LECTURE 5: THE ONE WHERE WE ACTUALLY CONSTRUCT THE COMPLEX

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1. Strict Dieudonné algebras

Let's recall where we are at. We have discussed the notion of a Dieudonné algebra and discussed how if R is a p-torsionfree ring with a lift of Frobenius φ, then the de Rham complex Ω^*_{R} is canonically a Dieudonné algebra. We have also described two processes: saturation which adds in elements of the form $p^n t^{1/p^n}$ or, more precisely, ensures that any element x such that dx is p-divisible is actually F-divisible. We have also produce a new operation on a saturated Dieudonné complex called V. For any such complex, we can speak of the quotients $\mathcal{W}_r(\mathrm{M})^*$ obtained by quotening out $\mathrm{V}^r x$ and $d\mathrm{V}^r x$. We can then speak of the V-completion of a Dieudonné complex; a saturated complex which is also V-complete (implicitly p-torsionfree) is said to be **strict**. The main theorem of the previous lectures says that the functor

$$\mathrm{M} \mapsto \mathcal{W}(\mathrm{Sat}(\mathrm{M}))^*$$

is a left adjoint to the inclusion of strict Dieudonné complexes. Now if M^* is a Dieudonné algebra, we want to say that $\mathcal{W}(\mathrm{Sat}(\mathrm{M}))^*$ is canonically a Dieudonné algebra whose underlying complex is strict.

Anyway, let us remind ourselves of what it means to be a Dieudonné algebra

Definition 1.0.1. A **Dieudonné algebra** is a commutative algebra object in **DC** satisfying:

(1) $\mathrm{A}^n = 0$ for $n < 0$;
(2) for each $x \in \mathrm{A}^0$, Fx is congruent to x^p modulo p;
(3) every homogeneous element of odd degree satisfies $x^2 = 0$.

We make a couple of remarks about Definition 1.0.1:

Remark 1.0.2. Firstly, we can always extract an underlying strict cdga from a Dieudonné algebra. From now on we will not use the words strict anymore in front of cdga as all of them will be strict and they mean something else in our context. Unwinding the definitions, the map F is always a homomorphism of underlying cga's (of course not commuting with the d). Morphisms of Dieudonné-algebras must commute with the F.

Remark 1.0.3. Let A^* be a saturated Dieudonné algebra. Then, from the fact that $\mathrm{A}^{-1} = 0$, we get that

$$\mathcal{W}_1(\mathrm{A})^0 = \mathrm{A}^0/\mathrm{VA}^0.$$

Let us see that $\mathcal{W}_1(\mathrm{A})^0$ is an $\mathbb{F}_p$-algebra. Indeed we see that $\mathrm{V}(1) = \mathrm{V}(\mathrm{F}(1)) = p$ and thus we have killed p.

Lemma 1.0.4. *Let A^* be a saturated Dieudonné algebra, then there is a unique ring structure on $\mathcal{W}(\mathrm{Sat}(\mathrm{A}))^*$ and a canonical map $\mathrm{A}^* \to \mathcal{W}(\mathrm{Sat}(\mathrm{A}))^*$ which is also a map of strict cdga's. Furthermore $\mathcal{W}(\mathrm{Sat}(\mathrm{A}))^*$ is a strict Dieudonné algebra.*

Proof. For the first claim, the only bit is to prove that $\mathrm{Im}(\mathrm{V}^r) + \mathrm{Im}(d\mathrm{V}^r)$ is a dg-ideal for each $r \geqslant 0$. Let us see this for $r = 1$: we need to compute $x\mathrm{V}(y)$; the thing to do is to prove the projection formula

$$x\mathrm{V}y = \mathrm{V}(\mathrm{F}(x)y).$$

With this and the identity

$$d(x\mathrm{V}y) = xd\mathrm{V}y \pm dx\mathrm{V}y,$$

we get

$$xd\mathrm{V}y = d(x\mathrm{V}y) \pm dx\mathrm{V}y = d(\mathrm{V}((\mathrm{F}x)y)) \pm \mathrm{V}(\mathrm{F}(dx)y).$$

The projection formula itself is proved in this way: we may check it after applying F since it is injective. Then

$$\mathrm{F}(x\mathrm{V}y) = \mathrm{F}(x)\mathrm{F}(\mathrm{V}(y)) = p\mathrm{F}(x)y = \mathrm{F}(\mathrm{V}(\mathrm{F}(x)y)).$$

At this point we conclude that $\mathcal{W}(\mathrm{Sat}(\mathrm{A}))^*$ is a strict cdga. It is also evident that the negative degree parts vanish.

For the "furthermore" part we need to verify part (2) of Definition 1.0.1 for $\mathcal{W}(\mathrm{Sat}(\mathrm{A}))^*$. In fact this condition is the same thing thing as verifying that $\mathrm{F}x = x^p$ modulo V (exercise). So we need to prove that F is the usual Frobenius on $\mathcal{W}_1$.

□

Therefore we say that A^* is a **strict Dieudonné algebra** if it is saturated and the map $\mathrm{A}^* \to \mathcal{W}(\mathrm{A})^*$ is an isomorphism.

Definition 1.0.5. The R be a $\mathbb{F}_p$-algebra, then the **saturated de Rham-Witt complex** of R is the initial strict Dieudonné algebra, denoted by $\mathcal{W}\Omega^*_{\mathrm{R}}$ equipped with a map $\mathrm{R} \to \mathrm{A}^0/\mathrm{VA}^0$.

Theorem 1.0.6. *The saturated de Rham-Witt complex of* R *exists.*

Proof. By Lemma 1.0.7, any map $\mathrm{R} \to \mathrm{A}^0/\mathrm{VA}^0$ factors through its reduction, hence we may assume that R is reduced. In this situation, the ring of witt vectors W(R) is p-torsion free. Hence, we are allowed to regard $\Omega^*_{\mathrm{W(R)}}$ as a strict Dieudonné algebra under $\mathrm{W}(\varphi) = \mathrm{F}$, the Witt vector Frobenius of W(R). Let A^* be given by

$$\mathcal{W}\mathrm{Sat}(\Omega^*_{\mathrm{W(R)}}).$$

Let B^* be a strict Dieudonné algebra. Then the following are equivalence pieces of data:

(1) a map $\mathcal{W}\mathrm{Sat}(\Omega^*_{\mathrm{W(R)}}) \to \mathrm{B}^*$ of Dieduonné algebras;
(2) a map $\mathrm{Sat}(\Omega^*_{\mathrm{W(R)}}) \to \mathrm{B}^*$ of Dieudonné algebras;
(3) a map $\Omega^*_{\mathrm{W(R)}} \to \mathrm{B}^*$ of Dieudonné algebras;
(4) a map $\mathrm{W(R)} \to \mathrm{B}$ which intertwines the frobenii;
(5) a map $\mathrm{R} \to \mathrm{B}^0/\mathrm{VB}^0$ of rings.

The equivalence of (1), (2) and (3) follows from universal properties. The equivalence of (3) and (4) follows from the universal properties of the de Rham complex. The equivalence of (4) and (5) is then Lemma 1.0.8 which is the key point.

□

Lemma 1.0.7. *Let* A^* *be a saturated Dieudonné algebra, then* $\mathrm{A}^0/\mathrm{VA}^0$ *is a reduced* $\mathbb{F}_p$*-algebra.*

Proof. Since $\mathrm{V}(1) = \mathrm{V}(\mathrm{F}(1)) = p$, we have that $p \in \mathrm{VA}^0$ and thus $\mathrm{A}^0/\mathrm{VA}^0$ is an $\mathbb{F}_p$-algebra. We now claim that $\mathrm{A}^0/\mathrm{VA}^0$ is reduced: let $x \in \mathrm{A}^0$ and assume that $\overline{x}^p = 0$ so that $x^p \in \mathrm{VA}^0$; we claim that $x \in \mathrm{VA}^0$. Since A^* is a Dieudonné algebra, and $\mathrm{F}x = x^p$ modulo p, we see that $\mathrm{F}x = \mathrm{V}y$ for some y in A^0. Applying the differential we get that $d\mathrm{V}y = d\mathrm{F}x = p\mathrm{F}dx$. But since A^* is saturated, we get that y is in the image of F. To prove that $x \in \mathrm{VA}^0$ it then suffices to prove that $\mathrm{F}x$ is since F is injective; this follows from:

$$\mathrm{F}x = \mathrm{V}y = \mathrm{VF}z = \mathrm{FV}z.$$

□

The next lemma is absolutely key.

Lemma 1.0.8. *Let* B^* *be a strict Dieudonné algebra and* R *a commutative* $\mathbb{F}_p$*-algebra. Then the following two pieces of data are equivalent:*

(1) a ring map $\mathrm{R} \to \mathrm{B}^0/\mathrm{VB}^0$*;*

(2) a ring map $\mathrm{W}(\mathrm{R}) \to \mathrm{B}^0$ *such that*

$$\begin{array}{ccc} \mathrm{W}(\mathrm{R}) & \longrightarrow & \mathrm{B}^0 \\ \downarrow & & \downarrow \\ \mathrm{R} & \longrightarrow & \mathrm{B}^0/\mathrm{VB}^0; \end{array}$$

and there is commutation with the Frobenius

$$\begin{array}{ccc} \mathrm{W}(\mathrm{R}) & \longrightarrow & \mathrm{B}^0 \\ \downarrow{\scriptstyle \mathrm{F}} & & \downarrow{\scriptstyle \mathrm{F}} \\ \mathrm{W}(\mathrm{R}) & \longrightarrow & \mathrm{B}^0. \end{array}$$

Proof. We observe two weird, interrelated things about this Lemma: first we are trying to map *out* of the Witt vectors of an $\mathbb{F}_p$-algebra (usually we map in) and secondly the claims only involve B^0 and ungraded rings (or rings places in degree zero). So something is really happening.

Here's the magic claim: B^0 must be isomorphic to $\mathrm{W}(\mathrm{B}^0/\mathrm{VB}^0)$ and the B^0-Frobenius must be the Witt vector Frobenius; this is forced by the strict Dieudonné algebra structure. Having this we are done: the map $\mathrm{W}(\mathrm{R}) \to \mathrm{B}^0$ is W of $\mathrm{R} \to \mathrm{B}^0/\mathrm{VB}^0$; to prove uniqueness we just need to characterize maps between W of $\mathbb{F}_p$-algebras of the form $\mathrm{W}(f)$; we skip this and refer the reader to [BLM21, Lemma 3.6.4].

Let's prove this: write $\mathrm{S} = \mathrm{B}^0/\mathrm{VB}^0$ and from Lemma 1.0.7, S is a reduced $\mathbb{F}_p$-algebra. Again, since it is reduced, W(S) is p-torsionfree. By universal properties of the Witt vectors, we get map $u : \mathrm{B}^0 \to \mathrm{W}(\mathrm{S})$ such that:

$$\begin{array}{ccc} \mathrm{W}(\mathrm{S}) & \xleftarrow[u]{} & \mathrm{B}^0 \\ \downarrow & & \downarrow \\ \mathrm{S} & \xleftarrow[=]{} & \mathrm{B}^0/\mathrm{VB}^0; \end{array}$$

and

$$\begin{array}{ccc} \mathrm{W}(\mathrm{S}) & \xleftarrow[u]{} & \mathrm{B}^0 \\ \downarrow{\scriptstyle \mathrm{F}} & & \downarrow{\scriptstyle \mathrm{F}} \\ \mathrm{W}(\mathrm{S}) & \xleftarrow[u]{} & \mathrm{B}^0. \end{array}$$

It then suffices to prove that u is an isomorphism.

Step 1 We claim that $u\mathrm{V} = \mathrm{V}u$. We can check this after applying F; using the second diagram above we get

$$\mathrm{F}(u\mathrm{V}x) = u\mathrm{FV}x.$$

But $\mathrm{FV}x = px$ so the above is equal to $upx = pux = \mathrm{FV}ux$ and we are done. Therefore the map u preserves the V-filtration on both sides.

Step 2 We thus get maps

$$u_r : \mathrm{B}^0/\mathrm{V}^r\mathrm{B}^0 \to \mathrm{W}_r(\mathrm{S})$$

compatible with u. Since B^* is strict, it suffices to prove that each u_r is an isomorphism. To prove this we consider the diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{B}^0/\mathrm{VB}^0 & \xrightarrow{\mathrm{V}^{r-1}} & \mathrm{B}^0/\mathrm{V}^r\mathrm{B}^0 & \longrightarrow & \mathrm{B}^0/\mathrm{V}^{r-1}\mathrm{B}^0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u_r} & & \downarrow{\scriptstyle u_{r-1}} & & \\ 0 & \longrightarrow & \mathrm{S} & \xrightarrow{\mathrm{V}^{r-1}} & \mathrm{W}_r(\mathrm{S}) & \longrightarrow & \mathrm{W}_{r-1}(\mathrm{S}) & \longrightarrow & 0. \end{array}$$

The left most map is the identity, the right most map is an isomorphism by inductive hypothesis.

□

1.1. Summary of construction. We thus have constructed a functor

$$\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p} \to \mathbf{DA} \qquad \mathrm{R} \mapsto \mathrm{W}\Omega^*_{\mathrm{R}};$$

by construction it factors through a small subcategory which we like

$$\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p} \to \mathbf{DA}_{\mathrm{str}}.$$

It is well-defined because of universal properties: the only thing we ever need to map into an object of $\mathbf{DA}_{\mathrm{str}}$ is a morphism of $\mathbb{F}_p$-algebras $\mathrm{R}^0 \to \mathrm{B}/\mathrm{VB}^0$. The engine behind this computation is the realization that we can recover B^0 from B/VB^0 via

$$\mathrm{W}(\mathrm{B}/\mathrm{VB}^0) = \mathrm{B}^0,$$

which is assured by the completed-saturated nature of B^*. One additional thing to keep in mind: $\mathrm{W}\Omega^*_{\mathrm{R}}$ is derived p-complete, since is V-complete, by a result discussed in the previous lecture. We also note that, by construction, $\mathrm{W}\Omega^*_{\mathrm{R}}$ acquires the structure of an algebra over $\mathrm{W}\Omega^*_{\mathbb{F}_p} = \mathbb{Z}_p$. To reflect this dependence on $\mathbb{F}_p$, we write

$$\mathrm{W}\Omega^*_{\mathrm{R}/\mathbb{F}_p} \in \mathbf{K}(\mathbb{Z}_p).$$

By the usual Kan extension maneuvers we can globalize our construction. I will mention two caveats (which were already implicit in our discussion of de Rham cohomology) in doing this:

1. we should only really consider the de Rham Witt complex on smooth $\mathbb{F}_p$-schemes; the rest of values are determined by *left* Kan extension.
2. While the de Rham Witt complex appears naturally as a cochain complex, its globalization (which is a *right* Kan extension) would require that we think of it as an object in the derived ∞-category; the cochain complex structure is then captured by an analog of the Hodge filtration on the global values.

Formally, we perform the following two procedures: first we take a left Kan extension

$$\begin{array}{ccc} \mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p} & \xrightarrow{\mathrm{W}\Omega^*_{-/\mathbb{F}_p}} & \mathbf{K}(\mathbb{Z}_p) \\ \downarrow & & \downarrow \\ \mathrm{CAlg}_{\mathbb{F}_p} & \longrightarrow & \mathbf{D}(\mathbb{Z}_p); \end{array}$$

and then take the right Kan extension to all animated $\mathbb{F}_p$-schemes

$$\begin{array}{ccc} \mathrm{CAlg}_{\mathbb{F}_p} & \longrightarrow & \mathbf{K}(\mathbb{Z}_p) \\ \downarrow & & \downarrow \\ \mathrm{AniSch}^{\mathrm{op}}_{\mathbb{F}_p} & \longrightarrow & \mathbf{D}(\mathbb{Z}_p); \end{array}$$

The resulting functor

$$\mathrm{LR\Gamma}_{\mathrm{crys}}(-/\mathbb{Z}_p) : \mathrm{AniSch}^{\mathrm{op}}_{\mathbb{F}_p} \to \mathbf{D}(\mathbb{Z}_p).$$

is called **derived crystalline cohomology**. If X is a smooth $\mathbb{F}_p$-scheme,

1.2. An example: $\mathbb{G}^n_m$. We will use Lemma 1.3.2 in order to calculate $\mathrm{W}\Omega^*_{\mathbb{G}^n_m}$. Let $\mathrm{R} = \mathbb{Z}[x_1^{\pm 1}, \cdots, x_r^{\pm 1}]$. Let's try to compute $\mathrm{W}\Omega^*_{\mathrm{R}/p}$. First, we package Ω^*_{R} in a compact way:

$$\Omega^*_{\mathrm{R}} = \bigwedge_{\mathrm{R}} \mathrm{R} \cdot \{\mathrm{dlog} x_i\} \qquad dx_i^n = n x_i^n \mathrm{dlog} x_i.$$

To make it into a Dieudonné algebra we define its Frobenius as $\mathrm{F}(x_i) = x_i^p$; we are then forced to have that $\mathrm{F}(dx) = x^{p-1}dx + d(\delta(x))$. But then this forces

$$\mathrm{F}(\mathrm{dlog} x_i) = \mathrm{F}(\tfrac{1}{x_i})\mathrm{F}(dx_i) = \tfrac{1}{x_i^p}(x_i^{p-1}dx_i + d(0)) = \mathrm{dlog} x_i.$$

We now describe its saturation. Set $\mathrm{R}_\infty := \mathbb{Z}[x_1^{\pm\frac{1}{p^\infty}}, \cdots, x_r^{\pm\frac{1}{p^\infty}}]$. Then $\Omega^*_{\mathrm{R}}[\mathrm{F}^{-1}]$ is the exterior algebra

$$\Omega^*_{\mathrm{R}}[\mathrm{F}^{-1}] = \bigwedge_{\mathrm{R}_\infty} \mathrm{R}_\infty \cdot \{\mathrm{dlog} x_i\}.$$

Remember that its differential:

$$\Omega^*_{\mathrm{R}}[\mathrm{F}^{-1}] \xrightarrow{d} \Omega^*_{\mathrm{R}}[\mathrm{F}^{-1}, p^{-1}]$$

is given by

$$d(\mathrm{F}^{-n}x) = p^{-n}\mathrm{F}^{-n}dx;$$

hence:

$$d(x_i^{\frac{a}{p^n}}) = d(\mathrm{F}^{-n}x_i^a) = \tfrac{1}{p^n}\mathrm{F}^{-n}dx_i^a = \tfrac{1}{p^n}\mathrm{F}^{-n}(ax_i^a\mathrm{dlog}(x_i)) = \tfrac{a}{p^n}\mathrm{F}^{-n}(x_i^a)\mathrm{F}^{-n}(\mathrm{dlog}(x_i));$$

which means

$$d(x_i^{\frac{a}{p^n}}) = \tfrac{a}{p^n}x_i^{\frac{a}{p^n}}(\mathrm{dlog}(x_i)).$$

Now, recall that the saturation is smaller: we must look at those forms ω for which $d\omega$ is also integral, i.e., has no denominators! Hence we conclude that at degree zero, we must look at finite sums of $bx_i^{\frac{a}{p^n}}$ such that p^n divides b. So this means that elements like

$$p^j x_i^{\frac{1}{p^n}} \qquad j \geqslant n$$

are allowed. Whereas things like

$$p^j x_i^{\frac{1}{p^n}} \qquad j < n$$

or

$$\ell x_i^{\frac{1}{p^n}} \qquad (\ell, p) = 1$$

are never allowed. Of course the elements of integral powers x_i^k ($k \in \mathbb{Z}$) are allowed anyway. How about in degree one? If $r = 1$, then we do have that

$$\mathrm{Sat}(\Omega^*_{\mathrm{R}})^1 \cong \mathrm{R} \cdot \mathrm{dlog} x_i.$$

In general, however, we only have an inclusion:

$$\mathrm{Sat}(\Omega^*_{\mathrm{R}})^1 \cong \bigoplus_{0 \leqslant i \leqslant r} \mathrm{R} \cdot \mathrm{dlog} x_i.$$

Now we want to compute the action of V in order to compute $\mathrm{W}_k\Omega^*_{\mathbb{F}_p[x_1^{\pm 1}, \cdots, x_r^{\pm 1}]}$. Let us assume, for simplicity that $r = 1$ and write the variable as T. We compute V now: we have that $\mathrm{FV} = p$ and so we must have that (assuming $(a, p) = 1$)

$$\mathrm{V}(b\mathrm{T}^{\frac{a}{p^k}}) = pb\mathrm{T}^{\frac{a}{p^{k+1}}} \qquad \mathrm{V}(\mathrm{dlog}\mathrm{T}) = p\mathrm{dlog}\mathrm{T}.$$

where n is an integer. Now, let's see what

$$\mathrm{W}_k(\Omega^*_{\mathbb{F}_p[\mathrm{T}^{\pm 1}]})^0 = \mathrm{E}^0/\mathrm{V}^k\mathrm{E}^0 \qquad k \geqslant 1.$$

look like. Well, we see that this object is a $\mathbb{Z}/p^k$-module since $p^k = \mathrm{V}^k(1)$ and that must be killed. Furthermore we see that T^j where $j \in \mathbb{Z}$ is an integral power is never possibly killed. Hence there is a summand

$$\bigoplus_{j \in \mathbb{Z}} \mathbb{Z}/p^k \cdot \mathrm{T}^j \subset \mathrm{W}_k(\Omega^*_{\mathbb{F}_p[\mathrm{T}^{\pm 1}]})^0.$$

So we should now look at those components which has non-integral powers. Before we mod out by V, this can be indexed as

$$\bigoplus_{a \in \mathbb{Z} \smallsetminus \{0\}, (a,p)=1, n \geqslant 1} \mathbb{Z} \cdot \{p^n\mathrm{T}^{\frac{a}{p^n}}\};$$

Since $\mathrm{V}^n(\mathrm{T}^a) = p^n \mathrm{T}^{\frac{a}{p^n}}$ we can rearrange:

$$\bigoplus_{a \in \mathbb{Z} \smallsetminus \{0\}, (a,p)=1, n \geqslant 1} \mathbb{Z} \cdot \{\mathrm{V}^n(\mathrm{T}^a)\};$$

Hence, for a fixed k, the complementary summand is

$$\bigoplus_{a \in \mathbb{Z} \smallsetminus \{0\}, (a,p)=1, n \geqslant 1} \mathbb{Z}/p^{k-n} \cdot \{\mathrm{V}^n(\mathrm{T}^a)\} \subset \mathrm{W}_k(\Omega^*_{\mathbb{F}_p[\mathrm{T}^{\pm 1}]})^0.$$

So anytime we have that $n > k$, elements with T-powers given by $\mathrm{T}^{\frac{a}{p^n}}$ all die off since they are forced to have zero coefficients.

Let us have a look at E^1; this is just:

$$\mathbb{Z}[\mathrm{T}^{\pm 1/p^\infty}] \cdot \mathrm{dlogT}$$

Noting that (by the formula for d above):

$$d(\mathrm{V}^n(\mathrm{T}^a)) = d(p^n \mathrm{T}^{\frac{a}{p^n}}) = p^n \tfrac{a}{p^n} \mathrm{T}^{\frac{a}{p^n}} \mathrm{dlogT};$$

we can rewrite the above as

$$\mathbb{Z}[\mathrm{T}^{\pm 1}] \cdot \mathrm{dlogT} \oplus \bigoplus_{a \in \mathbb{Z} \smallsetminus \{0\}, (a,p)=1, n \geqslant 1} \mathbb{Z} \cdot \{d\mathrm{V}^n(\mathrm{T}^a)\};$$

Therefore we get that

$$(\mathrm{W}_k\Omega^*)^1 = \mathbb{Z}/p^k[\mathrm{T}^{\pm 1}] \cdot \mathrm{dlogT} \oplus \bigoplus_{a \in \mathbb{Z} \smallsetminus \{0\}, (a,p)=1, n \geqslant 1} \mathbb{Z}/p^{k-n} \cdot \{d\mathrm{V}^n(\mathrm{T}^a)\};$$

Proceeding like this, we can prove the following result of Deligne's:

Lemma 1.2.1. *Let* $\mathrm{R} = \mathbb{F}_p[\mathrm{T}_1^{\pm 1}, \cdots \mathrm{T}_r^{\pm 1}, \mathrm{T}_{r+1}, \cdots, \mathrm{T}_{r+s}]$. *Then* $\mathrm{W}_k\Omega^*_{\mathrm{R}/\mathbb{F}_p}$ *contains the complex*

$$\Omega^*_{\mathbb{Z}/p^k[\mathrm{T}_1^{\pm 1}, \cdots \mathrm{T}_r^{\pm 1}, \mathrm{T}_{r+1}, \cdots, \mathrm{T}_{r+s}]/(\mathbb{Z}/p^k)}$$

as a direct summand. Its complement is acyclic.

1.3. **Smooth and de Rham comparison.** Our next goal is to prove the following key result in the theory: it compares crystalline cohomology with the de Rham cohomology of a smooth lift.

Theorem 1.3.1. [BLM21, Theorems 4.2.3-4, lifting comparison] *Let* R *be a commutative ring which is* p*-torsion free such that* R/p *is a smooth over a perfect ring* κ *of characteristic* $p > 0$, *and* $\varphi : \mathrm{R} \to \mathrm{R}$ *a lift of Frobenius. Then there is a map of Dieudonné algebras:*

$$\mu : \widehat{\Omega}^*_{\mathrm{R}} \to \mathcal{W}\Omega^*_{\mathrm{R}/p},$$

whose degree 0 *part fits into a diagram:*

$$\begin{array}{ccccc} \mathrm{R} & \longrightarrow & \widehat{\Omega}^0_{\mathrm{R}} & \xrightarrow{\mu} & \mathcal{W}\Omega^0_{\mathrm{R}/p} \\ \downarrow & & & & \downarrow \\ \mathrm{R}/p & & \longrightarrow & & \mathcal{W}_1\Omega^0_{\mathrm{R}/p}. \end{array}$$

Furthermore, μ *is a quasi-isomorphism.*

First, let us construct μ. Recall that the completed de Rham complex is defined to be

$$\widehat{\Omega}^*_{\mathrm{R}} := \lim \Omega^*_{\mathrm{R}}/p^n\Omega^*_{\mathrm{R}}.$$

In general, the canonical map $\Omega^*_{\mathrm{R}} \to \widehat{\Omega}^*_{\mathrm{R}}$ is not an isomorphism: for example if we look at at $\mathrm{R} = \mathbb{Z}[\mathrm{T}]$, the map

$$\mathbb{Z}[\mathrm{T}] \to \lim \mathbb{Z}[\mathrm{T}]/p^n$$

is not an isomorphism, even if $\mathbb{Z}$ is replaced by $\mathbb{Z}_p$.

Whenever R is p-torsionfree and comes equipped with a lift of the Frobenius, then we have the structure of a Dieudonné algebra $\mathrm{F} : \widehat{\Omega}^*_{\mathrm{R}} \to \eta_p \widehat{\Omega}^*_{\mathrm{R}}$, induced by the one on the uncompleted de Rham complex. We note that $\widehat{\Omega}^*_{\mathrm{R}}$ is p-adically complete though it is not p-torsion free in general. Yet, it has the following universal property: it is the initial p-complete and p-torsionfree[1] Dieudonné algebra equipped with a map $\mathrm{R} \to \mathrm{A}^0$ which intertwines φ and F. In particular, after Lemma 1.0.8, a map $\widehat{\Omega}^*_{\mathrm{R}} \to \mathrm{A}^*$ where A^* is assumed to be strict is the same thing as a map $\mathrm{R} \to \mathrm{A}^0/\mathrm{VA}^0$ (indeed: by the universal property of the Witt vectors we have a map $\mathrm{R} \to \mathrm{W}(\mathrm{R}/p)$ and so we might as well replace R with $\mathrm{W}(\mathrm{R}/p)$ from Lemma 1.0.8 applies).

So the map $\mu : \widehat{\Omega}^*_{\mathrm{R}} \to \mathcal{W}\Omega^*_{\mathrm{R}}$ is constructed via the map $\mathrm{R} \to \mathrm{R}/p \to \mathrm{W}\Omega^0_{\mathrm{R}}/\mathrm{VW}\Omega^0_{\mathrm{R}}$ where the second map comes from the universal property of the de Rham-Witt complex (even without choosing a model). We now claim:

Lemma 1.3.2. *The map μ induces an* isomorphism *of Dieudonné algebras after applying* $\mathcal{W}\mathrm{Sat}$.

Proof. Of course, $\mathcal{W}\mathrm{Sat}$ does not change the target of μ. Let A^* is a strict Dieudonné algebra, we need to prove that the map

$$\mathrm{Hom}(\mathcal{W}\Omega^*_{\mathrm{R}/p}, \mathrm{A}^*) \xrightarrow{\mu^*} \mathrm{Hom}(\mathcal{W}\mathrm{Sat}(\widehat{\Omega}^*_{\mathrm{R}}), \mathrm{A}^*),$$

is an isomorphism. Using the universal properties on both sides, we see the above map identifies with:

$$\mathrm{Hom}(\mathrm{R}/p\mathrm{R}, \mathrm{A}^0/\mathrm{VA}^0) \to \mathrm{Hom}(\mathrm{R}, \mathrm{A}^0/\mathrm{VA}^0).$$

But this is bijective since $\mathrm{A}^0/\mathrm{VA}^0$ is an $\mathbb{F}_p$-algebra. □

To proceed further, we need to discuss Cartier smoothness in this context.

1.3.3. *Cartier smoothness.* The following is an important result and axiomatizes what we have already seen with the Cartier isomorphism

Proposition 1.3.4 (Décalage)**.** *Let* M^* *be a complex of abelian groups which is p-torsionfree, Then we have a quasi-isomorphism of cochain complexes*

$$\gamma : (\eta_p \mathrm{M})^*/p(\eta_p \mathrm{M})^* \to \mathrm{H}^*(\mathrm{M}/p\mathrm{M});$$

where

$$\gamma(x) = [\tfrac{x}{p^n}] \qquad |x| = n;$$

and $\mathrm{H}^*(\mathrm{M}/p\mathrm{M})$ *is equipped with the Bockstein differential.*

Proof. We note that the source is a quotient of the complex $\eta_p\mathrm{M}$, hence any element in it is divisible by p to its degree. Hence the expression $\frac{x}{p^n}$ makes sense; it is even unique because M is p-torsionfree. Furthermore, $d(\frac{x}{p^n})$ is p-divisible because dx is divisible by p^{n+1}, therefore it is a cycle in $\mathrm{H}^*(\mathrm{M}/p\mathrm{M})$. This explains why the map of interest is well-defined.

Now, the Bockstein differential is produced by looking at the exact sequence of complexes

$$0 \to \mathrm{M}/p\mathrm{M} \to \mathrm{M}/p^2\mathrm{M} \to \mathrm{M}/p\mathrm{M} \to 0;$$

and the resulting boundary map on long exact sequences $\beta : \mathrm{H}^n(\mathrm{M}/p\mathrm{M}) \to \mathrm{H}^{n+1}(\mathrm{M}/p\mathrm{M})$. If we trace through the formula for β we see that $\beta([x]) = [p^{-1}dx]$. So we claim that we have a

[1] Assume that A^* is a p-torsionfree Dieudonné complex. We claim that the map $f : \Omega^*_{\mathrm{R}} \to \mathrm{A}^*$ induced by a map $\mathrm{R} \to \mathrm{A}^0$ intertwines the Frobenii on both sides. We can reduce to the case that $\omega = dx$ and we claim that $pf\mathrm{F}dx = p\mathrm{F}dfx$; but this is the case:

$$pf\mathrm{F}(dx) = pf(x^{p-1}dx + d(\delta(x)) = f(px^{p-1}dx + d(\varphi(x) - x^p)) = f(d\varphi(x)) = d\mathrm{F}\varphi(x) = p\mathrm{F}d(f(x)).$$

commutative diagram

$$\begin{array}{ccc} M^n/p & \longrightarrow & H^n(M/pM) \\ \downarrow d & & \downarrow \beta \\ M^{n+1}/p & \longrightarrow & H^{n+1}(M/pM); \end{array}$$

which follows from

$$\beta([\gamma(x)]) = \beta([p^{-n}x]) = [p^{-n-1}x] = [\gamma(dx)].$$

Now, we see that the map $(\eta_p M)^*/p(\eta_p M)^* \to H^*(M/pM)$ is clearly levelwise surjective. Let K^* be the kernel; we claim that it is acyclic. So what does it mean for $x \in (\eta_p M)^*/p(\eta_p M)^*$ to be in the kernel? Well this means that

$$\tfrac{x}{p^n} = dy + pz,$$

and thus

$$x = dp^n y + p^{n+1}z;$$

whence K^n is a quotient of $p^{n+1}M^n + dp^n M^{n-1}$. It is easy to then check that

$$K^n = p^{n+1}M^n + dp^n M^{n-1}/(p^{n+1}M^n \cap d^{-1}p^{n+2}M^{n+1}),$$

from the definition of $\eta_p M$. Let us now claim that any element in K^n which is a cocycle is also a coboundary. Let x be such an element, then

$$0 = dx = d^2(p^n y) + d(p^{n+1}z) = 0 + p^{n+1}dz \in K^{n+1},$$

whence $d(p^{n+1}z) \in p^{n+2}M^{n+1}$ and thus $p^{n+1}z \in d^{-1}p^{n+2}M^{n+1}$. Therefore, x can actually represented by just $dp^n y$ and hence a coboundary. □

We said that the above quasi-isomorphism is like the Cartier isomorphism. To make this relationship we examine the following triangle of complexes

(1.3.5)
$$\begin{array}{ccc} M^*/pM^* & \xrightarrow{\alpha_F} & (\eta_p M)^*/p(\eta_p M)^* \\ & \searrow^{F} \quad \swarrow_{\gamma} & \\ & H^*(M/pM) & \end{array}$$

We have seen that γ is a quasi-isomorphism, hence the map α_F is a quasi-isomorphism if and only if F is a quasi-isomorphism. We make the following definition

Definition 1.3.6. A Dieudonné complex is **Cartier smooth** if it is p-torsion-free and the map

$$(M^*/pM^*, d) \xrightarrow{F} (M^*/pM^*, 0)$$

is a quasi-isomorphism. In other words, the Frobenius induces an isomorphism of graded abelian groups

$$M^*/pM^* \cong H^*(M^*/pM^*).$$

We note the following:

Lemma 1.3.7. *Let* R *be a* p*-torsionfree ring and* φ *a lift of the Frobenius. Assume that* R/p *is smooth over a perfect field* κ *of characteristic* $p > 0$ *(or, more generally, a perfect ring). Then* $\widehat{\Omega}^*_R$ *is Cartier smooth.*

Proof. Note that $\widehat{\Omega}^*_R/p\widehat{\Omega}^*_R \cong \widehat{\Omega}^*_{R/p}$. We have already seen that the Cartier isomorphism holds in this setting. We need to say why $\widehat{\Omega}^*_R$ is p-torsion free. But then $\widehat{\Omega}^*_R/p^n\widehat{\Omega}^*_R \cong \Omega^*_{(R/p^n)/W_n(\kappa)}$ which is smooth since R/p is smooth by assumption. Hence, $\widehat{\Omega}^*_R/p^n\widehat{\Omega}^*_R$ is a finitely generated projective module of finite rank. From this we conclude p-torsionfree-ness.

□

Hence any Dieudonné complex which is Cartier smooth enjoys the quasi-isomorphism

$$\mathrm{M}/p\mathrm{M} \xrightarrow{\simeq} (\eta_p\mathrm{M})^*/p(\eta_p\mathrm{M})^*.$$

The upshot is that we can control the quasi-isomorphism type of the saturation of M^* which is Cartier smooth since it is given by a colimit of these η_p's.

Theorem 1.3.8. [BLM21, Theorem 2.4.2] *Let* M^* *be a Cartier smooth Dieudonné complex. Then the canonical map*

$$\mathrm{M}^* \to \mathrm{Sat}(\mathrm{M}^*)$$

induces a quasi-isomorphism

$$\mathrm{M}^*/p\mathrm{M}^* \to \mathrm{Sat}(\mathrm{M}^*)/p\mathrm{Sat}(\mathrm{M}^*).$$

Furthermore if each of the group M^* *is p-adically complete, then the map*

$$\mathrm{M}^* \to \mathcal{W}\mathrm{Sat}(\mathrm{M}^*)$$

is a quasi-isomorphism.

Proof. The first claim follows by the preceding discussion. By the hypothesis that M^* is p-adically complete, we need only prove that

$$\mathrm{M}^*/p\mathrm{M}^* \to \mathcal{W}\mathrm{Sat}(\mathrm{M}^*)/p\mathcal{W}\mathrm{Sat}(\mathrm{M}^*)$$

is a quasi-isomorphism. We factor this as

$$\mathrm{M}^*/p\mathrm{M}^* \to \mathrm{Sat}(\mathrm{M}^*)/p\mathrm{Sat}(\mathrm{M}^*) \to \mathcal{W}\mathrm{Sat}(\mathrm{M}^*)/p\mathcal{W}\mathrm{Sat}(\mathrm{M}^*);$$

the first map is a quasi-isomorphism by the first claim. The second map is a map between two saturated Dieudonné complexes, hence to check the desired quasi-isomorphism we need only check that the map

$$\mathcal{W}_1(\mathrm{Sat}(\mathrm{M}))^* \to \mathcal{W}_1(\mathcal{W}(\mathrm{Sat}(\mathrm{M}))^*$$

is an isomorphism which we have already seen. □

Proof of Theorem 1.3.1. After the above theorem, Theorem 1.3.1 essentially follows from noting that $\widehat{\Omega}^*_{\mathrm{R}}$ is Cartier smooth, which we have already seen. Indeed, it suffices, after Lemma 1.3.2 to prove that

$$\widehat{\Omega}^*_{\mathrm{R}} \to \mathcal{W}\mathrm{Sat}(\widehat{\Omega}^*_{\mathrm{R}})$$

is a quasi-isomorphism but this is a special case of the above theorem. □

Next, we formulate and sketch a proof of the de Rham comparison. In this situation, we begin not with a lift but R merely a $\mathbb{F}_p$-algebra. We have a tautological map $\mathrm{R} \to \mathcal{W}_1\Omega^*_{\mathrm{R}}$ which induces a map

$$\nu_{\mathrm{R}} : \Omega^*_{\mathrm{R}} \to \mathcal{W}_1\Omega^*_{\mathrm{R}},$$

by universal properties of the de Rham complex.

Theorem 1.3.9. [BLM21, Theorem 4.3.1, de Rham comparison] *Let* R *be a regular noetherian* $\mathbb{F}_p$*-algebra, then we have a canonical isomorphism*

$$\nu_{\mathrm{R}} : \Omega^*_{\mathrm{R}} \to \mathcal{W}_1\Omega^*_{\mathrm{R}}.$$

Proof. We break this proof into steps:

(1) starting life with R being smooth over a perfect field κ we may choose a formally smooth, p-complete lift A of R with a lift of the Frobenius φ. We then use Lemma 1.3.2 (rather, the discussion preceding it) to furnish a map

$$\mu : \widehat{\Omega}^*_{\mathrm{A}} \to \mathcal{W}\Omega^*_{\mathrm{R}},$$

whose degree zero part fits into:

$$\begin{array}{ccccc} A & \longrightarrow & \widehat{\Omega}^0_A & \xrightarrow{\mu} & \mathcal{W}\Omega^0_R \\ \downarrow & & & & \downarrow \\ A/p & & \longrightarrow & & \mathcal{W}_1\Omega^0_R. \end{array}$$

(2) From the above commutativity we conclude that we may factors ν_R as

$$\Omega^*_R \cong \widehat{\Omega}^*_A/p\widehat{\Omega}^*_A \xrightarrow{\nu/p} \mathcal{W}\Omega^*_R/p\mathcal{W}\Omega^*_R \to \mathcal{W}_1\Omega^*_R.$$

This makes our lives better.

(3) The map μ and the naturality of the Cartier map and the Frobenius on the de Rham-Witt complex induces a square

$$\begin{array}{ccc} \widehat{\Omega}^*_A/p\widehat{\Omega}^*_A & \longrightarrow & H^*(\widehat{\Omega}^*_A/p\widehat{\Omega}^*_A) \\ \downarrow & & \downarrow \\ \mathcal{W}\Omega^*_R/p\mathcal{W}\Omega^*_R & \longrightarrow & H^*(\mathcal{W}\Omega^*_R/p\mathcal{W}\Omega^*_R); \end{array}$$

but we can stick in ν_R as

$$\begin{array}{ccccc} \widehat{\Omega}^*_A/p\widehat{\Omega}^*_A & & \longrightarrow & & H^*(\widehat{\Omega}^*_A/p\widehat{\Omega}^*_A) \\ \downarrow & \overset{\nu}{\searrow} & & & \downarrow \\ \mathcal{W}\Omega^*_R/p\mathcal{W}\Omega^*_R & \longrightarrow & \mathcal{W}_1\Omega^*_R & \longrightarrow & H^*(\mathcal{W}\Omega^*_R/p\mathcal{W}\Omega^*_R). \end{array}$$

But now we have the Cartier isomorphism which is an isomorphism on the top horizontal map, the lifting comparison isomorphism for the right vertical map and the isomorphism at the bottom by virtue of the saturatedness of the de Rham-Witt complex.

(4) The general case follows by Néron-Popescu after noting that the formation of the de Rham-Witt complex preserves filtered colimits [BLM21, Corollary 4.3.5].

□

Remark 1.3.10. Theorem 1.3.9 says that whenever R is a regular $\mathbb{F}_p$-algebra, we have an isomorphism

$$\mathcal{W}_1\Omega^0_R = \mathcal{W}\Omega^0_R/V\mathcal{W}\Omega^0_R \cong \Omega^0_R = R.$$

But now we know from Lemma 1.0.8 (rather, its proof) that $\mathcal{W}\Omega^0_R \cong W(\mathcal{W}\Omega^0_R/V\mathcal{W}\Omega^0_R) = W(R)$. This is not obvious from just the construction of the de Rham witt complex.

References

[BLM21] B. Bhatt, J. Lurie, and A. Mathew, *Revisiting the de Rham–Witt complex*, no. 424, 2021, `https://doi.org/10.24033/ast`

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