | 1112 | $(\varnothing, \varnothing, \square),(\square, \square, \square)$ | 131 | 25 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 439 | $\begin{array}{lll} 0 & 5 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | $1{ }^{1} 4$ | $(\varnothing, \boxminus),(\varnothing, \theta),(\varnothing, \exists),(\varnothing, \boxminus),(\square, \square)$ | 141 | 19 |
| 440 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\square, \varnothing, \square)$ | 130 | 26 |
| 441 | $\begin{array}{lll} 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{array}$ | $1 \begin{array}{lll}1\end{array}$ | $(\square, \square),(\square, \square)$ | 129 | 41 |
| 442 | $\begin{array}{lll} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\varnothing, \square),(\varnothing, \square)$ | 256 | 0 |
| 443 | $\begin{array}{ll} 0 & 5 \\ 0 & 0 \end{array}$ | 12 | $(\square)$ | 192 | 16 |
| 444 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \square \square)$ | 288 | 24 |
| 445 | $\begin{array}{llll} 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \square)$ | 240 | 8 |
| 446 | $\begin{array}{llll} 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \square)$ | 218 | 20 |
| 447 | $\begin{array}{lllll} 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \square),(\varnothing, \square, \varnothing, \square \square)$ | 252 | 30 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 448 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 1 & 0 \end{array}$ | $1 \begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square, \square \square),(\varnothing, \square, \varnothing, \square \square)$ | 240 | 20 |
| 449 | $\begin{array}{llll} 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \varnothing, \square),(\square, \varnothing, \square),(\square, \square, \square)$ | 188 | 8 |
| 450 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1 & 2\end{array}$ | $(\varnothing, \varnothing, \boxminus),(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing)$ | 224 | 12 |
| 451 | $\begin{array}{llll} 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing)$ | 180 | 36 |
| 452 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 2 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square, \square \square),(\varnothing, \square, \square, \square \square)$ | 224 | 16 |
| 453 | $\begin{array}{llll} 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\varnothing, \square, \square)$ | 208 | 4 |
| 454 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 3 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \varnothing, \square),(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \square, \square, \square \square),(\square, \square, \varnothing, \square \square)$ | 204 | 22 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 455 |  | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\varnothing, \square, \square)$ | 176 | 24 |
| 456 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \square)$ | 270 | 21 |
| 457 |  | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, ■),(\varnothing, \square, \varnothing),(\square, \varnothing, \boldsymbol{\square})$ | 208 | 24 |
| 458 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, ■),(\varnothing, \square, \square)$ | 206 | 13 |
| 459 | $\begin{array}{llllll} 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$ |  | $(\varnothing, \varnothing, \square, \varnothing),(\square, \square, \varnothing, \square \square)$ | 202 | 28 |
| 460 | $\begin{array}{lllll} 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}$ | 1122 | $(\boldsymbol{m}, ~ \exists, \varnothing),(\boldsymbol{m}, \boxminus, \varnothing),(\boldsymbol{m}, \boxminus, \varnothing),(\square, \varnothing, \varnothing)$ | 208 | 12 |
| 461 | $\begin{array}{llllll} 0 & 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{array}$ |  | $(\varnothing, \varnothing, \varnothing, \square \square \square),(\varnothing, \varnothing, \square, \square, \square),(\square, \square, \varnothing, \square, \varnothing)$ | 198 | 24 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 462 | $\begin{array}{lllll} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \end{array}$ |  | $(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \square, \varnothing, \varnothing),(\square, \square, \varnothing, \boldsymbol{\square})$ | 188 | 19 |
| 463 | $\begin{array}{lllll} 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \varnothing, \square)$ | 172 | 20 |
| 464 | $\begin{array}{llll}0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{array}$ | 123 | $(\varnothing, \boxminus),(\varnothing, \nabla),(\square, \varnothing),(\square, \varnothing)$ | 162 | 14 |
| 465 | $\begin{array}{llll} 0 & 0 & 0 & 2 \\ 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} 0$ |  | $(\square, \square, \varnothing, ■),(\square, \varnothing, \varnothing, \boldsymbol{\square})$ | 188 | 24 |
| 466 | $\begin{array}{llll} 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 \end{array}$ | $11_{1} 11$ | $(\varnothing, \square, \square),(\varnothing, \square, \square)$ | 185 | 17 |
| 467 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, ■),(\varnothing, \square, \varnothing),(\square, \square, \boldsymbol{\Pi})$ | 172 | 22 |
| 468 | $\begin{array}{lllll} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 3 & 0 \end{array}$ | 11111 | $(\varnothing, \square, ■),(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing),(\square, \varnothing, \Pi \square)$ | 182 | 22 |
| 469 | $\begin{array}{llll}0 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0\end{array}$ | $\begin{array}{lllll}1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \varnothing)$ | 172 | 24 |

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| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 470 | 0 0 0 0 2 <br> 0 0 0 1 0 <br> 0 0 0 1 0 <br> 0 0 0 0 0 <br> 0 2 3 1 0 |  | $(\varnothing, \varnothing, \square, \boldsymbol{\square}),(\varnothing, \square, \varnothing, \varnothing)$ | 182 | 20 |
| 471 |  |  | $(\varnothing, \square, \varnothing, \varnothing),(\varnothing, \square, \varnothing, \varnothing)$ | 172 | 22 |
| 472 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}$ | 1112 | $(\varnothing, \varnothing, \square),(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing)$ | 162 | 18 |
| 473 | $\begin{array}{llll}0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{array}$ | 122 | $(\square, \square)$ | 146 | 20 |
| 474 | $\begin{array}{llll}0 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}$ | 122 | $(\varnothing, \boxminus),(\varnothing, \boxminus),(\square, \boxminus)$ | 165 | 6 |
| 475 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & 4 \end{array}$ |  | $(\varnothing, \varnothing, \square, \square),(\varnothing, \square, \varnothing, \varnothing),(\varnothing, \square, \square, \square \square),(\square, \varnothing, \square, \square)$ | 162 | 22 |
| 476 | $\begin{array}{llll} 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ |  | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 156 | 20 |
| 477 |  0 1 2 2 <br> 0 0 0 0 0 <br> 0 0 0 0 0 <br> 0 0 1 0 0 <br> 0 3 1 0 0 |  | $(\square, \square, \varnothing, ■),(\square, \square, \varnothing, ■)$ | 167 | 22 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 478 |  | 1123 |  | 162 | 16 |
| 479 |  | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\square, \varnothing, \square),(\square, \varnothing, \square)$ | 146 | 24 |
| 480 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, ■),(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing),(\square, \square, \boldsymbol{\square})$ | 146 | 22 |
| 481 | $\begin{array}{llll} 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \square, \varnothing)$ | 150 | 32 |
| 482 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & y^{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$ | $1 \begin{array}{lllll}1 & 1 & 1 & 2\end{array}$ | $(\varnothing, \varnothing, \square, \boxminus),(\square, \square, \Pi \square, \square)$ | 162 | 15 |
| 483 | $\begin{array}{lll}0 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{array}$ | 122 | $(\varnothing, \boxminus),(\square, \varnothing),(\boxminus, \varnothing),(\square, \varnothing)$ | 146 | 16 |
| 484 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, ■),(\varnothing, \square, \varnothing),(\varnothing, \square, \square \square),(\square, \varnothing, \varnothing)$ | 146 | 20 |
| 485 | $\begin{array}{llll} 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\square, \varnothing, \square)$ | 144 | 34 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 486 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \square, \square)$ | 140 | 36 |
| 487 | $\begin{array}{ll} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} 0$ | $1{ }_{1} 111$ | $(\square, \square, \varnothing)$ | 133 | 35 |
| 488 | $\begin{array}{llll} 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\square, \varnothing, \square)$ | 142 | 20 |
| 489 |  |  | $(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing),(\square, \square, \boldsymbol{\square})$ | 136 | 22 |
| 490 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} 0_{0}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\square, \square, \square)$ | 130 | 26 |
| 491 | $\begin{array}{ll} 0 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} 0$ | 11111 | $(\varnothing, \square, \square),(\square, \square, \varnothing)$ | 130 | 34 |
| 492 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array} 0$ |  | $(\varnothing, \square, \varnothing),(\square, \square, \square)$ | 125 | 34 |
| 493 | $\begin{array}{llll} 0 & 2 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$ |  | $(\varnothing, \varnothing, \square),(\varnothing, \square, \square)$ | 140 | 24 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 494 | $\begin{array}{lll} 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{array}$ | 111 | $(\varnothing, \square),(\square, \varnothing)$ | 128 | 40 |
| 495 | $\begin{array}{lll} 0 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\square, \square),(\square \square, \varnothing)$ | 98 | 52 |
| 496 | $\begin{array}{ll} 0 & 5 \\ 0 & 0 \end{array}$ | 12 | $(\square),(\square)$ | 160 | 28 |
| 497 | $\begin{array}{lllll} 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \varnothing, \square),(\varnothing, \square, \varnothing, \square \square)$ | 216 | 36 |
| 498 | $\begin{array}{llll} 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \varnothing)$ | 224 | 16 |
| 499 | $\begin{array}{lllll} 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \square),(\square, \varnothing, \square, \square)$ | 240 | 28 |
| 500 | $\begin{array}{lllll} 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \varnothing),(\square, \varnothing, \square, \square)$ | 208 | 16 |
| 501 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\square, \varnothing, \varnothing),(\square, \square, \square \square \square)$ | 192 | 12 |

Continued from previous page.


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 510 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \square),(\square, \square, \varnothing, \square \square)$ | 176 | 20 |
| 511 | $\begin{array}{llll} 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \square)$ | 176 | 4 |
| 512 | $\begin{array}{llll} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 1 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1 & 2\end{array}$ | $(\varnothing, ■ \square, \boxminus),(\varnothing, \square, \boxminus),(\square, \square, \varnothing),(\square \square, \square \square, \varnothing)$ | 152 | 16 |
| 513 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \varnothing),(\square, \square, \square \square),(\square, \square, \square \square)$ | 152 | 22 |
| 514 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \varnothing),(\varnothing, \square, \square, \square)$ | 156 | 24 |
| 515 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \square, \varnothing)$ | 144 | 24 |
| 516 | $\begin{array}{lllll} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}$ | $1 \begin{array}{llllll}1 & 1 & 1 & 2 & 2\end{array}$ | $(\varnothing, \boldsymbol{\square}, \boxminus, \varnothing),(\varnothing, \square \square, \exists, \varnothing),(\varnothing, \square, \boxminus, \varnothing),(\square, \square, \varnothing, \varnothing)$ | 141 | 20 |
| 517 | $\begin{array}{lll}0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{array}$ | $1 \begin{array}{lll}1 & 1 & 2\end{array}$ | $(\varnothing, \boxminus),(\varnothing, \boxminus),(\square, \square)$ | 131 | 18 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 518 | $\begin{array}{lllll} 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}$ | 11111 | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 134 | 38 |
| 519 | $\begin{array}{llll} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 1 & 0 & 0 \end{array}$ | 11212 | $(\varnothing, \boldsymbol{\square}, ~ \exists),(\square, \boldsymbol{\square} \boldsymbol{\square}, \exists),(\square, \square, \varnothing),(\square, \varnothing, \varnothing)$ | 141 | 21 |
| 520 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 3 & 2 \end{array}$ |  | $(\varnothing, \square, \square, \boldsymbol{\square}),(\varnothing, \square, \square, ■)$ | 146 | 25 |
| 521 | $\begin{array}{llll} 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | 1112 | $(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing),(\varnothing, \square, \nabla),(\square, \varnothing, В)$ | 136 | 22 |
| 522 |  | 11111 | $(\varnothing, \varnothing, \square),(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing),(\square, \square, \boldsymbol{\square})$ | 136 | 20 |
| 523 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing)$ | 131 | 22 |
| 524 |  | 1112 | $(\varnothing, \varnothing, \boxplus),(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing),(\square, \square, \varnothing)$ | 136 | 8 |
| 525 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}$ | $1 \begin{array}{lllll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 124 | 27 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 526 | $\begin{array}{lllll} 0 & 4 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}$ | 1112 | $(\varnothing, \square, \varnothing),(\square, \varnothing, \boxminus),(\square, \square, \square)$ | 116 | 25 |
| 527 | $\begin{array}{lll} 0 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$ | 112 | $(\varnothing, \boxminus),(\varnothing, \boxminus),(\varnothing, \boxminus),(\varnothing, \boxminus)$ | 114 | 23 |
| 528 | $\begin{array}{lll}0 & 0 & 5 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 3 & 0\end{array}$ | 111 | $(\square, ■),(\square, \square)$ | 112 | 68 |
| 529 | $\begin{array}{lll} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 112 | $(\varnothing, \square),(\square, \varnothing)$ | 110 | 23 |
| 530 | $\begin{array}{llll}0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0\end{array}$ | 111 | ( $\varnothing, \square$ ), (■,■) | 192 | -12 |
| 531 |  | $1{ }_{1} 111$ | $(\varnothing, \varnothing, \square),(\square, \square, \square \square \square)$ | 240 | 28 |
| 532 | 0 0 0 0 3 <br> 0 0 0 0 0 <br> 0 0 0 1 0 <br> 0 0 0 0  <br> 0 2 3 0 0 <br>     0 | $1 \begin{array}{lllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square, \boldsymbol{\square}),(\square, \varnothing, \varnothing, \square)$ | 210 | 35 |
| 533 | $\begin{array}{lllll} 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}$ | 11111 | $(\varnothing, \varnothing, \square),(\varnothing, \square, \varnothing)$ | 180 | 40 |
| 534 | $\begin{array}{lllll} 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}$ | 11811 | $(\varnothing, \varnothing, \square \square),(\square, \square, \square)$ | 141 | -3 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 535 |  | 11111 | $(\varnothing, \square, \square),(\square, \varnothing, ■)$ | 176 | -4 |
| 536 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \square)$ | 154 | 30 |
| 537 |  | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \square),(\varnothing, \square, \varnothing),(\varnothing, \square, \square \square)$ | 192 | 16 |
| 538 |  | $\begin{array}{llllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square, \square),(\square, \varnothing, \square, \square)$ | 192 | 16 |
| 539 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & y_{3} & 1 & 0 \end{array}$ |  | $(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \square, \varnothing, ■)$ | 162 | 19 |
| 540 | $\begin{array}{ll} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{array} 0$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \varnothing, \varnothing)$ | 156 | 24 |
| 541 |  |  | $(\varnothing, \square, \square, ■),(\varnothing, \square, \varnothing, \boldsymbol{\square})$ | 156 | 28 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 542 | $\begin{array}{llll} 0 & 0 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}$ |  | $(\varnothing, \varnothing, \varnothing, \square),(\square, \varnothing, \varnothing, \varnothing)$ | 146 | 24 |
| 543 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} 0_{0}^{0}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \square, \square)$ | 125 | 29 |
| 544 |  |  | $(\varnothing, \square, \square),(\varnothing, \square, \varnothing)$ | 134 | 34 |
| 545 |  | $1{ }_{1} 1111$ | $(\varnothing, \varnothing, \square),(\square, \square, \Pi \square)$ | 160 | 24 |
| 546 | $\begin{array}{lllll} 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 2 & 0 \end{array}$ | $\begin{array}{lllllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing, \square),(\square, \varnothing, \square, \square)$ | 155 | 24 |
| 547 | $\begin{array}{llll}0 & 3 & 4 \\ 0 & & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{array}$ | 112 | ( $\square, \square),(\square, \square)$ | 116 | 21 |
| 548 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 2 & 2 \end{array}$ |  | $(\varnothing, \varnothing, \square, \varnothing),(\varnothing, \square, \square, \square \square \square)$ | 146 | 23 |
| 549 | $\begin{array}{lll} 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | $1{ }^{1} 2$ | $(\varnothing, \boxminus),(\varnothing, \boxminus),(\varnothing, \boxminus),(\square, \varnothing)$ | 126 | 16 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 550 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \square, \square \square)$ | 130 | 23 |
| 551 | $\begin{array}{lllll} 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 \end{array}$ | $\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square, \varnothing),(\square, \varnothing, \square, \varnothing)$ | 126 | 27 |
| 552 | $\begin{array}{lll} 0 & 5 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 11  | $(\varnothing, \boxminus),(\varnothing, \boxminus),(\square, \square),(\square, \square)$ | 116 | 19 |
| 553 | $\begin{array}{llll} 0 & 2 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\varnothing, \square, \varnothing)$ | 120 | 26 |
| 554 | $\begin{array}{lll} 0 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | $1 \begin{array}{lll}1 & \\ \end{array}$ | $(\square, \square)$ | 90 | 18 |
| 555 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 2\end{array}$ | $(\varnothing, \varnothing, \boxminus),(\varnothing, \square, \boxminus)$ | 120 | 32 |
| 556 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 119 | 34 |
| 557 | $\begin{array}{llll} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, ■),(\varnothing, \square, \square)$ | 113 | 45 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 558 | $\begin{array}{lll} 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | 122 | $(\varnothing, \boxminus),(\boxplus, \varnothing)$ | 114 | 32 |
| 559 | $\begin{array}{llll}0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{array}$ | 112 | $(\varnothing, \boxminus),(\varnothing, \boxminus),(\square, \boxminus)$ | 110 | 30 |
| 560 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, ■),(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing),(\square, \square, ■)$ | 110 | 34 |
| 561 |  | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \square, \varnothing)$ | 110 | 34 |
| 562 | $\begin{array}{llll}0 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0\end{array}$ | 112 | $(\varnothing, \square)$ | 99 | 33 |
| 563 | $\begin{array}{llll} 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 113 | 40 |
| 564 | $\begin{array}{ll} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 3 \end{array} 0_{0}^{0}$ | 11111 | $(\square, \varnothing, \square),(\square, \square, \square)$ | 103 | 63 |
| 565 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \end{array}$ | 11111 | $(\varnothing, \square, ■),(\square, \square, \varnothing)$ | 98 | 55 |
| 566 | $\begin{array}{lll} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 112 | $(\varnothing, \square),(\varnothing, \square),(\square, \varnothing)$ | 94 | 36 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 567 | $\begin{array}{lll} 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 111 | $(\square, \square)$ | 81 | 101 |
| 568 | $\begin{array}{lll}0 & 8 \\ 0 & 0\end{array}$ | 11 | ( $\square$ ), ( $\square$ ), ( $\square$ ) | 128 | 48 |
| 569 | $\begin{array}{lll} 0 & 1 & 7 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | 111 | $(\varnothing, \square),(\varnothing, \square),(\square, \square)$ | 124 | 36 |
| 570 |  |  | $(\varnothing, \varnothing, \square, \boldsymbol{\square}),(\varnothing, \square, \varnothing, ■)$ | 180 | 42 |
| 571 | $\begin{array}{lll} 0 & 2 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 111 | $(\varnothing, \square),(\square, \square),(\square, \square)$ | 114 | 42 |
| 572 |  |  | $(\varnothing, \square, \square, \square),(\square, \varnothing, \square, \square)$ | 176 | 12 |
| 573 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 \end{array}$ | 11111 | $(\varnothing, \square, \square),(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing)$ | 160 | 0 |
| 574 | $\begin{array}{llll} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} 0_{0} 0$ | $1 \begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square, \boldsymbol{\square}),(\varnothing, \square, \varnothing, \square)$ | 146 | 16 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 575 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \mathbf{\square}),(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing)$ | 160 | 4 |
| 576 |  | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \varnothing),(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 136 | 24 |
| 577 | $\begin{array}{llll} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} 0$ |  | $(\varnothing, \square, \varnothing, \varnothing),(\varnothing, \square, \square, \square \square \square)$ | 140 | 26 |
| 578 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, ■),(\varnothing, \square, \varnothing),(\square, \square, \square \square \square)$ | 132 | 22 |
| 579 | $\begin{array}{lll} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{array}$ | 112 | $(\square, \varnothing),(\square, \varnothing),(\square, \boxminus)$ | 101 | 31 |
| 580 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \square, \square \square)$ | 144 | -8 |
| 581 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \square),(\square, \square, \boldsymbol{\square}),(\square, \square, \square)$ | 120 | 36 |
| 582 | $\begin{array}{llll}0 & 0 & 5 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{array}$ | 113 | $(\varnothing, \boxminus),(\varnothing, \boxminus),(\varnothing, \exists),(\square, \varnothing)$ | 116 | 16 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector |  |  |  | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 583 | $\begin{array}{lll} 0 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 1 | 2 | 2 |  | $(\square, \boxminus),(\square, \square)$ | 106 | 21 |
| 584 | $\begin{array}{llll} 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \end{array}$ | 1 | 1 | 1 | 1 | $(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing)$ | 115 | 27 |
| 585 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 0 \end{array}$ | 1 | 1 | 1 | 1 | $(\varnothing, \square, \square),(\varnothing, \square, \square),(\square, \varnothing, \varnothing),(\square, \varnothing, \varnothing)$ | 116 | 22 |
| 586 | $\begin{array}{lll} 0 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | 1 | 1 | 1 |  | $(\square, \square),(\square, \square),(\square, \square)$ | 99 | 51 |
| 587 | $\begin{array}{llll} 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \end{array}$ | 1 | 1 | 1 | 1 | $(\varnothing, \varnothing, \square),(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing),(\square, \square, \square \square)$ | 120 | 24 |
| 588 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 0 \end{array}$ | 1 | 1 | 1 | 1 | $(\varnothing, \square, \square),(\varnothing, \square, \square),(\square, \varnothing, \varnothing),(\square, \square, \square \square)$ | 110 | 26 |
| 589 | $\begin{array}{lll} 0 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 1 | 2 | 2 |  | $(\square, \boxminus),(\square, \boxminus)$ | 85 | 28 |
| 590 | $\begin{array}{lll} 0 & 2 & 6 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | 1 | 1 | 1 |  | $(\varnothing, \square),(\varnothing, \square),(\square, \varnothing)$ | 120 | 24 |
| 591 | $\begin{array}{llll} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{array}$ | 1 | 1 | 1 | 2 | $(\varnothing, \Pi \square, \boxminus),(\varnothing, \Pi \square, \boxminus),(\varnothing, \Pi \square, \boxminus),(\varnothing, \Pi, \boxminus)$ | 160 | 8 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 592 | $\begin{array}{llll} 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \square, \square)$ | 134 | 36 |
| 593 | $\begin{array}{ll} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array} 0_{0}^{0}$ | 1112 | $(\varnothing, \boldsymbol{m}, \boxminus),(\varnothing, \boldsymbol{m}, ~ \exists),(\varnothing, \boldsymbol{m}, ~ \exists),(\square, \boldsymbol{m}, ~ \exists)$ | 130 | 20 |
| 594 | $\begin{array}{lllll} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 \end{array}$ |  | $(\varnothing, \square, \square, \square),(\square, \varnothing, \square, \square)$ | 125 | 28 |
| 595 | $\begin{array}{llll} 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}$ | $1 \begin{array}{llllll}1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \square)$ | 112 | 56 |
| 596 | $\begin{array}{lll} 0 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | 111 | $(\varnothing, \square),(\square, \square),(\square, \varnothing)$ | 104 | 40 |
| 597 | $\begin{array}{lll}0 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0\end{array}$ | 112 | $(\square, \square),(\square, \square)$ | 100 | 32 |
| 598 | $\begin{array}{lllll} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}$ | 1111 | $(\varnothing, \varnothing, \square),(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing)$ | 110 | 26 |
| 599 | $\begin{array}{llll}0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0\end{array}$ |  | $(\varnothing, \square, \square),(\square, \varnothing, \square)$ | 104 | 37 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 600 | $\begin{array}{lll} 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | 112 | $(\varnothing, \square),(\square, \square),(\square, \varnothing)$ | 100 | 24 |
| 601 |  |  | $(\varnothing, \square, ■),(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing),(\square, \square, \square)$ | 100 | 36 |
| 602 | $\begin{array}{llll} 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \varnothing)$ | 99 | 40 |
| 603 | $\begin{array}{llll}0 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0\end{array}$ | 122 | $(\varnothing, \boxminus),(\varnothing, \boxminus),(\varnothing, \boxminus),(\varnothing, \boxminus)$ | 100 | 22 |
| 604 | $\begin{array}{llll}0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{array}$ | 113 | $(\varnothing, \boxminus),(\varnothing, \boxminus),(\square, \varnothing),(\square, \boxminus)$ | 95 | 31 |
| 605 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\square, \square, ■),(\square, \square, ■)$ | 90 | 45 |
| 606 | $\begin{array}{lll} 0 & 4 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}$ | 111 | $(\varnothing, \square),(\square, \square),(\square, \square)$ | 84 | 52 |
| 607 | $\begin{array}{lllll} 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 \end{array}$ |  | $(\varnothing, \square, \varnothing, \square),(\square, \varnothing, \square, \square)$ | 150 | 49 |
| 608 | $\begin{array}{llll}0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 3 & 0\end{array}$ | $1{ }^{1} 11$ | $(\varnothing, \square),(\varnothing, \square)$ | 216 | 24 |

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| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 609 | $\begin{array}{lll} 0 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$ | $1 \quad 11$ | $(\varnothing, \square),(\varnothing, \square)$ | 132 | 12 |
| 610 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \square),(\varnothing, \square, \square \square),(\square, \square, \square)$ | 120 | 40 |
| 611 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\varnothing, \square, \varnothing),(\varnothing, \square, \square \square),(\square, \varnothing, \square)$ | 116 | 20 |
| 612 | $\begin{array}{lll} 0 & 6 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | $1 \begin{array}{lll}1 & \\ \end{array}$ | $(\square, \varnothing),(\square \square \square)$ | 113 | -11 |
| 613 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square \square, \square, \square \square \square)$ | 102 | 21 |
| 614 | $\begin{array}{lll} 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 4\end{array}$ | $(\square, \square)$ | 86 | 34 |
| 615 | $\begin{array}{llll} 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \square),(\varnothing, \square, \varnothing),(\square, \square, \square)$ | 100 | 34 |
| 616 | $\begin{array}{lll} 0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | $1 \begin{array}{lll}1 & \\ \end{array}$ | $(\varnothing, \square),(\square, \square),(\square, \varnothing)$ | 96 | 8 |
| 617 | $\begin{array}{lll} 0 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1 & 2\end{array}$ | $(\varnothing, \square),(\square, \square)$ | 90 | 25 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 618 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array} 0$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \varnothing, \square)$ | 192 | 32 |
| 619 | $\begin{array}{lll} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 7 & 0 \end{array}$ | 111 | $(\square, \square),(\square, \varnothing),(\square, \square \square),(\square, \Pi \square)$ | 128 | 4 |
| 620 | $\begin{array}{lllll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \square)$ | 144 | 4 |
| 621 | $\begin{array}{lll} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{array}$ | 111 | $(\varnothing, \square),(\square, \square)$ | 112 | 24 |
| 622 | $\begin{array}{llllll} 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 \end{array}$ |  | $(\varnothing, \square, \square, \varnothing),(\square, \varnothing, \varnothing, \varnothing)$ | 114 | 31 |
| 623 | $\begin{array}{llll} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{array}$ | $11_{1} 12$ | $(\varnothing, \boldsymbol{m}, ~ \exists),(\varnothing, \boldsymbol{m}, ~ \exists),(\varnothing, \boldsymbol{\square}, \square),(\square, \boldsymbol{m m}, ~ \exists)$ | 105 | 25 |
| 624 | $\begin{array}{llll}0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0\end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \square),(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing),(\square, \varnothing, \square)$ | 95 | 29 |
| 625 | $\begin{array}{lll} 0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 112 | ( $\square, \square),(\square, \square),(\square, \square)$ | 85 | 42 |
| 626 | $\begin{array}{lll}0 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{array}$ | 112 | $(\varnothing, \square),(\square, \square),(\square, \varnothing)$ | 90 | 32 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 627 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} 0_{0} 0$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \square, \square \Pi)$ | 83 | 72 |
| 628 | $\begin{array}{lll} 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 113 | $(\square, \square),(\square, \varnothing)$ | 80 | 36 |
| 629 | $\begin{array}{llll} 0 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 122 | $(\varnothing, \boxminus),(\square, \square),(\square, \square)$ | 80 | 35 |
| 630 | $\begin{array}{ll} 0 & 7 \\ 0 & 0 \end{array}$ | 11 | (■),(■) | 96 | 90 |
| 631 | $\begin{array}{lll} 0 & 1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | 111 | ( $\varnothing, \square \square),(\square, \square)$ | 93 | 51 |
| 632 | $\begin{array}{lll} 0 & 1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 111 | ( $\varnothing, \square$ ),(■,Ш) | 92 | 78 |
| 633 | $\begin{array}{lll} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 6 & 6 \end{array}$ | 1111 | $(\varnothing, \square),(\square, \square),(\square, \square \Pi),(\square, \boldsymbol{\square})$ | 192 | 32 |
| 634 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \varnothing, \square),(\varnothing, \square, ■),(\square, \square, \boldsymbol{\square})$ | 168 | 40 |
| 635 | $\begin{array}{lll} 0 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 111 | $(\square, \square),(\square, \square)$ | 83 | 77 |
| 636 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} 0_{0}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\varnothing, \square, \square)$ | 128 | -8 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 637 |  | 1112 | $(\varnothing, \boldsymbol{m}, \boxminus),(\varnothing, \boldsymbol{m}, ~ \exists),(\varnothing, \boldsymbol{m m}, \square),(\square, \boldsymbol{m m}, \square)$ | 100 | 32 |
| 638 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \square, \square)$ | 104 | 40 |
| 639 |  |  | $(\varnothing, \square, \square),(\varnothing, \square, \boldsymbol{m})$ | 100 | 26 |
| 640 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \end{array}$ | 11111 | $(\varnothing, \square, \varnothing),(\varnothing, \square, \square)$ | 94 | 32 |
| 641 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 \end{array}$ | $1{ }_{1} 111$ | $(\square, \square, \boldsymbol{m}),(\square, \square, \varnothing)$ | 84 | 54 |
| 642 | $\begin{array}{llll}0 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0\end{array}$ | 112 | $(\varnothing, \boxminus),(\varnothing, \boxplus)$ | 82 | 54 |
| 643 | $\begin{array}{lll} 0 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 111 | $(\varnothing, \square \square),(\square, \varnothing)$ | 90 | 12 |
| 644 | $\begin{array}{lllll} 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \varnothing, \square),(\varnothing, \square, \boldsymbol{\square}),(\square, \square, \boldsymbol{m})$ | 144 | 48 |
| 645 | $\begin{array}{llll}0 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{array}$ | 111 | $(\varnothing, \square),(\square, \square)$ | 82 | 84 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 646 | $\begin{array}{lllll} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\varnothing, \square, \square)$ | 89 | 40 |
| 647 | $\begin{array}{lll} 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | 111 | $(\square, \square),(\square, \varnothing)$ | 74 | 68 |
| 648 | $\begin{array}{lll} 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}$ | $1{ }^{1} 1$ | $(\square, \square),(\square, \square)$ | 68 | 93 |
| 649 |  | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\varnothing, \square, \varnothing)$ | 108 | 20 |
| 650 | $\begin{array}{lllll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}$ | $1 \begin{array}{lllll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \square)$ | 86 | 21 |
| 651 | $\begin{array}{lll} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 111 | $(\varnothing, \square),(\square, \varnothing),(\square, \varnothing),(\square, \square \square)$ | 80 | 40 |
| 652 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \square, \square)$ | 80 | 40 |
| 653 | $\begin{array}{lll} 0 & 3 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{array}$ | 112 |  | 75 | 42 |
| 654 | $\begin{array}{lll} 0 & 3 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{array}$ | $1 \begin{array}{ll}1 & 1\end{array}$ | $(\varnothing, \square),(\varnothing, \square)$ | 68 | 64 |


| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 655 | $\begin{array}{llll} 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\varnothing, \varnothing, \square),(\varnothing, \square, \square),(\square, \square, \square \square \square)$ | 120 | 56 |
| 656 | $\begin{array}{llll} 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square \square),(\varnothing, \square, \square \square)$ | 128 | 0 |
| 657 | $\begin{array}{llll} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 6 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\varnothing, \square, \square \square),(\varnothing, \square, \square \square),(\square, \varnothing, \varnothing)$ | 104 | 12 |
| 658 | $\begin{array}{lll} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 5 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\square, \varnothing),(\square, \varnothing),(\square, \varnothing),(\square, \varnothing)$ | 70 | 60 |
| 659 | $\begin{array}{lll} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 6 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\square, \varnothing),(\square, \varnothing),(\square, \varnothing),(\square, \square \square)$ | 70 | 52 |
| 660 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \square, \square \square)$ | 84 | 58 |
| 661 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}$ | $\begin{array}{llll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\square, \square, \varnothing)$ | 74 | 64 |
| 662 | $\begin{array}{lll} 0 & 3 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{array}$ | $1 \begin{array}{lll}1 & \\ 1\end{array}$ | $(\square \square \square \square, \square),(\square \square, \boxminus),(\varnothing, \square)$ | 74 | 38 |
| 663 | $\begin{array}{lll}0 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}$ | $1{ }^{1} 13$ | $(\varnothing, \square),(\square, \varnothing)$ | 112 | -12 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 664 | $\begin{array}{lll} 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | 112 | $(\varnothing, \boxminus),(\square, \boxminus)$ | 69 | 68 |
| 665 | $\begin{array}{lll}0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{array}$ | 112 | $(\square, \boxplus)$ | 64 | 77 |
| 666 | $\begin{array}{lll} 0 & 4 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$ | 111 | $(\varnothing, \square),(\varnothing, \square),(\square, \varnothing)$ | 88 | 40 |
| 667 |  |  | $(\varnothing, \varnothing, \square),(\varnothing, \varnothing, \square)$ | 84 | 32 |
| 668 | $\begin{array}{lllll} 0 & 0 & 0 & 3 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$ | 1112 | $(\boldsymbol{\square}, \square, \square),(\boldsymbol{\square}, \square, \varnothing),(\square, \varnothing, \varnothing)$ | 89 | -3 |
| 669 | $\begin{array}{llll}0 & 3 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 5 \\ 0 & 0 & 0\end{array}$ | 112 | $(\boldsymbol{m}, ~ \exists),(\boldsymbol{m}, ~ \exists),(\boldsymbol{\Pi}, ~ \exists),(\varnothing, \boxminus)$ | 70 | 49 |
| 670 | $\begin{array}{llll}0 & 3 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0\end{array}$ | 111 | $(\varnothing, \square),(\varnothing, \square),(\square, \square)$ | 64 | 56 |
| 671 | $\begin{array}{lll} 0 & 7 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{array}$ | 112 | $(\varnothing, \square),(\square, \square)$ | 55 | 59 |
| 672 | $\begin{array}{llll} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square),(\square, \varnothing, \square)$ | 112 | -8 |



Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 682 | $\begin{array}{lll} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 6 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\square, ■),(\square, \varnothing),(\square, \varnothing),(\square \square \square)$ | 55 | 85 |
| 683 | $\begin{array}{ll}0 & 6 \\ 0 & 0\end{array}$ | 11 | $(\square \square)$ | 64 | 188 |
| 684 | $\begin{array}{lll} 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\square, \square \square)$ | 61 | 149 |
| 685 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\square, \varnothing, \varnothing)$ | 68 | 62 |
| 686 | $\begin{array}{lll} 0 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 2\end{array}$ | $(\varnothing, \boxminus),(\varnothing, \square),(\square, \varnothing)$ | 96 | -20 |
| 687 | $\begin{array}{lll} 0 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\square, \square)$ | 52 | 168 |
| 688 | $\begin{array}{llll} 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \varnothing, \square)$ | 65 | 37 |
| 689 | $\begin{array}{lll} 0 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 2\end{array}$ | $(\square, \square)$ | 51 | 51 |
| 690 | $\begin{array}{lll} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\varnothing, \square \square),(\square, \square)$ | 162 | 27 |
| 691 | $\begin{array}{lll} 0 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\varnothing, \square \square)$ | 99 | 27 |

Continued from previous page.
$\left.\begin{array}{ccccccc}\hline \text { Period ID } & \text { Adjacency matrix } & \text { Dimension vector } & \text { Generalized partitions } & \text { Degree } & \text { Euler Number } \\ \hline 692 & 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 5 & 0\end{array}\right)$

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 700 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square \square),(\varnothing, \square, \boldsymbol{\square})$ | 108 | 54 |
| 701 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} 0_{0}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \square, ■),(\square, \square, \square)$ | 54 | 105 |
| 702 | $\begin{array}{llll}0 & 0 & 3 \\ 0 & 0 & \\ 0 & 0 & 0 \\ 0 & 7 & 0\end{array}$ | 111 | $(\square, \square),(\square, \varnothing),(\square \square \square \square),(\square, \square)$ | 50 | 86 |
| 703 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} 0_{0} 0$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square \square),(\square, \square, \square \square \square)$ | 90 | 63 |
| 704 | $\begin{array}{lll} 0 & 0 & 3 \\ 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square \square),(\square, \square, \square)$ | 81 | 36 |
| 705 | $\begin{array}{llll}0 & 2 & 3 \\ 0 & 0 & \\ 0 & 0 & 0 \\ 0 & 2 & 0\end{array}$ | 111 | $(\square \square, \varnothing)$ | 51 | 135 |
| 706 | $\begin{array}{llll} 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\varnothing, \varnothing, \square \square),(\varnothing, \square, \varnothing)$ | 96 | -12 |
| 707 | $\begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array} 0_{0}^{0} 0$ | $1 \begin{array}{lllll}1 & 1 & 1\end{array}$ | $(\varnothing, \varnothing, \square \square),(\varnothing, \square, \varnothing)$ | 78 | 0 |
| 708 | $\begin{array}{lll}0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 4 & 0\end{array}$ | 111 | ( $\square, \square),(\square, \square)$ | 46 | 93 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 709 | $\begin{array}{lll} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 5 & 0 \end{array}$ | 111 | $(\varnothing, \square),(\varnothing, \square),(\square, \square),(\square \square, \square \square \square)$ | 72 | 72 |
| 710 | $\begin{array}{llll}0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0\end{array}$ | 111 | $(\varnothing, \square),(\square \square, \varnothing)$ | 66 | 84 |
| 711 | $\begin{array}{lllll} 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \end{array}$ | $1 \begin{array}{llll}1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\varnothing, \square \square, \square \square \square)$ | 63 | 48 |
| 712 | $\begin{array}{lll} 0 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$ | 111 | $(\varnothing, \square \square),(\square, \square)$ | 48 | 99 |
| 713 | $\begin{array}{lll} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 111 | $(\varnothing, \square),(\square, \square),(\square, \varnothing),(\square \square, \square ா \square)$ | 60 | 12 |
| 714 | $\begin{array}{lll} 0 & 3 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 5 \end{array}$ | 112 | $(\boldsymbol{\square m m}, \boxplus),(\boldsymbol{m}, \boxminus),(\boldsymbol{m}, \boxminus),(\square, \varnothing)$ | 80 | -32 |
| 715 | $\begin{array}{ll} 0 & 6 \\ 0 & 0 \end{array}$ | 14 | $(\boxminus)$ | 42 | 73 |
| 716 | $\begin{array}{lll} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{array}$ | 111 | (■, $\square$ ), ( $\square, \square)$ | 41 | 109 |
| 717 | $\begin{array}{lll} 0 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 112 | $(\varnothing, \boxminus),(\varnothing, \boxplus),(\square, \varnothing),(\square, \square)$ | 50 | 52 |
| 718 | $\begin{array}{llll}0 & 0 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 7 & 0\end{array}$ | 111 | $(\square, ■),(\square, \varnothing),(\square, \varnothing),(\square \square, \square \square \square)$ | 45 | 63 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 719 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\varnothing, \square, \square)$ | 80 | -28 |
| 720 | $\begin{array}{llll} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \square),(\square, \square, \square \square)$ | 62 | 44 |
| 721 | $\begin{array}{lll} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\square, \varnothing),(\square, \varnothing)$ | 40 | 152 |
| 722 | $\begin{array}{llll} 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \end{array}$ | $\begin{array}{lllll}1 & 1 & 1 & 1\end{array}$ | $(\varnothing, \square, \varnothing),(\square, \varnothing, \square)$ | 53 | 80 |
| 723 | $\begin{array}{lll} 0 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 2\end{array}$ | $(\varnothing, \boxminus),(\varnothing, \boxminus),(\square, \varnothing),(\square, \square)$ | 45 | 94 |
| 724 | $\begin{array}{lll} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\varnothing, \square),(\square, \varnothing)$ | 40 | 144 |
| 725 | $\begin{array}{lll} 0 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\square, \square \square)$ | 38 | 191 |
| 726 | $\begin{array}{lll} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\varnothing, \square \square),(\square, \square)$ | 42 | 27 |
| 727 | $\begin{array}{lll} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{array}$ | $1 \begin{array}{lll}1 & 1\end{array}$ | $(\varnothing, \square \square),(\square \square, \square \square \square)$ | 54 | 81 |

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| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 728 | $\begin{array}{lll} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 111 | $(\varnothing, \square),(\square \square, \square \square)$ | 44 | 116 |
| 729 | $\begin{array}{lll} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 111 | $(\square, \varnothing),(\square, \square)$ | 35 | 155 |
| 730 | $\begin{array}{lll}0 & 6 \\ 0 & 0\end{array}$ | 12 | ( B$),(\mathrm{\square}),(\square)$ | 33 | 90 |
| 731 | $\begin{array}{lll} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 7 & 0 \end{array}$ | 111 | $(\varnothing, \square),(\square, \boldsymbol{\square}),(\square, \boldsymbol{\square}),(\square, \square \Pi)$ | 64 | -48 |
| 732 | $\begin{array}{lll} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 8 & 0 \end{array}$ | 111 | $(\square, \varnothing),(\square, \boldsymbol{\square}),(\square, \boldsymbol{\square}),(\square, \boldsymbol{\square})$ | 40 | 72 |
| 733 | $\begin{array}{lll} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 8 & 0 \end{array}$ | 111 | $(\square, \boldsymbol{\square}),(\square, \boldsymbol{\Pi}),(\square, \boldsymbol{\Pi}),(\square, \boldsymbol{\square})$ | 36 | 92 |
| 734 | $\begin{array}{ll}0 & 6 \\ 0 & 0\end{array}$ | 12 |  | 28 | 140 |
| 735 | $\begin{array}{lll} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{array}$ | 111 | $(\square, ■),(\square, \square)$ | 26 | 251 |
| 736 | $\begin{array}{lll} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 111 | $(\varnothing, \square),(\varnothing, \square \square)$ | 48 | -72 |
| 737 | $\begin{array}{lll} 0 & 2 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$ | 111 | $(\varnothing, \square),(\varnothing, \square \square)$ | 30 | 114 |
| 738 | $\begin{array}{llll}0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 2 & 0\end{array}$ | $1{ }^{1} 11$ | $(\varnothing, \square \square),(\square, \square)$ | 27 | 99 |

Continued from previous page.

| Period ID | Adjacency matrix | Dimension vector | Generalized partitions | Degree | Euler Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 739 | $\begin{array}{lll} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 111 | ( $\varnothing, \square$ ), (■,Ш) | 26 | 186 |
| 740 | $\begin{array}{lll}0 & 5 \\ 0 & 0\end{array}$ | 12 | ( $\boxplus$ ), ( $\boxplus$ ) | 20 | 176 |
| 741 | $\begin{array}{ll} 0 & 9 \\ 0 & 0 \end{array}$ | 11 | ( $\square$ ), ( $\square$ ), ( $\square$ ), ( $\square$ ) | 16 | 224 |
| 742 | $\begin{array}{lll}0 & 5 \\ 0 & 0\end{array}$ | 12 | ( $\square$ ), ( $\square$ ) | 15 | 318 |
| 743 | $\begin{array}{lll} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{array}$ | 111 | $(\varnothing, \square \square),(\square, \boldsymbol{\square})$ | 32 | -112 |
| 744 | $\begin{array}{lll} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{array}$ | 111 | $(\varnothing, \square \square),(\square, \varnothing)$ | 20 | 212 |
| 745 | $\begin{array}{lll} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ | 111 | $(\square, \square),(\square, \square \square)$ | 17 | 293 |
| 746 | $\begin{array}{lll}0 & 8 \\ 0 & 0\end{array}$ | 11 | ( $\square$ ),(■),(■■) | 12 | 324 |
| 747 | $\begin{array}{ll}0 & 7 \\ 0 & 0\end{array}$ | 11 | (■),(■) | 9 | 369 |
| 748 | $\begin{array}{ll} 0 & 7 \\ 0 & 0 \end{array}$ | 11 | (■),(■■) | 8 | 552 |
| 749 | $\begin{array}{ll}0 & 6 \\ 0 & 0\end{array}$ | 11 | (■ए) | 5 | 825 |

Table A.3: Some regularized period sequences obtained from 4-dimensional Fano manifolds that arise as quiver flag zero loci.

| Period ID | Name |  | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | $\mathrm{B} \mathrm{S}_{124}^{4}, \mathbb{P}^{4}, \mathrm{CKP}_{1}$ |  | 1 | 0 | 0 | 0 | 0 | 120 | 0 | 0 |
| 2 | BØS ${ }_{115}^{4}$, $\mathrm{CKP}_{2}$ |  | 1 | 0 | 0 | 0 | 24 | 120 | 0 | 0 |
| 3 | $\mathrm{CKP}_{3}, Q^{4}$ |  | 1 | 0 | 0 | 0 | 48 | 0 | 0 | 0 |
| 4 | $\mathrm{CKP}_{4}$ |  | 1 | 0 | 0 | 0 | 48 | 120 | 0 | 0 |
| 5 | BØS ${ }_{118}^{4}, \mathrm{CKP}_{8}$ |  | 1 | 0 | 0 | 6 | 0 | 120 | 90 | 0 |
| 6 | BØS ${ }_{47}^{4}, \mathrm{CKP}_{10}$ |  | 1 | 0 | 0 | 6 | 24 | 0 | 90 | 2520 |
| 7 | $\mathrm{B} \emptyset \mathrm{S}_{94}^{4}, \mathrm{CKP}_{11}$ |  | 1 | 0 | 0 | 6 | 24 | 120 | 90 | 1260 |
| 8 | BØS ${ }_{37}^{4}, \mathrm{CKP}_{12}$ |  | 1 | 0 | 0 | 6 | 24 | 120 | 90 | 2520 |
| 9 | $\mathrm{CKP}_{13}, \mathrm{~B}_{1} \mathrm{~S}_{74}^{4}$ |  | 1 | 0 | 0 | 6 | 48 | 0 | 90 | 2520 |
| 10 | $\mathrm{CKP}_{14}$ |  | 1 | 0 | 0 | 6 | 48 | 0 | 90 | 3780 |
| 11 | $\mathrm{CKP}_{15}, \mathrm{~B}_{1} \mathrm{~S}_{86}^{4}$ |  | 1 | 0 | 0 | 6 | 48 | 120 | 90 | 2520 |
| 12 | $\mathrm{CKP}_{16}$ |  | 1 | 0 | 0 | 6 | 48 | 120 | 90 | 3780 |
| 13 | $\mathrm{CKP}_{18}$ |  | 1 | 0 | 0 | 6 | 72 | 120 | 90 | 5040 |
| 14 | $\begin{aligned} & \mathbb{P}^{2} \times \mathbb{P}^{2}, \quad \mathrm{CKP}_{20}, \\ & {\mathrm{~B} \emptyset \mathrm{~S}_{123}^{4}}^{2} \end{aligned}$ | $\mathrm{FI}_{6}^{4},$ | 1 | 0 | 0 | 12 | 0 | 0 | 900 | 0 |
| 15 |  |  | 1 | 0 | 0 | 12 | 0 | 120 | 540 | 0 |
| 16 | $\mathrm{BOS}_{114}^{4}, \mathrm{CKP}_{21}$ |  | 1 | 0 | 0 | 12 | 0 | 120 | 900 | 0 |
| 17 | $\mathrm{CKP}_{23}, \mathrm{~B}_{\mathrm{CS}}^{46} 4$ |  | 1 | 0 | 0 | 12 | 24 | 0 | 900 | 3780 |

Continued from previous page

| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 18 | $\mathrm{CKP}_{25}, \mathrm{~B} \emptyset \mathrm{~S}_{32}^{4}$ | 1 | 0 | 0 | 12 | 24 | 240 | 900 | 5040 |
| 19 | $\mathrm{CKP}_{26}$ | 1 | 0 | 0 | 12 | 48 | 0 | 540 | 7560 |
| 20 |  | 1 | 0 | 0 | 12 | 48 | 0 | 900 | 7560 |
| 21 |  | 1 | 0 | 0 | 12 | 48 | 120 | 540 | 7560 |
| 22 | $\mathrm{CKP}_{29}$ | 1 | 0 | 0 | 12 | 72 | 120 | 540 | 10080 |
| 23 | $\mathrm{CKP}_{30}$ | 1 | 0 | 0 | 12 | 96 | 120 | 540 | 15120 |
| 24 | $\mathrm{FI}_{5}^{4}$ | 1 | 0 | 0 | 18 | 0 | 0 | 1710 | 0 |
| 25 |  | 1 | 0 | 0 | 18 | 48 | 0 | 1710 | 11340 |
| 26 |  | 1 | 0 | 0 | 18 | 48 | 120 | 2430 | 11340 |
| 27 | $\mathrm{CKP}_{33}, \mathrm{FI}_{4}^{4}$ | 1 | 0 | 0 | 24 | 0 | 0 | 3240 | 0 |
| 28 | $\mathrm{CKP}_{34}$ | 1 | 0 | 0 | 24 | 48 | 0 | 3240 | 15120 |
| 29 |  | 1 | 0 | 0 | 24 | 48 | 120 | 3600 | 15120 |
| 30 | $\mathrm{CKP}_{35}$ | 1 | 0 | 0 | 24 | 96 | 120 | 3240 | 30240 |
| 31 | $\mathrm{CKP}_{36}$ | 1 | 0 | 0 | 24 | 120 | 120 | 3240 | 40320 |
| 32 | Str ${ }_{1}$ | 1 | 0 | 0 | 30 | 120 | 240 | 5850 | 50400 |
| 33 | $\mathrm{FI}_{3}^{4}, \mathrm{CKP}_{37}$ | 1 | 0 | 0 | 36 | 0 | 0 | 8100 | 0 |
| 34 | $\mathrm{CKP}_{39}$ | 1 | 0 | 0 | 36 | 144 | 120 | 8100 | 75600 |
| 35 | $\mathrm{CKP}_{47}, \mathrm{~B} \emptyset \mathrm{~S}_{121}^{4}$ | 1 | 0 | 2 | 0 | 6 | 120 | 20 | 2520 |

Continued from previous page

| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 36 | $\begin{aligned} & \mathbb{P}^{1} \times \mathbb{P}^{3}, \mathrm{CKP}_{51}, \mathrm{~B}_{12} \mathrm{~S}_{122}^{4}, \\ & \mathrm{MW}_{14}^{4} \end{aligned}$ | 1 | 0 | 2 | 0 | 30 | 0 | 740 | 0 |
| 37 | $B \emptyset S_{109}^{4}, \mathrm{CKP}_{54}$ | 1 | 0 | 2 | 0 | 30 | 120 | 380 | 2520 |
| 38 | $\mathrm{CKP}_{55}, \mathrm{~B} \emptyset \mathrm{~S}_{104}^{4}$ | 1 | 0 | 2 | 0 | 30 | 120 | 740 | 2520 |
| 39 | $\mathrm{CKP}_{60}, \mathrm{MW}_{13}^{4}$ | 1 | 0 | 2 | 0 | 54 | 0 | 740 | 0 |
| 40 | $\mathrm{CKP}_{61}, \mathrm{MW}_{12}^{4}$ | 1 | 0 | 2 | 0 | 54 | 0 | 1100 | 0 |
| 41 | $\mathrm{CKP}_{64}$ | 1 | 0 | 2 | 0 | 54 | 120 | 740 | 2520 |
| 42 | $\mathrm{CKP}_{65}$ | 1 | 0 | 2 | 0 | 54 | 120 | 1100 | 2520 |
| 43 | $\mathrm{CKP}_{67}$ | 1 | 0 | 2 | 0 | 54 | 240 | 1460 | 5040 |
| 44 | $\mathrm{B} \emptyset \mathrm{S}_{111}^{4}, \mathrm{CKP}_{76}$ | 1 | 0 | 2 | 6 | 6 | 180 | 110 | 2940 |
| 45 | $\mathrm{CKP}_{78}$ | 1 | 0 | 2 | 6 | 6 | 240 | 110 | 3780 |
| 46 | BØS ${ }_{106}^{4}$, $\mathrm{CKP}_{79}$ | 1 | 0 | 2 | 6 | 30 | 60 | 470 | 2940 |
| 47 | $\mathrm{CKP}_{80}, \mathrm{~B}_{1} \mathrm{~S}_{45}^{4}$ | 1 | 0 | 2 | 6 | 30 | 60 | 830 | 2940 |
| 48 | $\mathrm{CKP}_{81}, \mathrm{~B}_{1} \mathrm{~S}_{41}^{4}$ | 1 | 0 | 2 | 6 | 30 | 120 | 470 | 3780 |
| 49 | $\begin{aligned} & \mathbb{P}^{1} \times \mathrm{MM}_{2-33}^{3}, \quad \mathrm{~B} \emptyset \mathrm{~S}_{110}^{4}, \\ & \mathrm{CKP}_{83} \end{aligned}$ | 1 | 0 | 2 | 6 | 30 | 120 | 830 | 2520 |
| 50 | BØS $\mathrm{S}_{82}^{4}, \mathrm{CKP}_{84}$ | 1 | 0 | 2 | 6 | 30 | 180 | 470 | 4200 |
| 51 | $\mathrm{CKP}_{85}, \mathrm{~B} \emptyset \mathrm{~S}_{113}^{4}$ | 1 | 0 | 2 | 6 | 30 | 180 | 470 | 5460 |
| 52 | BØS ${ }_{92}^{4}$, $\mathrm{CKP}_{86}$ | 1 | 0 | 2 | 6 | 30 | 180 | 830 | 5460 |

Continued from previous page

| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 53 | $\mathrm{CKP}_{89}$ | 1 | 0 | 2 | 6 | 30 | 240 | 830 | 5040 |
| 54 | $B \emptyset S_{52}^{4}, \mathrm{CKP}_{91}$ | 1 | 0 | 2 | 6 | 54 | 60 | 830 | 2940 |
| 55 | $\mathrm{CKP}_{92}$ | 1 | 0 | 2 | 6 | 54 | 60 | 830 | 4200 |
| 56 | $\mathrm{CKP}_{93}$ | 1 | 0 | 2 | 6 | 54 | 60 | 1190 | 4200 |
| 57 | $\mathrm{CKP}_{96}, \mathrm{~B}_{6} \mathrm{~S}_{91}^{4}$ | 1 | 0 | 2 | 6 | 54 | 180 | 830 | 5460 |
| 58 | $\mathrm{CKP}_{98}$ | 1 | 0 | 2 | 6 | 54 | 180 | 1190 | 6720 |
| 59 | $\mathrm{CKP}_{99}$ | 1 | 0 | 2 | 6 | 54 | 180 | 1190 | 7980 |
| 60 | $B \emptyset S_{81}^{4}, \mathrm{CKP}_{100}$ | 1 | 0 | 2 | 6 | 54 | 240 | 1190 | 6300 |
| 61 | $\mathrm{CKP}_{101}$ | 1 | 0 | 2 | 6 | 54 | 240 | 1190 | 7560 |
| 62 | $\mathrm{CKP}_{102}$ | 1 | 0 | 2 | 6 | 54 | 360 | 1550 | 8820 |
| 63 | $\mathrm{CKP}_{103}$ | 1 | 0 | 2 | 6 | 78 | 180 | 1190 | 7980 |
| 64 | $\mathrm{CKP}_{104}$ | 1 | 0 | 2 | 6 | 78 | 360 | 1910 | 11340 |
| 65 | $\mathrm{CKP}_{107}$ | 1 | 0 | 2 | 6 | 102 | 600 | 2990 | 17640 |
| 66 | $\mathrm{CKP}_{109}$ | 1 | 0 | 2 | 12 | 6 | 120 | 920 | 840 |
| 67 | $\mathrm{B} \mathrm{\emptyset S}_{112}^{4}, \mathrm{CKP}_{110}, \mathbb{P}^{2} \times S_{8}^{2}$ | 1 | 0 | 2 | 12 | 6 | 180 | 920 | 1680 |
| 68 | $\mathrm{CKP}_{111}, \mathbb{P}^{1} \times Q^{3}$ | 1 | 0 | 2 | 12 | 6 | 240 | 560 | 2520 |
| 69 | $\mathrm{CKP}_{113}$ | 1 | 0 | 2 | 12 | 6 | 300 | 920 | 4200 |
| 70 | $\mathrm{CKP}_{114}$ | 1 | 0 | 2 | 12 | 6 | 360 | 560 | 5040 |
| 71 | B ¢ $\mathrm{S}_{60}^{4}$, $\mathrm{CKP}_{116}$ | 1 | 0 | 2 | 12 | 30 | 120 | 920 | 4620 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 72 | $\mathrm{B} \emptyset \mathrm{S}_{88}^{4}, \mathrm{CKP}_{117}$ | 1 | 0 | 2 | 12 | 30 | 180 | 1280 | 5460 |
| 73 | $\mathrm{B} \emptyset \mathrm{S}_{35}^{4}, \mathrm{CKP}_{118}$ | 1 | 0 | 2 | 12 | 30 | 180 | 1280 | 5460 |
| 74 | $\mathrm{CKP}_{119}$ | 1 | 0 | 2 | 12 | 30 | 180 | 1640 | 5460 |
| 75 | $\mathbb{P}^{1} \times \mathrm{MM}_{2-30}^{3}, \mathrm{CKP}_{120}$ | 1 | 0 | 2 | 12 | 30 | 240 | 1280 | 5040 |
| 76 |  | 1 | 0 | 2 | 12 | 30 | 300 | 920 | 9240 |
| 77 | B ¢ $\mathrm{S}_{93}^{4}, \mathrm{CKP}_{121}$ | 1 | 0 | 2 | 12 | 30 | 300 | 1280 | 7980 |
| 78 |  | 1 | 0 | 2 | 12 | 30 | 300 | 1640 | 7980 |
| 79 | $\mathrm{CKP}_{122}$ | 1 | 0 | 2 | 12 | 30 | 360 | 1280 | 7560 |
| 80 | $\mathrm{CKP}_{123}$ | 1 | 0 | 2 | 12 | 30 | 420 | 1280 | 11760 |
| 81 | $\mathrm{CKP}_{124}$ | 1 | 0 | 2 | 12 | 54 | 120 | 1280 | 8400 |
| 82 | $\mathrm{CKP}_{125}$ | 1 | 0 | 2 | 12 | 54 | 120 | 1640 | 8400 |
| 83 | $\mathrm{CKP}_{126}$ | 1 | 0 | 2 | 12 | 54 | 180 | 1640 | 9240 |
| 84 | $B \emptyset S_{85}^{4}, \mathrm{CKP}_{127}$ | 1 | 0 | 2 | 12 | 54 | 240 | 1280 | 9660 |
| 85 | $\mathrm{CKP}_{128}$ | 1 | 0 | 2 | 12 | 54 | 240 | 1280 | 10080 |
| 86 | $\mathrm{CKP}_{130}$ | 1 | 0 | 2 | 12 | 54 | 300 | 2000 | 11760 |
| 87 | $\mathrm{CKP}_{131}$ | 1 | 0 | 2 | 12 | 54 | 360 | 1640 | 12600 |
| 88 | $\mathrm{CKP}_{132}$ | 1 | 0 | 2 | 12 | 54 | 420 | 2000 | 15540 |
| 89 | $\mathrm{CKP}_{134}$ | 1 | 0 | 2 | 12 | 78 | 240 | 2000 | 14700 |
| 90 | $\mathrm{CKP}_{135}$ | 1 | 0 | 2 | 12 | 78 | 300 | 2000 | 14280 |

Continued from previous page

| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 91 | $\mathrm{CKP}_{136}$ | 1 | 0 | 2 | 12 | 78 | 300 | 2720 | 16800 |
| 92 | $\mathrm{CKP}_{137}$ | 1 | 0 | 2 | 12 | 78 | 360 | 2000 | 15120 |
| 93 | $\mathrm{CKP}_{138}$ | 1 | 0 | 2 | 12 | 78 | 480 | 2360 | 17640 |
| 94 | $\mathrm{CKP}_{139}$ | 1 | 0 | 2 | 12 | 102 | 240 | 2000 | 18480 |
| 95 | $\mathrm{CKP}_{141}$ | 1 | 0 | 2 | 12 | 102 | 480 | 2720 | 20160 |
| 96 | $\mathrm{CKP}_{142}$ | 1 | 0 | 2 | 12 | 102 | 480 | 2720 | 22680 |
| 97 | $\mathrm{CKP}_{144}$ | 1 | 0 | 2 | 12 | 126 | 720 | 3800 | 30240 |
| 98 | $\mathrm{CKP}_{145}$ | 1 | 0 | 2 | 12 | 198 | 1200 | 6320 | 52920 |
| 99 | $\mathrm{B} \mathrm{S}_{51}^{4}, \mathrm{CKP}_{146}$ | 1 | 0 | 2 | 18 | 6 | 180 | 1370 | 1260 |
| 100 | $\mathrm{CKP}_{147}$ | 1 | 0 | 2 | 18 | 6 | 240 | 1730 | 2100 |
| 101 |  | 1 | 0 | 2 | 18 | 6 | 300 | 1730 | 2940 |
| 102 |  | 1 | 0 | 2 | 18 | 6 | 420 | 1730 | 5460 |
| 103 | $\mathrm{CKP}_{148}$ | 1 | 0 | 2 | 18 | 30 | 240 | 2090 | 7140 |
| 104 |  | 1 | 0 | 2 | 18 | 30 | 360 | 2450 | 9660 |
| 105 | $\mathrm{CKP}_{151}, \mathrm{~B} \emptyset \mathrm{~S}_{73}^{4}$ | 1 | 0 | 2 | 18 | 54 | 180 | 2090 | 11340 |
| 106 | $\mathrm{CKP}_{152}$ | 1 | 0 | 2 | 18 | 54 | 240 | 2810 | 13440 |
| 107 | $\mathrm{CKP}_{153}$ | 1 | 0 | 2 | 18 | 78 | 300 | 2450 | 18900 |
| 108 | $\mathrm{CKP}_{154}$ | 1 | 0 | 2 | 18 | 78 | 360 | 3170 | 21000 |
| 109 |  | 1 | 0 | 2 | 18 | 102 | 300 | 3170 | 26460 |

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| Period ID |  | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 110 | $\mathrm{CKP}_{155}$ |  | 1 | 0 | 2 | 18 | 102 | 360 | 3890 | 28560 |
| 111 | $\mathrm{CKP}_{158}$ |  | 1 | 0 | 2 | 24 | 6 | 240 | 3260 | 1680 |
| 112 | $\mathrm{CKP}_{159}$ |  | 1 | 0 | 2 | 24 | 6 | 360 | 3260 | 3360 |
| 113 | $\mathrm{CKP}_{160}$ |  | 1 | 0 | 2 | 24 | 6 | 540 | 3260 | 6720 |
| 114 | $\mathrm{CKP}_{161}$ |  | 1 | 0 | 2 | 24 | 54 | 360 | 3980 | 18480 |
| 115 |  |  | 1 | 0 | 2 | 24 | 54 | 360 | 4340 | 18480 |
| 116 |  |  | 1 | 0 | 2 | 24 | 54 | 480 | 4700 | 21000 |
| 117 | $\mathrm{CKP}_{162}$ |  | 1 | 0 | 2 | 24 | 54 | 540 | 4340 | 21840 |
| 118 | $\mathrm{CKP}_{163}$ |  | 1 | 0 | 2 | 24 | 102 | 420 | 4700 | 35280 |
| 119 | $\mathrm{CKP}_{164}$ |  | 1 | 0 | 2 | 24 | 102 | 480 | 4700 | 35280 |
| 120 | $\mathrm{CKP}_{165}$ |  | 1 | 0 | 2 | 24 | 126 | 660 | 5780 | 49560 |
| 121 | $\mathrm{CKP}_{166}$ |  | 1 | 0 | 2 | 24 | 150 | 720 | 6140 | 55440 |
| 122 | $\mathrm{CKP}_{167}$ |  | 1 | 0 | 2 | 24 | 174 | 960 | 7220 | 70560 |
| 123 | CKP ${ }_{168}$ |  | 1 | 0 | 2 | 24 | 246 | 1440 | 9740 | 105840 |
| 124 | Str ${ }_{2}$ |  | 1 | 0 | 2 | 30 | 54 | 600 | 6590 | 26040 |
| 125 |  |  | 1 | 0 | 2 | 30 | 78 | 960 | 7670 | 46200 |
| 126 |  |  | 1 | 0 | 2 | 30 | 126 | 540 | 7670 | 56700 |
| 127 | $\mathrm{CKP}_{169}$ |  | 1 | 0 | 2 | 36 | 6 | 360 | 8120 | 2520 |
| 128 | $\mathrm{CKP}_{170}$ |  | 1 | 0 | 2 | 36 | 6 | 720 | 8120 | 8400 |

Continued from previous page.

| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 129 | $\mathrm{CKP}_{171}$ | 1 | 0 | 2 | 36 | 150 | 840 | 11000 | 86520 |
| 130 | $\mathrm{CKP}_{172}$ | 1 | 0 | 2 | 36 | 294 | 1680 | 15320 | 178920 |
| 131 | $\mathrm{CKP}_{174}$ | 1 | 0 | 2 | 36 | 438 | 2640 | 20360 | 287280 |
| 132 | $\mathrm{CKP}_{175}$ | 1 | 0 | 2 | 42 | 150 | 900 | 14690 | 99540 |
| 133 | $\mathrm{B}_{\square} \mathrm{S}_{43}^{4}, \mathrm{CKP}_{181}$ | 1 | 0 | 4 | 0 | 36 | 120 | 400 | 5040 |
| 134 | $\begin{aligned} & \mathrm{B} \mathrm{~S}_{117}^{4}, \mathrm{CKP}_{183}, \mathrm{MW}_{17}^{4}, \\ & \mathbb{P}^{1} \times \mathrm{MM}_{2-35}^{3}, \end{aligned}$ | 1 | 0 | 4 | 0 | 60 | 0 | 1480 | 0 |
| 135 | $\mathrm{CKP}_{185}, \mathrm{~B} \mathrm{~S}_{36}^{4}$ | 1 | 0 | 4 | 0 | 60 | 120 | 1480 | 5040 |
| 136 | MW ${ }_{11}^{4}$ | 1 | 0 | 4 | 0 | 84 | 0 | 2200 | 0 |
| 137 | $\mathrm{MW}_{10}^{4}, \mathrm{CKP}_{186}$ | 1 | 0 | 4 | 0 | 84 | 0 | 2560 | 0 |
| 138 | $\mathrm{CKP}_{187}$ | 1 | 0 | 4 | 0 | 84 | 240 | 2560 | 10080 |
| 139 | $\mathrm{CKP}_{189}, \mathrm{MW}_{7}^{4}$ | 1 | 0 | 4 | 0 | 108 | 0 | 3280 | 0 |
| 140 | $\begin{aligned} & \mathrm{B}_{1} \mathrm{~S}_{122}^{4}, \mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{2}, \mathbb{P}^{2} \times \\ & \mathbb{P}^{1} \times \mathbb{P}^{1}, \mathbb{P}^{1} \times \mathrm{MM}_{2-34}^{3}, \\ & \mathrm{CKP}_{195} \end{aligned}$ | 1 | 0 | 4 | 6 | 36 | 240 | 490 | 7560 |
| 141 | $\mathrm{CKP}_{197}$ | 1 | 0 | 4 | 6 | 36 | 300 | 490 | 9240 |
| 142 |  | 1 | 0 | 4 | 6 | 36 | 360 | 490 | 12600 |
| 143 | $\begin{aligned} & \mathbb{P}^{1} \times \mathrm{MM}_{3-30}^{3}, \quad \mathrm{CKP}_{200}, \\ & \mathrm{~B}_{2}, \end{aligned}$ | 1 | 0 | 4 | 6 | 60 | 180 | 1570 | 5460 |

Continued from previous page

| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 144 | $\mathrm{CKP}_{201}, \mathrm{~B}_{2} \mathrm{~S}_{34}^{4}$ | 1 | 0 | 4 | 6 | 60 | 180 | 1570 | 6720 |
| 145 | $\mathrm{CKP}_{203}$ | 1 | 0 | 4 | 6 | 60 | 240 | 1210 | 10080 |
| 146 | $\begin{aligned} & \mathbb{P}^{1} \times \mathrm{MM}_{3-26}^{3}, \quad \mathrm{CKP}_{204}, \\ & \mathrm{~B}_{1}, \end{aligned}$ | 1 | 0 | 4 | 6 | 60 | 240 | 1570 | 8820 |
| 147 | $\mathrm{CKP}_{205}, \mathrm{~B}_{2} \mathrm{~S}_{102}^{4}$ | 1 | 0 | 4 | 6 | 60 | 240 | 1570 | 9660 |
| 148 | $\mathrm{CKP}_{206},{\mathrm{~B} \emptyset \mathrm{~S}_{44}^{4}}^{4}$ | 1 | 0 | 4 | 6 | 60 | 240 | 1930 | 9660 |
| 149 | $\mathrm{CKP}_{207}$ | 1 | 0 | 4 | 6 | 60 | 300 | 1210 | 11760 |
| 150 | $\mathrm{CKP}_{208}$ | 1 | 0 | 4 | 6 | 60 | 300 | 1570 | 10500 |
| 151 | $\mathrm{CKP}_{209}$ | 1 | 0 | 4 | 6 | 60 | 360 | 1570 | 13860 |
| 152 | $\mathrm{CKP}_{214}$ | 1 | 0 | 4 | 6 | 84 | 240 | 2650 | 10080 |
| 153 | $\mathrm{CKP}_{215}$ | 1 | 0 | 4 | 6 | 84 | 240 | 2650 | 12180 |
| 154 |  | 1 | 0 | 4 | 6 | 84 | 300 | 2290 | 13020 |
| 155 | $\mathrm{B} \emptyset \mathrm{S}_{29}^{4}, \mathrm{CKP}_{217}$ | 1 | 0 | 4 | 6 | 84 | 360 | 2650 | 15120 |
| 156 | $\mathrm{CKP}_{218}$ | 1 | 0 | 4 | 6 | 84 | 360 | 3010 | 17220 |
| 157 | $\mathrm{CKP}_{219}$ | 1 | 0 | 4 | 6 | 84 | 420 | 2650 | 16800 |
| 158 | $\mathrm{CKP}_{220}$ | 1 | 0 | 4 | 6 | 108 | 240 | 3370 | 13860 |
| 159 | $\mathrm{CKP}_{222}$ | 1 | 0 | 4 | 6 | 108 | 300 | 3370 | 15540 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
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| 160 | $\mathrm{CKP}_{224}$ | 1 | 0 | 4 | 6 | 132 | 660 | 4810 | 30660 |
| 161 | $\begin{aligned} & \mathbb{P}^{1} \times \mathrm{MM}_{3-31}^{3}, \quad \mathrm{CKP}_{225}, \\ & \mathrm{~B}_{2}, \end{aligned}$ | 1 | 0 | 4 | 12 | 36 | 360 | 940 | 8400 |
| 162 | $S_{8}^{2} \times S_{8}^{2}, \mathrm{~B} \mathrm{\emptyset S}_{83}^{4}, \mathrm{CKP}_{226}$ | 1 | 0 | 4 | 12 | 36 | 360 | 1300 | 8400 |
| 163 | $\mathrm{CKP}_{227}, \mathrm{~B}_{1} \mathrm{~S}_{101}^{4}, \mathbb{P}^{2} \times S_{7}^{2}$ | 1 | 0 | 4 | 12 | 36 | 360 | 1300 | 9660 |
| 164 | $\mathrm{CKP}_{228}, \mathrm{P}^{1} \times \mathrm{MM}_{2-31}^{3}$ | 1 | 0 | 4 | 12 | 36 | 420 | 940 | 11760 |
| 165 | $\mathrm{CKP}_{230}$ | 1 | 0 | 4 | 12 | 36 | 480 | 1300 | 13440 |
| 166 | $\mathrm{CKP}_{231}$ | 1 | 0 | 4 | 12 | 36 | 480 | 1300 | 14700 |
| 167 | $\mathrm{CKP}_{233}$ | 1 | 0 | 4 | 12 | 36 | 720 | 940 | 25200 |
| 168 | $\begin{aligned} & \mathbb{P}^{1} \times \mathrm{MM}_{3-25}^{3}, \quad \mathrm{CKP}_{236}, \\ & \mathrm{~B} \mathrm{~S}_{108}^{4} \end{aligned}$ | 1 | 0 | 4 | 12 | 60 | 360 | 2020 | 10920 |
| 169 | $\mathrm{CKP}_{239}$ | 1 | 0 | 4 | 12 | 60 | 360 | 2380 | 13440 |
| 170 | $\mathrm{CKP}_{240}, \mathbb{P}^{1} \times \mathrm{MM}_{3-23}^{3}$ | 1 | 0 | 4 | 12 | 60 | 420 | 2020 | 14280 |
| 171 | $\mathrm{CKP}_{243}$ | 1 | 0 | 4 | 12 | 60 | 480 | 2020 | 17220 |
| 172 | $\mathrm{CKP}_{244}$ | 1 | 0 | 4 | 12 | 60 | 480 | 2380 | 18480 |
| 173 |  | 1 | 0 | 4 | 12 | 60 | 480 | 2740 | 18480 |
| 174 | $\mathrm{CKP}_{246}$ | 1 | 0 | 4 | 12 | 60 | 540 | 2020 | 19320 |
| 175 | $\mathrm{CKP}_{247}$ | 1 | 0 | 4 | 12 | 60 | 600 | 2020 | 23520 |
| 176 | $\mathrm{CKP}_{248}$ | 1 | 0 | 4 | 12 | 84 | 360 | 3100 | 15960 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
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| 177 | $\mathrm{B}_{\square} \mathrm{S}_{80}^{4}, \mathrm{CKP}_{251}$ | 1 | 0 | 4 | 12 | 84 | 420 | 2740 | 17640 |
| 178 |  | 1 | 0 | 4 | 12 | 84 | 420 | 2740 | 19320 |
| 179 | $\mathrm{CKP}_{252}$ | 1 | 0 | 4 | 12 | 84 | 420 | 3100 | 18900 |
| 180 | $\mathrm{CKP}_{255}$ | 1 | 0 | 4 | 12 | 84 | 480 | 2740 | 19740 |
| 181 | $\mathbb{P}^{1} \times \mathrm{MM}_{3-19}^{3}, \mathrm{CKP}_{256}$ | 1 | 0 | 4 | 12 | 84 | 480 | 3100 | 20160 |
| 182 |  | 1 | 0 | 4 | 12 | 84 | 480 | 3100 | 23520 |
| 183 | $\mathrm{CKP}_{258}$ | 1 | 0 | 4 | 12 | 84 | 480 | 3460 | 22260 |
| 184 | $\mathrm{CKP}_{259}$ | 1 | 0 | 4 | 12 | 84 | 600 | 3460 | 28560 |
| 185 | $\mathrm{CKP}_{260}$ | 1 | 0 | 4 | 12 | 84 | 600 | 3820 | 27300 |
| 186 | $\mathrm{CKP}_{262}$ | 1 | 0 | 4 | 12 | 84 | 720 | 3100 | 32760 |
| 187 | $\mathrm{CKP}_{265}$ | 1 | 0 | 4 | 12 | 108 | 540 | 4180 | 27720 |
| 188 |  | 1 | 0 | 4 | 12 | 108 | 600 | 3820 | 28560 |
| 189 | $\mathrm{CKP}_{266}$ | 1 | 0 | 4 | 12 | 108 | 600 | 4180 | 31080 |
| 190 |  | 1 | 0 | 4 | 12 | 108 | 600 | 4900 | 32340 |
| 191 | $\mathrm{CKP}_{267}$ | 1 | 0 | 4 | 12 | 108 | 720 | 4180 | 38640 |
| 192 | $\mathrm{CKP}_{268}$ | 1 | 0 | 4 | 12 | 108 | 720 | 4900 | 37380 |
| 193 | $\mathrm{CKP}_{269}$ | 1 | 0 | 4 | 12 | 132 | 600 | 4900 | 33600 |
| 194 | $\mathrm{CKP}_{270}$ | 1 | 0 | 4 | 12 | 132 | 720 | 5260 | 37800 |
| 195 |  | 1 | 0 | 4 | 18 | 36 | 720 | 2110 | 21000 |


| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
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| 196 | $\mathrm{CKP}_{277}, \mathbb{P}^{1} \times \mathrm{MM}_{2-27}^{3}$ | 1 | 0 | 4 | 18 | 60 | 600 | 2830 | 19740 |
| 197 | $\mathrm{CKP}_{279}$ | 1 | 0 | 4 | 18 | 60 | 840 | 3910 | 32340 |
| 198 | $\mathrm{CKP}_{280}, \mathrm{~B} \emptyset \mathrm{~S}_{53}^{4}$ | 1 | 0 | 4 | 18 | 84 | 480 | 3190 | 20580 |
| 199 | $\mathrm{CKP}_{282}$ | 1 | 0 | 4 | 18 | 84 | 540 | 3910 | 25200 |
| 200 | $\mathrm{CKP}_{283}, \mathrm{~B}_{2} \mathrm{~S}_{84}^{4}$ | 1 | 0 | 4 | 18 | 84 | 600 | 3550 | 25620 |
| 201 |  | 1 | 0 | 4 | 18 | 84 | 600 | 4270 | 26880 |
| 202 |  | 1 | 0 | 4 | 18 | 84 | 720 | 3910 | 32340 |
| 203 | $\mathrm{CKP}_{284}$ | 1 | 0 | 4 | 18 | 84 | 720 | 4630 | 32340 |
| 204 | $\mathrm{CKP}_{285}$ | 1 | 0 | 4 | 18 | 84 | 780 | 4270 | 34020 |
| 205 | $\mathrm{CKP}_{286}$ | 1 | 0 | 4 | 18 | 108 | 600 | 4270 | 30660 |
| 206 | $\mathrm{CKP}_{287}$ | 1 | 0 | 4 | 18 | 108 | 600 | 4630 | 31920 |
| 207 | $\mathrm{CKP}_{288}$ | 1 | 0 | 4 | 18 | 108 | 660 | 4990 | 34020 |
| 208 | $\mathrm{CKP}_{289}$ | 1 | 0 | 4 | 18 | 108 | 720 | 4990 | 38220 |
| 209 | $\mathrm{CKP}_{290}$ | 1 | 0 | 4 | 18 | 108 | 780 | 4990 | 39060 |
| 210 | $\mathrm{CKP}_{291}$ | 1 | 0 | 4 | 18 | 108 | 780 | 5350 | 40320 |
| 211 |  | 1 | 0 | 4 | 18 | 108 | 840 | 4990 | 44940 |
| 212 |  | 1 | 0 | 4 | 18 | 108 | 960 | 6070 | 49980 |
| 213 | $\mathrm{CKP}_{292}$ | 1 | 0 | 4 | 18 | 132 | 780 | 5350 | 42840 |
| 214 | $\mathrm{CKP}_{293}$ | 1 | 0 | 4 | 18 | 132 | 840 | 5710 | 48720 |

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| Period ID |  | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
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| 215 | $\mathrm{CKP}_{294}$ |  | 1 | 0 | 4 | 18 | 132 | 960 | 7150 | 55020 |
| 216 | $\mathrm{CKP}_{295}$ |  | 1 | 0 | 4 | 18 | 132 | 960 | 7510 | 57540 |
| 217 | $\mathrm{CKP}_{296}$ |  | 1 | 0 | 4 | 18 | 156 | 840 | 7150 | 56280 |
| 218 | $\mathrm{CKP}_{297}$ |  | 1 | 0 | 4 | 18 | 156 | 1020 | 7870 | 63000 |
| 219 |  |  | 1 | 0 | 4 | 18 | 180 | 1020 | 7870 | 66780 |
| 220 | $\mathrm{CKP}_{298}$ |  | 1 | 0 | 4 | 18 | 180 | 1080 | 9310 | 77700 |
| 221 | $\mathrm{CKP}_{299}$ |  | 1 | 0 | 4 | 24 | 36 | 720 | 3640 | 16800 |
| 222 | $\mathrm{CKP}_{300}$ |  | 1 | 0 | 4 | 24 | 36 | 1080 | 3640 | 33600 |
| 223 | $\mathrm{CKP}_{301}$ |  | 1 | 0 | 4 | 24 | 84 | 720 | 5800 | 31920 |
| 224 |  |  | 1 | 0 | 4 | 24 | 84 | 840 | 5800 | 38220 |
| 225 | $\mathrm{CKP}_{302}$ |  | 1 | 0 | 4 | 24 | 84 | 1080 | 5800 | 48720 |
| 226 |  |  | 1 | 0 | 4 | 24 | 84 | 1140 | 5800 | 51660 |
| 227 |  |  | 1 | 0 | 4 | 24 | 108 | 960 | 6880 | 49560 |
| 228 | $\mathrm{CKP}_{303}$ |  | 1 | 0 | 4 | 24 | 108 | 1080 | 6520 | 58800 |
| 229 | $\mathrm{CKP}_{304}$ |  | 1 | 0 | 4 | 24 | 132 | 840 | 6880 | 53340 |
| 230 |  |  | 1 | 0 | 4 | 24 | 132 | 840 | 7240 | 53760 |
| 231 |  |  | 1 | 0 | 4 | 24 | 132 | 840 | 7960 | 54600 |
| 232 |  |  | 1 | 0 | 4 | 24 | 132 | 1020 | 7600 | 60480 |
| 233 | $\mathrm{CKP}_{305}$ |  | 1 | 0 | 4 | 24 | 156 | 960 | 7960 | 63420 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
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| 234 | $\mathrm{CKP}_{306}$ | 1 | 0 | 4 | 24 | 156 | 1080 | 9040 | 72240 |
| 235 |  | 1 | 0 | 4 | 24 | 180 | 1440 | 11560 | 99120 |
| 236 | $\mathrm{CKP}_{307}$ | 1 | 0 | 4 | 24 | 204 | 1260 | 10480 | 95760 |
| 237 | $\mathrm{CKP}_{308}$ | 1 | 0 | 4 | 24 | 228 | 1440 | 12280 | 110880 |
| 238 |  | 1 | 0 | 4 | 24 | 276 | 1680 | 13720 | 137760 |
| 239 | $\mathrm{CKP}_{309}$ | 1 | 0 | 4 | 30 | 84 | 840 | 6610 | 36540 |
| 240 | $\mathrm{CKP}_{310}$ | 1 | 0 | 4 | 30 | 84 | 1200 | 8050 | 54600 |
| 241 | $\mathrm{CKP}_{311}$ | 1 | 0 | 4 | 30 | 132 | 960 | 8770 | 61740 |
| 242 |  | 1 | 0 | 4 | 30 | 132 | 1140 | 9490 | 70560 |
| 243 | $\mathrm{CKP}_{312}$ | 1 | 0 | 4 | 30 | 156 | 1320 | 11650 | 92400 |
| 244 |  | 1 | 0 | 4 | 30 | 204 | 1440 | 12730 | 113820 |
| 245 | $\mathrm{CKP}_{313}$ | 1 | 0 | 4 | 30 | 228 | 1440 | 12370 | 116340 |
| 246 | $\mathrm{CKP}_{314}$ | 1 | 0 | 4 | 36 | 36 | 1800 | 8500 | 58800 |
| 247 | $\mathrm{CKP}_{315}$ | 1 | 0 | 4 | 36 | 84 | 1440 | 10660 | 64680 |
| 248 | $\mathrm{CKP}_{316}$ | 1 | 0 | 4 | 36 | 156 | 1200 | 12820 | 90720 |
| 249 | $\mathrm{CKP}_{318}$ | 1 | 0 | 4 | 36 | 324 | 2160 | 20740 | 223440 |
| 250 | $\mathrm{CKP}_{319}$ | 1 | 0 | 4 | 42 | 156 | 1680 | 16510 | 119700 |
| 251 | $\mathrm{CKP}_{320}$ | 1 | 0 | 4 | 42 | 180 | 2040 | 19390 | 155400 |
| 252 | $\mathrm{CKP}_{321}$ | 1 | 0 | 4 | 42 | 252 | 2040 | 21190 | 196980 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
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| 253 |  | 1 | 0 | 4 | 48 | 180 | 1920 | 22000 | 156240 |
| 254 | $\mathrm{CKP}_{322}$ | 1 | 0 | 4 | 60 | 204 | 2640 | 33340 | 231840 |
| 255 | $\mathrm{CKP}_{323}$ | 1 | 0 | 4 | 60 | 564 | 4140 | 49900 | 648480 |
| 256 | $\mathrm{CKP}_{324}, \mathrm{~B} \mathrm{~S}_{38}^{4}$ | 1 | 0 | 6 | 0 | 90 | 120 | 1860 | 7560 |
| 257 | $\begin{aligned} & \mathbb{P}^{1} \times \mathrm{MM}_{2-32}^{3}, \quad \mathrm{MW}_{16}^{4}, \\ & \mathrm{CKP}_{325} \end{aligned}$ | 1 | 0 | 6 | 0 | 114 | 0 | 3300 | 0 |
| 258 | $\mathrm{CKP}_{326}, \mathrm{MW}_{8}^{4}$ | 1 | 0 | 6 | 0 | 138 | 0 | 4740 | 0 |
| 259 | $\mathrm{MW}_{5}^{4}, \mathrm{CKP}_{327}$ | 1 | 0 | 6 | 0 | 186 | 0 | 7980 | 0 |
| 260 | $\begin{aligned} & \mathbb{P}^{1} \times \mathbb{P}^{1} \times S_{8}^{2}, \mathrm{CKP}_{328}, \mathbb{P}^{1} \times \\ & \mathrm{MM}_{3-28}^{3}, \mathrm{~B}_{1} \mathrm{~S}_{107}^{4} \end{aligned}$ | 1 | 0 | 6 | 6 | 90 | 300 | 1950 | 13020 |
| 261 | $\mathrm{CKP}_{330}, \mathbb{P}^{1} \times \mathrm{MM}_{3-24}^{3}$ | 1 | 0 | 6 | 6 | 114 | 300 | 3390 | 14280 |
| 262 | $\mathrm{CKP}_{332}$ | 1 | 0 | 6 | 6 | 114 | 360 | 3390 | 18480 |
| 263 | $\mathrm{CKP}_{334}$ | 1 | 0 | 6 | 6 | 138 | 300 | 4830 | 15540 |
| 264 | $\mathrm{CKP}_{335}$ | 1 | 0 | 6 | 6 | 138 | 360 | 4830 | 21000 |
| 265 |  | 1 | 0 | 6 | 6 | 138 | 420 | 4830 | 24360 |
| 266 | $\mathrm{CKP}_{336}$ | 1 | 0 | 6 | 6 | 186 | 360 | 8070 | 24780 |
| 267 | $\mathrm{B}_{\square} \mathrm{S}_{79}^{4}, S_{8}^{2} \times S_{7}^{2}, \mathrm{CKP}_{340}$ | 1 | 0 | 6 | 12 | 90 | 540 | 2760 | 21420 |
| 268 | $\mathrm{CKP}_{341}, \mathbb{P}^{1} \times \mathrm{MM}_{2-29}^{3}$ | 1 | 0 | 6 | 12 | 90 | 600 | 2400 | 26040 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
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| 269 | $\begin{aligned} & \mathbb{P}^{1} \times \mathrm{MM}_{4-10}^{3}, \quad \mathrm{~B} \emptyset \mathrm{~S}_{90}^{4} \\ & \mathrm{CKP}_{345} \end{aligned}$ | 1 | 0 | 6 | 12 | 114 | 540 | 3840 | 23940 |
| 270 | $\mathrm{CKP}_{346}, \mathbb{P}^{1} \times \mathrm{MM}_{3-20}^{3}$ | 1 | 0 | 6 | 12 | 114 | 600 | 3840 | 28560 |
| 271 | $\mathrm{CKP}_{347}$ | 1 | 0 | 6 | 12 | 114 | 660 | 3840 | 32760 |
| 272 | $\mathrm{CKP}_{348}$ | 1 | 0 | 6 | 12 | 114 | 660 | 4200 | 32760 |
| 273 | $\mathrm{CKP}_{349}$ | 1 | 0 | 6 | 12 | 114 | 720 | 4200 | 36960 |
| 274 | $\mathbb{P}^{1} \times \mathrm{MM}_{3-17}^{3}, \mathrm{CKP}_{351}$ | 1 | 0 | 6 | 12 | 138 | 600 | 5280 | 31080 |
| 275 | $\mathrm{CKP}_{352}$ | 1 | 0 | 6 | 12 | 138 | 600 | 5280 | 33600 |
| 276 | $\mathrm{CKP}_{354}$ | 1 | 0 | 6 | 12 | 138 | 600 | 5640 | 35280 |
| 277 | $\mathrm{CKP}_{355}$ | 1 | 0 | 6 | 12 | 138 | 660 | 5280 | 35280 |
| 278 | $\mathrm{CKP}_{356}$ | 1 | 0 | 6 | 12 | 138 | 660 | 5640 | 36540 |
| 279 | $\mathrm{CKP}_{357}$ | 1 | 0 | 6 | 12 | 138 | 780 | 5640 | 45360 |
| 280 | $\mathrm{CKP}_{359}$ | 1 | 0 | 6 | 12 | 162 | 600 | 7080 | 38640 |
| 281 | $\mathrm{CKP}_{360}$ | 1 | 0 | 6 | 12 | 162 | 720 | 7080 | 44520 |
| 282 | $\mathrm{CKP}_{361}$ | 1 | 0 | 6 | 12 | 186 | 720 | 8520 | 51240 |
| 283 | $\mathrm{CKP}_{363}$ | 1 | 0 | 6 | 12 | 186 | 900 | 8880 | 63000 |
| 284 | $\mathbb{P}^{2} \times S_{6}^{2}, \mathrm{~B}_{6} \mathrm{~S}_{99}^{4}, \mathrm{CKP}_{365}$ | 1 | 0 | 6 | 18 | 90 | 720 | 3570 | 28980 |
| 285 | $\mathrm{CKP}_{367}$ | 1 | 0 | 6 | 18 | 114 | 780 | 5010 | 34860 |
| 286 | $\mathrm{CKP}_{368}, \mathbb{P}^{1} \times \mathrm{MM}_{3-18}^{3}$ | 1 | 0 | 6 | 18 | 114 | 840 | 4650 | 38220 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
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| 287 | $\mathrm{CKP}_{369}$ | 1 | 0 | 6 | 18 | 114 | 960 | 5010 | 47040 |
| 288 |  | 1 | 0 | 6 | 18 | 114 | 1140 | 4650 | 61740 |
| 289 | $\mathrm{CKP}_{372}$ | 1 | 0 | 6 | 18 | 138 | 780 | 6090 | 39900 |
| 290 | $\mathrm{CKP}_{373}, \mathrm{P}^{1} \times \mathrm{MM}_{3-16}^{3}$ | 1 | 0 | 6 | 18 | 138 | 900 | 6090 | 46620 |
| 291 | $\mathrm{CKP}_{374}$ | 1 | 0 | 6 | 18 | 138 | 900 | 6090 | 47460 |
| 292 | $\mathrm{CKP}_{376}$ | 1 | 0 | 6 | 18 | 138 | 960 | 5730 | 52080 |
| 293 | $\mathrm{CKP}_{377}$ | 1 | 0 | 6 | 18 | 138 | 960 | 7170 | 56700 |
| 294 |  | 1 | 0 | 6 | 18 | 138 | 1020 | 6450 | 57960 |
| 295 |  | 1 | 0 | 6 | 18 | 138 | 1080 | 7890 | 66780 |
| 296 | $\mathrm{CKP}_{378}$ | 1 | 0 | 6 | 18 | 162 | 960 | 7530 | 58380 |
| 297 |  | 1 | 0 | 6 | 18 | 162 | 960 | 7890 | 58380 |
| 298 | $\mathrm{CKP}_{380}$ | 1 | 0 | 6 | 18 | 162 | 1080 | 8250 | 65940 |
| 299 | $\mathrm{CKP}_{381}$ | 1 | 0 | 6 | 18 | 162 | 1080 | 8970 | 71820 |
| 300 | $\mathrm{CKP}_{382}$ | 1 | 0 | 6 | 18 | 186 | 1080 | 8970 | 69720 |
| 301 | $\mathrm{CKP}_{383}$ | 1 | 0 | 6 | 18 | 186 | 1140 | 8970 | 74340 |
| 302 | $\mathrm{CKP}_{384}$ | 1 | 0 | 6 | 18 | 186 | 1140 | 9690 | 76440 |
| 303 |  | 1 | 0 | 6 | 18 | 210 | 1320 | 11850 | 96180 |
| 304 | $\mathrm{CKP}_{388}, \mathbb{P}^{1} \times \mathrm{MM}_{2-25}^{3}$ | 1 | 0 | 6 | 24 | 114 | 1200 | 5820 | 57120 |
| 305 | $\mathrm{CKP}_{392}$ | 1 | 0 | 6 | 24 | 138 | 1080 | 7980 | 57960 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
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| 306 | $\mathrm{CKP}_{393}$ | 1 | 0 | 6 | 24 | 138 | 1260 | 7980 | 67620 |
| 307 | $\mathrm{CKP}_{394}$ | 1 | 0 | 6 | 24 | 138 | 1320 | 9060 | 78120 |
| 308 | $\mathrm{CKP}_{395}$ | 1 | 0 | 6 | 24 | 138 | 1440 | 7980 | 82320 |
| 309 | $\mathrm{CKP}_{397}$ | 1 | 0 | 6 | 24 | 162 | 1140 | 8700 | 67200 |
| 310 | $\mathrm{CKP}_{399}$ | 1 | 0 | 6 | 24 | 162 | 1320 | 9780 | 80640 |
| 311 | $\mathrm{CKP}_{400}$ | 1 | 0 | 6 | 24 | 186 | 1200 | 9780 | 76440 |
| 312 | $\mathrm{CKP}_{401}$ | 1 | 0 | 6 | 24 | 186 | 1200 | 10860 | 82320 |
| 313 | $\mathbb{P}^{1} \times \mathrm{MM}_{2-24}^{3}, \mathrm{CKP}_{402}$ | 1 | 0 | 6 | 24 | 186 | 1260 | 10140 | 78120 |
| 314 | $\mathrm{CKP}_{403}$ | 1 | 0 | 6 | 24 | 186 | 1320 | 10500 | 85680 |
| 315 | $\mathrm{CKP}_{404}$ | 1 | 0 | 6 | 24 | 186 | 1560 | 12660 | 110880 |
| 316 | $\mathrm{CKP}_{405}$ | 1 | 0 | 6 | 24 | 210 | 1440 | 11940 | 107520 |
| 317 | $\mathrm{CKP}_{406}$ | 1 | 0 | 6 | 24 | 210 | 1500 | 12660 | 107940 |
| 318 |  | 1 | 0 | 6 | 24 | 210 | 1500 | 12660 | 110460 |
| 319 | $\mathrm{CKP}_{407}$ | 1 | 0 | 6 | 24 | 210 | 1620 | 13020 | 115500 |
| 320 |  | 1 | 0 | 6 | 24 | 210 | 1800 | 13380 | 133980 |
| 321 |  | 1 | 0 | 6 | 24 | 210 | 1800 | 15180 | 138600 |
| 322 |  | 1 | 0 | 6 | 24 | 234 | 1800 | 16620 | 146160 |
| 323 | $\mathrm{CKP}_{408}$ | 1 | 0 | 6 | 24 | 234 | 1920 | 16980 | 153720 |
| 324 | $\mathrm{CKP}_{409}$ | 1 | 0 | 6 | 24 | 282 | 1920 | 19140 | 169260 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
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| 325 | $\mathrm{CKP}_{410}$ | 1 | 0 | 6 | 24 | 282 | 2280 | 21300 | 199080 |
| 326 |  | 1 | 0 | 6 | 30 | 162 | 1680 | 11670 | 103320 |
| 327 |  | 1 | 0 | 6 | 30 | 186 | 1800 | 13470 | 122220 |
| 328 | $\mathrm{CKP}_{411}$ | 1 | 0 | 6 | 30 | 210 | 1620 | 13470 | 116340 |
| 329 | $\mathrm{CKP}_{412}$ | 1 | 0 | 6 | 30 | 210 | 1740 | 14550 | 126420 |
| 330 | $\mathrm{CKP}_{413}$ | 1 | 0 | 6 | 30 | 234 | 1680 | 15270 | 133560 |
| 331 | $\mathrm{CKP}_{414}$ | 1 | 0 | 6 | 30 | 234 | 1980 | 16710 | 159600 |
| 332 | $\mathrm{CKP}_{415}$ | 1 | 0 | 6 | 30 | 282 | 1980 | 18150 | 168420 |
| 333 | $\mathrm{CKP}_{416}$ | 1 | 0 | 6 | 30 | 282 | 2160 | 19950 | 186480 |
| 334 | $\mathrm{CKP}_{417}$ | 1 | 0 | 6 | 36 | 186 | 1560 | 12480 | 97440 |
| 335 | $\mathrm{CKP}_{418}$ | 1 | 0 | 6 | 36 | 186 | 1920 | 15360 | 131880 |
| 336 | $\mathrm{CKP}_{419}$ | 1 | 0 | 6 | 36 | 186 | 2040 | 15720 | 138600 |
| 337 | $\mathrm{CKP}_{420}$ | 1 | 0 | 6 | 36 | 186 | 2520 | 16080 | 180600 |
| 338 | $\mathrm{CKP}_{421}$ | 1 | 0 | 6 | 36 | 210 | 1800 | 16440 | 136920 |
| 339 |  | 1 | 0 | 6 | 36 | 210 | 2100 | 16800 | 154980 |
| 340 | $\mathrm{CKP}_{422}$ | 1 | 0 | 6 | 36 | 234 | 1800 | 16080 | 137760 |
| 341 |  | 1 | 0 | 6 | 36 | 234 | 2520 | 19680 | 201600 |
| 342 |  | 1 | 0 | 6 | 36 | 258 | 2280 | 20400 | 191520 |
| 343 |  | 1 | 0 | 6 | 36 | 258 | 2340 | 21120 | 196980 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
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| 344 |  | 1 | 0 | 6 | 36 | 282 | 2520 | 22920 | 224280 |
| 345 | $\mathrm{CKP}_{423}$ | 1 | 0 | 6 | 36 | 306 | 2280 | 23280 | 221760 |
| 346 |  | 1 | 0 | 6 | 36 | 330 | 2640 | 27600 | 274680 |
| 347 | $\mathrm{CKP}_{424}$ | 1 | 0 | 6 | 36 | 330 | 2880 | 27600 | 278040 |
| 348 | $\mathrm{CKP}_{425}$ | 1 | 0 | 6 | 36 | 330 | 3240 | 30480 | 312480 |
| 349 | $\mathrm{CKP}_{426}$ | 1 | 0 | 6 | 36 | 378 | 3480 | 34080 | 352800 |
| 350 |  | 1 | 0 | 6 | 42 | 162 | 2760 | 17610 | 178920 |
| 351 | $\mathrm{CKP}_{427}$ | 1 | 0 | 6 | 42 | 306 | 2460 | 24090 | 229320 |
| 352 | $\mathrm{CKP}_{428}$ | 1 | 0 | 6 | 42 | 306 | 2820 | 26970 | 270060 |
| 353 | $\mathrm{CKP}_{429}$ | 1 | 0 | 6 | 48 | 282 | 2760 | 27420 | 253680 |
| 354 | $\mathrm{CKP}_{430}$ | 1 | 0 | 6 | 48 | 282 | 2760 | 28500 | 257040 |
| 355 | $\mathrm{CKP}_{432}$ | 1 | 0 | 6 | 48 | 426 | 3360 | 37860 | 406560 |
| 356 | $\mathrm{CKP}_{433}$ | 1 | 0 | 6 | 48 | 522 | 4800 | 51180 | 595560 |
| 357 | $\mathrm{CKP}_{435}$ | 1 | 0 | 6 | 54 | 378 | 3480 | 38670 | 392700 |
| 358 | $\mathrm{CKP}_{436}$ | 1 | 0 | 6 | 60 | 354 | 4080 | 44520 | 441840 |
| 359 | $\mathrm{CKP}_{437}$ | 1 | 0 | 6 | 60 | 474 | 3960 | 45600 | 503160 |
| 360 | $\mathrm{CKP}_{438}$ | 1 | 0 | 6 | 66 | 474 | 4860 | 57930 | 637560 |
| 361 |  | 1 | 0 | 6 | 84 | 714 | 6840 | 96360 | 1211280 |
| 362 | CKP $_{439}$ | 1 | 0 | 6 | 120 | 1146 | 11280 | 192300 | 2817360 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 363 | $\begin{aligned} & \mathrm{MW}_{18}^{4}, \mathrm{~B}_{1} \mathrm{~S}_{119}^{4}, \mathrm{CKP}_{440}, \\ & \mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{1}, \mathbb{P}^{1} \times \\ & \mathrm{MM}_{3-27}^{3} \end{aligned}$ | 1 | 0 | 8 | 0 | 168 | 0 | 5120 | 0 |
| 364 | $\mathrm{MW}_{9}^{4}, \mathbb{P}^{1} \times B_{5}^{3}$ | 1 | 0 | 8 | 0 | 192 | 0 | 6920 | 0 |
| 365 | $V_{14}^{4}$ | 1 | 0 | 8 | 0 | 288 | 0 | 15200 | 0 |
| 366 | $\begin{aligned} & \mathbb{P}^{1} \times \mathrm{MM}_{4-11}^{3}, \quad \mathrm{~B}_{\mathrm{O}}^{97} \mathrm{~S}_{97}^{4}, \\ & \mathrm{CKP}_{442}, \mathbb{P}^{1} \times \mathbb{P}^{1} \times S_{7}^{2} \end{aligned}$ | 1 | 0 | 8 | 6 | 168 | 360 | 5210 | 19740 |
| 367 | $\mathbb{P}^{1} \times \mathrm{MM}_{3-21}^{3}, \mathrm{CKP}_{443}$ | 1 | 0 | 8 | 6 | 192 | 360 | 7010 | 21000 |
| 368 | $\mathrm{CKP}_{444}$ | 1 | 0 | 8 | 6 | 216 | 360 | 8810 | 22260 |
| 369 | $\mathbb{P}^{1} \times \mathrm{MM}_{4-9}^{3}, \mathrm{CKP}_{445}$ | 1 | 0 | 8 | 12 | 168 | 720 | 5660 | 39480 |
| 370 | $\mathrm{CKP}_{446}, S_{7}^{2} \times S_{7}^{2}, \mathrm{~B} \mathrm{\emptyset S}_{75}^{4}$ | 1 | 0 | 8 | 12 | 168 | 720 | 6020 | 39480 |
| 371 | $\mathrm{CKP}_{447}, \mathrm{P}^{1} \times \mathrm{MM}_{4-8}^{3}$ | 1 | 0 | 8 | 12 | 192 | 720 | 7460 | 42000 |
| 372 | $\mathbb{P}^{1} \times \mathrm{MM}_{2-26}^{3}$ | 1 | 0 | 8 | 12 | 192 | 780 | 7460 | 47880 |
| 373 | $\mathrm{CKP}_{448}$ | 1 | 0 | 8 | 12 | 216 | 840 | 9620 | 57960 |
| 374 |  | 1 | 0 | 8 | 12 | 216 | 1440 | 8540 | 126000 |
| 375 | $\mathrm{CKP}_{449}$ | 1 | 0 | 8 | 12 | 288 | 1080 | 16100 | 92400 |
| 376 | $\mathrm{CKP}_{450}$ | 1 | 0 | 8 | 12 | 360 | 1200 | 23300 | 117600 |
| 377 | $S_{8}^{2} \times S_{6}^{2}, \mathrm{~B}_{6} \mathrm{~S}_{78}^{4}, \mathrm{CKP}_{451}$ | 1 | 0 | 8 | 18 | 168 | 1020 | 6830 | 54600 |
| 378 | $\mathrm{CKP}_{452}$ | 1 | 0 | 8 | 18 | 168 | 1080 | 6470 | 59220 |


| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 379 | $\mathrm{CKP}_{454}, \mathbb{P}^{1} \times \mathrm{MM}_{4-7}^{3}$ | 1 | 0 | 8 | 18 | 192 | 1080 | 8270 | 63000 |
| 380 | $\mathrm{CKP}_{455}$ | 1 | 0 | 8 | 18 | 216 | 1080 | 10070 | 69300 |
| 381 | $\mathrm{CKP}_{456}, \mathbb{P}^{1} \times \mathrm{MM}_{3-15}^{3}$ | 1 | 0 | 8 | 18 | 216 | 1140 | 10070 | 72660 |
| 382 | $\mathrm{CKP}_{457}$ | 1 | 0 | 8 | 18 | 216 | 1200 | 10430 | 79380 |
| 383 | $\mathrm{CKP}_{458}$ | 1 | 0 | 8 | 18 | 216 | 1260 | 11150 | 87360 |
| 384 | $\mathrm{CKP}_{459}$ | 1 | 0 | 8 | 18 | 240 | 1380 | 13310 | 105000 |
| 385 |  | 1 | 0 | 8 | 18 | 288 | 1560 | 16910 | 136500 |
| 386 | $\mathrm{CKP}_{460}$ | 1 | 0 | 8 | 24 | 168 | 1440 | 8360 | 78960 |
| 387 | $\mathbb{P}^{1} \times \mathrm{MM}_{4-5}^{3}, \mathrm{CKP}_{461}$ | 1 | 0 | 8 | 24 | 216 | 1440 | 10880 | 89040 |
| 388 | $\mathbb{P}^{1} \times \mathrm{MM}_{2-22}^{3}$ | 1 | 0 | 8 | 24 | 216 | 1560 | 11240 | 100800 |
| 389 | $\mathrm{CKP}_{462}$ | 1 | 0 | 8 | 24 | 216 | 2160 | 11240 | 168000 |
| 390 | $\mathbb{P}^{1} \times \mathrm{MM}_{3-13}^{3}, \mathrm{CKP}_{463}$ | 1 | 0 | 8 | 24 | 240 | 1560 | 13040 | 105840 |
| 391 | $\mathrm{CKP}_{464}$ | 1 | 0 | 8 | 24 | 240 | 1560 | 13400 | 110460 |
| 392 | $\mathrm{CKP}_{465}$ | 1 | 0 | 8 | 24 | 240 | 1740 | 13760 | 126420 |
| 393 | $\mathrm{CKP}_{466}$ | 1 | 0 | 8 | 24 | 264 | 1680 | 15200 | 126840 |
| 394 | $\mathrm{CKP}_{467}$ | 1 | 0 | 8 | 24 | 264 | 1740 | 15920 | 133980 |
| 395 | $\mathrm{CKP}_{468}$ | 1 | 0 | 8 | 24 | 264 | 1920 | 17360 | 154560 |
| 396 | $\mathrm{CKP}_{469}$ | 1 | 0 | 8 | 24 | 288 | 1920 | 18440 | 159600 |
| 397 | $\mathrm{CKP}_{470}$ | 1 | 0 | 8 | 24 | 312 | 2160 | 22040 | 194880 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 398 |  | 1 | 0 | 8 | 24 | 360 | 2160 | 25640 | 218400 |
| 399 | $\mathrm{CKP}_{471}$ | 1 | 0 | 8 | 30 | 216 | 1800 | 13490 | 116340 |
| 400 | $\mathbb{P}^{1} \times \mathrm{MM}_{3-11}^{3}, \mathrm{CKP}_{472}$ | 1 | 0 | 8 | 30 | 264 | 1980 | 16370 | 142800 |
| 401 | $\mathrm{CKP}_{473}$ | 1 | 0 | 8 | 30 | 264 | 1980 | 16730 | 147000 |
| 402 | $\mathrm{CKP}_{474}$ | 1 | 0 | 8 | 30 | 264 | 2160 | 17090 | 165900 |
| 403 |  | 1 | 0 | 8 | 30 | 288 | 2100 | 19250 | 171360 |
| 404 |  | 1 | 0 | 8 | 30 | 288 | 2220 | 20330 | 185640 |
| 405 | $\mathrm{CKP}_{475}$ | 1 | 0 | 8 | 30 | 312 | 2580 | 23930 | 229320 |
| 406 |  | 1 | 0 | 8 | 30 | 336 | 2520 | 25010 | 233940 |
| 407 |  | 1 | 0 | 8 | 36 | 216 | 3600 | 16100 | 294000 |
| 408 | $\mathrm{CKP}_{476}$ | 1 | 0 | 8 | 36 | 264 | 2280 | 19340 | 173880 |
| 409 | $\mathrm{CKP}_{477}$ | 1 | 0 | 8 | 36 | 264 | 2880 | 20420 | 232680 |
| 410 | $\mathrm{CKP}_{478}$ | 1 | 0 | 8 | 36 | 288 | 2700 | 21500 | 216720 |
| 411 | $\mathrm{CKP}_{479}$ | 1 | 0 | 8 | 36 | 312 | 2520 | 22940 | 210420 |
| 412 | $\mathrm{CKP}_{480}$ | 1 | 0 | 8 | 36 | 312 | 2760 | 24380 | 239820 |
| 413 |  | 1 | 0 | 8 | 36 | 312 | 2760 | 25100 | 243600 |
| 414 | $\mathrm{CKP}_{481}$ | 1 | 0 | 8 | 36 | 336 | 2760 | 26180 | 251160 |
| 415 | $\mathrm{CKP}_{482}$ | 1 | 0 | 8 | 36 | 360 | 2760 | 27260 | 261240 |
| 416 | $\mathrm{CKP}_{483}$ | 1 | 0 | 8 | 36 | 360 | 2940 | 28340 | 277200 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 417 | $\mathrm{CKP}_{484}$ | 1 | 0 | 8 | 36 | 360 | 3060 | 29780 | 295680 |
| 418 | $\mathrm{CKP}_{485}$ | 1 | 0 | 8 | 36 | 360 | 3300 | 31220 | 320040 |
| 419 | $\mathrm{CKP}_{486}$ | 1 | 0 | 8 | 36 | 408 | 3360 | 35180 | 358680 |
| 420 | $\mathrm{CKP}_{487}$ | 1 | 0 | 8 | 36 | 432 | 3780 | 39500 | 413280 |
| 421 |  | 1 | 0 | 8 | 42 | 312 | 3000 | 27350 | 263340 |
| 422 |  | 1 | 0 | 8 | 42 | 360 | 3540 | 32750 | 346080 |
| 423 |  | 1 | 0 | 8 | 42 | 408 | 3480 | 36350 | 368340 |
| 424 |  | 1 | 0 | 8 | 42 | 456 | 4320 | 44270 | 479220 |
| 425 | $\mathrm{CKP}_{488}$ | 1 | 0 | 8 | 48 | 264 | 4320 | 27440 | 366240 |
| 426 |  | 1 | 0 | 8 | 48 | 336 | 4680 | 35000 | 467040 |
| 427 | $\mathrm{CKP}_{489}, \mathbb{P}^{1} \times \mathrm{MM}_{2-18}^{3}$ | 1 | 0 | 8 | 48 | 360 | 3360 | 31040 | 295680 |
| 428 |  | 1 | 0 | 8 | 48 | 384 | 3960 | 37160 | 400680 |
| 429 | $\mathrm{CKP}_{490}$ | 1 | 0 | 8 | 48 | 408 | 3960 | 38960 | 410760 |
| 430 |  | 1 | 0 | 8 | 48 | 432 | 4140 | 42560 | 454440 |
| 431 |  | 1 | 0 | 8 | 48 | 432 | 4320 | 44720 | 487200 |
| 432 | $\mathrm{CKP}_{491}$ | 1 | 0 | 8 | 48 | 504 | 4800 | 51560 | 572040 |
| 433 | $\mathrm{CKP}_{492}$ | 1 | 0 | 8 | 48 | 504 | 4920 | 53000 | 613200 |
| 434 | $\mathrm{CKP}_{493}$ | 1 | 0 | 8 | 54 | 360 | 4200 | 39770 | 406980 |
| 435 | $\mathrm{CKP}_{494}$ | 1 | 0 | 8 | 54 | 480 | 5160 | 53810 | 608580 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 436 | $\mathrm{CKP}_{495}$ | 1 | 0 | 8 | 60 | 360 | 5160 | 45260 | 514080 |
| 437 | $\mathrm{CKP}_{496}$ | 1 | 0 | 8 | 60 | 552 | 5280 | 60740 | 685440 |
| 438 |  | 1 | 0 | 8 | 60 | 672 | 7200 | 83060 | 1032360 |
| 439 |  | 1 | 0 | 8 | 66 | 456 | 6000 | 61550 | 699300 |
| 440 | $\mathrm{CKP}_{497}$ | 1 | 0 | 8 | 72 | 792 | 8460 | 104120 | 1339800 |
| 441 | $\mathrm{CKP}_{498}$ | 1 | 0 | 8 | 84 | 408 | 8040 | 78740 | 887040 |
| 442 | $\mathbb{P}^{1} \times B_{4}^{3}, \mathrm{CKP}_{500}, \mathrm{MW}_{6}^{4}$ | 1 | 0 | 10 | 0 | 318 | 0 | 15220 | 0 |
| 443 | $V_{12}^{4}$ | 1 | 0 | 10 | 0 | 438 | 0 | 28900 | 0 |
| 444 | $\begin{aligned} & \mathbb{P}^{1} \times \mathrm{MM}_{5-3}^{3}, \quad \mathrm{~B}_{1} \mathrm{~S}_{98}^{4}, \\ & \mathrm{CKP}_{501}, \mathbb{P}^{1} \times \mathbb{P}^{1} \times S_{6}^{2} \end{aligned}$ | 1 | 0 | 10 | 12 | 270 | 840 | 11080 | 55440 |
| 445 | $\mathbb{P}^{1} \times \mathrm{MM}_{2-23}^{3}, \mathrm{CKP}_{502}$ | 1 | 0 | 10 | 12 | 318 | 960 | 15760 | 74760 |
| 446 | $\mathrm{CKP}_{503}$ | 1 | 0 | 10 | 12 | 366 | 960 | 20800 | 82320 |
| 447 | $\mathrm{CKP}_{504}, S_{7}^{2} \times S_{6}^{2}, \mathrm{~B}_{6} \mathrm{~S}_{76}^{4}$ | 1 | 0 | 10 | 18 | 270 | 1320 | 12610 | 91560 |
| 448 | $\mathrm{CKP}_{505}, \mathbb{P}^{1} \times \mathrm{MM}_{4-4}^{3}$ | 1 | 0 | 10 | 24 | 318 | 1800 | 17380 | 135240 |
| 449 | $\mathrm{CKP}_{506}$ | 1 | 0 | 10 | 24 | 318 | 2400 | 18460 | 215040 |
| 450 | $\mathbb{P}^{1} \times \mathrm{MM}_{2-21}^{3}$ | 1 | 0 | 10 | 24 | 342 | 1920 | 19900 | 154560 |
| 451 | $\mathrm{CKP}_{507}$ | 1 | 0 | 10 | 24 | 462 | 2640 | 35740 | 287280 |
| 452 | $\mathbb{P}^{1} \times \mathrm{MM}_{3-12}^{3}, \mathrm{CKP}_{508}$ | 1 | 0 | 10 | 30 | 342 | 2340 | 21070 | 186060 |
| 453 | $\mathrm{CKP}_{509}, \mathbb{P}^{1} \times \mathrm{MM}_{2-19}^{3}$ | 1 | 0 | 10 | 30 | 342 | 2520 | 21430 | 208740 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 454 | $\mathrm{CKP}_{510}$ | 1 | 0 | 10 | 30 | 366 | 2520 | 24670 | 221760 |
| 455 | $\mathrm{CKP}_{511}$ | 1 | 0 | 10 | 30 | 462 | 2760 | 35110 | 290640 |
| 456 | $\mathrm{CKP}_{512}, \mathbb{P}^{2} \times S_{5}^{2}$ | 1 | 0 | 10 | 36 | 270 | 2160 | 15040 | 134400 |
| 457 | $\mathrm{CKP}_{513}$ | 1 | 0 | 10 | 36 | 366 | 2760 | 25840 | 235200 |
| 458 | $\mathrm{CKP}_{514}$ | 1 | 0 | 10 | 36 | 366 | 2880 | 28000 | 271740 |
| 459 | $\mathrm{CKP}_{515}$ | 1 | 0 | 10 | 36 | 366 | 3000 | 26200 | 260400 |
| 460 | $\mathbb{P}^{1} \times \mathrm{MM}_{2-20}^{3}$ | 1 | 0 | 10 | 36 | 390 | 2940 | 27640 | 255360 |
| 461 | $\mathrm{CKP}_{516}$ | 1 | 0 | 10 | 36 | 390 | 3000 | 28720 | 273000 |
| 462 | $\mathrm{CKP}_{517}$ | 1 | 0 | 10 | 36 | 414 | 3180 | 31960 | 306600 |
| 463 | $\mathrm{CKP}_{518}$ | 1 | 0 | 10 | 36 | 414 | 3480 | 33400 | 351960 |
| 464 |  | 1 | 0 | 10 | 36 | 486 | 3720 | 42400 | 420000 |
| 465 | $\mathrm{CKP}_{520}$ | 1 | 0 | 10 | 42 | 414 | 3480 | 33850 | 334320 |
| 466 | $\mathrm{CKP}_{521}$ | 1 | 0 | 10 | 42 | 414 | 3840 | 38530 | 407820 |
| 467 | $\mathrm{CKP}_{522}$ | 1 | 0 | 10 | 42 | 462 | 4080 | 41770 | 436800 |
| 468 | $\mathrm{CKP}_{523}$ | 1 | 0 | 10 | 48 | 414 | 4080 | 36460 | 387240 |
| 469 | $\mathrm{CKP}_{524}$ | 1 | 0 | 10 | 48 | 414 | 4320 | 38260 | 425040 |
| 470 | $\mathrm{CKP}_{525}$ | 1 | 0 | 10 | 48 | 462 | 4200 | 41140 | 425880 |
| 471 | $\mathrm{CKP}_{526}$ | 1 | 0 | 10 | 48 | 486 | 4320 | 44740 | 465360 |
| 472 |  | 1 | 0 | 10 | 48 | 486 | 4680 | 47260 | 519960 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 473 |  | 1 | 0 | 10 | 48 | 486 | 5400 | 49420 | 631680 |
| 474 |  | 1 | 0 | 10 | 48 | 510 | 5160 | 56260 | 632100 |
| 475 | $\mathrm{CKP}_{527}$ | 1 | 0 | 10 | 48 | 534 | 4920 | 53020 | 578760 |
| 476 | $\mathrm{CKP}_{528}$ | 1 | 0 | 10 | 48 | 558 | 5280 | 57700 | 646800 |
| 477 | $\mathrm{CKP}_{529}$ | 1 | 0 | 10 | 54 | 486 | 4920 | 49150 | 534240 |
| 478 |  | 1 | 0 | 10 | 54 | 534 | 5100 | 54550 | 594300 |
| 479 | $\mathrm{CKP}_{530}$ | 1 | 0 | 10 | 54 | 582 | 5580 | 63910 | 711060 |
| 480 | $\mathrm{CKP}_{531}$ | 1 | 0 | 10 | 54 | 606 | 6180 | 68230 | 805140 |
| 481 | $\mathrm{CKP}_{532}$ | 1 | 0 | 10 | 60 | 510 | 6120 | 59680 | 714000 |
| 482 |  | 1 | 0 | 10 | 60 | 582 | 5760 | 61480 | 683760 |
| 483 |  | 1 | 0 | 10 | 60 | 582 | 6360 | 68680 | 807240 |
| 484 | $\mathrm{CKP}_{533}$ | 1 | 0 | 10 | 60 | 654 | 6840 | 77680 | 924840 |
| 485 | $\mathrm{CKP}_{534}$ | 1 | 0 | 10 | 60 | 654 | 7080 | 77320 | 945840 |
| 486 | $\mathrm{CKP}_{535}$ | 1 | 0 | 10 | 66 | 750 | 7920 | 93970 | 1156680 |
| 487 | $\mathrm{CKP}_{536}$ | 1 | 0 | 10 | 66 | 846 | 10080 | 125290 | 1619940 |
| 488 | $\mathrm{CKP}_{538}$ | 1 | 0 | 10 | 72 | 558 | 7200 | 74620 | 890400 |
| 489 | $\mathrm{CKP}_{539}$ | 1 | 0 | 10 | 72 | 726 | 8280 | 97660 | 1212120 |
| 490 | $\mathrm{CKP}_{540}$ | 1 | 0 | 10 | 72 | 846 | 9360 | 113860 | 1475880 |
| 491 | $\mathrm{CKP}_{541}$ | 1 | 0 | 10 | 78 | 750 | 8700 | 107110 | 1328880 |


| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 492 | $\mathrm{CKP}_{542}$ | 1 | 0 | 10 | 78 | 846 | 10140 | 124030 | 1643880 |
| 493 | $\mathrm{CKP}_{543}$ | 1 | 0 | 10 | 84 | 750 | 8520 | 102880 | 1244040 |
| 494 | $\mathrm{CKP}_{544}$ | 1 | 0 | 10 | 96 | 702 | 10560 | 124660 | 1538880 |
| 495 | $\mathrm{CKP}_{545}$ | 1 | 0 | 10 | 168 | 1566 | 23040 | 402940 | 6002640 |
| 496 | $V_{10}^{4}$ | 1 | 0 | 12 | 0 | 684 | 0 | 58800 | 0 |
| 497 | $S_{6}^{2} \times S_{6}^{2}, \mathrm{CKP}_{546}, \mathrm{~B} \emptyset \mathrm{~S}_{77}^{4}$ | 1 | 0 | 12 | 24 | 396 | 2160 | 23160 | 186480 |
| 498 | $\mathrm{P}^{1} \times \mathrm{MM}_{4-3}^{3}, \mathrm{CKP}_{547}$ | 1 | 0 | 12 | 24 | 444 | 2160 | 26760 | 191520 |
| 499 | $S_{8}^{2} \times S_{5}^{2}, \mathrm{CKP}_{548}$ | 1 | 0 | 12 | 36 | 396 | 2820 | 24060 | 219240 |
| 500 | $\mathbb{P}^{1} \times \mathrm{MM}_{3-10}^{3}, \mathrm{CKP}_{549}$ | 1 | 0 | 12 | 36 | 492 | 3360 | 35220 | 319200 |
| 501 | $\mathrm{CKP}_{550}$ | 1 | 0 | 12 | 36 | 492 | 3540 | 38460 | 371700 |
| 502 | $\mathrm{CKP}_{551}$ | 1 | 0 | 12 | 36 | 540 | 5400 | 41700 | 705600 |
| 503 | $\mathbb{P}^{1} \times \mathrm{MM}_{2-17}^{3}$ | 1 | 0 | 12 | 42 | 540 | 4140 | 43230 | 423360 |
| 504 | $\mathrm{CKP}_{552}$ | 1 | 0 | 12 | 42 | 540 | 4560 | 49710 | 528360 |
| 505 | $\mathrm{CKP}_{553}, \mathbb{P}^{1} \times \mathrm{MM}_{3-7}^{3}$ | 1 | 0 | 12 | 48 | 564 | 4680 | 48000 | 486360 |
| 506 | $\mathrm{CKP}_{554}$ | 1 | 0 | 12 | 48 | 588 | 5040 | 54480 | 577920 |
| 507 | $\mathrm{CKP}_{555}$ | 1 | 0 | 12 | 48 | 588 | 5040 | 55200 | 588000 |
| 508 |  | 1 | 0 | 12 | 48 | 636 | 5940 | 68880 | 780780 |
| 509 |  | 1 | 0 | 12 | 54 | 732 | 7680 | 84810 | 1136520 |
| 510 | $\mathrm{CKP}_{556}$ | 1 | 0 | 12 | 60 | 636 | 6000 | 64020 | 698460 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 511 | $\mathrm{CKP}_{557}, \mathbb{P}^{1} \times \mathrm{MM}_{2-16}^{3}$ | 1 | 0 | 12 | 60 | 636 | 6120 | 63300 | 693000 |
| 512 |  | 1 | 0 | 12 | 60 | 684 | 6840 | 76620 | 893760 |
| 513 | $\mathrm{CKP}_{558}$ | 1 | 0 | 12 | 60 | 684 | 6840 | 77340 | 898800 |
| 514 | $\mathrm{CKP}_{559}$ | 1 | 0 | 12 | 60 | 708 | 6840 | 77700 | 893760 |
| 515 | $\mathrm{CKP}_{560}$ | 1 | 0 | 12 | 60 | 780 | 8400 | 101460 | 1254960 |
| 516 |  | 1 | 0 | 12 | 66 | 804 | 8400 | 100830 | 1237740 |
| 517 |  | 1 | 0 | 12 | 66 | 828 | 8880 | 108030 | 1369620 |
| 518 | CKP $_{561}$ | 1 | 0 | 12 | 72 | 708 | 9120 | 93000 | 1254960 |
| 519 |  | 1 | 0 | 12 | 72 | 756 | 8580 | 97320 | 1209180 |
| 520 | $\mathrm{CKP}_{562}$ | 1 | 0 | 12 | 72 | 780 | 8340 | 97320 | 1178520 |
| 521 |  | 1 | 0 | 12 | 72 | 828 | 9000 | 108480 | 1354920 |
| 522 | $\mathrm{CKP}_{563}$ | 1 | 0 | 12 | 72 | 876 | 9600 | 118200 | 1501920 |
| 523 | $\mathrm{CKP}_{564}$ | 1 | 0 | 12 | 78 | 876 | 10440 | 125490 | 1649340 |
| 524 |  | 1 | 0 | 12 | 84 | 876 | 9960 | 122700 | 1540560 |
| 525 | $\mathrm{CKP}_{565}$ | 1 | 0 | 12 | 90 | 1116 | 13860 | 184350 | 2553600 |
| 526 |  | 1 | 0 | 12 | 96 | 1140 | 14400 | 193080 | 2721600 |
| 527 |  | 1 | 0 | 12 | 96 | 1356 | 17640 | 245640 | 3609480 |
| 528 | $\mathrm{CKP}_{566}$ | 1 | 0 | 12 | 108 | 756 | 16320 | 155100 | 2494800 |
| 529 |  | 1 | 0 | 12 | 120 | 1284 | 17700 | 253200 | 3671640 |


| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 530 | $\mathbb{P}^{1} \times B_{3}^{3}, \mathrm{MW}_{3}^{4}, \mathrm{CKP}_{567}$ | 1 | 0 | 14 | 0 | 690 | 0 | 50900 | 0 |
| 531 | $\begin{aligned} & \mathbb{P}^{1} \times \mathbb{P}^{1} \times S_{5}^{2}, \mathbb{P}^{1} \times \mathrm{MM}_{6-1}^{3}, \\ & \mathrm{CKP}_{568}, \end{aligned}$ | 1 | 0 | 14 | 30 | 546 | 2760 | 33350 | 246540 |
| 532 | $S_{7}^{2} \times S_{5}^{2}, \mathrm{CKP}_{569}$ | 1 | 0 | 14 | 36 | 546 | 3480 | 37040 | 330540 |
| 533 | $\mathrm{CKP}_{570}$ | 1 | 0 | 14 | 36 | 690 | 3960 | 57200 | 468720 |
| 534 | $\mathrm{CKP}_{571}$ | 1 | 0 | 14 | 36 | 690 | 5760 | 59000 | 821520 |
| 535 | $\mathrm{CKP}_{572}, \mathbb{P}^{1} \times \mathrm{MM}_{2-15}^{3}$ | 1 | 0 | 14 | 36 | 714 | 4320 | 59720 | 519120 |
| 536 | $\mathrm{CKP}_{573}$ | 1 | 0 | 14 | 36 | 858 | 4560 | 83840 | 637560 |
| 537 | $\mathrm{CKP}_{574}, \mathbb{P}^{1} \times \mathrm{MM}_{4-1}^{3}$ | 1 | 0 | 14 | 48 | 690 | 5280 | 59540 | 594720 |
| 538 | $\mathbb{P}^{1} \times \mathrm{MM}_{3-8}^{3}, \mathrm{CKP}_{575}$ | 1 | 0 | 14 | 54 | 690 | 5700 | 61070 | 631260 |
| 539 | $\mathrm{CKP}_{576}$ | 1 | 0 | 14 | 60 | 786 | 7140 | 82760 | 933240 |
| 540 | $\mathrm{CKP}_{577}$ | 1 | 0 | 14 | 60 | 786 | 7320 | 84920 | 981120 |
| 541 | $\mathrm{CKP}_{578}$ | 1 | 0 | 14 | 66 | 834 | 8160 | 95450 | 1126440 |
| 542 | $\mathrm{CKP}_{579}$ | 1 | 0 | 14 | 72 | 882 | 9240 | 109940 | 1355760 |
| 543 | $\mathrm{CKP}_{580}$ | 1 | 0 | 14 | 72 | 1002 | 10800 | 138020 | 1807680 |
| 544 | $\mathrm{CKP}_{581}$ | 1 | 0 | 14 | 72 | 1026 | 10560 | 136220 | 1733760 |
| 545 | $\mathrm{CKP}_{582}$ | 1 | 0 | 14 | 78 | 834 | 8880 | 98870 | 1177260 |
| 546 | $\mathrm{CKP}_{583}$ | 1 | 0 | 14 | 78 | 906 | 9600 | 112190 | 1363320 |
| 547 |  | 1 | 0 | 14 | 78 | 1146 | 12780 | 174830 | 2377620 |

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| Period ID |  | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 548 | CKP $_{584}$ | 1 | 0 | 14 | 84 | 930 | 10320 | 122720 | $\alpha_{7}$ |
| 549 |  | 1 | 0 | 14 | 84 | 1074 | 12600 | 161600 | 2189640 |
| 550 | CKP $_{585}$ | 1 | 0 | 14 | 84 | 1074 | 12720 | 163040 | 2202060 |
| 551 | CKP $_{586}$ | 1 | 0 | 14 | 96 | 1170 | 14040 | 184820 | 2526720 |
| 552 |  | 1 | 0 | 14 | 96 | 1194 | 15360 | 201740 | 2897160 |
| 553 | CKP $_{587}$ | 1 | 0 | 14 | 96 | 1266 | 15240 | 207860 | 2918160 |
| 554 |  | 1 | 0 | 14 | 96 | 1434 | 18600 | 276620 | 4253760 |
| 555 |  | 1 | 0 | 14 | 102 | 1242 | 15720 | 211910 | 2994600 |
| 556 | CKP $_{588}$ | 1 | 0 | 14 | 102 | 1338 | 17280 | 237830 | 3452400 |
| 557 | CKP $_{589}$ | 1 | 0 | 14 | 102 | 1530 | 19800 | 284990 | 4270980 |
| 558 |  | 1 | 0 | 14 | 108 | 1218 | 17400 | 224600 | 3334800 |
| 559 |  | 1 | 0 | 14 | 108 | 1314 | 18240 | 245120 | 3690960 |
| 560 | CKP $_{590}$ | 1 | 0 | 14 | 120 | 1506 | 20640 | 296420 | 4484760 |
| 561 | CKP $_{591}$ | 1 | 0 | 14 | 120 | 1554 | 20520 | 298940 | 4515000 |
| 562 |  | 1 | 0 | 14 | 138 | 2106 | 30120 | 474530 | 7913220 |
| 563 | CKP $_{592}$ | 1 | 0 | 14 | 144 | 1506 | 21480 | 311900 | 4544400 |
| 564 | CKP $_{593}$ | 1 | 0 | 14 | 144 | 1506 | 24480 | 349700 | 5456640 |
| 565 | CKP $_{594}$ | 1 | 0 | 14 | 156 | 2226 | 33000 | 534200 | 9067800 |
| 566 |  | 1 | 0 | 14 | 180 | 2082 | 33480 | 560480 | 9276960 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 567 | $\mathrm{CKP}_{595}$ | 1 | 0 | 14 | 288 | 2994 | 58440 | 1220900 | 21414960 |
| 568 | $\mathrm{CKP}_{596}, V_{8}^{4}$ | 1 | 0 | 16 | 0 | 1296 | 0 | 160000 | 0 |
| 569 | $\mathrm{CKP}_{597}$ | 1 | 0 | 16 | 24 | 1296 | 4320 | 163240 | 840000 |
| 570 | $S_{6}^{2} \times S_{5}^{2}, \mathrm{CKP}_{598}$ | 1 | 0 | 16 | 42 | 720 | 4920 | 58390 | 567840 |
| 571 | $\mathrm{CKP}_{599}$ | 1 | 0 | 16 | 60 | 1344 | 11520 | 192940 | 2347800 |
| 572 | $\mathrm{CKP}_{600}, \mathbb{P}^{1} \times \mathrm{MM}_{3-6}^{3}$ | 1 | 0 | 16 | 66 | 936 | 8280 | 97630 | 1086540 |
| 573 | $\mathbb{P}^{1} \times \mathrm{MM}_{2-12}^{3}, \mathrm{CKP}_{601}$ | 1 | 0 | 16 | 72 | 1056 | 9840 | 122920 | 1428000 |
| 574 | $\mathrm{CKP}_{602}$ | 1 | 0 | 16 | 78 | 1080 | 11040 | 138490 | 1725780 |
| 575 | $\mathrm{CKP}_{603}, \mathbb{P}^{1} \times \mathrm{MM}_{2-13}^{3}$ | 1 | 0 | 16 | 84 | 1104 | 11400 | 137860 | 1685040 |
| 576 | $\mathrm{CKP}_{604}$ | 1 | 0 | 16 | 84 | 1152 | 12600 | 162700 | 2132760 |
| 577 | $\mathrm{CKP}_{605}$ | 1 | 0 | 16 | 90 | 1176 | 12900 | 164590 | 2139060 |
| 578 | $\mathrm{CKP}_{606}$ | 1 | 0 | 16 | 90 | 1200 | 13440 | 175750 | 2332680 |
| 579 |  | 1 | 0 | 16 | 96 | 1632 | 19320 | 302200 | 4447800 |
| 580 | $\mathbb{P}^{1} \times \mathrm{MM}_{2-11}^{3}, \mathrm{CKP}_{607}$ | 1 | 0 | 16 | 108 | 1248 | 15600 | 188260 | 2538480 |
| 581 | $\mathrm{CKP}_{608}$ | 1 | 0 | 16 | 108 | 1488 | 18600 | 261700 | 3797640 |
| 582 |  | 1 | 0 | 16 | 108 | 1488 | 18960 | 267460 | 3922800 |
| 583 |  | 1 | 0 | 16 | 108 | 1632 | 20640 | 309220 | 4640160 |
| 584 | $\mathrm{CKP}_{609}$ | 1 | 0 | 16 | 114 | 1488 | 19440 | 268990 | 3973620 |
| 585 | $\mathrm{CKP}_{610}$ | 1 | 0 | 16 | 114 | 1488 | 19740 | 275470 | 4077780 |


| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 586 | $\mathrm{CKP}_{611}$ | 1 | 0 | 16 | 114 | 1512 | 23640 | 303190 | 5285700 |
| 587 | $\mathrm{CKP}_{612}$ | 1 | 0 | 16 | 120 | 1488 | 19440 | 268360 | 3894240 |
| 588 | $\mathrm{CKP}_{613}$ | 1 | 0 | 16 | 126 | 1752 | 23940 | 355570 | 5509980 |
| 589 |  | 1 | 0 | 16 | 204 | 3264 | 52680 | 952180 | 18086880 |
| 590 | $\mathrm{CKP}_{614}$ | 1 | 0 | 18 | 48 | 1494 | 9120 | 206820 | 1864800 |
| 591 | $\mathbb{P}^{1} \times \mathrm{MM}_{2-14}^{3}$ | 1 | 0 | 18 | 90 | 1302 | 13260 | 168570 | 2089080 |
| 592 | $\mathrm{CKP}_{615}$ | 1 | 0 | 18 | 102 | 1398 | 16200 | 212670 | 2919420 |
| 593 |  | 1 | 0 | 18 | 108 | 1542 | 18180 | 249480 | 3486420 |
| 594 | $\mathrm{CKP}_{616}$ | 1 | 0 | 18 | 114 | 1542 | 19200 | 262890 | 3780420 |
| 595 | $\mathrm{CKP}_{617}$ | 1 | 0 | 18 | 120 | 1878 | 23400 | 351180 | 5323080 |
| 596 | $\mathrm{CKP}_{618}$ | 1 | 0 | 18 | 120 | 1878 | 25200 | 379980 | 6032880 |
| 597 |  | 1 | 0 | 18 | 120 | 2022 | 26160 | 421020 | 6607440 |
| 598 | $\mathrm{CKP}_{619}$ | 1 | 0 | 18 | 132 | 1926 | 25800 | 388800 | 6041280 |
| 599 | $\mathrm{CKP}_{620}$ | 1 | 0 | 18 | 138 | 2166 | 30240 | 478170 | 7777560 |
| 600 |  | 1 | 0 | 18 | 144 | 2118 | 30960 | 481860 | 7971600 |
| 601 | $\mathrm{CKP}_{621}$ | 1 | 0 | 18 | 156 | 2190 | 32760 | 513720 | 8536080 |
| 602 | $\mathrm{CKP}_{622}$ | 1 | 0 | 18 | 156 | 2310 | 33240 | 537120 | 8919960 |
| 603 |  | 1 | 0 | 18 | 156 | 2358 | 34920 | 564120 | 9502920 |
| 604 |  | 1 | 0 | 18 | 174 | 2454 | 38880 | 636030 | 11007780 |


| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 605 | $\mathrm{CKP}_{623}$ | 1 | 0 | 18 | 192 | 2862 | 46440 | 802980 | 14515200 |
| 606 | $\mathrm{CKP}_{624}$ | 1 | 0 | 18 | 228 | 2934 | 55320 | 969840 | 18061680 |
| 607 | $\mathrm{CKP}_{625}, S_{5}^{2} \times S_{5}^{2}$ | 1 | 0 | 20 | 60 | 1140 | 9120 | 121700 | 1377600 |
| 608 | $\mathrm{CKP}_{626}, \mathbb{P}^{2} \times S_{4}^{2}$ | 1 | 0 | 20 | 102 | 1188 | 11760 | 123050 | 1391880 |
| 609 | $\mathrm{CKP}_{627}$ | 1 | 0 | 20 | 120 | 1668 | 21120 | 303320 | 4519200 |
| 610 | $\mathrm{CKP}_{628}$ | 1 | 0 | 20 | 120 | 1860 | 23280 | 342200 | 5115600 |
| 611 | $\mathrm{CKP}_{629}$ | 1 | 0 | 20 | 126 | 1908 | 24480 | 361010 | 5470920 |
| 612 |  | 1 | 0 | 20 | 144 | 2148 | 31800 | 505280 | 8329440 |
| 613 | $\mathrm{CKP}_{630}$ | 1 | 0 | 20 | 156 | 2340 | 34080 | 540740 | 8942640 |
| 614 | $\mathrm{Str}_{3}$ | 1 | 0 | 20 | 156 | 2700 | 41040 | 697700 | 12503400 |
| 615 | $\mathrm{CKP}_{631}$ | 1 | 0 | 20 | 168 | 2580 | 38400 | 629120 | 10709160 |
| 616 |  | 1 | 0 | 20 | 168 | 2580 | 39600 | 648920 | 11239200 |
| 617 |  | 1 | 0 | 20 | 198 | 3228 | 52260 | 925130 | 17075100 |
| 618 | $\mathrm{CKP}_{633}, S_{8}^{2} \times S_{4}^{2}$ | 1 | 0 | 22 | 102 | 1434 | 13740 | 160510 | 1881180 |
| 619 | $\mathrm{CKP}_{634}$ | 1 | 0 | 22 | 120 | 1914 | 23280 | 347980 | 5206320 |
| 620 | $\mathbb{P}^{1} \times \mathrm{MM}_{3-3}^{3}, \mathrm{CKP}_{635}$ | 1 | 0 | 22 | 132 | 2058 | 24360 | 345280 | 4867800 |
| 621 | $\mathrm{CKP}_{636}$ | 1 | 0 | 22 | 144 | 2394 | 34200 | 557140 | 9241680 |
| 622 | $\mathrm{CKP}_{637}$ | 1 | 0 | 22 | 162 | 2490 | 34260 | 531490 | 8504160 |
| 623 |  | 1 | 0 | 22 | 168 | 2634 | 38040 | 613660 | 10263120 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 624 | $\mathrm{CKP}_{638}$ | 1 | 0 | 22 | 186 | 3090 | 47880 | 824530 | 14728980 |
| 625 |  | 1 | 0 | 22 | 186 | 3354 | 52980 | 960970 | 17852100 |
| 626 |  | 1 | 0 | 22 | 192 | 3258 | 51720 | 914620 | 16742880 |
| 627 | $\mathrm{CKP}_{639}$ | 1 | 0 | 22 | 246 | 4290 | 74280 | 1433830 | 28650720 |
| 628 |  | 1 | 0 | 22 | 264 | 4122 | 77880 | 1476220 | 29789760 |
| 629 |  | 1 | 0 | 22 | 264 | 4554 | 82200 | 1613740 | 33027120 |
| 630 | $\mathrm{CKP}_{640}, V_{6}^{4}$ | 1 | 0 | 24 | 0 | 3240 | 0 | 672000 | 0 |
| 631 | $\mathrm{CKP}_{641}$ | 1 | 0 | 24 | 36 | 3240 | 10800 | 680100 | 3528000 |
| 632 | $\mathrm{CKP}_{642}$ | 1 | 0 | 24 | 72 | 3288 | 21600 | 720600 | 7101360 |
| 633 | $\begin{aligned} & \mathbb{P}^{1} \times \mathrm{MM}_{7-1}^{3}, \mathbb{P}^{1} \times \mathbb{P}^{1} \times S_{4}^{2}, \\ & \mathrm{CKP}_{643}, \end{aligned}$ | 1 | 0 | 24 | 96 | 1704 | 14400 | 193920 | 2150400 |
| 634 | $S_{7}^{2} \times S_{4}^{2}, \mathrm{CKP}_{644}$ | 1 | 0 | 24 | 102 | 1704 | 15720 | 205530 | 2452380 |
| 635 | $\mathrm{CKP}_{645}$ | 1 | 0 | 24 | 144 | 3480 | 46920 | 909600 | 16450560 |
| 636 | $\mathrm{CKP}_{646}, \mathbb{P}^{1} \times \mathrm{MM}_{2-9}^{3}$ | 1 | 0 | 24 | 174 | 2784 | 37680 | 578490 | 9059820 |
| 637 |  | 1 | 0 | 24 | 186 | 3144 | 47280 | 804390 | 14118720 |
| 638 | $\mathrm{CKP}_{647}$ | 1 | 0 | 24 | 192 | 3048 | 45840 | 757680 | 13077120 |
| 639 | $\mathrm{CKP}_{648}$ | 1 | 0 | 24 | 192 | 3192 | 48120 | 816000 | 14306040 |
| 640 | $\mathrm{CKP}_{649}$ | 1 | 0 | 24 | 234 | 3648 | 60780 | 1060350 | 19603500 |
| 641 | $\mathrm{CKP}_{650}$ | 1 | 0 | 24 | 264 | 4632 | 83040 | 1611960 | 32664240 |

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| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 642 |  | 1 | 0 | 24 | 264 | 5352 | 101040 | 2040360 | 43219680 |
| 643 | $\mathrm{CKP}_{651}$ | 1 | 0 | 26 | 72 | 3534 | 22320 | 787580 | 7514640 |
| 644 | $\mathrm{CKP}_{652}, S_{6}^{2} \times S_{4}^{2}$ | 1 | 0 | 26 | 108 | 1998 | 19080 | 270440 | 3435600 |
| 645 | $\mathrm{CKP}_{653}$ | 1 | 0 | 26 | 216 | 4302 | 72480 | 1371500 | 27676320 |
| 646 | $\mathrm{CKP}_{654}$ | 1 | 0 | 26 | 246 | 4302 | 72120 | 1339550 | 25814460 |
| 647 | $\mathrm{CKP}_{655}$ | 1 | 0 | 26 | 288 | 5166 | 102960 | 2038580 | 44530080 |
| 648 | $\mathrm{CKP}_{656}$ | 1 | 0 | 26 | 396 | 6222 | 151080 | 3168440 | 74446680 |
| 649 | $\mathrm{CKP}_{657}$ | 1 | 0 | 28 | 240 | 3996 | 62400 | 1067680 | 19007520 |
| 650 | $\mathrm{CKP}_{658}$ | 1 | 0 | 28 | 258 | 4764 | 82200 | 1573390 | 31316460 |
| 651 | $\mathrm{CKP}_{659}$ | 1 | 0 | 28 | 288 | 5484 | 100800 | 2038960 | 42887040 |
| 652 | $\mathrm{CKP}_{660}$ | 1 | 0 | 28 | 306 | 5580 | 104100 | 2099350 | 44273880 |
| 653 |  | 1 | 0 | 28 | 342 | 6540 | 129540 | 2770570 | 61901700 |
| 654 | $\mathrm{CKP}_{661}$ | 1 | 0 | 28 | 432 | 9660 | 210240 | 5004640 | 126134400 |
| 655 | $S_{5}^{2} \times S_{4}^{2}, \mathrm{CKP}_{662}$ | 1 | 0 | 30 | 126 | 2658 | 27720 | 439590 | 6247500 |
| 656 | $\mathbb{P}^{1} \times \mathrm{MM}_{2-10}^{3}, \mathrm{CKP}_{663}$ | 1 | 0 | 30 | 216 | 3858 | 54000 | 891660 | 14726880 |
| 657 | $\mathrm{CKP}_{664}$ | 1 | 0 | 30 | 240 | 4338 | 66960 | 1182900 | 21408240 |
| 658 | $\mathrm{CKP}_{665}$ | 1 | 0 | 30 | 300 | 6690 | 124920 | 2778600 | 61790400 |
| 659 | $\mathrm{CKP}_{666}$ | 1 | 0 | 30 | 372 | 7314 | 153720 | 3385200 | 79195200 |


| Period ID |  | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 660 | CKP $_{667}$ | 1 | 0 | 32 | 318 | 6144 | 113280 | 2304770 | $\alpha_{7}$ |
| 661 | CKP $_{668}$ | 1 | 0 | 32 | 384 | 7728 | 157800 | 3492320 | 48799800 |
| 662 |  | 1 | 0 | 32 | 384 | 8112 | 167520 | 3766640 | 80806320 |
| 663 | $\mathbb{P}^{1} \times V_{14}^{3}$ | 1 | 0 | 34 | 312 | 5910 | 97920 | 1820140 | 88438560 |
| 664 |  | 1 | 0 | 34 | 390 | 8694 | 179520 | 4180750 | 34520640 |
| 665 |  | 1 | 0 | 34 | 498 | 10278 | 245040 | 5923330 | 100127580 |
| 666 | CKP $_{669}$ | 1 | 0 | 36 | 336 | 6708 | 119520 | 2419200 | 153543600 |
| 667 | CKP $_{670}$ | 1 | 0 | 36 | 360 | 7188 | 134400 | 2795400 | 50507520 |
| 668 |  | 1 | 0 | 36 | 396 | 7572 | 143160 | 2921580 | 60459840 |
| 669 |  | 1 | 0 | 36 | 456 | 9876 | 214680 | 5072760 | 62324640 |
| 670 | $\mathrm{CKP}_{671}$ | 1 | 0 | 36 | 552 | 12852 | 304080 | 7828200 | 125137740 |
| 671 |  | 1 | 0 | 36 | 768 | 18996 | 500640 | 14713200 | 210966000 |
| 672 | $\mathrm{CKP}_{672}, \mathbb{P}^{1} \times \mathrm{MM}_{2-7}^{3}$ | 1 | 0 | 38 | 348 | 6954 | 117840 | 2268560 | 450203040 |
| 673 | $\mathrm{CKP}_{673}$ | 1 | 0 | 38 | 384 | 8106 | 156480 | 3390500 | 44336040 |
| 674 | $\mathrm{CKP}_{674}$ | 1 | 0 | 38 | 396 | 8010 | 150600 | 3136160 | 76130880 |
| 675 | $\mathrm{CKP}_{675}, S_{4}^{2} \times S_{4}^{2}$ | 1 | 0 | 40 | 192 | 4776 | 59520 | 1120000 | 67735080 |
| 676 | $\mathrm{CKP}_{676}$ | 1 | 0 | 44 | 516 | 11580 | 248880 | 5903540 | 19138560 |
| 677 | $\mathrm{CKP}_{677}$ | 1 | 0 | 44 | 636 | 15804 | 393480 | 10666340 | 145945800 |
| 678 | $\mathrm{CKP}_{678}$ | 1 | 0 | 44 | 696 | 17388 | 445680 | 12371480 | 301939680 |


| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 679 |  | 1 | 0 | 44 | 744 | 18396 | 492360 | 14028200 | 419215440 |
| 680 | $\mathrm{CKP}_{679}$ | 1 | 0 | 44 | 888 | 23052 | 649200 | 19904120 | 635293680 |
| 681 | $\mathbb{P}^{1} \times \mathrm{MM}_{2-6}^{3}, \mathrm{CKP}_{680}$ | 1 | 0 | 46 | 528 | 11826 | 238560 | 5341780 | 122340960 |
| 682 | $\mathrm{CKP}_{681}$ | 1 | 0 | 46 | 714 | 18618 | 496560 | 14203810 | 428469300 |
| 683 | $V_{4}^{4}, \mathrm{CKP}_{682}$ | 1 | 0 | 48 | 0 | 15120 | 0 | 7392000 | 0 |
| 684 | $\mathrm{CKP}_{683}$ | 1 | 0 | 48 | 216 | 15408 | 151320 | 7959000 | 117482400 |
| 685 | $\mathrm{CKP}_{684}$ | 1 | 0 | 48 | 660 | 15552 | 367320 | 9396300 | 251895000 |
| 686 | $\mathbb{P}^{1} \times V_{12}^{3}$ | 1 | 0 | 50 | 600 | 13758 | 288480 | 6659420 | 157802400 |
| 687 | $\mathrm{CKP}_{685}$ | 1 | 0 | 50 | 792 | 21078 | 635760 | 18069260 | 600739440 |
| 688 | $\mathrm{CKP}_{686}$ | 1 | 0 | 52 | 696 | 17412 | 424440 | 11365000 | 317604000 |
| 689 |  | 1 | 0 | 52 | 1044 | 29124 | 874080 | 28285540 | 956113200 |
| 690 | $\mathbb{P}^{2} \times S_{3}^{2}, \mathrm{CKP}_{687}$ | 1 | 0 | 54 | 498 | 9882 | 162000 | 2938770 | 54057780 |
| 691 | $\mathrm{CKP}_{688}$ | 1 | 0 | 54 | 528 | 11178 | 207720 | 4427820 | 98491680 |
| 692 | $\mathrm{CKP}_{689}$ | 1 | 0 | 54 | 744 | 19194 | 481680 | 13279500 | 381906000 |
| 693 | $\mathrm{CKP}_{690}$ | 1 | 0 | 54 | 888 | 24378 | 677520 | 20447820 | 644873040 |
| 694 | $\mathrm{CKP}_{692}, S_{8}^{2} \times S_{3}^{2}$ | 1 | 0 | 56 | 498 | 10536 | 171900 | 3240110 | 60897480 |
| 695 | $\mathrm{CKP}_{693}$ | 1 | 0 | 56 | 528 | 11832 | 217920 | 4748600 | 106293600 |
| 696 | $\mathrm{CKP}_{694}$ | 1 | 0 | 56 | 600 | 14424 | 317100 | 7961600 | 207233040 |


| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 697 | $\mathbb{P}^{1} \times \mathbb{P}^{1} \times S_{3}^{2}, \mathrm{CKP}_{695}, \mathbb{P}^{1} \times$ | 1 | 0 | 58 | 492 | 11214 | 178440 | 3502120 | 65938320 |
|  | $\mathrm{MM}_{8-1}^{3}$ |  |  |  |  |  |  |  |  |
| 698 | $\mathrm{CKP}_{696}, S_{7}^{2} \times S_{3}^{2}$ | 1 | 0 | 58 | 498 | 11214 | 181800 | 3561250 | 68151720 |
| 699 |  | 1 | 0 | 58 | 888 | 23694 | 632400 | 18393340 | 559525680 |
| 700 | $S_{6}^{2} \times S_{3}^{2}, \mathrm{CKP}_{697}$ | 1 | 0 | 60 | 504 | 11916 | 195120 | 3962040 | 78104880 |
| 701 | $\mathrm{CKP}_{698}$ | 1 | 0 | 60 | 1068 | 30156 | 893280 | 28423860 | 948659040 |
| 702 | $\mathrm{CKP}_{699}$ | 1 | 0 | 60 | 1212 | 35916 | 1134480 | 38512860 | 1368087000 |
| 703 | $\mathrm{CKP}_{700}, S_{5}^{2} \times S_{3}^{2}$ | 1 | 0 | 64 | 522 | 13392 | 225720 | 4887190 | 102194400 |
| 704 | $\mathrm{CKP}_{701}$ | 1 | 0 | 66 | 852 | 21510 | 504000 | 13009080 | 347891040 |
| 705 | $\mathrm{CKP}_{702}$ | 1 | 0 | 66 | 1356 | 47574 | 1614240 | 58420920 | 2223985680 |
| 706 | $\mathbb{P}^{1} \times \mathrm{MM}_{2-5}^{3}, \mathrm{CKP}_{703}$ | 1 | 0 | 68 | 816 | 21012 | 465960 | 11662880 | 297392760 |
| 707 | $\mathrm{CKP}_{704}$ | 1 | 0 | 68 | 852 | 22308 | 520680 | 13640900 | 368091360 |
| 708 | $\mathrm{CKP}_{705}$ | 1 | 0 | 68 | 1320 | 43236 | 1421040 | 51100520 | 1914785040 |
| 709 | $S_{4}^{2} \times S_{3}^{2}, \mathrm{CKP}_{706}$ | 1 | 0 | 74 | 588 | 17550 | 319560 | 7862600 | 185440080 |
| 710 | $\mathrm{CKP}_{707}$ | 1 | 0 | 78 | 1140 | 32706 | 877320 | 26208960 | 814453920 |
| 711 | $\mathrm{CKP}_{708}$ | 1 | 0 | 78 | 1176 | 34002 | 937080 | 28577940 | 909170640 |
| 712 | $\mathrm{CKP}_{709}$ | 1 | 0 | 78 | 1680 | 60066 | 2142720 | 82424580 | 3324124440 |
| 713 | $\mathrm{CKP}_{710}$ | 1 | 0 | 80 | 1212 | 36240 | 1020360 | 31974020 | 1043489160 |
| 714 | $\mathbb{P}^{1} \times V_{10}^{3}$ | 1 | 0 | 80 | 1320 | 38688 | 1078320 | 32604200 | 1016215200 |


| Period ID | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $\alpha_{7}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 715 |  | 1 | 0 | 84 | 1932 | 69636 | 2622480 | 106446900 | 4526098920 |
| 716 | $\mathrm{CKP}_{711}$ | 1 | 0 | 84 | 2148 | 77316 | 3051480 | 128188740 | 5649930720 |
| 717 |  | 1 | 0 | 90 | 1788 | 59886 | 2032920 | 74950920 | 2894154480 |
| 718 | $\mathrm{CKP}_{712}$ | 1 | 0 | 90 | 2040 | 76014 | 2873160 | 117404820 | 5023514160 |
| 719 | $\mathbb{P}^{1} \times \mathrm{MM}_{2-4}^{3}, \mathrm{CKP}_{713}$ | 1 | 0 | 92 | 1518 | 47172 | 1357680 | 42774050 | 1385508600 |
| 720 | $\mathrm{CKP}_{714}$ | 1 | 0 | 92 | 1626 | 51492 | 1574580 | 52448150 | 1816414320 |
| 721 | $\mathrm{CKP}_{715}$ | 1 | 0 | 92 | 2112 | 83820 | 3281280 | 141863600 | 6368328960 |
| 722 | $\mathrm{CKP}_{716}$ | 1 | 0 | 102 | 1950 | 67002 | 2266320 | 83881470 | 3245543280 |
| 723 |  | 1 | 0 | 102 | 2274 | 84330 | 3207480 | 132223890 | 5710371660 |
| 724 | $\mathrm{CKP}_{717}$ | 1 | 0 | 102 | 2688 | 106410 | 4495680 | 203447460 | 9658434240 |
| 725 | $\mathrm{CKP}_{718}$ | 1 | 0 | 102 | 3408 | 146250 | 6695280 | 334814340 | 17506424880 |
| 726 | $\mathrm{CKP}_{719}$ | 1 | 0 | 104 | 2472 | 97944 | 3940320 | 171825080 | 7840793520 |
| 727 | $\mathrm{CKP}_{720}, S_{3}^{2} \times S_{3}^{2}$ | 1 | 0 | 108 | 984 | 37260 | 848880 | 26609400 | 804368880 |
| 728 | $\mathrm{CKP}_{721}$ | 1 | 0 | 128 | 2976 | 120960 | 4959840 | 221633120 | 10369947840 |
| 729 | $\mathrm{CKP}_{722}$ | 1 | 0 | 138 | 4650 | 222918 | 11448480 | 632940330 | 36647730000 |
| 730 |  | 1 | 0 | 150 | 4866 | 241002 | 12623040 | 711272850 | 42024975300 |
| 731 | $\mathrm{CKP}_{723}, \mathbb{P}^{1} \times V_{8}^{3}$ | 1 | 0 | 154 | 3840 | 159486 | 6504960 | 284808340 | 12889551360 |
| 732 | $\mathrm{CKP}_{724}$ | 1 | 0 | 168 | 4752 | 219624 | 10383840 | 531501360 | 28511659680 |
| 733 | $\mathrm{CKP}_{725}$ | 1 | 0 | 184 | 5688 | 286008 | 14876160 | 837897160 | 49505030400 |


| Period ID |  | Name | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ |
| :---: | :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 734 |  | 1 | 0 | 224 | 9312 | 580704 | 38555520 | 2752140320 | 206084027520 |
| 735 | CKP $_{726}$ | 1 | 0 | 272 | 13560 | 952176 | 73148160 | 5996559080 | 516454715280 |
| 736 | CKP $_{727}, \mathbb{P}^{1} \times V_{6}^{3}$ | 1 | 0 | 398 | 17616 | 1221810 | 85572960 | 6386359700 | 493612489440 |
| 737 | CKP $_{728}$ | 1 | 0 | 420 | 19992 | 1488708 | 114603120 | 9497959800 | 824518956240 |
| 738 | CKP $_{729}$ | 1 | 0 | 444 | 22404 | 1771596 | 146305440 | 13047797460 | 1221757064640 |
| 739 | CKP $_{730}$ | 1 | 0 | 468 | 24852 | 2065764 | 180367920 | 17014559940 | 1685867765400 |
| 740 |  | 1 | 0 | 540 | 37632 | 3836268 | 420664320 | 49565795760 | 6131551910400 |
| 741 | CKP $_{731}$ | 1 | 0 | 1040 | 105984 | 15564048 | 2472668160 | 422070022400 | 75673543680000 |
| 742 |  | 1 | 0 | 1386 | 166284 | 28575342 | 5322513240 | 1065056580360 | 223880895211680 |
| 743 | $\mathbb{P}^{1} \times V_{4}^{3}, \mathrm{CKP}_{732}$ | 1 | 0 | 1946 | 215808 | 35318526 | 5981882880 | 1074550170260 | 200205416839680 |
| 744 | $\mathrm{CKP}_{733}$ | 1 | 0 | 1992 | 227472 | 38459880 | 6796332000 | 1282447706160 | 252711084477600 |
| 745 | $\mathrm{CKP}_{734}$ | 1 | 0 | 2136 | 262896 | 48275736 | 9412519800 | 1975803279600 | 435882277192320 |
| 746 | $\mathrm{CKP}_{735}$ | 1 | 0 | 2664 | 466368 | 115475112 | 31137505920 | 9021039724800 | 2746619333498880 |
| 747 | $\mathrm{CKP}_{736}$ | 1 | 0 | 6804 | 2040912 | 852143652 | 389608626240 | 191430924575040 | 98894833331535360 |
| 748 | $\mathrm{CKP}_{737}$ | 1 | 0 | 12816 | 5435904 | 3188239632 | 2051802731520 | 1419118168838400 | 1032164932439531520 |
| 749 | $\mathrm{CKP}_{738}$ | 1 | 0 | 99000 | 130800000 | 233995275000 | 462392774925120 | 982577026659240000 | 2197113382189414080000 |


[^0]:    *Another proof of this, using different methods, has recently been given by Rachel Webb [49].

[^1]:    *Note that usually the I-function is written as a function in $(\tau, z)$, just like the J-function. This is what you obtain if you set $\tau=0$ (the only case we need).

[^2]:    *This is the Euler characteristic of $X$ as a topological space.

[^3]:    ${ }^{\dagger}$ To be precise: we find at least 141 four-dimensional Fano manifolds for which the regularised quantum period was not previously known. The regularised quantum period of a Fano manifold $X$ is expected to completely determine $X$. See $[16,14]$ for known quantum periods.

[^4]:    *They are chosen to minimize the quantity $\sum_{i=0}^{\rho} r_{i}^{2}$, which is a rough proxy for the complexity of the Chow ring of the Abelianization.

