LECTURE 6: THE ONE IN WHICH WE HIT THE SLOPES

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1. The slope spectral sequence

Let us now fix a perfect field κ of characteristic $p > 0$. As a result of what we have done, for any smooth κ-scheme X we obtain from the **slope filtration**:

$$\mathrm{Fil}^{\geqslant *}_{\mathrm{slope}}\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) := \mathrm{R}\Gamma_{\mathrm{Zar}}(\mathrm{X}; \mathrm{W}\Omega^{\geqslant *}_{\mathrm{X}}) \to \mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}),$$

the **slope spectral sequence**:

$$\mathrm{E}_1^{ij} = \mathrm{H}^j(\mathrm{X}; \mathrm{W}\Omega^i)(=: \mathrm{H}^i(\mathrm{W}\Omega^j_{\mathrm{X}})) \Rightarrow \mathrm{H}^{i+j}_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}); \tag{1.0.1}$$

the differentials have bidegree $d_r = (r, 1-r)$. This spectral sequence displays as:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^{i+2}(\mathrm{W}\Omega^{j-1}_{\mathrm{X}}) \arrow[r, "d_1"] \arrow[drr, "d_2"] & \mathrm{H}^{i+2}(\mathrm{W}\Omega^{j}_{\mathrm{X}}) \arrow[r, "d_1"] & \mathrm{H}^{i+2}(\mathrm{W}\Omega^{j+1}_{\mathrm{X}}) \arrow[r, "d_1"] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^{i+1}(\mathrm{W}\Omega^{j-1}_{\mathrm{X}}) \arrow[r, "d_1"] \arrow[drr, "d_2"] & \mathrm{H}^{i+2}(\mathrm{W}\Omega^{j}_{\mathrm{X}}) \arrow[r, "d_1"] & \mathrm{H}^{i+1}(\mathrm{W}\Omega^{j+1}_{\mathrm{X}}) \arrow[r, "d_1"] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^{i}(\mathrm{W}\Omega^{j-1}_{\mathrm{X}}) \arrow[r, "d_1"] & \mathrm{H}^{i}(\mathrm{W}\Omega^{j}_{\mathrm{X}}) \arrow[r, "d_1"] & \mathrm{H}^{i}(\mathrm{W}\Omega^{j+1}_{\mathrm{X}}) \arrow[r, "d_1"] & \cdots
\end{tikzcd}$$

As usual we have the induced filtration

$$\mathrm{Fil}^{\geqslant j}\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) := \mathrm{Im}(\mathrm{H}^i(\mathrm{X}, \mathrm{W}\Omega^{\geqslant j}_{\mathrm{X}}) \to \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}));$$

with the graded pieces being

$$\mathrm{Fil}^{\geqslant j}\mathrm{H}^i/\mathrm{Fil}^{\geqslant j+1}\mathrm{H}^i = \mathrm{E}^{j, i-j}_{\infty}.$$

The following theorem is one of the key points of the slope spectral sequence

Theorem 1.0.2 (Illusie, after Bloch)**.** *Assume further that* X *is proper. For all* $r \geqslant 1$*, the differentials* $d_r \otimes \mathrm{K}$ *are all zero.*

We will briefly discuss a proof of the above theorem after we have been introduced into the formalism of slopes. From now on we will use the following notation for the **Hodge-Witt cohomology groups**:

$$\mathrm{H}^i(\mathrm{W}\Omega^j_{\mathrm{X}}) := \mathrm{H}^i(\mathrm{X}; \mathrm{W}\Omega^j_{\mathrm{X}}),$$

and similar such notation for cohomology with values in the sheaves $\mathrm{W}_r\Omega^j$; we abbreviate crystalline cohomology via:

$$\mathrm{H}^i(\mathrm{X}) := \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}).$$

1.1. **Some basic analysis.** One of the early achievements of crystalline cohomology are results of Nygaard [Nyg79b, Nyg79a]:

(1) if X is a proper, smooth variety over $\mathbb{Q}$ we have seen that the Hodge-to-de Rham spectral sequence degenerates. An older incarnation of this is the fact that "all regular 1-forms on X" are closed: the map
$$d : \mathrm{H}^0(\mathrm{X}; \Omega^1_{\mathrm{X}}) \to \mathrm{H}^0(\mathrm{X}, \Omega^2_{\mathrm{X}})$$
is zero. This turns out to not work out in positive characteristics and, when X is a surface, controls Hodge-to-de Rham degeneration. In [Nyg79a], Nygaard proved that as soon as we know that $\underline{\mathrm{Pic}}(\mathrm{X})$ is reduced, we actually *do* have this degeneration.

(2) Nygaard further proved in [Nyg79b] that a K3 surface over an algebraically closed field of characteristic $p > 0$ has no global vector fields, i.e.,
$$\mathrm{H}^0(\Omega^1_{\mathrm{X}}) = 0.$$
This result was proved by Rudakov and Shafarevich first.

We will not try to explain his proofs, but use it as an excuse to perform some basic analysis about the crystalline cohomology of low dimensional varieties using the slope spectral sequence. For a surface the slope spectral sequence looks like (and there is no more space for groups or differentials):

$$\begin{tikzcd}
\mathrm{H}^2(\mathrm{W}\mathcal{O}_{\mathrm{X}}) \arrow[r] \arrow[drr] & \mathrm{H}^2(\mathrm{W}\Omega^1_{\mathrm{X}}) \arrow[r] & \mathrm{H}^2(\mathrm{W}\Omega^2_{\mathrm{X}}) \\
\mathrm{H}^1(\mathrm{W}\mathcal{O}_{\mathrm{X}}) \arrow[r] \arrow[drr] & \mathrm{H}^1(\mathrm{W}\Omega^1_{\mathrm{X}}) \arrow[r] & \mathrm{H}^1(\mathrm{W}\Omega^2_{\mathrm{X}}) \\
\mathrm{H}^0(\mathrm{W}\mathcal{O}_{\mathrm{X}}) \arrow[r] & \mathrm{H}^0(\mathrm{W}\Omega^1_{\mathrm{X}}) \arrow[r] & \mathrm{H}^0(\mathrm{W}\Omega^2_{\mathrm{X}})
\end{tikzcd}$$

Let's investigate H^1. In fact, the bottom row's differentials are all zero (of course the next result works for more than just surfaces)

Lemma 1.1.1. *For any smooth κ-variety* X*, the maps* $d_1 : \mathrm{H}^0(\mathrm{W}\Omega^j_{\mathrm{X}}) \to \mathrm{H}^0(\mathrm{W}\Omega^{j+1}_{\mathrm{X}})$ *are all zero.*

Proof. This is a combination of two facts: that $d_1 \otimes \mathrm{K}$ is zero by Theorem 1.0.2 and that the 0-th cohomology groups are torsion free (only thing we need to check is that they are p-torsion free). But, by construction these $\mathrm{W}\Omega^j$'s are p-torsion free on each affine open of X and thus the global sections are p-torsion free (since taking global sections is a left exact functor and preserves injective maps!). □

From this display, we get:

Lemma 1.1.2. *For any κ-smooth, proper variety* X*,* $\mathrm{H}^0(\mathrm{W}\mathcal{O}_{\mathrm{X}}) \cong \mathrm{H}^0(\mathrm{X})$ *which computes* $\mathrm{W}^{\pi_0(\mathrm{X})}$*, where* $\pi_0(\mathrm{X})$ *is the set of geometric connected components of* X*.*

Now let us examine $\mathrm{H}^1(\mathrm{W}\mathcal{O}_{\mathrm{X}})$. The first nontrivial result is the claim that
$$\mathrm{H}^1(\mathrm{W}\mathcal{O}_{\mathrm{X}}) \to \mathrm{H}^1(\mathrm{W}\Omega^1_{\mathrm{X}})$$
is still always zero whenever X is furthermore proper; **let's assume this for the rest of these notes**. By the way this already gives us:

Lemma 1.1.3. *If* X *is smooth and proper, then* $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{X}/\mathrm{W})$ *is a perfect complex of* W*-modules. In particular, the cohomology groups are finitely generated* W*-modules.*

Proof. It suffices to prove that $\mathrm{R\Gamma}_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) \otimes^{\mathrm{L}} \mathrm{W}/p$ is perfect. We claim that: $\mathrm{R\Gamma}_{\mathrm{dR}}(\mathrm{X}_\kappa) \simeq \mathrm{R\Gamma}_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) \otimes^{\mathrm{L}} \mathrm{W}/p$. Indeed, recall that a morphism of saturated Dieudonné complexes is a mod-p quasi-isomorphism if and only if the map on $\mathcal{W}_1$ is an isomorphism [BLM21, Corollary 2.7.4]. Along the same lines, one can prove that [BLM21, Remark 2.7.3] for any such complex, $\mathrm{M}/p \simeq \mathcal{W}_1(\mathrm{M})^*$ is a quasi-isomorphism. Hence, the claim follows from de Rham comparison. Finally, the result follows from perfectness of de Rham cohomology which is well-known. □

Remark 1.1.4. We also have Poincaré duality [Ber74, Théorème VII.2.3]. Again X is smooth and proper of dimension d and furthermore geometrically connected, then we have a trace map (which is an isomorphism) [Ber74, Proposition 2.1]

$$\mathrm{Tr} : \mathrm{H}^{2d}(\mathrm{X}) \to \mathrm{W}.$$

This induces a perfect pairing

$$\mathrm{H}^i(\mathrm{X}) \otimes \mathrm{H}^{2d-i}(\mathrm{X}) \to \mathrm{W}.$$

These results can be proved by de Rham comparison as in Lemma 1.1.3.

This is a combination of two results. First, we have the following result due to Serre [Ser58, Proposition 4]:

Lemma 1.1.5 (Serre)**.** *The* W*-module* $\mathrm{H}^1(\mathrm{W}\mathcal{O}_\mathrm{X})$ *is free of finite rank.*

Proof. This is the analog of the following fact: if X is a manifold (or anytime singular cohomology is computed via sheaf cohomology [Pet22]), we have that $\mathrm{H}^1(\mathrm{X};\mathbb{Z})$ is torsion-free. Indeed, the short exact sequence of sheaves $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$ induces an exact sequence

$$\mathrm{H}^0(\mathrm{X};\mathbb{Z}) \to \mathrm{H}^0(\mathrm{X};\mathbb{Z}/n) \to \mathrm{H}^1(\mathrm{X};\mathbb{Z})[n] \to 0;$$

but then the first map is actually surjective since they compute connected components.

Now the same proof sort of works: we have an exact sequence of sheaves

$$0 \to \mathrm{W}\mathcal{O} \xrightarrow{\mathrm{V}^k} \mathrm{W}\mathcal{O} \to \mathrm{W}_k\mathcal{O} \to 0,$$

which leads us to examine

$$\mathrm{H}^0(\mathrm{W}\mathcal{O}_\mathrm{X}) \to \mathrm{H}^0(\mathrm{W}_k\mathcal{O}_\mathrm{X}) \to \mathrm{H}^1(\mathrm{W}\mathcal{O}_\mathrm{X})[\mathrm{V}^k] \to 0,$$

and the first map is surjective since H^0 computes connected components. This means that $\mathrm{H}^1(\mathrm{W}\mathcal{O}_\mathrm{X})$ is actually V^k-torsionfree.

We want to now conclude that it is p^k-torsion free as well. Let us assume that the finiteness is true (see [Ill79, Proposition II 2.17] for a proof). This means that the module of interest has bounded V^∞-torsion. Since $p = \mathrm{FV}$ we have that

$$\mathrm{H}^1(\mathrm{W}\mathcal{O}_\mathrm{X})[\mathrm{V}^k] \subset \mathrm{H}^1(\mathrm{W}\mathcal{O}_\mathrm{X})[p^k];$$

the goal is to prove that these two subgroups are equal for $k = \infty$ (which means that it is actually equal for some finite k since we have assumed finiteness). For $k = \infty$, we write the inclusion above as $\mathrm{T} \subset \mathrm{T}'$ and it suffices to prove that $\mathrm{V}^\mathrm{M}\mathrm{T}' = 0$ for some $\mathrm{M} \gg 0$. We have an induced injective map

$$\mathrm{V} : \mathrm{T}'/\mathrm{T} \to \mathrm{T}'/\mathrm{T}.$$

But since T'/T is a finitely generated free W-module, the map above is an isomorphism. We can thus write

$$\mathrm{T}' = \mathrm{V}\mathrm{T}' + \mathrm{T},$$

whence $\mathrm{V}^\mathrm{N}\mathrm{T}' = \mathrm{V}^{\mathrm{N}+1}\mathrm{T}' = \cdots$ for N big enough such that $\mathrm{V}^\mathrm{N}\mathrm{T} = 0$. We have the the V-filtration is separated, i.e., $\cap \mathrm{V}^k\mathrm{H}^1(\mathrm{W}\mathcal{O}_\mathrm{X}) = 0$ [Ill79, Corollaire II 2.5]. Thus we must have that $\mathrm{V}^\mathrm{N}\mathrm{T} = 0$ and therefore $\mathrm{T}' \subset \mathrm{T}$. □

Next, Nygaard gave a criterion for when d is zero using structural features of these Hodge-Witt cohomology groups.

Lemma 1.1.6. [Nyg79a, Lemma 2.5] *Let $d : \mathrm{L} \to \mathrm{M}$ be a map of* W*-modules and* F *(resp.* V*) is a φ-linear (resp. φ^{-1}-linear) endomorphism of* M *(resp.* L*). Equip,* L *and* M *with topologies compatibly with* W *such that d is continuous, the one on* M *is separated and the topology on* L *is weaker than the* V*-adic topology. If $\mathrm{F}d\mathrm{V} = d$ and the chains*

$$\ker d \subset \ker \mathrm{F}d \subset \cdots \ker \mathrm{F}^n d \subset \cdots \mathrm{L}$$

$$\mathrm{Im}\, d \subset \mathrm{Im}\, \mathrm{F}d \subset \cdots \mathrm{Im}\, \mathrm{F}^n d \subset \cdots \mathrm{M}$$

stabilize then $d = 0$.

Serre's lemma proves that the chain in the domain stabilizes. To apply Nygaard's criterion we examine

$$\mathrm{Im}\, d \subset \mathrm{Im}\, \mathrm{F}d \subset \cdots \mathrm{Im}\, \mathrm{F}^n d \subset \cdots \subset \mathrm{H}^1(\mathrm{W}\Omega^1_{\mathrm{X}})$$

We first observe that $\mathrm{E}_2^{1,1} = \mathrm{E}_\infty^{1,1} = \ker(d : \mathrm{H}^1(\mathrm{W}\Omega^1_{\mathrm{X}}) \to \mathrm{H}^1(\mathrm{W}\Omega^2_{\mathrm{X}}))/\,\mathrm{Im}\, d$; this is a subquotient of $\mathrm{H}^2(\mathrm{X})$ and is thus finitely generated. We further have that $\mathrm{Im}\, \mathrm{F}^n d \subset \ker d$ for all n and thus we have a chain

$$\mathrm{Im}\, \mathrm{F}d/\,\mathrm{Im}\, d \subset \mathrm{Im}\, \mathrm{F}^2 d/\,\mathrm{Im}\, d \subset \cdots \mathrm{Im}\, \mathrm{F}^n d/\,\mathrm{Im}\, d \subset \cdots \mathrm{E}_\infty^{1,1}$$

which does stabilize by the finite generation of $\mathrm{E}_\infty^{1,1}$. We conclude:

Proposition 1.1.7. *The group $\mathrm{H}^1(\mathrm{X})$ always decomposes as*

$$0 \to \mathrm{H}^0(\mathrm{W}\Omega^1_{\mathrm{X}}) \to \mathrm{H}^1(\mathrm{X}) \to \mathrm{H}^1(\mathrm{W}\mathcal{O}_{\mathrm{X}}) \to 0.$$

One conclusion is that:

Corollary 1.1.8. *Assume further that* X *is proper, the* W*-module $\mathrm{H}^1_{\mathrm{crys}}(\mathrm{X}/\mathrm{W})$ is always torsionfree.*

Hence, for X a smooth proper, geometrically connected curve: we get that

$$\mathrm{H}^0(\mathrm{X}) \cong \mathrm{H}^0(\mathrm{W}\Omega_{\mathrm{X}});$$

$$\mathrm{H}^2(\mathrm{X}) \cong \mathrm{H}^1(\mathrm{W}\Omega^1_{\mathrm{X}});$$

$$0 \to \mathrm{H}^0(\mathrm{W}\Omega^1_{\mathrm{X}}) \to \mathrm{H}^1(\mathrm{X}) \to \mathrm{H}^1(\mathrm{W}\mathcal{O}_{\mathrm{X}}) \to 0. \tag{1.1.9}$$

By Poincaré duality for crystalline cohomology such that H^0 is dual to H^2 in this case, mutually isomorphic to W.

For a surface, Nygaard analyzed that the only possible differential which is nonzero is

$$\mathrm{H}^2(\mathrm{W}\mathcal{O}_{\mathrm{X}}) \to \mathrm{H}^2(\mathrm{W}\Omega^1_{\mathrm{X}}),$$

which will be zero as soon as we know that it is finitely generated. In fact, the slope spectral sequence is controlled by what happens at E_1: for its collapse it suffices to prove that the d_1-differentials are zero [Nyg79a, Proposition 2.2].

2. Slopes and applications

Let us fix some notation:

1. let κ be a perfect field of characteristic $p > 0$;
2. we write $\varphi : (\mathrm{W} :=)\mathrm{W}\kappa \to \mathrm{W}\kappa$ to be the Witt-vector Frobenius just to limit the confusion with all the Frobenii floating around;
3. We write $\mathrm{K} := \mathrm{W}[\frac{1}{p}]$, the fraction field of W.

An **F-isocrystal** is a pair (M, F) where

1. M is a finitely generated W-module;
2. $\mathrm{F} : \mathrm{M} \to \mathrm{M}$ a φ-linear endomorphism of M such that $\mathrm{F}[\frac{1}{p}]$ is invertible, i.e., a map

   $$\varphi_* \mathrm{M} \xrightarrow{\mathrm{F}} \mathrm{M}$$

   which is an invertible after $\otimes_{\mathrm{W}}\mathrm{K}$.

The category of F-isocrystals will be denoted by $\mathbf{Isoc}_{\mathrm{F}}(\kappa)$; maps are appropriate commutative diagrams. We might also consider the **isogeny category** of isocrystals $\mathbf{Isoc}_{\mathrm{F}}(\kappa)_{\mathbb{Q}}$: this is the localization at maps of isocrystals which are isomorphisms after inverting p.

Remark 2.0.1. We have followed the definition given by Katz in [Kat79, Basic Definition]. This might differ with what other people might mean by an isocrystal: some people say that such a thing is a finite dimensional F-vector space V equipped with a Frobenius-semilinear map F which is also a bijection. In other words, the datum of the lattice is not accounted for. This latter notion is closer to the isogeny category.

Let us give examples of some F-isocrystals

Example 2.0.2. We can make one up: let $\lambda = \frac{n}{m}$ be a fraction expressed in its lowest term. Write

$$\mathrm{M}(\lambda) = \mathrm{W}[\mathrm{T}]/(\mathrm{T}^m - p^n) \cong \mathbb{Z}_p[\mathrm{T}]/(\mathrm{T}^m - p^n) \otimes_{\mathbb{Z}_p} \mathrm{W}.$$

The F is given by multiplication by $\cdot \mathrm{T} \otimes \varphi$. Explicitly: we can choose a W-basis for $\mathrm{M}(\frac{n}{m})$ given by $\{1, \cdots, \mathrm{T}^{m-1}\}$ so that the action of F is given by:

$$\mathrm{F}(x_1, \cdots, x_m) = (p^n\varphi(x_m), \varphi(x_1), \cdots, \varphi(x_{m-1})).$$

Example 2.0.3. Let X be a smooth and projective. The absolute Frobenius on X given by $\mathrm{F}_{\mathrm{X}} : \mathrm{X} \to \mathrm{F}_{\mathrm{X}}$ induces the structure on an isocrystal on $\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W})$ for all $i \geqslant 0$; we denote the endomorphism by

$$\Phi_{\mathrm{X}} : \varphi_*\mathrm{R}\Gamma(\mathrm{X}/\mathrm{W}) \to \mathrm{R}\Gamma(\mathrm{X}/\mathrm{W}) \qquad \Phi^i_{\mathrm{X}} : \varphi_*\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) \to \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}).$$

We note that Φ_{X} differs from the "internal" or "Dieudonné-theoretic" Frobenius by exactly p^i in degree i:

$$\Phi_{\mathrm{X}} = p^i\mathrm{F} : \mathrm{W}\Omega^i_{\mathrm{X}} \to \mathrm{W}\Omega^i_{\mathrm{X}}.$$

We will see that this is a key invariant of X. In any case, Φ^i_{X} endows each $\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W})$ the structure of an F-isocrystal: the point is to check that the map Φ_{X} is injective modulo torsion as a consequence of Poincaré duality[1].

The following is a key structural result in the theory of isocrystals.

Theorem 2.0.4 (Dieudonné-Manin)**.** *Let κ be an algebraically closed field. Then the isogeny category of isocrystals is semisimple*[2] *where the simple objects*[3] *are exactly*

$$\mathrm{M}(\tfrac{n}{m}) \qquad n \text{ and } m \text{ are coprime, } n > 0.$$

In particular, any object can be written as

$$\mathrm{M} \cong \bigoplus_{\lambda \in \mathbb{Q}} \mathrm{M}(\lambda).$$

If κ was not algebraically closed and (M, F) is an isocrystal, then we can consider $\overline{\mathrm{K}}$, the fraction field of $\mathrm{W}\overline{\kappa}$ where $\overline{\kappa}$ is a fixed closure of κ. Then $(\mathrm{M} \otimes_{\mathrm{W}} \overline{\mathrm{K}}, \mathrm{F} \otimes \varphi)$ defines an F-isocrystal over $\overline{\kappa}$ up to isogeny, whence, by descent, it can be decomposed (up to isomorphism) by Theorem 2.0.4

$$\mathrm{M} \cong \bigoplus \mathrm{M}_\lambda,$$

where M_λ is the largest subobject with $\mathrm{M}_\lambda \otimes_{\mathrm{W}} \overline{\mathrm{K}} \cong \mathrm{M}(\lambda) \otimes \overline{\mathrm{K}}$.

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[1] Proof: on top cohomology, the map Φ_{X} agrees with φ the Frobenius of W. Let $x \in \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W})$ be a nonzero class, then we claim that Φ_{X} is a nonzero class; indeed by Poincaré duality we may choose β such that $\alpha \cup \beta$ is the class 1 in W but then $\Phi_{\mathrm{X}}(\alpha) \cup \Phi_{\mathrm{X}}(\beta) = \Phi_{\mathrm{X}}(\alpha \cup \beta)$ hence nonzero.

[2] A semisimple category is an abelian category where every object is a finite direct sum of simples.

[3] Recall that an object is simple if 0 and the object itself are the only quotients.

Definition 2.0.5. The **(Newton) slopes** of an F-isocrystal M is the sequence of rational numbers (defined up to isogeny)

$$(\lambda_1, \cdots, \lambda_r),$$

where

$$0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_r,$$

given by

$$(n_1/m_1, \cdots, n_1/m_1, n_2/m_2, \cdots, n_2/m_2, \cdots)$$

extracted from a decomposition of $\mathrm{M} \otimes \mathrm{W}(\overline{\kappa})$; where n_j/m_j is repeated according to the number of copies of $\mathrm{E}(n_j/m_j)$.

Definition 2.0.6. For any F-isocrystal M and any interval $\mathrm{I} \subset \mathbb{Q}$, let us write

$$\mathrm{M}_{\mathrm{I}} = \bigoplus_{\lambda \in \mathrm{I}} \mathrm{M}_{\lambda}$$

where M_{λ} is a component of slope λ.

Example 2.0.7. A **Dieudonné module** is an F-isocrystal of slope $[0, 1]$; more precisely: it is given by a finitely generated free W-module M equipped with a semilinear Frobenius endomorphsim $\mathrm{F} : \mathrm{M} \to \mathrm{M}$ such that $p\mathrm{M} \subset \mathrm{FM}$ such that F is an isomorphism after inverting p. In particular V is defined on such objects and we have that $\mathrm{VF} = \mathrm{FV} = p$ just like the definition of a Dieudonné complex. There is an equivalence of categories between p-divisible formal groups and Dieudonné modules [Gro74]. This equivalence, under the functor called M, can be described explicitly in some cases:

(1) if G is the the p-divisible group associated to the p^{∞}-torsion of an abelian scheme, then it is precisely given by $\mathrm{H}^1_{\mathrm{crys}}(\mathrm{A}/\mathrm{W})$ [BBM82, Théorme 2.5.6];
(2) if $\mathrm{G} = \mathbb{Q}_p/\mathbb{Z}_p$, then $\mathrm{M}(\mathrm{G}) = \mathrm{W}$ with $\mathrm{F} = p \cdot \varphi$ [BBM82, Proposition 4.2.1.6];
(3) if $\mathrm{G} = \mu_{p^{\infty}}$, then $\mathrm{M}(\mathrm{G}) = \mathrm{W}$ with $\mathrm{F} = \varphi$ [BBM82, Proposition 4.2.1.6];

The formalism of slopes helps us collapse the slope spectral sequence. This is not immediate and it requires the following input:

(1) the action of V is topologically nilpotent on $\mathrm{H}^j(\mathrm{X}; \mathrm{W}\Omega^i)$ [Ill79, Corollaire II.2.5];
(2) $\mathrm{H}^j(\mathrm{X}; \mathrm{W}\Omega^i)$ modulo p^{∞}-torsion is free of finite type [Ill79, Théoromè II.2.13].

What goes into the collapse is to use a modified version of the operator V on each term of the spectral sequence which turns out to have different slopes: the differentials cannot ever preserve slopes for combinatorial reasons. The full argument is carried out in Bloch's paper [Blo77]. For us, the key result about crystals is the following:

Theorem 2.0.8. [Ill79, Corollaire 3.5] *Let* X *be a smooth proper* κ*-variety. Then, the canonical map* $\mathrm{H}^*_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) \to \mathrm{H}^*(\mathrm{X}; \mathrm{W}\Omega^{\leqslant i})$ *induces an isomorphism:*

$$\mathrm{H}^*_{\mathrm{crys}}(\mathrm{X}/\mathrm{W})_{[0,i[} \to \mathrm{H}^*(\mathrm{W}\Omega^{\leqslant i-1}_{\mathrm{X}}).$$

Example 2.0.9. Let us try to understand the isocrystal structure of an elliptic curve. First, we have that, abstractly: $\mathrm{H}^1(\mathrm{X}) \cong \mathrm{W} \oplus \mathrm{W}$ just like in usual algebraic topology. The exact sequence (1.1.9) rationally decomposes $\mathrm{H}^1(\mathrm{X})$ into $\mathrm{H}^1(\mathrm{W}\mathcal{O}_{\mathrm{X}})$ and $\mathrm{H}^0(\mathrm{W}\Omega^1_{\mathrm{X}})$; in characteristic zero we have that

$$h^{1,0} = h^{0,1} = 1,$$

However, it turns out that an elliptic curve in positive characteristics can have either:

(ordinary) $h^{0,1} = \dim_{\mathrm{W}}(\mathrm{H}^1(\mathrm{W}\mathcal{O}_{\mathrm{X}})) = 1$, Frobenius acts by p^0; $h^{1,0} = \dim_{\mathrm{W}}(\mathrm{H}^0(\mathrm{W}\Omega^1_{\mathrm{X}})) = 1$, Frobenius acts by p
(supersingular) $h^{0,1} = 2$, $h^{1,0} = 0$ and $\mathrm{H}^1(\mathrm{X}) \cong \mathrm{H}^1(\mathrm{X})_{[1/2]}$.

2.1. **An application: Esnault's theorem.** We now discuss one of the best results of all time, it is essentially a one page paper in Inventiones [Esn03]. Let $\kappa = \mathbb{F}_q$, $q = p^a$. The following is a simple, yet very powerful, observation.

Lemma 2.1.1. *Assume that* X *is geometrically connected, proper and smooth over* κ *and that*

$$\mathrm{H}^i(\mathrm{X}; \mathrm{W}\mathcal{O}) = 0 \qquad i > 0.$$

Then:

$$|\mathrm{X}(\kappa)| \equiv 1 \qquad \bmod p.$$

In particular a κ *rational point must exist.*

Proof. We have the Lefschetz trace formula for crystalline cohomology [É88, Théorème 1.6]. In our case, this says that:

$$|\mathrm{X}(\kappa)| = \sum (-1)^i \mathrm{Trace}(\mathrm{Frob}^a | \mathrm{H}^i(\mathrm{X}) \otimes \mathrm{K}).$$

Hence if $\mathrm{H}^i(\mathrm{X}; \mathrm{W}\mathcal{O}) = 0$, then Theorem 2.0.8 says that all Frobenius eigenvalues have positive p-adic valuations, i.e., no slope zero part. Furthermore since X is geometrically connected, the action of the Frobenius is trivial. Therefore, except for cohomological degree zero part, every other eigenvalues are divisible by p. Hence, we get the result. □

Remark 2.1.2. By using the Riemann hypothesis for crystalline cohomology [KM74], we get an expression for the zeta function of X:

$$\exp\Big(\sum_{n>0} |\mathrm{X}(\mathbb{F}_{q^n})| \tfrac{t^n}{n}\Big) = \zeta_{\mathrm{X}}(t) = \prod_{0 \leqslant i \leqslant \dim(\mathrm{X})} \det(1 - \mathrm{F}^a | \mathrm{H}^i(\mathrm{X}) \otimes \mathrm{K})^{(-1)^{i+1}}.$$

This gives a slightly stronger result: for any finite extension $\mathbb{F}_{q^n}$ of κ, we have

$$|\mathrm{X}(\mathbb{F}_{q^n})| \equiv 1 \quad \bmod p \qquad \forall n.$$

Recall that a Fano variety over a field κ is a geometrically connected, projective, smooth variety whose dualizing sheaf ω is antiample (that is to say, $-\omega_{\mathrm{X}}$ is ample).

Theorem 2.1.3 (Esnault; Lang's conjecture)**.** *Let* X *be a Fano variety over a finite field* $\kappa = \mathbb{F}_{q=p^n}$*. Then* X *admits a rational point.*

Proof. We begin with a (substantial) result of Kollar, Miyaoka and Mori [KMM92] and Campana [Cam92]: for any field k, a Fano variety is **rationally chain connected**: roughly, over a closure any two points can be connected by a chain of rational curves. This has the following consequence: X has a zero cycle of degree one and $\mathrm{CH}_0(\mathrm{X} \times_\kappa \overline{k(\mathrm{X})})$ is $\mathbb{Z}$ via the degree map (in fact this is true for L in place of $\overline{k(\mathrm{X})}$ for L algebraically closed). This forces (rational) decomposition of the diagonal [BS83]; see Lemma 3.0.8:

$$\mathrm{N}[\Delta_{\mathrm{X}}] = \alpha \times \mathrm{X} + \Gamma \in \mathrm{CH}_{\dim(\mathrm{X})}(\mathrm{X} \times \mathrm{X})$$

where ξ is zero cycle of degree one, and Z is narrow (as defined below); say Z is supported on $\mathrm{X} \times \mathrm{D}$ where $\mathrm{D} \hookrightarrow \mathrm{X}$ is a proper closed reduced subscheme.

There is the "usual formalism" of correspondences acting on crystalline cohomology and the action of Δ, as a correspondence, is given by Γ_* up to torsion (because the action of $\alpha \times \mathrm{X}$ is trivial since it factors through the crystalline cohomology noting that α is supported in subscheme of dimension zero). But now Γ_* applied to $\mathrm{H}^i(\mathrm{X}) \otimes \mathrm{K}$ sends this group to the kernel of the map $\mathrm{H}^i(\mathrm{X}) \otimes \mathrm{K} \to \mathrm{H}^i(\mathrm{X} \smallsetminus \mathrm{D}) \otimes \mathrm{K}$ and hence, the image of Γ_* is in the image of the pushforward $\mathrm{H}^i_{\mathrm{D}}(\mathrm{X}) \otimes \mathrm{K} \to \mathrm{H}^i(\mathrm{X}) \otimes \mathrm{K}$. But now we claim that the Frobenius acts on $\mathrm{H}^i_{\mathrm{D}}(\mathrm{X}) \otimes \mathrm{K}$ acts by slopes $\geqslant 1$; in fact if Z is any non-empty closed subvariety of dimension $\geqslant 1$ then $\mathrm{H}^i_{\mathrm{Z}}(\mathrm{X})$ enjoys this property. Indeed, if Z is smooth and codimension c, then we have the purity isomorphism $\mathrm{H}^i_{\mathrm{Z}}(\mathrm{X}) \otimes \mathrm{K} \cong \mathrm{H}^{i-c}(\mathrm{Z}) \otimes \mathrm{K}$ which intertwines Frob with $p^c\mathrm{Frob}$[4]; since $c \geqslant 1$ the slopes must be $\geqslant 1$ since the slopes of $\mathrm{H}^*(\mathrm{Z})$ must be $\geqslant 0$. In general, one carries out a standard stratification argument noting that in the argument, the slopes cannot decrease.

□

3. Digression: decomposition of the diagonal

Let k be a field; unless otherwise stated every variety here is smooth and proper. We say that X is **universally** CH_0**-trivial** if the map

$$\deg : \mathrm{CH}_0(\mathrm{X}_\mathrm{F}) \to \mathbb{Z}$$

is an isomorphism for any field extension F of k. Here deg is the map that sends

$$\sum n_i[p_i] \mapsto \sum n_i \deg(k(p_i)/k).$$

It is useful to adopt the following notation as well for any variety X:

$$\mathrm{CH}_0(\mathrm{X})^0 := \ker(\deg) \subset,$$

the group of **zero cycles of degree zero**. So X is universally CH_0-trivial whenever $\mathrm{CH}_0(\mathrm{X})^0 = 0$.

Example 3.0.1. If $\mathrm{X} = \operatorname{Spec} \mathrm{L} \to \operatorname{Spec} k$ is a finite Galois extension, then $\mathrm{CH}_0(\mathrm{X}_\mathrm{L}) \cong \bigoplus_{|\mathrm{G}|} \mathbb{Z}$ so it is not universally CH_0-trivial.

Example 3.0.2. Since $\mathrm{CH}_0(\mathbb{P}^n_\mathrm{K})$ for any field K is isomorphic to $\mathbb{Z}$ via the degree map, projective space are universally

Being universally CH_0-trivial is one of those conditions where one can check "by hand." Here's one way to verify this condition:

Lemma 3.0.3. *Assume further that* X *is geometrically connected. Then the following are equivalent:*

(1) X *is universally* CH_0*-trivial;*
(2) X *has a zero cycle of degree one and* $\mathrm{CH}_0(\mathrm{X} \times k(\mathrm{X}))^0 = 0$.

We will prove this lemma after we add on one more equivalent conditions in the above list. It is the condition that we are ultimately interested and is "motivic" in nature in that it has consequences across all cohomology theories that one might associate to X.

Definition 3.0.4 (Bloch-Srinivas)**.** A smooth proper variety X over k, of dimension d. We say that a cycle $\mathrm{Z} \in \mathrm{CH}_k(\mathrm{X} \times \mathrm{X})$ is **narrow** if it is supported on $\mathrm{X} \times \mathrm{V}$ for some $\mathrm{V} \hookrightarrow \mathrm{X}$ an reduced subscheme of codimension $\geqslant 1$, i.e., for some V, under the map

$$\mathrm{CH}_k(\mathrm{X} \times \mathrm{X}) \to \mathrm{CH}_k(\mathrm{X} \times \mathrm{V}) \tag{3.0.5}$$

the cycle Z is zero. We say that X has a **decomposition of the diagonal** if the following equality holds in $\mathrm{CH}_d(\mathrm{X} \times \mathrm{X})$:

$$[\Delta_\mathrm{X}] = \alpha \times \mathrm{X} + \mathrm{Z} \in \mathrm{CH}_d(\mathrm{X} \times \mathrm{X})$$

where Z is narrow and α is of degree one. We say that X has a **rational decomposition of the diagonal** if there exists an integer N such that

$$\mathrm{N}[\Delta_\mathrm{X}] = \alpha \times \mathrm{X} + \mathrm{Z} \in \mathrm{CH}_d(\mathrm{X} \times \mathrm{X})$$

where N is narrow and α is of degree one. Equivalently, the equation (3.0.5) holds in the rationalization $\mathrm{CH}_d(\mathrm{X} \times \mathrm{X})_\mathbb{Q}$.

The following theorem is due to Bloch and Srinivas [BS83].

[4]This is quite annoying to find a reference for: in [Ber97, Proposition 1.9] it was proved that rationalized crystalline cohomology for a smooth proper scheme coincides with rigid cohomology and then we have the purity isomorphism for rigid cohomology [Chi98, Theorem 2.4].

Theorem 3.0.6. *Assume further that* X *is geometrically connected. Then the following are equivalent:*

(1) X *is universally* CH_0*-trivial;*
(2) X *has a zero cycle of degree one and* $\mathrm{CH}_0(\mathrm{X} \times k(\mathrm{X}))^0 = 0$*;*
(3) X *has a decomposition of the diagonal.*

Proof. Condition (2) is a special case of (1). Let us prove that (2) implies (3). Let $\mathrm{K} := k(\mathrm{X})$ and $j : \mathrm{X}_\mathrm{K} \to \mathrm{X} \times \mathrm{X}$ be the canonical map. Then the class $j^*[\Delta_\mathrm{X}]$ and $j^*(\alpha \boxtimes [\mathrm{X}])$ where α is of degree one are both degree 1 cycles (the reader is left to check this). Therefore, by the assumption of (2), both cycles are rationally equivalent in $\mathrm{CH}_0(\mathrm{X} \times k(\mathrm{X}))$. Since:

$$\mathrm{CH}_0(\mathrm{X} \times k(\mathrm{X})) \cong \mathrm{CH}^d(\mathrm{X} \times k(\mathrm{X})) \cong \operatorname*{colim}_{\mathrm{U} \subset \mathrm{X}} \mathrm{CH}^d(\mathrm{X} \times \mathrm{U})$$

where U is open, the cycle $j^*([\Delta_\mathrm{X}] - \alpha \boxtimes [\mathrm{X}])$ is zero for some U large enough. By the localization sequence in Chow theory, it the pushforward of a cycle β where β is narrow.

Let us prove (3) implies (1). Observe that correspondences act on Chow groups in the sense that we have a map

$$\mathrm{CH}_d(\mathrm{X} \times \mathrm{X}) \to \mathrm{End}(\mathrm{CH}_0(\mathrm{X}))$$

by doing "push-and-pull" as usual in such a way that the graph of any morphism f induces the pullback of cycles. We note that Δ_X induces the identity endomorphism since it is the graph of the identity map. We claim that for any cycle $\beta \in \mathrm{CH}_0(\mathrm{X})$:

$$\beta = \deg(\beta)\alpha,$$

where α is the degree one cycle coming from assumption (3).

To this end, compute $(\alpha \boxtimes [\mathrm{X}])^*(\beta)$: write $p_1, p_2 : \mathrm{X} \times \mathrm{X} \to \mathrm{X}$ for the projections, then using (mainly) the projection formula we get:

$$(\alpha \boxtimes [\mathrm{X}])^*(\beta) = p_{1*}(\alpha \cdot p_2^*(\beta)) = \alpha \cdot p_{1*}p_2^*(\beta) = \alpha \cdot \deg(\beta).$$

To this end, we need to prove that $\mathrm{Z}^*(\beta) = 0$. We can write $[\mathrm{Z}] = i_*\mathrm{Z}'$ where Z' is supported on $\mathrm{V} \times \mathrm{X}$ and $i : \mathrm{V} \times \mathrm{X} \subset \mathrm{X} \times \mathrm{X}$ is the immersion. The key is the small lemma below. Once we have this we note that for any elementary zero cycle $[p]$ we get:

$$\mathrm{Z}_*([p]) = (i_*(\mathrm{Z}'))_*([p]) = p_{2*}([\{p\} \times \mathrm{X}] \cdot i_*(\mathrm{Z}')) = 0$$

since we may assume that

$$(\{p\} \times \mathrm{X}) \cap (\mathrm{X} \setminus \mathrm{V}) \times \mathrm{X}.$$

□

Lemma 3.0.7. *As above, for any zero cycle* $z \in \mathrm{CH}_0(\mathrm{X})$*, then for any nonempty open* $\mathrm{U} \subset \mathrm{X}$*, there exists a cycle* z' *supported away from* $\mathrm{X} \setminus \mathrm{U}$ *such that* $z' = z$ *in* $\mathrm{CH}_0(\mathrm{X})$*.*

Proof sketch. Pass a curve through z such that it touches U; this lets us assume that X is a curve. Then we can (using quasi-projectivity of X) find a function φ which has poles described by z and zero's supported on U, i.e., $\mathrm{div}(\varphi) = z' - z$. This says that in $\mathrm{CH}_0(\mathrm{X})$ we have that $z' = z$. □

All this is nice, but having a decomposition of the diagonal on the nose is a restrictive condition. So we contend ourselves with the rational version which is much easier to check:

Lemma 3.0.8. *As soon as the following holds:*

- *there is a degree one zero cycle* α *and* $\mathrm{CH}_0(\mathrm{X} \times \overline{k(\mathrm{X})}) = 0$ *where* $\overline{k(\mathrm{X})}$ *is an algebraic closure of* $k(\mathrm{X})$*,*

X *admits a rational decomposition of the diagonal.*

Proof. Given any finite, field extension F'/F we have the transfer map on zero cycles

$$\mathrm{CH}_0(X_{F'}) \to \mathrm{CH}_0(X_F);$$

such that the composite

$$\mathrm{CH}_0(X_F) \to \mathrm{CH}_0(X_{F'}) \to \mathrm{CH}_0(X_F)$$

is multiplication by the degree. Therefore, in the argument of Theorem 3.0.6 we can only conclude that there exists some integer N such that

$$j^*(N([\Delta_X] - \alpha \boxtimes [X])) = 0$$

for N large enough and U large enough. The same argument lets us conclude. □

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