LECTURE -3: MOTIVATION AND BEGINNING OF PRISMATIC COHOMOLOGY

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1. Notes on absolute prismatic cohomology (mini-course)

The goal of this mini-course is to say something about the following result due to Bouis, following up on recent work by Bhatt and Mathew.

Theorem 1.0.1. *Let* R *be a perfectoid ring and* S *is a perfectoid valuation ring under* R*, then the map*

$$\mathbf{Z}/p(j)^{\mathrm{syn}}(\mathrm{S}) \to \mathrm{R}\Gamma(\mathrm{S}[\tfrac{1}{p}];\mu_p^{\otimes j}).$$

is an isomorphism in degrees $< i-1$ *and an injection in degree* $i-1$*.*

We will explain how this is a result about "resolution of singularities in mixed characteristics."

1.1. Genesis and motivation for prismatic cohomology.

The story begins with de Rham cohomology. If S is a base scheme, then we may associate to any S-scheme X its de Rham complex[1]

$$\Omega^{\bullet}_{\mathrm{X}/\mathrm{S}} := [\mathcal{O} \xrightarrow{d} \Omega^1_{\mathrm{X}/\mathrm{S}} \xrightarrow{d} \cdots \to \Omega^j_{\mathrm{X}/\mathrm{S}} \cdots].$$

This is perhaps one of the easiest invariant of a scheme that one can come up with which pops up naturally once one learns about differential forms in algebraic geometry:

$$\mathbb{H}^i_{\mathrm{Zar}}(\mathrm{X};\Omega^{\bullet}_{\mathrm{X}/\mathrm{S}}) =: \mathrm{H}^i_{\mathrm{dR}}(\mathrm{X}/\mathrm{S}).$$

Let us write $\mathrm{R}\Gamma_{\mathrm{Zar}}(\mathrm{X};\Omega^{\bullet}_{\mathrm{X}/\mathrm{S}})$ be the global sections. One of the most pleasant results in this direction is due to Grothendieck which gives a way to construct the $\mathbb{C}$-valued singular cohomology of an variety purely algebraically:

Theorem 1.1.1. *Let* X *be a smooth* $\mathbb{C}$*-variety, then there is a functorial equivalence*

$$\mathrm{R}\Gamma(\mathrm{X};\Omega^{\bullet}_{\mathrm{X}/\mathbb{C}}) \xrightarrow{\simeq} \mathrm{R}\Gamma(\mathrm{X}^{\mathrm{an}};\Omega^{\mathrm{an},\bullet}_{\mathrm{X}^{\mathrm{an}}/\mathbb{C}})$$

Combining this result with the de Rham-Poincaré theorem tells us that we have an isomorphism

$$\mathrm{H}^i(\mathrm{X}^{\mathrm{an}};\mathbb{C}) \cong \mathrm{H}^i_{\mathrm{dR}}(\mathrm{X}/\mathbb{C}).$$

Remark 1.1.2. We have the Hodge to de Rham spectral sequence

$$\mathrm{E}_1^{p,q} = \mathrm{H}^q(\mathrm{X};\Omega^p) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{dR}}(\mathrm{X}/\mathbb{C});$$

its E_2 page is given by $\mathrm{H}^p(\mathrm{X};\mathrm{R}^q\Omega^{\bullet})$. This spectral sequence is also available in the analytic setting. Serre's GAGA theorem says that if X is a smooth projective complex variety then coherent and coherent-analytic cohomologies coincide and therefore we have Theorem 1.1.1 in this case. For many local-to-global arguments, however, it is crucial that one can drop the projectivity assumption.

[1] While we can make sense of the de Rham complex of any morphism, it is usually a pathological object. Illusie's **derived de Rham** complex is a much more reasonable object in this setting.

Therefore, using the elementary definition of de Rham cohomology we are on our way to producing a good Weil/motivic cohomology which reproduces singular cohomology with $\mathbb{C}$-coefficients. Of course we are restricted here to characteristic zero varieties and $\mathbb{C}$-coefficients which suffers from a number of pathologies. Let us, however, think about this as a kind of a "baby case."

1.1.3. *Category of coefficients.* From the outset, Grothendieck wanted a "category of coefficients" to exist for his theory. Indeed, Poincaré duality is but a shadow of the existence of such a theory: given a smooth morphism f there should be two functors $f^!$ and f^* which are related by some kind of a "shift and twist" formula

$$f^*(d)[2d] \simeq f^!.$$

The correct category of coefficients for de Rham cohomology turned out to be algebraic D-modules. There are many ways to set up this theory, but we will consider the approach of **crystals** due to Grothendieck.

Definition 1.1.4. Let $f : \mathrm{X} \to \mathrm{S}$ be a scheme. The **infinitesimal site** of f, denoted by $\mathrm{Inf}(\mathrm{X}/\mathrm{S})$ has as objects pairs (U, g) where $\mathrm{U} \subset \mathrm{X}$ is a Zariski open and $g : \mathrm{U} \hookrightarrow \mathrm{T}$ is a closed immersion over S which is furthermore nilpotent. The morphisms are given by squares of S-schemes

$$\begin{array}{ccc} \mathrm{U} & \longrightarrow & \mathrm{T} \\ \downarrow{\scriptstyle j} & & \downarrow \\ \mathrm{U}' & \longrightarrow & \mathrm{T}', \end{array}$$

where j is an open immersion. We take the Grothendieck topology on $\mathrm{Inf}(\mathrm{X}/\mathrm{S})$ where covering sieves are generated by $\{(\mathrm{U}_i, \mathrm{T}_i) \to (\mathrm{U}, \mathrm{T})\}$ such that $\{\mathrm{T}_i \to \mathrm{T}\}$ is a Zariski open cover.

One of the key points of the infinitesimal site is that its category of sheaves is easy to describe. Recall that if $f : \mathrm{X} \to \mathrm{Y}$ is a morphism of schemes, then we can define a sheaf-theoretic pullback $f^{-1} : \mathrm{Shv}(\mathrm{Y}) \to \mathrm{Shv}(\mathrm{X})$; this is to be distinguished from f^* where we further tensor with some sheaf of rings. Then the datum of sets $\mathcal{F}$ on the infinitesimal site is a collection

$$\mathcal{F}_{(\mathrm{U},\mathrm{T})} \in \mathrm{Shv}(\mathrm{T}) \qquad (\mathrm{U}, \mathrm{T}) \in \mathrm{Inf}(\mathrm{X}/\mathrm{S})$$

and maps

$$f^{-1}\mathcal{F}_{(\mathrm{U}',\mathrm{T}')} \to \mathcal{F}_{(\mathrm{U},\mathrm{T})} \qquad (\mathrm{U}, \mathrm{T}) \to (\mathrm{U}', \mathrm{T}'),$$

and compatibilities among them subject to the following "local-constancy" condition: if $\mathrm{T} \to \mathrm{T}'$ is an open immersion then

$$f^{-1}\mathcal{F}_{(\mathrm{U}',\mathrm{T}')} \cong \mathcal{F}_{(\mathrm{U},\mathrm{T})}.$$

Grothendieck's idea of a *crystal* is that we further add on a sheaf of rings and demand that $\mathcal{F}$ forms a "cartesian section." Indeed, we have the following sheaf on the infinitesimal site which we call the **structure sheaf**

$$\mathcal{O}_{\mathrm{inf}} : \mathrm{Inf}(\mathrm{X}/\mathrm{S})^{\mathrm{op}} \to \mathrm{CAlg} \qquad (\mathrm{U} \to \mathrm{T}) \mapsto \mathcal{O}(\mathrm{T});$$

note that we also have a variant

$$\overline{\mathcal{O}_{\mathrm{inf}}}(\mathrm{U} \to \mathrm{T}) \mapsto \mathcal{O}(\mathrm{U}).$$

Definition 1.1.5. An **infinitesimal crystal** is a sheaf of $\mathcal{O}_{\mathrm{inf}}$-modules such that we have isomorphisms $f^*\mathcal{F}_{(\mathrm{U}',\mathrm{T}')} \simeq \mathcal{F}_{(\mathrm{U},\mathrm{T})}$. We denote this category by $\mathbf{Crys}(\mathrm{X}/\mathrm{S})$.

Grothendieck proved the following striking theorem. Recall that a morphism of sites $(\mathcal{C}, t) \to (\mathcal{D}, t')$, in particular, defines a functor $\mathcal{D} \to \mathcal{C}$. The functor

$$(\mathrm{X}/\mathrm{S})_{\mathrm{inf}} \to \mathrm{Sch}_{\mathrm{S}},$$

defines a morphism of sites $\lambda : (\mathrm{Sch}_{\mathrm{S}})_{\mathrm{Zar}} \to (\mathrm{X}/\mathrm{S})_{\mathrm{inf}}$; and restriction of a sheaf $\mathcal{E}$ on the Zariski site defines an infinitesimal sheaf $\lambda^*\mathcal{E}$; for example $\lambda^*\mathcal{O} = \mathcal{O}_{\mathrm{inf}}$. We have the stupid truncation functor $\Omega^\bullet \to \mathcal{O}$ which defines a morphism of complexes of sheaves of $\mathcal{O}$-modules on the infinitesimal site:

$$\lambda^*\Omega^\bullet \to \mathcal{O}_{\mathrm{inf}}.$$

Theorem 1.1.6. *Let* $\mathrm{S} = \mathbb{Q}$ *and assume that* X *is essentially smooth. Then the map* $\lambda^*\Omega^\bullet \to \mathcal{O}_{\mathrm{inf}}$ *is an equivalence. Concretely,*

$$\mathrm{H}^*_{\mathrm{inf}}(\mathrm{X}; \mathcal{O}_{\mathrm{inf}}) \cong \mathrm{H}^*_{\mathrm{dR}}(\mathrm{X}/\mathbb{Q}).$$

Theorem 1.1.6 tells us that, morally, **Crys**(X/S) has the de Rham complex a monoidal unit, whence it is a kind of "category of modules over de Rham cohomology." In fact, one can give a description of the category of crystals.

Theorem 1.1.7. *As in Theorem 1.1.6, there is a canonical equivalence of categories:*

$$\mathbf{Crys}(\mathrm{X}/\mathbb{Q}) \simeq \mathbf{Mod}_{\mathcal{D}_\mathrm{X}}$$

where $\mathcal{D}_\mathrm{X}$ *is the sheaf of different operators.*

1.2. **The de Rham stack.** To get feeling on what Theorem 1.1.7 is about let us adopt the following notion: suppose that X is a scheme over a field and R is a ring, we say that two points $x, y \in \mathrm{X}(\mathrm{R})$ are **infinitesimally close** if under the map $\mathrm{X}(\mathrm{R}) \to \mathrm{X}(\mathrm{R}_{\mathrm{red}})$, x and y goes to the same image. This captures the notion of a point being nearby. It is not hard to imagine that the notion of a crystal has something to do with identifying sheaves whose stalks are infinitesimally close. To make this more precise, we can associate to X a stack where we have identified infinitesimally close points

Definition 1.2.1 (Simpson)**.** Let $\mathrm{X} : \mathrm{CAlg} \to \mathrm{Set}$ be a functor. Its associated **de Rham prestack** is the functor X_{dR} where

$$\mathrm{X}_{\mathrm{dR}}(\mathrm{R}) = \mathrm{X}(\mathrm{R}_{\mathrm{red}})$$

We then have the following result.

Theorem 1.2.2. *As in Theorem 1.1.6, we have an equivalence of categories*

$$\mathbf{QCoh}(\mathrm{X}_{\mathrm{dR}}) \simeq \mathbf{Crys}(\mathrm{X}/\mathbb{Q}).$$

The de Rham prestack is a stack whenever X is one, whence we may think of it as a reasonably geometric object. We can unpack what it means to be a quasicoherent sheaf on the de Rham prestack in the following manner:

(1) a quasicoherent sheaf $\mathcal{F}$ on X;
(2) for every pair $x, y \in \mathrm{X}(\mathrm{R})$ which are infinitesimally close, an isomorphism of R-modules
$$\eta_{x,y} : x^*\mathcal{F} \simeq y^*\mathcal{F};$$
these isomorphisms are stable under base change;
(3) for three points $x, y, z \in \mathrm{X}(\mathrm{R})$ which are infinitesimally close, the requirement that $\eta_{x,y} = \eta_{x,z} \circ \eta_{z,y}$.

Combining the above theorem and Theorem 1.1.6 we may regard X_{dR} as a stack whose structure sheaf computes the de Rham cohomology of X:

$$\mathrm{R}\Gamma(\mathrm{X}_{\mathrm{dR}}, \mathcal{O}) \simeq \mathrm{R}\Gamma(\mathrm{X}, \Omega^\bullet_{\mathrm{X}/\mathbb{Q}}).$$

The identification with D-modules is then more believable if we believe in the Riemann-Hilbert correspondence: D-modules are the same thing as sheaves with a notion of parallel transport that identifies points which are close together (vector bundles with an integrable connection).

Thinking of crystals as quasicoherent sheaves on a particular stack is not merely a frivolous exercise. One of the key points of de Rham theory is that if X is furthermore projective, then the Hodge-to-de Rham spectral sequence

$$\mathrm{E}_1^{p,q} = \mathrm{H}^q(\mathrm{X}; \Omega^p_{\mathrm{X}/\mathbb{Q}}) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{dR}}(\mathrm{X})$$

degenerates and endows de Rham cohomology with the **Hodge filtration**. This is a split filtration after base change to $\mathbb{C}$ and, in conjunction with Theorem 1.1.6 we endow the $\mathbb{C}$-singular cohomology of smooth projective variety with a Hodge decomposition. Of course all of this story is classical, but the de Rham stacks lets us see more structure:

Example 1.2.3. The following elaboration will not do the whole story justice. Let X be a smooth projective $\mathbb{C}$-variety. As already explained above, we have the Hodge decomposition

$$\mathrm{H}^n(\mathrm{X};\mathbb{C}) \cong \bigoplus_{i+j=n} \mathrm{H}^i(\mathrm{X};\Omega^j).$$

Now, there is a dictionary between grading and an action of the group $(\mathbb{C}^\times)^\delta$ (the multiplicative group viewed as a discrete group). The work of Simpson and Katzarkov, Pantev, Toen constructs this action on the level of X_{dR}; the induced action recovers the usual Hodge decomposition. This lets us, for example, speak of Hodge structures on fundamental group and other invariants that one can extract out of X_{dR}.

Prismatic cohomology is actually more accurately related to **crystalline cohomology** which is a cohomology theory for $\mathbb{F}_p$-schemes taking values in $\mathbb{Z}_p$-algebras; one of the main features of this theory is that it agrees with the de Rham cohomology of a (smooth) lift. We will not elaborate more on this theory (noting that one can use prismatic cohomology to reproduce to crystalline theory) and instead provided the reader with the following table.

	de Rham cohomology	**Crystalline cohomology**
Usual input	$\mathbb{Q}$-scheme	$\mathbb{F}_p$-scheme
Output	$\mathbf{D}(\mathbb{Q})$	$\mathbf{D}(\mathbb{Z}_p)$ + Frobenius
Thickening datum	infinitesimal thickenings	PD-thickenings
Hodge-to-de Rham/slope	$\mathrm{E}_1^{p,q} = \mathrm{H}^q(\mathrm{X};\Omega^p)$	$\mathrm{E}^1_{p,q} = \mathrm{H}^q(\mathrm{X}, \mathrm{W}\Omega^p)$
Conjugate	$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{X};\mathcal{H}^q(\Omega^\bullet))$	$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{X};\mathcal{H}^q(\mathrm{W}\Omega^\bullet))$
Stacky approach	X_{dR}	$\mathrm{W}(\mathrm{X}_{\mathrm{perf}})/\mathcal{G}$ (Drinfeld).

1.3. **A sketch of the construction of crystalline cohomology.** The following is a theorem:

Theorem 1.3.1. *Let* X *be a smooth* $\mathbb{F}_p$*-algebra such that* X' *is a lift of* X *to a smooth scheme over* $\mathbb{Z}_p$*. Then* $\mathrm{R}\Gamma_{\mathrm{dR}}(\mathrm{X}) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{X}/\mathbb{Z}_p)$.

We can also turn Theorem 1.3.1 into a construction of crystalline cohomology. I learned this from A. Raksit and it is apparently due to Drinfeld and written by by Li-Mondal [].

Theorem 1.3.2. *The functor of p-completed derived de Rham complex into p-complete* $\mathbb{E}_\infty$*-*$\mathbb{Z}_p$*-algebras:*

$$\widehat{\mathrm{L}\Omega}_p : \mathrm{Ani}_{\mathbb{Z}_p} \to \left(\widehat{\mathrm{CAlg}_{\mathbb{Z}_p}}\right)_p$$

factors through the functor of mod-p reduction

$$\mathrm{Ani}_{\mathbb{Z}_p} \to \mathrm{Ani}_{\mathbb{F}_p}.$$

The resulting functor

$$\mathrm{Ani}_{\mathbb{F}_p} \to \left(\widehat{\mathrm{CAlg}_{\mathbb{Z}_p}}\right)_p$$

is equivalent to (derived) crystalline cohomology.

Proof. The functor $\widehat{\mathrm{L}\Omega}_p$ is a symmetric monoidal functor (which amounts to the Künneth formula and is equivalent to saying that it preserves finite coproducts) which preserves colimits (since, by construction, it preserves sifted colimits). Hence it is determined completely by what happens on the level of $\widehat{\mathrm{L}\Omega}_p$ on $\mathrm{Poly}_{\mathbb{Z}_p}$. Any such functor $\mathrm{F} : \mathrm{Poly}_{\mathbb{Z}_p} \to \left(\widehat{\mathrm{CAlg}_{\mathbb{Z}_p}}\right)_p$ will factor through $\mathrm{Poly}_{\mathbb{F}_p}$ if and only if when F is applied to the two maps:

$$t \mapsto 0, t \mapsto p : \mathbb{Z}_p[t] \to \mathbb{Z}_p,$$

the induced maps are homotopic. In order to check this, we consider the pushout diagram

$$\begin{array}{ccc} \widehat{\mathrm{L}\Omega}_p(\mathbb{Z}_p[t]) & \xrightarrow{t\mapsto p} & \widehat{\mathrm{L}\Omega}_p(\mathbb{Z}_p) \\ \downarrow{\scriptstyle t\mapsto 0} & & \downarrow \\ \widehat{\mathrm{L}\Omega}_p(\mathbb{Z}_p) & \longrightarrow & \widehat{\mathrm{L}\Omega}_p(\mathbb{F}_p). \end{array}$$

where we have used that the functor preserves finite coequalizers. The resulting pushout is the derived de Rham complex of $\mathbb{F}_p$ relative to $\mathbb{Z}_p$. Bhatt has computed this as

$$\left(\widehat{\mathrm{L}\Omega_{\mathbb{F}_p/\mathbb{Z}_p}}\right)_p \simeq \mathbb{Z}_p\langle t\rangle/(t=p),$$

where $\langle t\rangle$ are the divided power variables. But now it suffices to construct splittings of the right vertical and bottom horizontal maps, i.e., maps

$$\mathbb{Z}_p\langle t\rangle/(t=p) \to \mathbb{Z}_p \simeq \widehat{\mathrm{L}\Omega}_p(\mathbb{Z}_p),$$

which we can do because of the universal properties of $\mathbb{Z}_p\langle t\rangle$ and the natural divided power structure on $\mathbb{Z}_p$. □

1.4. **Prismatic cohomology.** We would like to construct a cohomology theory of schemes over $\mathbb{Z}_p$ which, morally, interpolates between the de Rham cohomology of the special fiber and the étale cohomology of the generic fiber. As a first approximation, one should think of it as a fancy version of "integral" de Rham cohomology. Furthermore, this cohomology theory will be the derived global sections of the structure sheaf on a stack.

Definition 1.4.1. Let R be a $\mathbf{Z}_{(p)}$-algebra. A **δ-ring** is the datum of an endomorphism $\varphi : \mathrm{R} \to \mathrm{R}$ and a path $h \in \pi_1(\mathrm{End}(\mathrm{R}/^{\mathrm{L}}p))$ between $\varphi/^{\mathrm{L}}p$ and $\mathrm{Frob}/^{\mathrm{L}}_p$.

Here, we note that the definition of a δ-ring can easily be extended to one involving an animated ring or even a derived ring (a non-connective generalization of animated rings). One could also have worked with a more explicit definition of a δ-ring where we are given a set map $\delta : \mathrm{R} \to \mathrm{R}$ such that the map

$$\varphi(x) := x^p + p\delta(x)$$

defines a Frobenius after reduction modulo p. We will not elaborate on what δ is supposed to specify but it not too hard to reinvent them. We will use the following terminology:

(1) we say that an element $x \in \mathrm{R}$ is **distinguished** if $\delta(x)$ is a unit;
(2) it is **rank one** if $\delta(x) = 0$.

Rank one elements satisfy $\varphi(x) = x^p$ so it is some kind of "fixed points" of the Frobenius. That the two notions coincide is proved in [BS19, Remark 2.5].

Definition 1.4.2. A **prism** is a pair (R, I) where R is a δ-ring and I is a Cartier divisor on Spec R, subject to two conditions:

(1) it is (p, I)-complete;
(2) $p \in \mathrm{I} + \varphi(\mathrm{I})\mathrm{R}$.

1.4.3. *Examples.* We now give some examples of prisms before we proceed. We will make us of the following lemma to check the prism condition:

Lemma 1.4.4. *Let* A *be a δ-ring such that p is in the radical of* A*. Then for an element d in the radical, the following are equivalent:*

(1) $p \in (d, \varphi(d))$;
(2) d is distinguished.

Proof. If d is distinguished, then the equation $\varphi(d) = d^p + p\delta(d)$ implies that $p = \frac{\varphi(d)+d^p}{\delta(d)}$, whence the claim. The converse is an exercise in unpacking definitions and is [BS19, Lemma 2.24-25]. □

Example 1.4.5. A prism is said to be **crystalline** if $\mathrm{I} = (p)$. The simplest example is given by taking a characteristic p ring R which is **perfect** and forming the Witt vectors, W(R); it has a universal property as a *unique* (derived) p-adically complete, flat $\mathbf{Z}_p$-algebra for which $\mathrm{W(R)}/p = \mathrm{R}$. The kernel of the natural map $\mathrm{W(R)} \to \mathrm{R}$ is given by p and the Frobenius on R (which is an automorphism) lifts to an automorphism of W(R); this is an example of a **perfect prism**: one in which the Frobenius map is an automorphism. For concreteness if $\mathrm{R} = \mathbf{F}_p$ then $\mathrm{W(R)} = \mathbf{Z}_p$.

Example 1.4.6. Let $\mathrm{A} = \mathbf{Z}_p[[q-1]]$, and consider

$$[p]_q := 1 + q + \cdots + q^{p-1} = \tfrac{q^p-1}{q-1}.$$

The element $[p]_q$ has the property that $\mathrm{A}/[p]_q = \mathbf{Z}_p[\zeta_p]$, the ring of integers of the cyclotomic extension $\mathbf{Q}_p[\zeta_p]$. Furthermore it has the property that setting $q = 1$ gets us

$$[p]_q = p \qquad \mod (q-1).$$

This proves that $[p]_q$ is distinguished since $\delta([p]_q) = \delta(p)$ modulo $q-1$ and the element $q-1$ is in the radical of A (whence we may detect units after modding it out). Note that we have used that the map $\mathrm{A} \xrightarrow{q=1} \mathbf{Z}_p$ is a δ-map.

Example 1.4.7. Let $\mathrm{K}/\mathbf{Q}_p$ be a discretely valued extension of $\mathbf{Q}_p$; this means that $\mathcal{O}_\mathrm{K} \subset \mathrm{K}$ is a discrete valuation ring (with residue field κ) for which we may choose a uniformizer π. We have the maximal unramified subring $\mathrm{W} \cong \mathrm{W}(\kappa) \subset \mathcal{O}_\mathrm{K}$ and we have $\mathrm{W}[[u]]$ which is a δ-ring whose lift of Frobenius is determined by $u \mapsto u^p$. We have a map $\mathrm{W}[[u]] \to \mathcal{O}_\mathrm{K}$ determined by setting $u \mapsto \pi$. In fact, its kernel is principal and generated by $\mathrm{E}(u)$, the minimal polynomial for π over W (it is usually called the Eisenstein polynomial of π). Then the pair $(\mathrm{W}[[u]], (\mathrm{E}(u)))$ is a prism called the **Breuil-Kisin prism**.

It is good for intuition to see why this is a prism. As above, we may do so after reducing modulo u; now $\mathrm{E}(u)$ is characterized by the fact that the constant coefficient is p-adic valuation 1 (not divisible by p^2). One of the things that the δ-structure does is that it lowers the p-adic valuation of a non-unit by one (one checks this on the initial δ-ring $\mathbf{Z}_{(p)}$) and thus modulo u, $\delta(\mathrm{E}(u))$ is a unit. One of the motivation for integral p-adic Hodge theory is to construct a cohomology theory valued in Breuil-Kisin-Fargues modules which are essentially $\mathrm{W}[[u]]$-modules with extra structure.

Remark 1.4.8. One of the most remarkable properties of a prism is the following "rigidity" lemma:

Lemma 1.4.9. [BS19, Lemma 3.5] *Suppose that* $(\mathrm{A}, \mathrm{I}) \to (\mathrm{B}, \mathrm{J})$ *is a map of prisms (defined as a map of the underlying* δ*-rings preserving the ideal), then we have an isomorphism* $\mathrm{I} \otimes_\mathrm{A} \mathrm{B} \cong \mathrm{J}$. *In particular,* $\mathrm{IB} = \mathrm{J}$.

Therefore, once we fix a base prism that we are working over, everything else under it is determined. This observation has led to the definition of the prismatic cohomology of a δ-ring (as constructed by Antieau-Krause-Nikolaus) which interpolates between absolute and relative prismatic cohomology.

The following property will be assumed for many of the results to hold (all the examples above satisfy this condition):

Definition 1.4.10. A prism (A, I) is said to be **bounded** if A/I has bounded p^∞-torsion.

We are now ready to define prismatic cohomology.

Construction 1.4.11. Let (A, I) be a prism and R be a formally smooth A/I-algebra. The **relative prismatic site** of R over A, denoted by $(\mathrm{R}/\mathrm{A})_{\Delta}$ is the category whose objects are bounded prisms (B, J) (necessarily of the form $(\mathrm{B}, \mathrm{IB})$) together with a map $\mathrm{R} \to \mathrm{B}/\mathrm{IB}$. It is equipped with the *indiscrete* topology so that

$$\mathrm{Shv}((\mathrm{R}/\mathrm{A})_{\Delta}) \simeq \mathrm{PSh}((\mathrm{R}/\mathrm{A})_{\Delta}).$$

The **prismatic structure sheaf** is the functor

$$\mathcal{O}(\mathrm{R} \to (\mathrm{B}, \mathrm{J})) = \mathrm{B},$$

while the **Hodge-Tate sheaf** is the functor

$$\mathcal{O}(\mathrm{R} \to (\mathrm{B}, \mathrm{J})) = \mathrm{B}/\mathrm{J}.$$

Remark 1.4.12. There is a variant called the absolute prismatic site (this is not to be confused with the prismatic site relative to the crystalline prism $(\mathbf{Z}_p, (p))$. The input is a p-adically complete, formally smooth ring R. The objects of this site are maps $\mathrm{R} \to \mathrm{A}/\mathrm{I}$ where (A, I) is a prism. This definition is very abstract, but the resulting prismatic cohomology Δ_{R} is, in some sense, the most important object in the theory. The Cartier-Witt stack is a geometric incarnation of this construction which will be more useful.

Remark 1.4.13. If $\mathcal{C}$ is a small category with the indiscrete topology and $\mathcal{F}$ is a presheaf (hence a sheaf) on $\mathcal{C}$ we have that the sheaf cohomology of $\mathcal{F}$ is computed by

$$\mathrm{R}\Gamma(\mathcal{C}, \mathcal{F}) \simeq \lim_{\mathrm{A} \in \mathcal{C}} \mathcal{F}(\mathrm{A}).$$

Hence, we should think of defining sheaf cohomology in this case as something elementary (as one only needs the notion of a limit), but can be quite unwieldy.

Construction 1.4.14. As in Construction 2.0.11, the **prismatic complex** of R relative to A is defined as

$$\Delta_{\mathrm{R}/\mathrm{A}} := \mathrm{R}\Gamma((\mathrm{R}/\mathrm{A})_{\Delta}, \mathcal{O}_{\Delta}) \simeq \lim_{(\mathrm{R}/\mathrm{A})_{\Delta}} \mathrm{B}.$$

Here the inverse limit is taken in the ∞-category of (p, I)-complete derived A-modules. On the other hand, we define

$$\overline{\Delta}_{\mathrm{R}/\mathrm{A}} := \Delta_{\mathrm{R}/\mathrm{A}} \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{A}/\mathrm{I},$$

as the **Hodge-Tate complex**. It is also easy to check that the Hodge-Tate complex is the global sections of the Hodge-Tate sheaf.

Remark 1.4.15. There is a frobenius action on $\mathcal{O}_{\Delta}$ by definition and thus we have a frobenius action on the prismatic complex, which is semilinear for the frobenius on A:

$$\varphi^*_{\mathrm{A}} \Delta_{\mathrm{R}/\mathrm{A}} \xrightarrow{\varphi} \Delta_{\mathrm{R}/\mathrm{A}}.$$

It is common to denote $\Delta^{(1)}_{\mathrm{R}/\mathrm{A}} := \varphi^*_{\mathrm{A}} \Delta_{\mathrm{R}/\mathrm{A}}$.

References

[BS19] B. Bhatt and P. Scholze, *Prisms and Prismatic Cohomology*, preprint arXiv:1905.08229

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