LECTURE -2: SOME FEATURES OF PRISMATIC COHOMOLOGY

ELDEN ELMANTO

1. Notes on absolute prismatic cohomology (mini-course)

One of the things which is important in the prismatic theory is how simple this definition is. To get a feel for what it is, we make the following remark:

Remark 1.0.1. [Stacks, Tag 07HL] Recall that a divided power algebra on a pair (A, I) is the datum γ of "elements that look like $x^n/n!$" which are denoted by $\gamma_n(x)$. For example we can look at the pair $(\mathbb{Z}_{(p)}, (p))$ and set $\gamma_n(x = pa) = \frac{p^n a^n}{n!}$, noting that $p^n/n! \in p\mathbb{Z}_{(p)}$. A divided power algebra whose underlying ring is a $\mathbb{Z}_{(p)}$-algebra is said to be **compatible** if the divided power structure is compatible with the one from $(\mathbb{Z}_{(p)}, (p))$. Let us define the the **affine crystalline site**: for A a smooth $\mathbb{F}_p$-algebra, an object is a pair $((\mathrm{B}, \mathrm{J}, \gamma), \alpha)$ where $(\mathrm{B}, \mathrm{J}, \gamma)$ is a compatible divided power algebra and α is a map $\alpha : \mathrm{A} \to \mathrm{B}/\mathrm{J}$. We denote this site by $(\mathrm{A}/\mathbb{F}_p)_{\mathrm{crys}}$. We endow this again with the indiscrete topology. The **crystalline structure sheaf** is defined via

$$\mathcal{O}_{\mathrm{crys}}((\mathrm{B}, \mathrm{J}, \gamma), \alpha) = \mathrm{B}.$$

Then, whatever definition of crystalline cohomology you might have defined satisfy

$$\mathrm{R}\Gamma((\mathrm{A}/\mathbb{F}_p)_{\mathrm{crys}}, \mathcal{O}_{\mathrm{crys}}) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{A}/\mathbb{F}_p).$$

In particular, there is no real need to speak of Grothendieck topologies when one speaks about defining crystalline cohomology.

Now, I want to explain what is the engine of the theory. At heart, everything we do relies on a very basic lemma called the **derived Nakayama**, but first an official definition of what we mean to be complete.

Definition 1.0.2. Let A be a ring and I an ideal of A. Let $\mathrm{M} \in \mathbf{D}(\mathrm{A})$, then we say that M is **derived I-complete**, if the the limit

$$\xrightarrow{f} \mathrm{M} \xrightarrow{f} \mathrm{M} \xrightarrow{f} \cdots,$$

is acyclic. Equivalently, the map

$$\mathrm{M} \to \lim(\mathrm{M} \otimes_{\mathbb{Z}[x]} \mathbb{Z}[x]/(x^n))$$

is an equivalence where M is treated as an A-module via the map classifying $f\colon \mathbb{Z}[x] \xrightarrow{f} \mathrm{A}$.

Remark 1.0.3. I find it useful to think about the inverse limit as above as "f-coperfection" (something like $\mathrm{M}^{f\flat}$ seems like a good notation) after what happens if you do this on the frobenius. This means that being f-complete is right orthogonal to being f-coperfect. I note that if f is invertible in M, then M is equivalent to its f-coperfection and whence it is the exact opposite of what it means to be complete.

Remark 1.0.4. One of the main points of working with derived completion is the following lemma:

Lemma 1.0.5. *The subcategory of* $\mathbf{Mod}_{\mathrm{A}}$ *spanned by derived* I*-complete* A*-modules forms an abelian category.*

This is not the case for the category of classically complete A-modules. However, the functor of derived completion is really derived: if M is discrete, the derived I-completion can produce higher homology groups. We also note that if M is derived complete and is bounded I-torsion, then M is classically complete.

Lemma 1.0.6. [Stacks, Tag 0G1U] *Let* I *be a finitely generated ideal of a ring* A*. If* M *is a derived* I*-complete module, and* $\mathrm{M} \otimes^{\mathrm{L}} \mathrm{A}/\mathrm{I} \simeq 0$ *then* $\mathrm{M} \simeq 0$.

Remark 1.0.7. The usual Nakayama's lemma requires some finite generation hypotheses on the module M. In lieu of this, we have the derived completeness assumption M, but M is allowed to be an arbitrary object in $\mathbf{D}(\mathrm{A})$. In the context of this theory, we really do not want to restrict ourselves with finiteness hypotheses.

One of the reasons why prismatic cohomology is accessible is because we can really understand the Hodge-Tate complex; in conjunction with the fact that it is derived I-complete, we can often reduce questions about prismatic cohomology to the Hodge-Tate complex.

Construction 1.0.8. Let (A, I) be a prism and assume that $\mathrm{M} \in \mathbf{D}(\mathrm{A}/\mathrm{I})$; we denote the **Breuil-Kisin twist** by

$$\mathrm{M}\{i\} := \mathrm{M} \otimes^{\mathrm{L}} \mathrm{I}^i/\mathrm{I}^{i+1}.$$

We have a Bockstein map

$$\beta : \mathrm{H}^i(\overline{\mathbb{\Delta}}_{\mathrm{R}/\mathrm{A}}\{i\}) \to \mathrm{H}^{i+1}(\mathbb{\Delta}_{\mathrm{R}/\mathrm{A}}\{i+1\}).$$

We have [BS19, Construction 4.8] a map of commutative dga:

$$(\Omega^{\bullet}_{\mathrm{R}/(\mathrm{A}/\mathrm{I})}, d) \to (\mathrm{H}^{\bullet}(\mathbb{\Delta}_{\mathrm{R}/\mathrm{A}}\{\bullet\}), \beta),$$

characterized by

(1) the zero-th term is an R-algebra and the map above is the structure map;
(2) $\Omega^1_{\mathrm{R}/(\mathrm{A}/\mathrm{I})} \to \mathrm{H}^1(\overline{\mathbb{\Delta}}_{\mathrm{R}/(\mathrm{A}/\mathrm{I})}\{1\})$ is given by

$$fdg \mapsto f\beta(g).$$

Theorem 1.0.9. *The above furnishes a canonical equivalence*

$$(\Omega^{\bullet}_{\mathrm{R}/(\mathrm{A}/\mathrm{I})}, d) \xrightarrow{\simeq} (\mathrm{H}^{\bullet}(\mathbb{\Delta}_{\mathrm{R}/\mathrm{A}}\{\bullet\}), \beta).$$

This also implies that

Remark 1.0.10. One key result in characteristic $p > 0$ algebraic geometry is the **Cartier isomorphism**: if R is a *smooth* algebra over a *perfect* field of characteristic $p > 0$ is that the map

$$\mathrm{C}^{-1} : \Omega^n_{\mathrm{R}/k} \to \mathrm{H}^n(\Omega^{\bullet}_{\mathrm{R}/k}) \qquad \mathrm{C}^{-1}(fdg_1 \wedge \cdots \wedge dg_n) = f^p g_1^{p-1} \cdots g_n^{p-1} dg_1 \wedge \cdots \wedge dg_n.$$

is an isomorphism; usually the map is defined but not an isomorphism unless R is smooth. One can treat this as a kind of formality statement and Theorem 1.0.9 is a kind of mixed characteristic analog of the Cartier isomorphism.

Here is an immediate application of Theorem 1.0.9 is the following "crystalline" property of prismatic cohomology.

Proposition 1.0.11. *Let* $g : (\mathrm{A}, \mathrm{I}) \to (\mathrm{B}, \mathrm{IB})$ *be a morphism of bounded prisms. Then the canonical map*

$$\mathbb{\Delta}_{\mathrm{R}/\mathrm{A}} \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B} \to \mathbb{\Delta}_{\mathrm{R}\otimes_{\mathrm{A}}\mathrm{B}/\mathrm{B}}$$

is an equivalence.

Proof. It suffices to prove the equivalence after $\otimes^{\mathrm{L}}\mathrm{B}/\mathrm{IB}$. This follows, after Theorem 1.0.9, the fact that de Rham cohomology is stable under base change:

$$\Omega^{\bullet}_{\mathrm{R}/(\mathrm{A}/\mathrm{I})} \otimes_{\mathrm{A}/\mathrm{I}} \mathrm{B}/\mathrm{IB} \simeq \Omega^{\bullet}_{(\mathrm{R}\otimes_{\mathrm{A}/\mathrm{I}}\mathrm{B}/\mathrm{IB})/(\mathrm{B}/\mathrm{IB})}.$$

□

1.1. **The comparison theorem.** We now run through the comparison results in prismatic cohomology without doing it much justice. I will formulate it in the simplest way possible

Theorem 1.1.1 (Bhatt-Morrow-Scholze). *Let* $X \to \mathbb{Z}_p$ *be smooth and projective. Then there is a natural complex*

$$R\Gamma_{\Delta}(X/\mathbb{Z}_p[[T]])/p \in \mathbf{Perf}(\mathbb{F}_p[[T]]);$$

where

(étale comparison) we have a natural equivalence:

$$R\Gamma_{\Delta}(X/\mathbb{Z}_p[[T]])/p[\tfrac{1}{T}] \simeq R\Gamma_{\text{ét}}(X_C; \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p((T));$$

(de Rham comparison) and another natural equivalence:

$$R\Gamma_{\Delta}(X/\mathbb{Z}_p[[T]])/(p, T) \simeq R\Gamma_{dR}(X_{\mathbb{F}_p}).$$

Consequently

$$\dim_{\mathbb{F}_p} H^n(X_C; \mathbb{F}_p) \leqslant \dim_{\mathbb{F}_p} H^n_{dR}(X_{\mathbb{F}_p}).$$

Let us give a "proof" of sort.

Proof Sketch. We work with the "trivial" Breuil-Kisin prism: $(\mathbb{Z}_p[[T]], (T - p))$ noting that the map $\mathbb{Z}_p[[T]] \to \mathbb{Z}_p$ given by sending T to p has kernel generated by $T - p$; so the format of prismatic cohomology will taken in a smooth $\mathbb{Z}_p$-algebra (or more generall, scheme) and outputs a complex

$$\Delta_{R/\mathbb{Z}_p[[T]]} \in \mathbf{D}(\mathbb{Z}_p[[T]]),$$

equipped with a Frobenius map which is semilinear with respect to the Frobenius $\mathbb{Z}_p[[T]]$ determined by $T \mapsto T^p$.

We begin by explaining why the complex is perfect. In general suppose that A is a commutative ring and I is a finitely generated ideal and $M \in \mathbf{D}(A)$ which is (derived) I-adically complete. Being perfect means that it is in the thick subcategory generated by the unit object A; such a condition can be checked after modding out by I. In fact, we can also check perfectness by verifying that the cohomology modules are perfect [Stacks, Tag 066U] Therefore, we need only verify this condition after modding out $\Delta_{R/\mathbb{Z}_p[[T]]}$ by I. Then the Hodge-Tate comparison Theorem 1.0.9 reduces us to checking the requisite statement on the level of cohomology modules, which is a classical fact.

Next we deal with the de Rham comparison. The usual one states

$$R\Gamma_{dR}(\operatorname{Spec} R/\mathbb{Z}_p) \simeq \Delta_{R/\mathbb{Z}_p[[T]]} \otimes_{\mathbb{Z}_p[[T]]} \varphi^* \mathbb{Z}_p.$$

So we if (derive) mod p-reduce both sides, we get the usual result. The de Rham comparison is actually a reflection of the crystalline nature of prismatic cohomology: indeed we have a morphism of prisms $(\mathbb{Z}_p[[T]], (p - T)) \to (\mathbb{Z}_p, (p))$ from which we base change to get

$$\Delta_{R/\mathbb{Z}_p[[T]]} \otimes_{\mathbb{Z}_p[[T]]} \mathbb{Z}_p \simeq \Delta_{R_{\mathbb{Z}_p}/\mathbb{Z}_p}.$$

Up to a Frobenius twist, $\Delta_{R_{\mathbb{Z}_p}/\mathbb{Z}_p}$ is equivalent to $R\Gamma_{crys}(R/p/\mathbb{Z}_p)$, the crystalline cohomology of the mod p reduction of R. But now, from the fact that crystalline cohomology is the de Rham cohomology of a smooth module, the result follows (again up to a twist).

Let now X be a proper and smooth over $\mathbb{Z}_p$. The usual form of the étale comparison theorem (which requires a base change to a perfect prism — so take the perfection of the underlying prism which has I generated by d) tells us that for any smooth $\mathbb{Z}_p$-algebra we then have an equivalence

$$R\Gamma_{\text{ét}}(X_C; \mathbb{F}_p) \simeq (R\Gamma_{\Delta}(X)/p[\tfrac{1}{d}])^{\varphi - 1}. \tag{1.1.2}$$

I will leave out the details but one can remove the φ from the above statement by using Lemma 1.1.3.

□

Lemma 1.1.3. *Let* K *be an algebraically closed field of characteristic* $p > 0$*. Let* M *be a perfect complex equipped with a frobenius* φ *(*M *is equipped with a map* $\mathrm{M} \to \mathrm{Frob}_*\mathrm{M}$ *where* Frob *is the Frobenius in* K*), then if* φ *is an equivalence, we have that* $\mathrm{H}^i(\mathrm{M}^{\varphi=1}) \otimes_{\mathbb{F}_p} \mathrm{K} \simeq \mathrm{H}^i(\mathrm{M})$.

1.2. **The Cartier-Witt stack and prismatic cohomology.** I now want to speak about the "geometry" of prismatic cohomology. More precisely, I want to explain the idea of the Cartier-Witt stack which is a tool to produce deeper structures within this cohomology theory. If the analogy between prismatic cohomology and de Rham cohomology is to be believed, then one should be able to associate to any (p-adic formal) scheme X a stack $\mathrm{WCart}_{\mathrm{X}}$ with the key property that

$$\mathrm{R}\Gamma(\mathrm{WCart}_{\mathrm{X}}, \mathcal{O}_{\mathrm{WCart}_{\mathrm{X}}}) \simeq \mathrm{R}\Gamma_{\Delta}(\mathrm{X}).$$

Here's a definition:

Definition 1.2.1. A **virtual divisor** on a scheme X is an $\mathcal{O}_{\mathrm{X}}$-linear morphism $\alpha : \mathcal{L} \to \mathcal{O}_{\mathrm{X}}$ where $\mathcal{L}$ is a line bundle. A **Cartier-Witt** divisor on an affine scheme Spec R, where R is p-nilpotent, is the datum of a generalized divisor on Spec W(R) subject to a **derived prism** condition:

1. the image of the map $\mathrm{I} \xrightarrow{\alpha} \mathrm{W(R)} \to \mathrm{R}$ is nilpotent;
2. the image of the map $\mathrm{I} \xrightarrow{\alpha} \mathrm{W(R)} \xrightarrow{\delta} \mathrm{W(R)}$ generated the unit ideal.

By sending a non-p-nilpotent ring to the empty set, we get a prestack

$$\mathrm{WCart} : \mathrm{CAlg} \to \mathrm{Spc}.$$

The generalized prism condition is a "loosening" of the prismatic condition: it defines a prism whenever α is actually an inclusion. However, it does impose certain restrictions.

Lemma 1.2.2. *Let* R *be a p-nilpotent ring and* $\mathrm{I} \subset \mathrm{W(R)}$ *is an invertible ideal. Then the following are equivalent:*

(1) $(\mathrm{W(R)}, \mathrm{I})$ *is a prism such that the* (p, I)*-adic topology refines the* V*-adic topology;*
(2) $(\mathrm{W(R)}, \mathrm{I})$ *defines Cartier-Witt prism under the inclusion map* $\mathrm{I} \hookrightarrow \mathrm{W(R)}$.

Remark 1.2.3. The fact that (p, I)-adic topology refines the V-adic topology on W(R) is quite restrictive: it basically says that the image of I under the map $\mathrm{W(R)} \to \mathrm{R}$ must be nilpotent. Let us see that there are lots of examples: let R be a perfect ring of characteristic $p > 0$, then we have $(\mathrm{W(R)}, (d))$ a prism. There are many choices of d which corresponds to various untilts of R. I claim, in the event that the derived prism is satisfied, that d must actually be p. Indeed, we see that $\overline{d} \in \mathrm{R}$ under the map $\mathrm{W(R)} \to \mathrm{R}$, which by Definition 1.2.1.(1) is assumed to be nilpotent, must actually be zero since R is perfect hence reduced. This means that $(d) \in p\mathrm{W(R)}$. But then this must mean that $d = p$ by the rigidity property of prisms.

Proof. That $\delta(\mathrm{I})$ generates the unit ideal corresponds to the "classical" prism condition that we have local monogenic generation.

Let $(\mathrm{W(R)}, \mathrm{I})$ be a prism with the above topological condition. Then I claim that the image of I in $\mathrm{W(R)} \to \mathrm{R}$ is nilpotent. To see this, the assumption on the topology means that there exists N such that $(p, \mathrm{I})^{\mathrm{N}} \in \mathrm{VW(R)}$. Therefore, since $\mathrm{W(R)}/\mathrm{V} = \mathrm{R}$ we see that $\mathrm{I}^{\mathrm{N}} = 0$ in R. Now, assume that $(\mathrm{W(R)}, \mathrm{I})$ defines a Cartier-Witt prism. I claim that the image of I under $\mathrm{W(R)} \to \mathrm{R}$ being nilpotent implies that W(R) is I-adically complete. Indeed, I claim that each finite level Witt-vector is I-nilpotent (then use that inverse limit of I-nilpotent modules are I-complete). Iterating the Witt-vector Frobenius

$$\mathrm{F}^{n-1} : \mathrm{W}_n(\mathrm{R}) \to \mathrm{R}$$

furnishes R with a $\mathrm{W}_n(\mathrm{R})$-module structure. We have an exact sequence of $\mathrm{W}_n(\mathrm{R})$-modules

$$0 \to \mathrm{F}_*^{n-1}\mathrm{R} \to \mathrm{W}_n(\mathrm{R}) \xrightarrow{\mathrm{F}} \mathrm{W}_{n-1}(\mathrm{R}) \to 0.$$

By induction $\mathrm{W}_{n-1}(\mathrm{R})$ is I-nilpotent (we have identifies the image of I under $\mathrm{W(R)} \to \mathrm{W}_n(\mathrm{R})$). It suffices to prove that $\mathrm{F}_*^{n-1}(\mathrm{I})$ is I-nilpotent (what I mean here is that the image of I under

$W \to W_n$ is nilpotent). Now we note that for any morphism of rings $f : R \to S$, then S is an I-nilpotent R-module if and only if the image of I under f is nilpotent. Using this, it then suffices to prove that $F_*^{n-1}R$ is I-nilpotent. By assumption R is a p-nilpotent ring, hence it suffices to prove that the reduction mod p of $F^{n-1}(I)$ is I-nilpotent. But

$$F^{n-1}(I) = I^{p^{n-1}} \quad \mod p.$$

Hence we are done since I was assumed to be nilpotent in R. The condition on □

I will now discuss several ways to think about the Cartier-Witt stack.

1.2.4. *As a quotient stack.* We give a Zariski atlas for the Cartier-Witt stack.

Definition 1.2.5. Let R be a p-nilpotent ring. Let

$$\mathrm{WCart}_0(R) \subset \mathrm{WCart}(R)$$

be the set of witt vectors with expression $\sum_{n\geqslant 0} V^n[a_n]$ such that a_0 is nilpotent and a_1 is a unit. Setting the value to be empty if R is not nilpotent we get a functor

$$\mathrm{WCart}_0 : \mathrm{CAlg} \to \mathrm{Set}.$$

On the other hand we have $W^\times$ which is an affine group scheme given by units of the Witt vectors. W acts on $W^\times$ by multiplication and preserves WCart_0.

Note that if $f \in \mathrm{WCart}_0(R)$, then we get a Cartier-Witt divisor prescribed by $I = W(R)$ and the map is given by a multiplication $W(R) \xrightarrow{f} W(R)$ with the conditions on $\mathrm{WCart}_0(R)$ rigged such that it indeed prescribes a Cartier-Witt divisor. This gives us a morphism of stacks

$$\mathrm{WCart}_0 \to \mathrm{WCart}$$

Lemma 1.2.6. *The morphism above exhibits* WCart *as a Zariski stack quotient*

$$\left[\mathrm{WCart}_0/W^\times\right] \simeq \mathrm{WCart}.$$

1.2.7. *As the moduli space of prisms.* One should think of WCart as the "classifying stack for prism structures." To formalize this, I claim that that there is a unique morphism $\mathrm{Spf}(A) \to \mathrm{WCart}$ "classifying" the prism structure. Indeed, to construct such a morphism, I must tell you what to do on an R-point of A: $A \to R$. By the universal property of W, we have a δ-ring map $A \to W(R)$. We then have $W(R) \otimes_A I$, a generalized Cartier divisor on W(R). One checks that, in fact, this is a Cartier-Witt divisor; all in all we get a morphism

$$\rho_A : \mathrm{Spf}(A) \to \mathrm{WCart}.$$

Example 1.2.8. In fact, WCart_0 is representable by the completed ring

$$A^0 := \mathbb{Z}[a_0, a_1^{\pm 1}, a_2, \cdots]_{(p,a_0)}.$$

The Witt vetor Frobenius on $\mathrm{WCart}_0(R)$ defines a lift of Frobenius on the ring $\mathbb{Z}[a_0, a_1^{\pm 1}, a_2, \cdots]_{(p,a_0)}$ and hence the structure of a δ-ring and the ideal $I^0 = (a_0)$ defines a prism (A^0, I^0). The map $\mathrm{WCart}_0 \to \mathrm{WCart}$ can be described as $\rho : \mathrm{Spf}(A^0) \to \mathrm{WCart}$.

Another sense in which WCart is like a classifying stack for prism structures is that it acts like a base in which fibered products translates into products; see [BL22, Proposition 3.2.8] for details. Another way is via its quasicoherent sheves.

The derived ∞-category of quasicoherent sheaves on WCart is easy to define:

$$\mathbf{D}(\mathrm{WCart}) := \lim_{\mathrm{Spec}\, R \to \mathrm{WCart}} \mathbf{D}(R). \tag{1.2.9}$$

Unpacking this: to define a complex of quasicoherent sheaves on WCart is to give for each map $\mathrm{Spec}\, R \to \mathrm{WCart}$ an object $M \in \mathbf{D}(R)$ satisfying various compatibilities.

Interpreting WCart as the moduli of prisms via 1.2.7, we also have an equivalence

$$\mathbf{D}(\mathrm{WCart}) \simeq \lim_{(A,I)} \mathbf{D}(A)_{(p,I)},$$

the derived ∞-category of (p, I)-complete A-modules; the limit here runs through all bounded prisms [BL22, Proposition 3.3.5]. This is much more satisfactory and we can exactly say what it means to specify an complex of quasicoherent sheaves on WCart: we have to give, for each bounded prism (A, I) a (p, I)-complete complex $\mathrm{M}_{(\mathrm{A},\mathrm{I})}$ and for each morphism of prisms $f : (\mathrm{A}, \mathrm{I}) \to (\mathrm{B}, \mathrm{J})$ an equivalence $f^*\mathrm{M}_{(\mathrm{A},\mathrm{I})} \simeq \mathrm{M}_{(\mathrm{B},\mathrm{J})}$. We will see how to specify such an object soon. With this we can define:

Definition 1.2.10. An **absolute prismatic crystal** is an object $\mathrm{M} \in \mathbf{D}(\mathrm{WCart})$.

1.2.11. *Via the geometry of the Hodge-Tate locus.* The story of WCart is loosely inspired by the stack $[\mathbb{A}^1/\mathbb{G}_m]$ classifying virtual Cartier divisors. One should think of this stack as a "thickening" of the stack $\mathrm{B}\mathbb{G}_m$; but this can be made more precise with the stack $\left[\widehat{\mathbb{A}}^1/\mathbb{G}_m\right]$ where we have taken the $\mathbb{G}_m$-quotient of the formal completion of $\mathbb{A}^1$ at 0. Indeed, the Cartier-Witt stack does have a map $\mathrm{WCart} \to \left[\widehat{\mathbb{A}}^1/\mathbb{G}_m\right]$ which refines the map to $[\mathbb{A}^1/\mathbb{G}_m]$.

The **Hodge-Tate locus** can be defined by the pullback:

$$\begin{array}{ccc} \mathrm{WCart}^{\mathrm{HT}} & \longrightarrow & \mathrm{WCart} \\ \downarrow & & \downarrow \\ \mathrm{B}\mathbb{G}_m & \longrightarrow & \left[\widehat{\mathbb{A}}^1/\mathbb{G}_m\right]. \end{array}$$

In particular, it is a Cartier divisor inside WCart and comes equipped with a line bundle given by the map to $\mathrm{B}\mathbb{G}_m$. Which line bundle is it?

Construction 1.2.12. Let X be a bounded p-adic formal scheme, then the stack $\mathrm{WCart}_{\mathrm{X}}$ is a functor from rings to groupoids, whose value on a p-nilpotent ring is the groupoid of pairs $(\mathrm{I} \xrightarrow{\alpha} \mathrm{W}(\mathrm{R}); \operatorname{Spec} \overline{\mathrm{W}(\mathrm{R})} \to \mathrm{X})$ where $\overline{\mathrm{W}(\mathrm{R})}$ is the *derived quotient*: formally a Cartier-Witt divisor classifies a map $\operatorname{Spec} \mathrm{W}(\mathrm{R}) \to [\mathbb{A}^1/\mathbb{G}_m]$ and $\operatorname{Spec}(\overline{\mathrm{W}(\mathrm{R})})$ is defined via pullback:

$$\operatorname{Spec}(\overline{\mathrm{W}(\mathrm{R})}) := \mathrm{B}\mathbb{G}_m \times_{[\mathbb{A}^1/\mathbb{G}_m]} \operatorname{Spec} \mathrm{W}(\mathrm{R}).$$

Another feature of the Hodge-Tate locus is that it is actually quite simple to describe. Here's a starting observation which states that a point surjects onto the Hodge-Tate locus.:

Lemma 1.2.13. *The map* $\eta : \operatorname{Spf}(\mathbb{Z}_p) \to \mathrm{WCart}$ *defined by sending a p-nilpotent ring* R *to the prism* $(\mathrm{W}(\mathrm{R}) \xrightarrow{\mathrm{V}(1)} \mathrm{W}(\mathrm{R}))$ *factors through the Hodge-Tate locus* $\operatorname{Spf}(\mathbb{Z}_p) \to \mathrm{WCart}^{\mathrm{HT}}$ *and defines a (fpqc-locally) surjective morphism of stacks.*

Proof. Indeed, $\mathrm{V}(1) = (0, 1, \cdots)$ is the most canonical element that goes to zero under the map $\mathrm{W}(\mathrm{R}) \to \mathrm{R}$. Now, we can reduce to the following situation: we have a Cartier-Witt divisor prescribed by a map $\mathrm{W}(\mathrm{R}) \xrightarrow{\mathrm{V}(u)} \mathrm{W}(\mathrm{R})$ where u is a unit in R; this is what typically an element of the Hodge-Tate locus looks like. Then use the fact that, as maps of group schemes, the Frobenius $\mathrm{F} : \mathrm{W}^\times \to \mathrm{W}^\times$ is faithfully flat [BL22, Proposition 3.4.7]. This then means that we may write u as $\mathrm{F}(u')$ where u' is a unit, whence the element of interest is given by $\mathrm{V}(\mathrm{F}(u')) = u'\mathrm{V}(1)$. But then this means that the Cartier-Witt divisor is isomorphic to $(\mathrm{W}(\mathrm{R}), \mathrm{V}(1))$ as desired. □

General results about stacks then tells us that $\mathrm{WCart}^{\mathrm{HT}}$ is equivalent, over $\operatorname{Spf}\mathbb{Z}_p$ to a classifying stack of a group: take $\mathbb{G}_m$ and take the divided power completion at 1; this is an affine group scheme denoted by $\mathbb{G}_m^\sharp$. It can be described as follows:

(1) its ring of functions is the subring of $\mathbb{Q}[t^{\pm 1}]$ together with t^{-1} and the divided powers $(t-1)^n/n!$;
(2) rationally: $\mathbb{G}_{m,\mathbb{Q}} \simeq \mathbb{G}_{m,\mathbb{Q}}^\sharp$;
(3) p-locally it is equivalent to the frobenius kernel of the units Witt vectors $\mathrm{W}^\times[\mathrm{F}]$.

Hence, p-locally, we have that $\mathrm{B}\mathbb{G}_m^\sharp \simeq \mathrm{WCart}^{\mathrm{HT}}$. This leads to a description of the derived ∞-category of quasicoherent sheaves

1.2.14. *The "sheafy" prismatic cohomology.* We now assemble relative prismatic cohomology into a prismatic crystal:

Construction 1.2.15. Let $\mathcal{X}$ be a formally smooth p-adic formal scheme, then we define

$$\mathcal{H}_{\mathbb{A}}(\mathrm{X}) \in \mathbf{D}(\mathrm{WCart})$$

as the prismatic crystal which assigns to a bounded prims $(\mathrm{A}, \mathrm{I}) \mapsto \mathbb{A}_{\mathcal{X} \times_{\mathrm{A/I}} \mathrm{A/A}}$; the crystallinity property is a consequence of the base change property from Proposition 1.0.11. The **absolute prismatic cohomology** of $\mathcal{X}$ is the global section

$$\mathrm{R}\Gamma_{\mathbb{A}}(\mathrm{WCart}, \mathcal{H}_{\mathbb{A}}(\mathrm{X})) =: \mathrm{R}\Gamma_{\mathbb{A}}(\mathcal{X}).$$

Remark 1.2.16. In general, just as in the story with de Rham cohomology, $\mathrm{R}\Gamma_{\mathbb{A}}(\mathcal{X})$ is only reasonable when $\mathcal{X}$ is formally smooth over $\mathbb{Z}_p$. One should take the derived version (obtained by left Kan extension) $\mathrm{L}^{\mathrm{Sm}}\mathrm{R}\Gamma_{\mathbb{A}}(-)(\mathcal{X})$ in general to get a well-behaved theory.

1.2.17. *Prismatization.* We note that there is also a relative variant of WCart which parallels the story of the de Rham prestack. The definition is kind of straightforward having seen the relative prismatic site: recall that we have a formally smooth algebra R over A/I where (A, I) is a base prism, then the relative prismatic site is constructed by probing R via prisms over (A, I): maps $\mathrm{R} \to \mathrm{B/IB}$. But we have seen that Cartier-Witt divisors constitute a generalization of prisms where I is allowed to be an invertible sheaf. Consequently, we need to take the quotient $\mathrm{A}/\mathcal{L}$ and this needs to be taken in the context of derived algebraic geometry. We can still do this but we need to invoke (very mild) amount of derived schemes.

Construction 1.2.18. Let X be a bounded p-adic formal scheme, then the stack $\mathrm{WCart}_{\mathrm{X}}$ is a functor from rings to groupoids, whose value on a p-nilpotent ring is the groupoid of pairs $(\mathrm{I} \xrightarrow{\alpha} \mathrm{W(R)}; \operatorname{Spec} \overline{\mathrm{W(R)}} \to \mathrm{X})$ where $\overline{\mathrm{W(R)}}$ is the *derived quotient*: formally a Cartier-Witt divisor classifies a map $\operatorname{Spec} \mathrm{W(R)} \to [\mathbb{A}^1/\mathbb{G}_m]$ and $\operatorname{Spec}(\overline{\mathrm{W(R)}})$ is defined via pullback:

$$\operatorname{Spec}(\overline{\mathrm{W(R)}}) := \mathrm{B}\mathbb{G}_m \times_{[\mathbb{A}^1/\mathbb{G}_m]} \operatorname{Spec} \mathrm{W(R)}.$$

References

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Department of Mathematics, Harvard University, 1 Oxford St. Cambridge, MA 02138, USA
E-mail address: elmanto@math.harvard.edu
URL: https://www.eldenelmanto.com/