LECTURE 2: IN WHICH WE DO SOME DERIVED LINEAR ALGEBRA

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We now discuss the "pure algebra" part of the Deligne-Illusie theorem, following ideas of Achinger and Suh [AS20]. The Cartier isomorphism says that we have the object in the derived category of the ringed topos $(\mathrm{X}^{(1)}, \mathcal{O})$ given by

$$\mathrm{K} = \mathrm{F}_{\mathrm{X/S}*}\Omega^{\bullet}_{\mathrm{X/S}}$$

which is an abstract Koszul complex. What comes next is actually very simple, at least conceptually. So let me say what we are trying to do:

Remark 0.0.1. What does it mean to decompose the simplest piece of K? Well $\tau^{\leqslant 1}\mathrm{K}$ only has two terms which are, up Frobenius twists which we will ignore:

$$\mathrm{H}^1 = \Omega^1 \qquad \mathrm{H}^0 = \mathcal{O}.$$

The result then asserts that lifting to W always ensures this splitting (since we are looking at the case of $1 = 2 - 1$). This is the purely geometric part of the theorem, and we will relate this to the lifting problem, so we postpone this to the next class. But what can we do assuming that we have this splitting? The point is that we want to spread this splitting throughout the de Rham complex, or at least as much as possible. The Koszulity of the de Rham complex wants us to say something like

$$\mathrm{Sym}^q(\tau^{\leqslant 1}\Omega) \simeq \tau^{\leqslant q}\Omega.$$

Then from knowing how Sym and $\oplus$ interacts, and the decomposition at $\tau^{\leqslant 1}$, we obtain the desired decomposition. This is not literally true, but will be if we are a bit more careful about what exact we mean by the symmetric powers. Also it cannot be literally true as there is a restriction on how much splitting we get relative to p.

To make Remark 0.0.1 precise, let us recall a construction in linear algebra and another in derived linear algebra:

Construction 0.0.2 (Divided powers)**.** Let A be a ring and M a *finitely generated projective* (we will not consider the underived construction on general modules) A-module. Then the **divided power algebra** on M is the commutative A-algebra generated by elements $x \in \mathrm{M}$ and elements $\gamma_n(x)$ for $n \geqslant 0$ subject to the divided power relations [Stacks, Tag 07GL]. Setting $|x| = 1, |\gamma_n(x)| = n$, we have a decomposition into homogeneous components

$$\Gamma_{\mathrm{A}}(\mathrm{M}) \cong \bigoplus \Gamma^d_{\mathrm{A}}(\mathrm{M}).$$

We note that there are isomorphisms $\mathrm{M} \cong \Gamma^1_{\mathrm{A}}(\mathrm{M}), \mathrm{A} \cong \Gamma^0_{\mathrm{A}}(\mathrm{M})$. Furthermore

$$(\mathrm{M}^{\otimes d})^{\Sigma_d} \cong \Gamma^d_{\mathrm{A}}(\mathrm{M})$$

for any $d \geqslant 0$; this should be taken in contrast with the more familiar isomorphism $(\mathrm{M}^{\otimes d})_{\Sigma d} \cong \mathrm{Sym}^d_{\mathrm{A}}(\mathrm{M})$.

Remark 0.0.3. We note that the axioms of $\gamma_n(x)$ ensures that they behave like $x^n/n!$. So we have $x^n = n!\gamma_n(x)$. However, γ_n is not really any kind of homomorphism from M so one should be careful about taking this relation too seriously.

Remark 0.0.4. We also recall that if M is a finitely generated free A-module, then we have the duality isomorphism

$$\Gamma^d_{\mathrm{A}}(\mathrm{M}^\vee) \cong (\mathrm{Sym}^d_{\mathrm{A}}(\mathrm{M}))^\vee.$$

We shall need the following basic lemma

Lemma 0.0.5. *Let $d!$ be invertible in* A *and* M *a finitely generated projective* A*-module. Then the "averaging map"*

$$\mathrm{Sym}^k_{\mathrm{A}}(\mathrm{M}) = (\mathrm{M}^{\otimes k})_{\Sigma_k} \to \Gamma^k_{\mathrm{A}}(\mathrm{M}) = (\mathrm{M}^{\otimes k})^{\Sigma_k}$$

is an isomorphism for $k \leqslant d$.

Proof. The point is that $x^k/k!$ are all defined as soon as $k \leqslant d$. □

The following procedure will be bread and butter.

Construction 0.0.6 (Animation)**.** Consider the category of finitely generated projective A-modules $\mathrm{Mod}^{\mathrm{fg.proj}}_{\mathrm{A}}$ and suppose that we have some functor

$$\mathrm{F} : \mathrm{Mod}^{\mathrm{fg.proj}}_{\mathrm{A}} \to \mathrm{Mod}_{\mathrm{A}}.$$

Then we can extend F to LF, a sifted colimits-preserving functor fitting into the following diagram

$$\begin{tikzcd} \mathrm{Mod}^{\mathrm{fg.proj}}_{\mathrm{A}} \arrow[r, "\mathrm{F}"] \arrow[d] & \mathrm{Mod}_{\mathrm{A}} \arrow[d] \\ \mathbf{D}(\mathrm{A})_{\geqslant 0} \arrow[r, "\mathrm{LF}"] & \mathbf{D}(\mathrm{A})_{\geqslant 0}. \end{tikzcd}$$

For us the functors we care about are

$$\mathrm{F} = \bigwedge^i, \mathrm{Sym}^i, \Gamma^i.$$

Remark 0.0.7. One of the features of defining constructions by animation is that they are manifestly "homotopy invariant" in that they are defined on the derived ∞-category. The constructions do not, *a priori* come with preferred representatives.

Remark 0.0.8. Say M is an arbitrary, discrete A-module. In practice how one computes LF is the following: we pick some simplicial resolution $\mathrm{P}_\bullet \to \mathrm{M}$, whose terms are filtered colimits of finitely generated projective modules. By the Dold-Kan correspondence [Lur17, Section 1.2.3], this is exactly picking (colimits of) projective resolutions. Then

$$\mathrm{LF}(\mathrm{M}) \simeq \operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathrm{F}(\mathrm{P}_n).$$

Definition 0.0.9 (Higher Koszul complexes)**.** Let $(\mathrm{X}, \mathcal{O})$ be a ringed topos and let $f : \mathrm{M} \to \mathrm{N}$ be a map of flat $\mathcal{O}$-modules and let F be the fiber of the map in the ∞-category $\mathbf{D}((\mathrm{X}, \mathcal{O}))$. Then the ***q*-th Koszul cohomology** of f is given by

$$\mathrm{Kos}_q(f) := \mathrm{L}\bigwedge^q(\mathrm{F}[1])[-q].$$

Remark 0.0.10. The object F[1], is just another name for C, the cofiber of f; the reason why we shift by [1] is purely technical: we have only defined these derived constructions on connective objects and the fiber might go into negative degrees[1]; but they actually *do* extend to non-connective ones as first observed by [Ill71, Chapter I.4]. In fact he does this in generality of bounded below objects (those whose cohomology vanishes in *large* enough degrees, equivalently those that vanish in *small* enough degrees). Moreoever, we have an equivalences (in this generality):

$$\mathrm{L}\bigwedge^q(\mathrm{F}[1])[-q] \simeq \mathrm{L}\Gamma^q(\mathrm{F}),$$

[1]To clarify the confusion in class: the cofiber is the complex $[\mathrm{M} \to \mathrm{N}]$ where M is placed in homological degree 1 and N is in degree zero; the fiber is cofiber shifted by -1 which means that M is in degree zero and N is in homological degree -1.

first proved by Illusie in [Ill71, Proposition 4.2.3.1] (see also [Lur18, Section 25.4.2]); this is also true for a (suitably defined extension) of $\mathrm{F} \in \mathbf{D}(\mathrm{A})$. There is also another equivalence:

$$\mathrm{LSym}^q(\mathrm{F}[1])[-q] \simeq \mathrm{L}\bigwedge^q(\mathrm{F}).$$

With this in mind, we define the **q-th Koszul homology** of f as

$$\mathrm{Kos}^q := \mathrm{LSym}^q(\mathrm{F}[1])[-q];$$

this could be more familiar to more people. Anyway, the above two equivalences basically states that we can produce the derived divided and exterior power from just the symmetric powers construction.

Remark 0.0.11. Let V be a free module over a ring R. Then we can look at the graded algebra whose terms are given by

$$\mathrm{Sym}^*(\mathrm{V}^\vee) \qquad \mathrm{Sym}^*(\mathrm{V}^\vee)\otimes \mathrm{V}^\vee \qquad \mathrm{Sym}^*(\mathrm{V}^\vee)\otimes \bigwedge^2 \mathrm{V}^\vee \qquad \cdots \qquad \mathrm{Sym}^*(\mathrm{V}^\vee)\otimes \bigwedge^k \mathrm{V}^\vee$$

There are the underlying dga's of both the de Rham complex of $\mathbb{A}(\mathrm{V}) \to \operatorname{Spec}\mathrm{R}$ and the Koszul complex of the map of $k[x_1, \cdots, x_n]^{\bigoplus n} \to k[x_1, \cdots, x_n]$ classifying the elements $x_1, \cdots, x_n$ (or the Koszul complex of the ideal $(x_1, \cdots, x_n)$). However, the de Rham complex has a differential that "goes cohomologically up":

$$\mathrm{Sym}^*(\mathrm{V}^\vee) \longrightarrow \mathrm{Sym}^*(\mathrm{V}^\vee)\otimes \mathrm{V}^\vee \longrightarrow \mathrm{Sym}^*(\mathrm{V}^\vee)\otimes \bigwedge^2 \mathrm{V}^\vee \longrightarrow \cdots \longrightarrow \mathrm{Sym}^*(\mathrm{V}^\vee)\otimes \bigwedge^k \mathrm{V}^\vee,$$

while the Koszul complex has one that goes "cohomologically down"

$$\mathrm{Sym}^*(\mathrm{V}^\vee) \longleftarrow \mathrm{Sym}^*(\mathrm{V}^\vee)\otimes \mathrm{V}^\vee \longleftarrow \mathrm{Sym}^*(\mathrm{V}^\vee)\otimes \bigwedge^2 \mathrm{V}^\vee \qquad \cdots \longleftarrow \mathrm{Sym}^*(\mathrm{V}^\vee)\otimes \bigwedge^k \mathrm{V}^\vee$$

The point here is that the de Rham complex is a case of the Koszul cochain complex.

What is the significance of $\mathrm{Kos}_q(f)$? Manifestly, the definition of the complex $\mathrm{Kos}_q(f)$ only depends only on the fiber of the map f. Therefore, if the object $\mathrm{Fib}(f)$ is decomposable as an object in the derived ∞-category, then so does $\mathrm{Kos}_q(f)$. Therefore, to accomplish our goal of splitting K, it is useful to express it in terms of the Koszul complex. This is the result of Achinger and Suh.

Theorem 0.0.12 (Achinger-Suh)**.** *Let m be an integer such that $m!$ be invertible in $\mathcal{O}$ and let $q \geqslant m$. Assume either:*

(1) that $q = m$ or
(2) $m + 1$ is a nonzero divisor in $\mathcal{O}$.

Let K *be an abstract Koszul complex and write:*

$$\tau^{\leqslant 1}\mathrm{K} = [\mathrm{K}^0 \xrightarrow{\partial} \mathrm{Z}^1\mathrm{K}],$$

such that $\mathrm{K}^0, \mathrm{B}^1\mathrm{K}, \mathrm{Z}^1\mathrm{K}, \mathcal{H}^1(\mathrm{K})$ *are all flat. Then we have an induced quasi-isomorphism*

$$\tau^{\geqslant q-m}\mathrm{Kos}^q(\partial) \simeq \tau^{[q-m,q]}\mathrm{K}.$$

In particular, if $\tau^{\leqslant 1}\mathrm{K}$ *is decomposable, then so is* $\tau^{[0,q]}\mathrm{K}$.

To prove this result, we pick a model of Kos_q; which we call $\mathrm{Kos}_q^\bullet$. We sort of have to — the proof of the theorem does rely on choosing an explicit representative of the truncated de Rham complex (even though the conclusions do not).

Construction 0.0.13. Let $f : \mathrm{M} \to \mathrm{N}$ be a morphism of flat $\mathcal{O}$-modules. Then construct a cochain (cohomologically nonnegative) complex of the form

$$\mathrm{Kos}_q^\bullet := 0 \to \Gamma^q_{\mathcal{O}}(\mathrm{M}) \to \bigwedge_{\mathcal{O}}^{1} \mathrm{N} \otimes \Gamma^{q-1}_{\mathcal{O}}(\mathrm{M}) \to \bigwedge_{\mathcal{O}}^{2} \mathrm{N} \otimes \Gamma^{q-2}_{\mathcal{O}}(\mathrm{M}) \to \cdots \bigwedge_{\mathcal{O}}^{q} \mathrm{N} \to 0,$$

where the leftmost term is placed in degree zero. We denote the differential by

$$d(y \otimes \gamma_{e_1}(x_1) \cdots \gamma_{e_r}(x_r)) = \sum_j y \wedge f(x_j) \otimes (\gamma_{e_1}(x_1) \cdots \gamma_{e_j - 1}(x_j) \cdots \gamma_{e_r}(x_r)).$$

Lemma 0.0.14. *Let $f : \mathrm{M} \to \mathrm{N}$ be a morphism of flat $\mathcal{O}$-modules. Then there is an equivalence:*

$$\mathrm{Kos}^\bullet_q(f) \simeq \mathrm{Kos}_q(f).$$

Proof sketch. We only make some comments on the proof and give a reference at the end. We start with some general comments. Assume that we have an exact sequence

$$0 \to \mathrm{M}' \to \mathrm{M} \to \mathrm{M}'' \to 0;$$

where everything in sight are finitely generated free modules. In this case, consider the following "Koszul-type" chain omplex

$$0 \to \bigwedge_{\mathcal{O}}^{q} \mathrm{M}' \to \bigwedge_{\mathcal{O}}^{q-1} \mathrm{M}' \otimes \mathrm{Sym}_{\mathcal{O}} \mathrm{M} \to \cdots \to \mathrm{M}' \otimes \mathrm{Sym}^{q-1}_{\mathcal{O}} \mathrm{M} \to \mathrm{Sym}^q_{\mathcal{O}} \mathrm{M} \to 0.$$

Where might have one seen this complex? Well we are writing down the q-homogeneous component of the Koszul complex [Stacks, Tag 0623] associated to elements $f_1, \cdots f_n$ in $\mathrm{Sym}^*_{\mathcal{O}}\mathrm{M}$ which are the image of the generators of M'. These form a regular sequences in $\mathrm{Sym}^*_{\mathcal{O}}\mathrm{M}$ and thus the Koszul complex is acyclic [Stacks, Tag 062D] (by regularity of the sequence) and computes $\mathrm{Sym}^*_{\mathcal{O}}\mathrm{M}/(f_1, \cdots, f_n)$. But the latter is nothing by $\mathrm{Sym}^*_{\mathcal{O}}\mathrm{M}''$. Therefore we have proved that the above complex computes $\mathrm{Sym}^*_{\mathcal{O}}\mathrm{M}'' \simeq \mathrm{LSym}^*_{\mathcal{O}}\mathrm{M}'' \simeq \mathrm{Sym}^*_{\mathcal{O}}(\mathrm{F}[1])$ where F is the fiber of the map $\mathrm{M}' \to \mathrm{M}$. Sorting out the homogeneous degree q piece we obtain a complex computing $\mathrm{Kos}^q_\bullet(\mathrm{M}' \to \mathrm{M})$.

Now we dualize the argument. recall that for any flat module M we have duality isomorphisms:

$$\Gamma^n_{\mathcal{O}}(\mathrm{M}^\vee) \cong \mathrm{Sym}^n_{\mathcal{O}}(\mathrm{M})^\vee \qquad \bigwedge_{\mathcal{O}}(\mathrm{M}^\vee) \cong (\bigwedge_{\mathcal{O}} \mathrm{M})^\vee.$$

Dualizing the above exact sequence

$$0 \to (\mathrm{M}'')^\vee \to \mathrm{M}^\vee \to (\mathrm{M}')^\vee \to 0;$$

we get a dual complex

$$0 \to \Gamma^q_{\mathcal{O}}(\mathrm{M}) \to \mathrm{M}'' \otimes \Gamma^{q-1}_{\mathcal{O}}(\mathrm{M}) \to \bigwedge_{\mathcal{O}}^{2} \mathrm{M}'' \otimes \Gamma^{q-2}_{\mathcal{O}}(\mathrm{M}) \to \cdots \bigwedge_{\mathcal{O}}^{q} \mathrm{M}'' \to 0.$$

Now, this complex computes $\mathrm{LSym}^*_{\mathcal{O}}((\mathrm{M}')^\vee)^\vee \simeq \mathrm{L}\Gamma^*(\mathrm{M}')$. In our situation, let us assume that we are in the special case where $\mathrm{M} \to \mathrm{N}$ is surjective so that the fiber is exactly just the kernel, denoted by F. We see that the complex $\mathrm{Kos}^\bullet_q(f)$ is exactly the one given as above.

An "official proof" of this result can be found in the references of [AS20, Proposition 2.5]; but we can also reduce to the above case by the techniques of the proof of [Lur18, Proposition 25.2.4.2].

□

Proof of Theorem 0.0.12. First, we begin by constructing a map on stupid truncations

$$\mu : \sigma^{\geqslant q-m}\mathrm{Kos}^\bullet_q \to \sigma^{\geqslant q-m}\tau^{\leqslant q}\mathrm{K},$$

which is of the form:

$$\begin{tikzcd} 0 \to \bigwedge_{\mathcal{O}}^{q-m} \mathrm{Z}^1\mathrm{K} \otimes \mathrm{Sym}_{\mathcal{O}}^m(\mathrm{K}^0) \arrow[r] \arrow[d] & \bigwedge_{\mathcal{O}}^{1} \mathrm{Z}^1\mathrm{K} \otimes \mathrm{Sym}_{\mathcal{O}}^{q-1}(\mathrm{K}^0) \arrow[r] \arrow[d] & \cdots & \bigwedge_{\mathcal{O}}^{q} \mathrm{Z}^1\mathrm{K} \arrow[r] \arrow[d] & 0 \\ 0 \to \mathrm{K}^{q-m} \arrow[r] & \mathrm{K}^1 \arrow[r] & \cdots & \mathrm{Z}^q\mathrm{K}. & \end{tikzcd}$$

Here we are already using Lemma 0.0.5 and the assumption that $m!$ is invertible in $\mathcal{O}$ to convert the divided powers into symmetric powers. Notice that when $q = m$, the map of complexes we are interested in is of the form

$$\mathrm{Kos}_m^\bullet(\partial) \simeq \mathrm{L}\Gamma^q(\mathcal{H}^1(\mathrm{K})[-1]) \simeq \mathrm{LSym}^q(\mathcal{H}^1(\mathrm{K})[-1]) \to \mathrm{K},$$

and such a map can be constructed without any problems using the universal property of LSym.

So now, let us assume that $m+1$ is a nonzero divisor. To promote the above to the clever truncation we need to prove that the image of

$$\bigwedge_{\mathcal{O}}^{q-m-1} \mathrm{Z}^1\mathrm{K} \otimes \Gamma_{\mathcal{O}}^{m+1}(\mathrm{K}^0) \to \bigwedge_{\mathcal{O}}^{q-m} \mathrm{Z}^1\mathrm{K} \otimes \mathrm{Sym}_{\mathcal{O}}^m(\mathrm{K}^0)$$

into $d\mathrm{K}^{m-n-1}$. We leave this as an exercise the reader, or see the proof of [AS20, Theorem 2.8]. Instead we explain the minimal failure of this (due to Achinger-Suh): set $m = p-1, p = q$. Then we can consider the de Rham complex of the polynomial algebra $\mathbb{F}_p[x]$ as a $\mathbb{F}_p[x^p]$-module. We are trying to understand the boundary cycles going into:

$$\mathrm{Kos}^p(\partial)^1 \to \mathrm{K}^1.$$

In other words we are contemplating the diagram

$$\begin{tikzcd} \Gamma^p_{\mathbb{F}_p[x^p]}(\mathbb{F}_p[x]) \arrow[r] \arrow[d, dashed] & (\Gamma^{p-1}_{\mathbb{F}_p[x^p]}(\mathbb{F}_p[x]) \cong) \, \mathrm{Sym}^{p-1}_{\mathbb{F}_p[x^p]}(\mathbb{F}_p[x]) \otimes \mathbb{F}_p[x]dx \arrow[d, "\mu"] \\ d(\mathbb{F}_p[x]) \arrow[r] & \mathbb{F}_p[x]dx; \end{tikzcd}$$

and we want the dashed arrow to exist. But it does not because the class $\gamma_p(x)$ gets mapped to $\gamma_{p-1}(x) \otimes dx$ is then equal to $\frac{x^{p-1}}{(p-1)!} \otimes dx = -x^{p-1} \otimes dx$ (modulo p of course!) and thus get mapped to $-x^{p-1}dx$, which is decidedly not a boundary since it does not have a primitive (which would involve dividing by $p!$).

Then using Lemma 0.0.15, at each j, the effect on taking cohomology is given by

$$\bigwedge_{\mathcal{O}}^{j}(\mathcal{H}^1(\mathrm{K})) \otimes_{\mathcal{O}} \Gamma_{\mathcal{O}}^{q-j}(\mathcal{H}^0(\mathrm{K})) \xrightarrow{\cong} \mathcal{H}^j(\mathrm{Kos}^\bullet(f)) \to \mathcal{H}^j(\mathrm{K}),$$

where the total composite is induced by multiplication. Since K is assumed to be abstract Koszul, the composite is an isomorphism, whence we are done.

□

We leave the next lemma as an exercise: they point is that $\mathrm{Kos}^\bullet$ converts sums to tensor products; see [AS20, Proposition 2.7] for the solution.

Lemma 0.0.15. *Given a map $f : \mathrm{M} \to \mathrm{N}$ of flat $\mathcal{O}$-modules, there exists, for each j, a unique arrow*

$$\bigwedge_{\mathcal{O}}^{j}(\mathrm{coker}(f)) \otimes \Gamma_{\mathcal{O}}^{q-j}(\ker(f)) \to \mathcal{H}^j(\mathrm{Kos}^\bullet(f)).$$

which is an isomorphism.

Remark 0.0.16. We remark on several interesting aspect of the proof. Firstly, we use the fact that the $\mathrm{Kos}_q^\bullet$ is manifestly a functor of ∞-categories to spread information from the $\tau^{\leqslant 1}$ part of K to the rest of K. However, we need an explicit description of the complex to relate it back to Sym^*, at least in the second case; this later maneuver is not "model-independent."

Corollary 0.0.17. *Let* X/S *be of relative dimension* $< p$*. Assume that the truncation* $\tau^{\leqslant 1}\mathrm{F}_{\mathrm{X}/\mathrm{S}*}\Omega^\bullet_{\mathrm{X}/\mathrm{S}}$ *decomposes. Then there is an quasi-isomorphism*

$$\bigoplus_j \Omega^j_{\mathrm{X}^{(1)}/\mathrm{S}}[-j] \xrightarrow{\simeq} \mathrm{F}_{\mathrm{X}/\mathrm{S}*}\Omega^\bullet_{\mathrm{X}/\mathrm{S}},$$

which induces the Cartier isomorphim:

$$\mathrm{C}^{-1} : \Omega^j_{\mathrm{X}^{(1)}/\mathrm{S}} \to \mathcal{H}^j(\mathrm{F}_{\mathrm{X}/\mathrm{S}*}\Omega^\bullet_{\mathrm{X}/\mathrm{S}}),$$

for each $j \geqslant 0$*. More generally, any truncation of the form* $[a, a+p-2]$ *for* $p > 2$ *and* $[a, a+1]$ *splits in the same fashion.*

Proof. The truncation of the de Rham complex takes the form

$$\tau^{\leqslant 1}\mathrm{F}_{\mathrm{X}/\mathrm{S}*}\Omega^\bullet_{\mathrm{X}/\mathrm{S}} \cong [\mathcal{O}_{\mathrm{X}^{(1)}} \to \mathrm{Z}^1\mathrm{F}_*\Omega^1_{\mathrm{X}/\mathrm{S}}].$$

Since $\mathrm{X} \to \mathrm{S}$ is smooth, the flatness assumption on the ringed topos of $(\mathrm{X}^{(1)}_{\mathrm{Zar}}, \mathcal{O}))$ is satisfied. So we may apply the Achinger-Suh theorem to conclude that $\mathrm{F}_{\mathrm{X}/\mathrm{S}*}\Omega^\bullet_{\mathrm{X}/\mathrm{S}}$ abstractly decomposes (after taking $\mathrm{Kos}_{\dim(\mathrm{X}/\mathrm{S})}$ which bounds above the de Rham complex). But let us be more precise about this; knowing that we have a decomposition on $\tau^{\leqslant 1}$, we may choose a quasi-isomorphism

$$\tau^{\leqslant 1}\mathrm{F}_{\mathrm{X}/\mathrm{S}*}\Omega^\bullet_{\mathrm{X}/\mathrm{S}} \simeq [\mathcal{H}^0(\mathrm{F}_{\mathrm{X}/\mathrm{S}*}\Omega^\bullet_{\mathrm{X}/\mathrm{S}}) \xrightarrow{0} \mathcal{H}^1(\mathrm{F}_{\mathrm{X}/\mathrm{S}*}\Omega^\bullet_{\mathrm{X}/\mathrm{S}})],$$

thus we have a map of complexes, induced by the Cartier isomorphism:

$$[\mathcal{O}_{\mathrm{X}^{(1)}} \xrightarrow{0} \Omega^1_{\mathrm{X}^{(1)}/\mathrm{S}}] \xrightarrow{\mathrm{C}^{-1}} [\mathcal{H}^0(\mathrm{F}_{\mathrm{X}/\mathrm{S}*}\Omega^\bullet_{\mathrm{X}/\mathrm{S}}) \xrightarrow{0} \mathcal{H}^1(\mathrm{F}_{\mathrm{X}/\mathrm{S}*}\Omega^\bullet_{\mathrm{X}/\mathrm{S}})].$$

Taking $\mathrm{Kos}_p^\bullet$ on this isomorphism we get a quasi-isomorphism

$$\bigoplus_j \Omega^j_{\mathrm{X}^{(1)}/\mathrm{S}} \simeq \mathrm{Kos}_p^\bullet([\mathcal{O}_{\mathrm{X}^{(1)}} \xrightarrow{0} \Omega^1_{\mathrm{X}^{(1)}/\mathrm{S}}]) \xrightarrow{\simeq} \mathrm{F}_{\mathrm{X}/\mathrm{S}*}\Omega^\bullet_{\mathrm{X}/\mathrm{S}},$$

The stronger claim follows from Theorem 0.0.12 by setting $m = p-2$ when $p > 2$.

□

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