LECTURE 3: THE ONE IN WHICH WE SPEAK SOME FRENCH

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Let us first sketch the idea of deformation theory. Suppose that $\widetilde{\mathrm{A}} \to \mathrm{A}$ is a surjective morphism of rings and suppose that X is a flat A-scheme. We want to ask the following questions:

(1) does there exists a diagram

$$\begin{array}{ccc} \widetilde{\mathrm{X}} & \longleftarrow & \mathrm{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\widetilde{\mathrm{A}}) & \longleftarrow & \mathrm{Spec}(\mathrm{A}); \end{array}$$

such that $\widetilde{\mathrm{X}} \to \mathrm{Spec}(\widetilde{\mathrm{A}})$ is flat and the map above is cartesian?

(2) If such a diagram exists, then how many (isomorphism classes) of them are there?

In the case that $\widetilde{\mathrm{A}} \to \mathrm{A}$ is **square zero extension**, i.e., the map is a surjection and the kernel is square zero, then the answer can be expressed in terms of (derived) linear algebra.

The goal of this class is to prove the following result which relates deformation theory to de Rham cohomology; for us a stack[1] is a presheaf of $(\infty-)$groupoids on the small étale site of a scheme.

Theorem 0.0.1 (Relèvements contre scindage)**.** *Let* $\mathrm{X} \to \mathrm{S}$ *be a morphism of schemes in characteristic* $p > 0$ *and let* $\widetilde{\mathrm{S}}$ *be a fixed, flat lift over* $\mathbb{Z}/p^2$ *of* S. *Then:*

(1) assume that $f : \mathrm{X} \to \mathrm{S}$ *is lci, then there exits a (higher)* $\mathrm{X}^{(1)}$*-stack, denoted by,*

$$\mathrm{Rel}(\mathrm{X}^{(1)}, \mathrm{S})$$

which parametrizes flat lifts of $\mathrm{X}^{(1)}$ *to* $\widetilde{\mathrm{S}}$ *and fits into a pullback square of stacks*

$$\begin{array}{ccc} \mathrm{Rel}(\mathrm{X}^{(1)}, \mathrm{S}) & \longrightarrow & \mathrm{Maps}(\tau_{\leqslant 1}\mathrm{L}_{\mathrm{X}^{(1)}/\widetilde{\mathrm{S}}}, \mathcal{O}[1]). \\ \downarrow & & \downarrow \\ \{\mathrm{id}\} & \longrightarrow & \mathrm{Maps}(\mathcal{O}, \mathcal{O}). \end{array}$$

(2) There exits a (higher) $\mathrm{X}^{(1)}$*-stack denoted by:*

$$\mathrm{Sci}(\tau^{\leqslant 1}\mathrm{F}_{\mathrm{X/S}*}\Omega^{\bullet}_{\mathrm{X/S}})$$

[1]More precisely; let $\mathrm{X}_{\text{ét}}$ be the small étale site on X whose objects are étale morphisms to X. The covers are given by joinly surjective, finite collection of étale morphisms. A prestack is functor $\mathrm{X}_{\text{ét}}^{\mathrm{op}} \to \mathrm{Ani}$ and a stack is one that satisfies the descent condition. If the prestack lands in $\mathrm{Gpd} \subset \mathrm{Ani}$ then the descent condition is more concrete and involves up to level 2 of the simplicial diagram.

which parametrizes splittings of the 1*-truncated de Rham complex and fits into a pullback square of stacks*

$$\begin{array}{ccc} \mathrm{Sci}(\tau^{\leqslant 1}\mathrm{F}_{\mathrm{X/S}*}\Omega^{\bullet}_{\mathrm{X/S}}) & \longrightarrow & \mathrm{Maps}(\tau^{\leqslant 1}\mathrm{F}_{\mathrm{X/S}*}\Omega^{\bullet}_{\mathrm{X/S}}, \mathcal{O}). \\ \downarrow & & \downarrow \\ \{\mathrm{id}\} & \longrightarrow & \mathrm{Maps}(\mathcal{O}, \mathcal{O}). \end{array}$$

(3) *In case that* $\mathrm{X} \to \mathrm{S}$ *is smooth, we have a canonical equivalence*

$$\mathrm{Rel}(\mathrm{X}^{(1)}, \mathrm{S}) \simeq \mathrm{Sci}(\tau^{\leqslant 1}\mathrm{F}_{\mathrm{X/S}*}\Omega^{\bullet}_{\mathrm{X/S}});$$

furthermore both stacks are $\mathrm{X}^{(1)}$*-gerbes banded by* $\mathrm{H}^1(\mathrm{X}^{(1)}, \mathrm{T}_{\mathrm{X}^{(1)}/\mathrm{S}})$ *and the above equivalence are equivariant.*

To prove the last point of Theorem 0.0.1, one can proceed via the following theorem of Illusie:

Theorem 0.0.2 (Illusie)**.** *As in Theorem 0.0.1, if* $\mathrm{X} \to \mathrm{S}$ *is furthermore smooth, there exists an equivalence*

$$(\tau_{\leqslant 1}\mathrm{L}_{\mathrm{X}^{(1)}/\widetilde{\mathrm{S}}})[-1] \xrightarrow{\simeq} \tau^{\leqslant 1}\mathrm{F}_{\mathrm{X/S}*}\Omega^{\bullet}_{\mathrm{X/S}}.$$

Having all the above ingredients we are done: since lifts correspond exactly to splittings we get the $[0, 1]$ case of the result. The proof of Theorem 0.0.2 uses more ideas from crystalline cohomology. However, let us indicate why it should be correct "by size." The reader unfamiliar with the cotangent complex is encouraged to skip this part and return; we will also make this idea precise in the final part of the lecture.

Anyway, if $\mathrm{X} \to \mathrm{S}$ is smooth, then we get a cofiber sequence

$$\mathcal{H}^0(\tau^{\leqslant 1}\mathrm{F}_{\mathrm{X/S}*}\Omega^{\bullet}_{\mathrm{X/S}}) \to \tau^{\leqslant 1}\mathrm{F}_{\mathrm{X/S}*}\Omega^{\bullet}_{\mathrm{X/S}} \to \mathcal{H}^1(\tau^{\leqslant 1}\mathrm{F}_{\mathrm{X/S}*}\Omega^{\bullet}_{\mathrm{X/S}})[-1].$$

On the other hand, we have the transitivity triangle for $\mathrm{X}^{(1)} \xrightarrow{f} \mathrm{S} \hookrightarrow \widetilde{\mathrm{S}}$:

$$f^*\mathrm{L}_{\mathrm{S}/\widetilde{\mathrm{S}}}[1] \to \mathrm{L}_{\mathrm{X}^{(1)}/\widetilde{\mathrm{S}}} \to \mathrm{L}_{\mathrm{X}^{(1)}/\mathrm{S}};$$

which, in the smooth case, unpacks to the following cofiber sequence after truncating and shifting by $[-1]$:

$$\mathcal{O}_{\mathrm{X}^{(1)}} \to (\tau_{\leqslant 1}\mathrm{L}_{\mathrm{X}^{(1)}/\widetilde{\mathrm{S}}})[-1] \to \Omega^1_{\mathrm{X}^{(1)}/\mathrm{S}}[-1].$$

We then have the Cartier isomorphisms

$$\mathrm{C}^{-1} : \mathcal{O}_{\mathrm{X}^{(1)}} \xrightarrow{\simeq} \mathcal{H}^0(\tau^{\leqslant 1}\mathrm{F}_{\mathrm{X/S}*}\Omega^{\bullet}_{\mathrm{X/S}}) \qquad \mathrm{C}^{-1} : \Omega^1_{\mathrm{X}^{(1)}/\mathrm{S}} \xrightarrow{\simeq} \mathcal{H}^1(\tau^{\leqslant 1}\mathrm{F}_{\mathrm{X/S}*}\Omega^{\bullet}_{\mathrm{X/S}}).$$

Hence all we need to do is to ensure that we have the correct map. In order to do this, we need to construct the following dashed arrow

$$\begin{array}{ccc} \tau_{\leqslant 1}\mathrm{L}_{\mathrm{X}^{(1)}/\widetilde{\mathrm{S}}} & \dashrightarrow & \tau^{\leqslant 1}\mathrm{F}_{\mathrm{X/S}*}\Omega^{\bullet}_{\mathrm{X/S}}[1] \\ \downarrow & & \downarrow \\ \Omega^1_{\mathrm{X}^{(1)}/\mathrm{S}} & \xrightarrow{\mathrm{C}^{-1}} & \mathcal{H}^1(\tau^{\leqslant 1}\mathrm{F}_{\mathrm{X/S}*}\Omega^{\bullet}_{\mathrm{X/S}}) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathrm{X}^{(1)}/\mathrm{S}}[2] & \xrightarrow{\mathrm{C}^{-1}} & \mathcal{H}^0(\tau^{\leqslant 1}\mathrm{F}_{\mathrm{X/S}*}\Omega^{\bullet}_{\mathrm{X/S}})[2], \end{array} \tag{0.0.3}$$

Actually following this strategy is not so easy. In [DI87], the authors proceed by an analysis of the stacks involved can be a bit unwieldy because one has to write explicit cocycles to glue certain maps together. We will give a proof of Theorem 0.0.2 to after our treatment of crystalline cohomology and we will also give the original proof of part (3) below.

1. Deformation theory

We work with animated rings throughout; more precisely we write $\mathrm{AniCAlg}_k$ for the ∞-category of animated k-algebras and $\mathrm{CAlg}_k \subset \mathrm{AniCAlg}_k$ to be the subcategory spanned by the discrete ones. Let k be a *fixed* based ring which we assume to be classical and let A be an animated k-algebra, then a **k-linear derivation** of A valued in an A-module M is a k-linear morphism $d : \mathrm{A} \to \mathrm{M}$ such that the map

$$s_d := (\mathrm{id}, d) : \mathrm{A} \to \mathrm{A} \oplus \mathrm{M},$$

provides a k-algebra section, where $\mathrm{A} \oplus \mathrm{M}$ is the trivial square-zero extension of A by M; recall that when everything in sight is discrete the multiplication is given by

$$(a, m) \cdot (a', m') = (aa', am' + a'm).$$

We write

$$\mathrm{Der}_k(\mathrm{A}, \mathrm{M})$$

as the ∞-groupoid of k-linear derivations of A valued in an A-module M. The (k-linear) **cotangent complex** is the universal (derived) k-linear derivation of A, i.e., it is a k-linear derivation $d : \mathrm{A} \to \mathrm{L}_{\mathrm{A}/k}$ such that

$$\mathrm{Maps}_{\mathrm{A}}(\mathrm{L}_{\mathrm{A}/k}, \mathrm{M}) \simeq \mathrm{Der}_k(\mathrm{A}, \mathrm{M}).$$

Note that this is the obvious "higher" analogue of the universal property of $\Omega^1_{\mathrm{A}/k}$. For now, all we need to know is that $\mathrm{L}_{\mathrm{A}/k}$ is an A-module satisfying the above universal property.

A **square-zero** extension of A is a k-algebra map $\widetilde{\mathrm{A}} \to \mathrm{A}$ which fits into the following pullback square in $\mathrm{AniCAlg}_k$

$$\begin{array}{ccc} \widetilde{\mathrm{A}} & \longrightarrow & \mathrm{A} \\ \downarrow & & \downarrow{\scriptstyle s_{d_\mathrm{M}}} \\ \mathrm{A} & \xrightarrow{0} & \mathrm{A} \oplus \mathrm{M}[1]; \end{array}$$

here M is assumed to be connective just to ensure that $\widetilde{\mathrm{A}}$ remains an animated ring. In this diagram, the bottom arrow is the map associated to the zero derivation $0 : \mathrm{A} \to \mathrm{M}[1]$; in particular the fiber of the map $\widetilde{\mathrm{A}} \to \mathrm{A}$ is given by M. Therefore the datum of a square-zero extension is entirely determined by a map $d_\mathrm{M} : \mathrm{A} \to \mathrm{M}[1]$; equivalently an A-linear map $\mathrm{L}_{\mathrm{A}/k} \to \mathrm{M}[1]$.

Definition 1.0.1. Let $\widetilde{\mathrm{A}} \to \mathrm{A}$ be a square zero extension. A **deformation** of $\mathrm{B} \in \mathrm{CAlg}_\mathrm{A}$ to $\widetilde{\mathrm{A}}$ is a pair $(\widetilde{\mathrm{B}}, \alpha)$ such that $\widetilde{\mathrm{B}}$ is a $\widetilde{\mathrm{A}}$-algebra and α is an equivalence $\alpha : \widetilde{\mathrm{B}} \otimes^{\mathrm{L}}_{\widetilde{\mathrm{A}}} \mathrm{A} \simeq \mathrm{B}$.

Suppose that $\mathrm{A}, \widetilde{\mathrm{A}}, \widetilde{\mathrm{B}}$ are discrete rings. We say that a deformation (A, α) of a discrete A-algebra B is a **flat deformation** if $\widetilde{\mathrm{B}}$ is a flat $\widetilde{\mathrm{A}}$-algebra.

Remark 1.0.2. Let $\mathrm{I} \to \widetilde{\mathrm{A}} \to \mathrm{A}$ be a fiber sequence; then if a $\widetilde{\mathrm{A}} \to \widetilde{\mathrm{B}}$ is the underlying algebra of the deformation, we get that

$$\mathrm{I} \otimes^{\mathrm{L}}_{\widetilde{\mathrm{A}}} \widetilde{\mathrm{B}} \to \widetilde{\mathrm{B}} \to \mathrm{B}$$

is a fiber sequence as well so that $\widetilde{\mathrm{B}}$ is the extension of B by $\mathrm{I} \otimes^{\mathrm{L}}_{\widetilde{\mathrm{A}}}$. In classical treatments of deformation theory, we usually prescribe how the kernel looks and we will see that this is related to some flatness hypotheses.

Some basics to get us started. As explained above, for each morphism of rings $f : \mathrm{A} \to \mathrm{B}$, we can associate to f the cotangent complex $\mathrm{L}_f = \mathrm{L}_{\mathrm{B}/\mathrm{A}} \in \mathbf{D}(\mathrm{B})$ which classifies derivations. It is functorial for pairs of morphisms in a way that we will not really spell out. Here are its key properties, suited for our needs:

Theorem 1.0.3 (Cotangent complex)**.** *Let* $\mathrm{A} \in \mathrm{AniCAlg}_k$, *then*

(1) for any animated A*-algebra* B*, the cotangent complex* $\mathrm{L}_{\mathrm{B}/\mathrm{A}}$ *is concentrated in homologically non-negative degrees in* $\mathbf{D}(\mathrm{B})$*;*
(2) $\pi_0(\mathrm{L}_{\mathrm{B}/\mathrm{A}}) = \Omega^1_{\pi_0(\mathrm{B})/\pi_0(\mathrm{A})}$*;*
(3) if $\mathrm{A} \to \mathrm{B}$ *is morphism of discrete rings, which is surjective with ideal kernel* I *then* $\pi_0(\mathrm{L}_{\mathrm{B}/\mathrm{A}}) = 0, \pi_1(\mathrm{L}_{\mathrm{B}/\mathrm{A}}) = \mathrm{I}/\mathrm{I}^2$*;*
(4) let $\mathrm{A} \to \mathrm{A}'$ *be a morphism, then* $\mathrm{L}_{\mathrm{B}/\mathrm{A}} \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{A}' \simeq \mathrm{L}_{\mathrm{B}\otimes^{\mathrm{L}}_{\mathrm{A}}\mathrm{A}'/\mathrm{A}'}$
(5) given a sequence $\mathrm{A} \to \mathrm{B} \to \mathrm{B}'$ *then we have a cofiber sequence*

$$\mathrm{L}_{\mathrm{B}/\mathrm{A}} \otimes_{\mathrm{B}} \mathrm{B}' \to \mathrm{L}_{\mathrm{B}'/\mathrm{A}} \to \mathrm{L}'_{\mathrm{B}'/\mathrm{B}}$$

(6) if B *is étale over* A *then* $\mathrm{L}_{\mathrm{B}/\mathrm{A}} \simeq 0$*; if it is smooth morphism of discrete rings then* $\mathrm{L}_{\mathrm{B}/\mathrm{A}} \simeq \Omega^1_{\mathrm{B}/\mathrm{A}}[0]$.
(7) if $\mathrm{A} \to \mathrm{B}$ *is surjective whose kernel ideal* I *is Koszul regular*[2]*, then* $\mathrm{L}_{\mathrm{B}/\mathrm{A}} \cong \mathrm{I}/\mathrm{I}^2[1]$.

Remark 1.0.4. While we will blackbox the cotangent complex, we can give the construction (rather, a formula or sort) via animation which we already discussed: it fits as the sifted-colimit extension of the functor of Kähler differentials

$$\begin{array}{ccc} \mathrm{Poly}_k & \xrightarrow{\Omega^1_{-/k}} & \mathrm{Mod}_k \\ \downarrow & & \downarrow \\ \mathrm{AniCAlg}_k & \xrightarrow{\mathrm{L}_{(-)/k}} & \mathbf{D}(k). \end{array}$$

Be warned, however, that $\mathrm{L}_{\mathrm{A}/k}$ does not immediately acquire the structure of an A-module from this formulation. In any case, we can define $\mathrm{L}_{\mathrm{A}/\mathrm{B}}$ as the cofiber of the map

$$\mathrm{B} \otimes_{\mathrm{A}} \mathrm{L}_{\mathrm{A}/k} \to \mathrm{L}_{\mathrm{B}/k}.$$

Remark 1.0.5. Explicitly, the cotangent complex can be computed as follows: pick a simplicial A-polynomial resolution $\mathrm{P}_\bullet \to \mathrm{A}$, i.e., P_n is a polynomial A-algebra for each n. Then

$$\mathrm{L}_{\mathrm{B}/\mathrm{A}} \simeq \mathrm{B} \otimes_{\mathrm{P}_\bullet} \Omega^1_{\mathrm{P}_\bullet/\mathrm{A}}.$$

1.1. **Gerbe of liftings.** Let's attempt to understand what it means to produce a deformation; we work for now in the context of animated rings so that we do not have to worry about flatness assumptions for a little while. We are staring at all the possible ways in which the following dashed arrows can be filled:

$$\begin{array}{ccc} \widetilde{\mathrm{A}} & \longrightarrow & \mathrm{A} \\ \downarrow & & \downarrow f \\ \widetilde{\mathrm{B}} & \dashrightarrow & \mathrm{B}, \end{array}$$

such that $\widetilde{\mathrm{B}}$ is a deformation of B. At the beginning of time, we have fixed the top horizontal arrow which corresponds to a k-derivation $\mathrm{D} : \mathrm{A} \to \mathrm{I}[1]$ where I is the kernel of the top map. This corresponds to an A-linear map

$$\mathrm{L}_{\mathrm{A}/k} \to \mathrm{I}[1].$$

We want to end up with a square zero extension given by the bottom arrow. One thing that one could do is to take the derived base change $\mathrm{I}[1] \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B}$ and we end up with a diagram of k-modules with only one dashed arrow left:

$$\begin{array}{ccc} \mathrm{A} & \longrightarrow & \mathrm{I}[1] \\ \downarrow f & & \downarrow \\ \mathrm{B} & \dashrightarrow & \mathrm{I}[1] \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B}, \end{array}$$

[2]By this we mean that the Koszul complex is acyclic [Stacks, Tag 062D]; this is implied by the sequence being regular [Stacks, Tag 062F].

where we ask that the dashed arrow is a B-linear morphism of modules. So the space of deformations is actually the space of all possible fillers of the above square; this has the advantage of linearizing the problem but also reducing the number of arrows. How does one package this in terms of the cotangent complex? Well we see that the above squares are the same thing as the space of fillers:

$$\begin{array}{ccc} \mathrm{L}_{\mathrm{A}/k} & \longrightarrow & \mathrm{I}[1] \\ \downarrow & & \downarrow \\ \mathrm{L}_{\mathrm{A}/k} \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B} & \longrightarrow & \mathrm{I}[1] \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B} \\ \downarrow & & \downarrow {=} \\ \mathrm{L}_{\mathrm{B}/k} & \dashrightarrow & \mathrm{I}[1] \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B}, \end{array}$$

where we might as well forget about the very top arrow since it is the same datum as the middle one. Now, we have the transitivity sequence

$$\mathrm{L}_{\mathrm{A}/k} \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B} \to \mathrm{L}_{\mathrm{B}/k} \to \mathrm{L}_{\mathrm{B}/\mathrm{A}},$$

plugging this in we are looking at

$$\begin{array}{ccc} \mathrm{L}_{\mathrm{B}/\mathrm{A}}[-1] & \overset{o(\mathrm{A},\mathrm{B})}{\searrow} & \\ \downarrow & & \\ \mathrm{L}_{\mathrm{A}/k} \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B} & \longrightarrow & \mathrm{I}[1] \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B} \\ \downarrow & & \downarrow {=} \\ \mathrm{L}_{\mathrm{B}/k} & \dashrightarrow & \mathrm{I}[1] \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B}, \end{array}$$

where the arrow $o(\widetilde{\mathrm{A}}, \mathrm{A}, \mathrm{B})$ is null if and only if the filler exists! This map is called the **Kodaira-Spencer** class of the map $\operatorname{Spec} \mathrm{B} \to \operatorname{Spec} \mathrm{A}$. It lives in the group

$$[\mathrm{L}_{\mathrm{B}/\mathrm{A}}[-1], \mathrm{I}[1] \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B}] = \mathrm{Ext}^2(\mathrm{L}_{\mathrm{B}/\mathrm{A}}, \mathrm{I}[1] \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B}).$$

Now, say that we want to parametrize all the possible lifts; rather we should try to find what the difference between two possible lifts look like. Well if $f, g : \mathrm{L}_{\mathrm{B}/k} \to \mathrm{I}[1] \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B}$ are lifts, then they are subject to the constrain that $f - g|_{\mathrm{L}_{\mathrm{A}/k} \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B}}$ must be null so that we get a map from $\mathrm{L}_{\mathrm{B}/\mathrm{A}} \to \mathrm{I}[1] \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B}$; actually this is not quite precise: it does not make sense to say that $f - g$ is nullhomotopic but, rather, it is nullhomotopic via some map which is parametrized by $\mathrm{L}_{\mathrm{B}/\mathrm{A}} \to \mathrm{I}[1] \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B}$. This is to say that all possible lifts form a *torsor* under

$$[\mathrm{L}_{\mathrm{B}/\mathrm{A}}, \mathrm{I}[1] \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B}] = \mathrm{Ext}^1(\mathrm{L}_{\mathrm{B}/\mathrm{A}}, \mathrm{I}[1] \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B}).$$

Continuing this trend, we see that automorphisms of such a lift is parametrized by

$$[\mathrm{L}_{\mathrm{B}/\mathrm{A}}[1], \mathrm{I}[1] \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B}] = \mathrm{Ext}^0(\mathrm{L}_{\mathrm{B}/\mathrm{A}}, \mathrm{I}[1] \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B}).$$

Therefore, we have that:

(1) the obstruction to finding a diagram is given by a class
$$o(\widetilde{\mathrm{A}}, \mathrm{A}, \mathrm{B}) \in \mathrm{Ext}^2(\mathrm{L}_{\mathrm{B}/\mathrm{A}}, \mathrm{I}[1] \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B});$$
(2) the set of diagrams form a torsor under
$$\mathrm{Ext}^1(\mathrm{L}_{\mathrm{B}/\mathrm{A}}, \mathrm{I}[1] \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B});$$
(3) automorphisms of a fixed lift is described by the group
$$\mathrm{Ext}^0(\mathrm{L}_{\mathrm{B}/\mathrm{A}}, \mathrm{I}[1] \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{B}).$$

1.2. **Flat deformations.** We will now do three things simulataneously: we do everything in the discrete setting, globalize everything to schemes and ask to classify flat deformations. So let us contemplate the following deformation problem: let us look at a particular diagram:

$$\begin{array}{ccc} \mathrm{X} & \longrightarrow & \widetilde{\mathrm{X}} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \widetilde{f}} \\ \mathrm{S} & \longrightarrow & \widetilde{\mathrm{S}}, \end{array}$$

where $\mathrm{S} \hookrightarrow \widetilde{\mathrm{S}}$ is a square-zero extension (no assumptions yet). Because we want everything to be discrete, let us assume that *all schemes in sight are classical* and understand what constrains we get.

First, every such picture means that we have an $\widetilde{\mathrm{S}}$-linear deformation of X given by $\widetilde{\mathrm{X}}$ (ignoring the S). This means that it is classified by a map

$$\mathrm{L}_{\mathrm{X}/\widetilde{\mathrm{S}}} \to \mathcal{J}[1]$$

for some $\mathcal{O}_{\mathrm{X}}$-module $\mathcal{J}$. Since we insist on everything being discrete, we learn that $\mathcal{J}$ must be discrete. At this point, we can ask for two conditions which are natural and inspired by the derived picture above:

(1) we ask that $\mathrm{L}\widetilde{f}^*\mathcal{I} \simeq \mathcal{J}$ (so, implicitly $\mathrm{L}f^* = f^*$, i.e., f is flat) where $\mathcal{I}$ is the ideal of definition of $\mathrm{S} \hookrightarrow \widetilde{\mathrm{S}}$;
(2) we ask that the above diagram is derived cartesian.

Indeed, by Remark 1.0.2 we see that condition (2) must imply (1). But we will soon see that they are, in fact, equivalent.

Lemma 1.2.1. *As in the situation above, the following are equivalent:*

(1) condition (1) is true;
(2) condition (2) is true;
(3) the morphism $\widetilde{f}$ is flat.

Proof. We can translate everything to algebra and contemplate the square of discrete rings

$$\begin{array}{ccc} \widetilde{\mathrm{A}} & \longrightarrow & \mathrm{A} \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \widetilde{\mathrm{B}} & \longrightarrow & \mathrm{B}, \end{array}$$

say the kernel of the top map is I and the kernel of the bottom is J.

Assume that (1) is true. We have a morphism $\widetilde{\mathrm{B}} \otimes^{\mathrm{L}}_{\widetilde{\mathrm{A}}} \mathrm{A} \to \mathrm{B}$ and we wish to prove that this map is an equivalence. Then we have a cofiber sequence of $\widetilde{\mathrm{B}}$-modules

$$\widetilde{\mathrm{B}} \otimes_{\widetilde{\mathrm{A}}} \mathrm{I} \to \widetilde{\mathrm{B}} \to \widetilde{\mathrm{B}} \otimes^{\mathrm{L}}_{\widetilde{\mathrm{A}}} \mathrm{A};$$

but comparing this to the cofiber sequence

$$\mathrm{J} \to \widetilde{\mathrm{B}} \to \mathrm{B},$$

we see that the desired map is an equivalence under the assumption $\widetilde{\mathrm{B}} \otimes_{\widetilde{\mathrm{A}}} \mathrm{I} \simeq \mathrm{J}$.

We have seen that (2) implies (1). Assume (2), let us see that $\widetilde{f}$ must be a flat morphism. Indeed let N be a $\widetilde{\mathrm{A}}$-module, whence we have an exact sequence

$$0 \to \mathrm{IN} \to \mathrm{N} \to \mathrm{N}/\mathrm{IN} \to 0;$$

this tells us that we need only check that $\widetilde{\mathrm{B}} \otimes^{\mathrm{L}}_{\widetilde{\mathrm{A}}} \mathrm{M}$ is discrete for $\mathrm{M} = \mathrm{IN}$ or N/IN. Since I is square zero, this tells us that we can assume that N is killed by I, whence N is naturally an A-module. IN this case, we have

$$\widetilde{\mathrm{B}} \otimes^{\mathrm{L}}_{\widetilde{\mathrm{A}}} \mathrm{N} \simeq \widetilde{\mathrm{B}} \otimes^{\mathrm{L}}_{\widetilde{\mathrm{A}}} \mathrm{A} \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{N}.$$

Now, discreteness happens if and only if $\widetilde{\mathrm{B}} \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{A} \simeq \widetilde{\mathrm{B}}$, which is exactly (2). This also proves that (3) implies (2). □

Hence, everything is governed by the condition that $\mathrm{L}f^*\mathcal{I} \simeq f^*\mathcal{I} \simeq \mathcal{J}$. So let us recap our discussion. We want to ask for all fillers:

$$\begin{array}{ccc} \mathrm{X} & \dashrightarrow & \widetilde{\mathrm{X}} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \widetilde{f}} \\ \mathrm{S} & \longrightarrow & \widetilde{\mathrm{S}}, \end{array}$$

where $\widetilde{f}$ (and hence f) is flat, hence we are looking for an S-linear square-zero extension of X which is classified by a map $\mathrm{L}_{\mathrm{X/S}} \to \mathcal{J}[1]$. By the discussion above, flatness forces the identity of $\mathcal{J}[1]$ as $\mathcal{J}[1] \simeq f^*\mathcal{I}[1] \simeq \mathrm{L}f^*\mathcal{I}[1]$. Now, I plug in the transitivity triangle for $\mathrm{X} \to \mathrm{S} \to \widetilde{\mathrm{S}}$ and get a diagram:

$$\begin{array}{ccccc} f^*\mathrm{L}_{\mathrm{S}/\widetilde{\mathrm{S}}} & \longrightarrow & \mathrm{L}_{\mathrm{X}/\widetilde{\mathrm{S}}} & \longrightarrow & \mathrm{L}_{\mathrm{X/S}} \\ \downarrow & & \downarrow & \swarrow & \\ f^*\mathcal{I}[1] & \longrightarrow & \mathcal{J}[1] & & . \end{array}$$

Here, the morphism $f^*\mathrm{L}_{\mathrm{S}/\widetilde{\mathrm{S}}} \to f^*\mathcal{I}[1]$ is induced by the "lowest homotopy group" map: $\mathrm{L}_{\mathrm{S}/\widetilde{\mathrm{S}}} \to \pi_1(\mathrm{L}_{\mathrm{S}/\widetilde{\mathrm{S}}}) = \mathcal{I}$ furnished by Theorem 1.0.3(3). Now let us truncate the transitivity sequence by $\tau_{\leqslant 1}$. It is not necessarily true that the $\tau_{\leqslant 1}$ preserves cofiber sequences. However, if $\mathrm{L}_{\mathrm{X/S}}$ is concentrated in degrees $\leqslant 1$ (for example if it is smooth, or even lci) we get a cofiber sequence[3]:

$$\tau_{\leqslant 1} f^*\mathrm{L}_{\mathrm{S}/\widetilde{\mathrm{S}}} \to \tau_{\leqslant 1}\mathrm{L}_{\mathrm{X}/\widetilde{\mathrm{S}}} \to \tau_{\leqslant 1}\mathrm{L}_{\mathrm{X/S}}.$$

With this assumption on $\mathrm{L}_{\mathrm{X/S}}$, the discussion of Section 1 tells us that the whole deformation problem only depends on $\tau_{\leqslant 1}\mathrm{L}_{\mathrm{X/S}} \simeq \mathrm{L}_{\mathrm{X/S}}$, hence we are at liberty to work with truncations of $\tau_{\leqslant 1}\mathrm{L}_{\mathrm{X}/\widetilde{\mathrm{S}}}$ and $\tau_{\leqslant 1} f^*\mathrm{L}_{\mathrm{S}/\widetilde{\mathrm{S}}}$. We note that, in general, $\tau_{\leqslant 1} f^*\mathrm{L}_{\mathrm{S}/\widetilde{\mathrm{S}}}$ need not be equivalent to $f^*\mathrm{L}_{\mathrm{S}/\widetilde{\mathrm{S}}}$. Indeed, by flatness of the map $g : \widetilde{\mathrm{S}} \to \operatorname{Spec} \mathbb{Z}/p^2$ and Theorem 1.0.3(4), $\mathrm{L}_{\mathrm{S}/\widetilde{\mathrm{S}}} \simeq \mathrm{L}g^*\mathrm{L}_{\mathbb{F}_p/(\mathbb{Z}/p^2)} \simeq g^*\mathrm{L}_{\mathbb{F}_p/(\mathbb{Z}/p^2)}$ which means that it is concentrated in homological degrees 1 and 2[4]. In any case, we get that $\tau_{\leqslant 1} f^*\mathrm{L}_{\mathrm{S}/\widetilde{\mathrm{S}}} \simeq f^*g^*\tau_{\leqslant 1}\mathrm{L}_{\mathbb{F}_p/(\mathbb{Z}/p^2)}$. We claim:

Lemma 1.2.2. *The canonical map* $\mathrm{L}_{\mathbb{F}_p/\mathbb{Z}_p} \to \mathrm{L}_{\mathbb{F}_p/(\mathbb{Z}/p^2)}$ *is a* $\tau_{\leqslant 1}$*-equivalence, whence* $\tau_{\leqslant 1}\mathrm{L}_{\mathbb{F}_p/(\mathbb{Z}/p^2)} \simeq \mathbb{F}_p[1]$.

Proof. The transitivity triangle for $\mathbb{Z}_p \to \mathbb{Z}/p^2 \to \mathbb{F}_p$ gives the first claim immediately. The second claim follow from Theorem 1.0.3(7) and the fact that p is a nonzero divisor in $\mathbb{Z}_p$. □

By flatness of g we also have that $\mathcal{I} = \mathcal{O}$. Hence we get a diagram (still under the assumption that $\mathrm{L}_{\mathrm{X/S}}$ is concentrated in degrees $\leqslant 1$):

$$\begin{array}{ccccc} \tau_{\leqslant 1} f^*\mathrm{L}_{\mathrm{S}/\widetilde{\mathrm{S}}} & \longrightarrow & \tau_{\leqslant 1}\mathrm{L}_{\mathrm{X}/\widetilde{\mathrm{S}}} & \longrightarrow & \mathrm{L}_{\mathrm{X/S}} \\ \downarrow{\scriptstyle \simeq} & & \downarrow & \swarrow & \\ f^*\mathcal{O}[1] & \longrightarrow & \mathcal{J}[1] & & . \end{array}$$

Hence the flatness condition, which is equivalent to saying that $f^*\mathcal{O} \simeq \mathcal{I}$, can be summarized as saying that the map $\mathrm{L}_{\mathrm{X}/\widetilde{\mathrm{S}}} \to f^*\mathcal{I}[1]$ classifying the flat deformation above is a *splitting* of the transtivity triangle. We conclude:

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[3]The key point here is that the map on H_1 of $f^*\mathrm{L}_{\mathrm{S}/\widetilde{\mathrm{S}}} \to \mathrm{L}_{\mathrm{X}/\widetilde{\mathrm{S}}}$ is injective.

[4]The transitivity triangle yields $\mathrm{L}_{\mathbb{F}_p/\mathbb{Z}_p} \to \mathrm{L}_{\mathbb{F}_p/(\mathbb{Z}/p^2)} \to \mathbb{F}_p \otimes_{\mathbb{Z}/p^2} \mathrm{L}_{(\mathbb{Z}/p^2)/\mathbb{Z}_p}[1]$.

Lemma 1.2.3. *As in Theorem 0.0.1, we have a pullback square*

$$\begin{array}{ccc} \mathrm{Rel}(\mathrm{X},\mathrm{S}) & \longrightarrow & \mathrm{Maps}(\tau_{\leqslant 1}\mathrm{L}_{\mathrm{X}/\widetilde{\mathrm{S}}}, f^*\mathcal{I}[1]). \\ \downarrow & & \downarrow \\ \{\mathrm{id}\} & \longrightarrow & \mathrm{Maps}(f^*\mathcal{I}, f^*\mathcal{I}). \end{array}$$

Here's a sample application

Proposition 1.2.4. *Let* R *be a perfect* $\mathbb{F}_p$*-algebra. There exists a unique, flat* $\mathbb{Z}_p$*-algebra* $\mathrm{W}(\mathrm{R})$ *such that* $\mathrm{W}(\mathrm{R})/p = 0$*. Furthermore it enjoys the following universal property: if* S *is a* p*-complete ring then any map* $\mathrm{R} \to \mathrm{S}/p$ *lifts uniquely to a* p*-adically continuous map* $\mathrm{W}(\mathrm{R}) \to \mathrm{S}$.

Proof. First, we solve the (derived) deformation problem of the following form:

$$\begin{array}{ccc} \mathbb{Z}/p^2 & \longrightarrow & \mathbb{F}_p \\ \downarrow & & \downarrow f \\ \widetilde{\mathrm{R}} & \dashrightarrow & \mathrm{R}, \end{array}$$

By the discussion above, this is controlled by the cotangent complex $\mathrm{L}_{\mathrm{R}/\mathbb{F}_p}$. But this object is acyclic: indeed the frobenius F_R induces the zero map on cotangent complexes on any $\mathbb{F}_p$-algebra (because $dx^p = 0$). But it is also an isomorphism because R is perfect. Hence, there is a *unique* solution to the above problem. Furthermore, we see that the sequence $\mathrm{R}[1] \to \mathrm{L}_{\mathrm{R}/\mathbb{Z}/p^2} \to \mathrm{L}_{\mathrm{R}/\mathbb{F}_p}$ splits since the last term is zero and hence the lift must be flat as explained above. We write $\mathrm{W}_2(\mathrm{R})$ to be $\widetilde{\mathrm{R}}$ We then successively solve the deformation problem by induction and produce $\mathrm{W}_n(\mathrm{R})$'s and take the inverse limit. We note that, at each stage, the obstruction to lifting is governed by the cotangent complex $\mathrm{L}_{\mathrm{W}_n(\mathrm{R})/\mathbb{Z}/p^n}$. But this complex is acyclic since it is acyclic after derived base change to $\mathbb{F}_p$ and appeal to derived Nakayama [Stacks, Tag 0G1U]. The universal property follows from another deformation theory argument and the vanishing of the cotangent complex, which we leave to the reader. □

This is a general case of Theorem 0.0.1(1). So what is this gerbe thing all about? Well the fiber $\mathrm{Rel}(\mathrm{X}^{(1)}, \mathrm{S}) \to \mathrm{id}$ is the fiber of the map $\mathrm{Maps}(\mathrm{L}_{\mathrm{X}^{(1)}/\widetilde{\mathrm{S}}}, f^*\mathcal{I}[1]) \to \mathrm{Maps}(f^*\mathcal{I}, f^*\mathcal{I})$ which, by the transitivity triangle, is exactly equivalent to

$$\mathrm{Maps}(\mathrm{L}_{\mathrm{X}^{(1)}/\mathrm{S}}, f^*\mathcal{I}[1]).$$

When X/S is smooth, then $\mathrm{L}_{\mathrm{X}^{(1)}/\mathrm{S}} \simeq \Omega^1_{\mathrm{X}^{(1)}/\mathrm{S}}[0]$ and the anima above is discrete and equivalent to the sheafy-ext (which is really a group scheme over X)

$$\mathcal{E}\mathrm{xt}(\Omega^1_{\mathrm{X}^{(1)}/\mathrm{S}}, f^*\mathcal{I});$$

and if $\mathcal{I}$ is the structure sheaf (as in the situation of the Deligne-Illusie theorem), then this is just the sheafy cohomology group

$$\mathcal{H}^1(\mathrm{X}^{(1)}, \mathrm{T}_{\mathrm{X}^{(1)}/\mathrm{S}}).$$

1.3. **Proof of Theorem 0.0.1.(3).** Let us place ourselves in the context where $\mathrm{X} \to \mathrm{S}$ is a smooth morphism. We have seen two gerbes:

(1) the gerbe of liftings, whose band is given by $\mathcal{H}^1(\mathrm{X}^{(1)}, \mathrm{T}_{\mathrm{X}^{(1)}/\mathrm{S}})$ as explained in the preceding paragraph;

(2) the gerbe of splittings. From the diagram

$$\begin{array}{ccc} \mathrm{Sci}(\tau^{\leqslant 1}\mathrm{F}_{\mathrm{X/S}*}\Omega^\bullet_{\mathrm{X/S}}) & \longrightarrow & \mathrm{Maps}(\tau^{\leqslant 1}\mathrm{F}_{\mathrm{X/S}*}\Omega^\bullet_{\mathrm{X/S}}, \mathcal{O}). \\ \downarrow & & \downarrow \\ \{\mathrm{id}\} & \longrightarrow & \mathrm{Maps}(\mathcal{O}, \mathcal{O}), \end{array}$$

the fiber of the map $\mathrm{Sci}(\tau^{\leqslant 1}\mathrm{F}_{\mathrm{X/S}*}\Omega^\bullet_{\mathrm{X/S}}) \to \{\mathrm{id}\}$ is the fiber of the map $\mathrm{Maps}(\tau^{\leqslant 1}\mathrm{F}_{\mathrm{X/S}*}\Omega^\bullet_{\mathrm{X/S}}, \mathcal{O}) \to \mathrm{Maps}(\mathcal{O}, \mathcal{O})$ which we once again see to be $\mathcal{H}^1(\mathrm{X}, \mathrm{T}_{\mathrm{X/S}})$.

That means that any choice of a splitting, and any choice of a lifting, differs by a section of $\mathcal{H}^1(\mathrm{X}, \mathrm{T}_{\mathrm{X/S}})$. Let us keep that in mind as we prove the result.

We want to make a map

$$\mathrm{Rel}(\mathrm{X}, \mathrm{S}) \to \mathrm{Sci}(\tau^{\leqslant 1}\mathrm{F}_{\mathrm{X/S}*}\Omega^\bullet_{\mathrm{X/S}});$$

we first make a map from auxlliary stacks

$$\widetilde{\mathrm{Rel}(\mathrm{X}^{(1)}}, \mathrm{S}) \to \mathrm{Sci}(\tau^{\leqslant 1}\widetilde{\mathrm{F}_{\mathrm{X/S}*}}\Omega^\bullet_{\mathrm{X/S}});$$

Here, the left hand side parametrizes the datum of 1) an étale map $\mathrm{Y} \to \mathrm{X}$, 2) a $\widetilde{\mathrm{S}}$-lift off $\mathrm{Y}^{(1)}$ called $\widetilde{\mathrm{Y}}^{(1)}$ (this is so far just a point of $\widetilde{\mathrm{Rel}(\mathrm{X}^{(1)}}, \mathrm{S})$), together 3) with a lift of the relative Frobenius: an $\widetilde{\mathrm{S}}$-lift of Y and a map

$$\varphi : \widetilde{\mathrm{Y}} \to \widetilde{\mathrm{Y}^{(1)}}$$

whose mod p reduction is

$$\mathrm{F}_{\mathrm{Y/S}} : \mathrm{Y} \to \mathrm{Y}^{(1)}.$$

To each such choice, we assign a splitting. The right hand side parametrizes splittings in the category of complexes: this means we pick a representative

$$\tau^{\leqslant 1}\mathrm{F}_{\mathrm{Y/S}*}\Omega^\bullet_{\mathrm{Y/S}} \simeq [\mathrm{Z}^0_{\mathrm{Y}} \to \mathrm{Z}^1_{\mathrm{Y}}]$$

and a map $\mathcal{H}^1(\tau^{\leqslant 1}\mathrm{F}_{\mathrm{Y/S}*}\Omega^\bullet_{\mathrm{Y/S}}) \to \mathrm{Z}^1_{\mathrm{Y}}$ splitting the canonical map. In fact, the resulting morphism of stacks is a certain morphism of gerbes (whose band we will see soon!).

Now, to each lift, by the previous lecture, we get divided Frobenius map

$$\varphi^*/p : \Omega^1_{\widetilde{\mathrm{Y}}^{(1)}/\widetilde{\mathrm{S}}} \to \mathrm{ZF}_{\mathrm{Y/S}*}\Omega^1_{\mathrm{Y/S}}$$

whose mod p-reduction is the diagram refining the inverse Cartier map:

$$\begin{array}{ccc} \Omega^1_{\mathrm{Y}^{(1)}/\mathrm{S}} & \xrightarrow{\varphi^*/p} & \mathrm{ZF}_{\mathrm{Y/S}*}\Omega^1_{\mathrm{Y/S}} \\ \downarrow{=} & \searrow^{\mathrm{C}^{-1}} & \downarrow \\ \Omega^1_{\mathrm{Y}^{(1)}/\mathrm{S}} & \xrightarrow{\cong} & \mathcal{H}^1(\mathrm{F}_{\mathrm{Y/S}*}\Omega^\bullet_{\mathrm{Y/S}}). \end{array}$$

Which means we can send this to the splitting (in the category of complexes) prescribed by the complex

$$[\mathrm{F}_*\mathcal{O} \xrightarrow{d} \mathrm{ZF}_{\mathrm{Y/S}*}\Omega^1_{\mathrm{Y/S}}]$$

by going $\mathcal{H}^1(\mathrm{F}_{\mathrm{Y/S}*}\Omega^\bullet_{\mathrm{Y/S}}) \cong \Omega^1_{\mathrm{Y}^{(1)}/\mathrm{S}} \xrightarrow{\varphi^*/p} \mathrm{ZF}_{\mathrm{Y/S}*}\Omega^1_{\mathrm{Y/S}}$. We need to check three things:

(1) we have a morphism of stacks;
(2) the map is independent of choices;
(3) we obtain a map of gerbes.

To check that we have a morphism of stacks rigorously we use the technology of cartesian fibrations, this will take us too far afield. To prove that it is independent of choices: deformation theory tells us that lifts with frobenii form a gerbe banded by $\mathrm{H}^0(\mathrm{Y}, \mathrm{F}^*\mathrm{T}_{\mathrm{Y}/\mathrm{S}}) = \mathrm{Hom}(\Omega^1_{\mathrm{Y}^{(1)}/\mathrm{S}}, \mathrm{F}_*\mathcal{O})$, i.e., any two lifts differ by map $\Omega^1_{\mathrm{Y}^{(1)}/\mathrm{S}} \to \mathrm{F}_*\mathcal{O}$. Each such a datum then defines a homotopy between any two lifts[5]. As a result we get a map of gerbes:

$$\mathrm{Rel}(\mathrm{X}^{(1)}, \mathrm{S}) \to \mathrm{Sci}(\tau^{\leqslant 1}\mathrm{F}_{\mathrm{X}/\mathrm{S}*}\Omega^\bullet_{\mathrm{X}/\mathrm{S}}).$$

To prove that the map is an isomorphism is then a local check or note that we have the dashed map map as in Theorem (0.0.3) which is an equivalence.

References

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[5] Any two strict splittings $s, s' : \mathcal{H}^1(\mathrm{F}_{\mathrm{Y}/\mathrm{S}*}\Omega^\bullet_{\mathrm{Y}/\mathrm{S}}) \to \mathrm{Z}^1\mathrm{K}$ will differ by a chain homotopy, which is a map $h : \mathcal{H}^1(\mathrm{F}_{\mathrm{Y}/\mathrm{S}*}\Omega^\bullet_{\mathrm{Y}/\mathrm{S}}) \to \mathrm{K}^0$ such that $dh = s - s'$. Hence a map $\Omega^1_{\mathrm{Y}^{(1)}/\mathrm{S}} \to \mathrm{F}_*\mathcal{O}$ can really be described as a morphism between two splittings.