LECTURE 8: THE ONE WHERE WE DO SOME GEOMETRY

ELDEN ELMANTO

1. Motivation: Weil reciprocity

Let Σ be a Riemann surface. Around a point $p \in \Sigma$ we may choose a **local parameter** which is simply a meromorphic function on Σ which has a simple zero around p; we call this π_p. Given any meromorphic function g on Σ, we may expand g around p and express itas

$$g = \sum_{m \geqslant k} a_m \pi_p^m = a_k \pi_p^k + a_{k+1}\pi_p^{k+1} + \cdots .$$

André Weil proved the following remarkable theorem which constrains the holomorphic functions that can appear on Σ. Let f, g be two rational functions, written as

$$f = a_k \pi_p^k + a_{k+1}\pi_p^{k+1} + \cdots ;$$

$$g = b_\ell \pi_p^\ell + a_{\ell+1}\pi_p^{\ell+1} + \cdots$$

then define the **local factor** at p to be

$$\partial_p(f, g) = (-1)^{k\ell} \tfrac{b_\ell^k}{a_k^\ell} \in \mathbb{C}.$$

The local factors turn out to not depend on choices and furthermore satisfy:

$$\partial_p(f, g_1 g_2) = \partial_p(f, g_1) \cdot \partial_p(f, g_2) \qquad \partial_p(f_1 f_2, g) = \partial_p(f_1, g) \cdot \partial_p(f_2, g)$$

as well as

$$\partial_p(f, 1 - f) = 1.$$

Weil then proved the following remarkable theorem:

Theorem 1.0.1 (Weil's reciprocity law)**.** *For any two nonconstant meromorphic functions f, g*

$$\prod_{p \in \Sigma} \partial_p(f, g) = 1.$$

An even more concrete result is the following consequence: let us write $\mathrm{div}(g)$ to be the formal sums of zero's and poles of a rational function; we write $f(\mathrm{div}(g))$ to be the *product* of the value of f at $\mathrm{div}(g)$ counted with multiplicities:

Corollary 1.0.2. *For any rational functions such that $\mathrm{div}(g)$ and $\mathrm{div}(f)$ have disjoint support, then $f(\mathrm{div}(g)) = g(\mathrm{div}(f))$.*

This is the most geometric, I think, of the various reciprocity laws that one encounters in arithmetic geometry. One of the points of today's lecture is to see this occur in Bloch's higher Chow groups and Milnor's version of algebraic K-theory.

2. Bloch's higher Chow groups and motivic cohomology

Our goal now is to introduce another one of the principal actors of this class: Bloch's higher Chow groups.

Definition 2.0.1. Let $q \geqslant 0$. We set

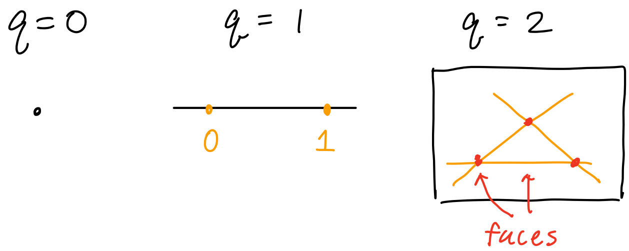
$$\Delta^q := \operatorname{Spec} \mathbb{Z}[\mathrm{T}_0, \cdots, \mathrm{T}_q] / \sum \mathrm{T}_i = 1.$$

This is a scheme, abstractly isomorphic to $\mathbb{A}^q$. For each fixed $q \geqslant 1$, looking at

$$\mathrm{V}(\mathrm{T}_i) \hookrightarrow \Delta^q \qquad i = 0, \cdots, q$$

defines $q+1$ divisors, abstractly isomorphic to $\Delta^{q-1} \cong \mathbb{A}^{q-1}$. We call an arbitrary intersections of subschemes of this form (for any $q \geqslant 1$) the **faces**. We display all of this as a (semi-)cosimplicial scheme which Δ^q in degree q, denoted by $\Delta^\bullet$.

Remark 2.0.2. Here are the first few algebraic simplices with the faces drawn on them:



We will work only with $\Delta_k^\bullet$, the base change to a field. This avoids many nightmares having to do with the right definition of dimension of intersections at different fibers.

Definition 2.0.3. If X is a k-variety and Z, W are two subvarieties of X, we say that Z interects W **properly** if every component of $\mathrm{Z} \cap \mathrm{W}$ has $\operatorname{codim}_\mathrm{X}(\mathrm{Z}) + \operatorname{codim}_\mathrm{X}(\mathrm{W})$ (note that for any component P, $\operatorname{codim}_\mathrm{X}(\mathrm{P}) \leqslant \operatorname{codim}_\mathrm{X}(\mathrm{Z}) + \operatorname{codim}_\mathrm{X}(\mathrm{W})$ always if the ambient scheme is regular). We write

$$\mathrm{Z} \pitchfork \mathrm{W},$$

whenever Z intersects W properly.

Remark 2.0.4. We give some easy examples:

(1) Say $\mathrm{X} = \mathbb{A}^2$, Z a point and W a curve. Then if $\mathrm{Z} \in \mathrm{W}$ we have that

$$\operatorname{codim}_\mathrm{X}(\mathrm{P}) = 2 < 3 = \operatorname{codim}_\mathrm{X}(\mathrm{Z}) + \operatorname{codim}_\mathrm{X}(\mathrm{W}).$$

This is not a proper intersection. The only way that a proper intersection can happen is if $\mathrm{P} = \emptyset$.

(2) If now Z and W are both curves, then if $\mathrm{Z} = \mathrm{W}$

$$\operatorname{codim}_\mathrm{X}(\mathrm{P}) = 1 < 2,$$

hence Z and W meets properly if and only if they meet at finite many points.

Construction 2.0.5. Let X be a k-variety. Then, for $j \geqslant 0$ define

$$z^j(\mathrm{X}, \bullet) := \mathbb{Z}\{\mathrm{W} \hookrightarrow \mathrm{X} \times \Delta^\bullet : \mathrm{W} \text{ is integral closed codim } j \text{ subscheme, } \mathrm{W} \pitchfork \mathrm{X} \times \mathrm{F}\}.$$

Under pullback of faces, we get a simplicial abelian group and Bloch's **higher Chow groups** is defined to be the homotopy groups (or, alternatively, the homology of the alternating sum):

$$\mathrm{H}_q(z^j(\mathrm{X}, \bullet)) := \mathrm{CH}^j(\mathrm{X}, q).$$

The following is either a definition (if you have not seen it before) or a lemma:

Lemma 2.0.6. *Let* X *be an equidimensional, reduced* k*-scheme. Then* $\mathrm{CH}^j(\mathrm{X}) \cong \mathrm{CH}^j(\mathrm{X}, 0)$.

Remark 2.0.7 (Functoriality)**.** We remark on the functoriality of Bloch's higher Chow groups. First, we have the proper pushforward: if $f : \mathrm{X} \to \mathrm{Y}$ is a proper morphism of relative dimension d, then we have a pushforward map

$$z^{j+d}(\mathrm{X}, \bullet) \xrightarrow{f_*} z^j(\mathrm{Y}, \bullet)$$

obtained by pushing-forward cycles along f. If f is a flat morphism, then we can pullback cycles

$$z^j(\mathrm{Y}, \bullet) \xrightarrow{f^*} z^j(\mathrm{X}, \bullet).$$

The pullback functoriality defines Bloch's higher Chow groups as a functor

$$z^j(-; \bullet) : \left(\mathrm{Var}^{\mathrm{flat}}\right)^{\mathrm{op}} \to \mathbf{K}(\mathbb{Z}).$$

One of the principal results (via the moving lemma), due to Bloch-Levine, is that this functor enhances to a functor out of *smooth schemes* but into the derived category, i.e., we have a commutative square

$$\begin{tikzcd} \left(\mathrm{Sm}_k^{\mathrm{flat}}\right)^{\mathrm{op}} \arrow[r, "z^j(-;\bullet)"] \arrow[d] & \mathbf{K}(\mathbb{Z}) \arrow[d] \\ \mathrm{Sm}_k^{\mathrm{op}} \arrow[r, "z^j(-;\bullet)"] & \mathbf{D}(\mathbb{Z}). \end{tikzcd}$$

2.1. **The theorem of Nesterenko-Suslin and Totaro.** As promised, our goal is to give a cohomological interpretation of the Milnor K-groups of fields. The main result is the following:

Theorem 2.1.1 (Nesterenko-Suslin, Totaro)**.** *Let* F *be a field. Then there is a natural isomorphism*

$$\mathrm{CH}^j(\mathrm{F}, j) \cong \mathrm{K}_j^{\mathrm{M}}(\mathrm{F}).$$

We will give an outline of the proof of Theorem 2.1.1, with the goal of acquainting ourselves with the Bloch's construction and some structural properties of Milnor K-theory. We follow the proof given by Totaro in his thesis [Tot92]. Before we proceed, we explain some more shadow of how Milnor K-theory behaves like a cohomology theory.

2.1.2. *Step 0: cubical constructions.* In algebraic topology, the simplicial formalism has been adopted as the standard way to encode homotopy coherence (quasicategories, simplicial homology etc.). However we note that the products of simplices are not simplices; this led to the method of barycentric subdivision which often poses its own complications. To define a map from Milnor K-theory to higher Chow groups, we first write a cubical model for the higher Chow groups so that we can just define a map in degree one and spread the effects via products.

Construction 2.1.3 (Cubical higher Chow groups)**.** By an n-cube $\boxplus^n$ we mean $\mathbb{A}^n$ thought of as $(\mathbb{P}^1 - \{1\})^n$; we coordinatize them by $\mathrm{T}_1, \cdots, \mathrm{T}_n$. Each n-cube has $2n$-divisors which are called **faces**

$$\boxplus^{n-1} \cong \mathrm{D}^{\epsilon}_{\mathrm{T}_i} \subset \boxplus^n;$$

where $\mathrm{D}^{\epsilon}_{\mathrm{T}_i}$ is the vanishing locus of $\mathrm{T}_i - \epsilon$ where $\epsilon \in \{0, \infty\}$. Now let X be an equidimensional k-scheme and $j \geqslant 0$ be fixed. We define, for each n:

$$z^j_{\boxplus}(\mathrm{X}, n) := \mathbb{Z}\{\mathrm{Z} \hookrightarrow \mathrm{X} \times \boxplus^n : \mathrm{W} \text{ is integral closed codim } j \text{ subscheme}, \mathrm{W} \pitchfork \mathrm{X} \times \mathrm{F}\}/d^j(\mathrm{X}, n)$$

where $d^j(\mathrm{X}, n)$ is the subgroup generated by **degenerate cycles**: these are cycles which are pulled back from the projections $\pi_k : \mathrm{X} \times \boxplus^n \to \mathrm{X} \times \boxplus^{n-1}$ given by $(x_1, \cdots, x_n) \mapsto (x_1, \cdots, \hat{x}_k, \cdots, x_n)$. We have a chain complex $z^j_{\boxplus}(\mathrm{X}, \bullet)$ where the differentials are given by

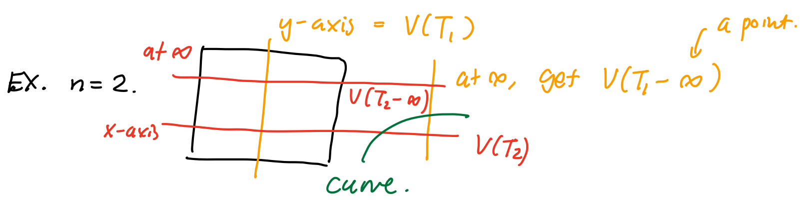
$$d := \sum (-1)^i ((\mathrm{D}^{\infty}_{\mathrm{T}_i} \cdot) - (\mathrm{D}^0_{\mathrm{T}_i} \cdot)).$$

We have the following lemma which follows from simplicial vs cubical formalism.

Lemma 2.1.4. *We have a canonical isomorphism*

$$\mathrm{H}_n(z^j_{\boxplus}(\mathrm{X},\bullet)) \cong \mathrm{CH}^j(\mathrm{X},n).$$

Remark 2.1.5. The picture for the cubical situation is (here we see an element of $z^1_{\boxplus}(\mathrm{X},2)$:)



The upshot of the working with cubical versions of higher Chow groups is the following:

Construction 2.1.6 (Products)**.** We have the **exterior product** on higher Chow groups

$$\mathrm{CH}^j(\mathrm{X},n) \times \mathrm{CH}^k(\mathrm{Y},m) \to \mathrm{CH}^{j+k}(\mathrm{X}\times\mathrm{Y},n+m);$$

defined via the isomorphism:

$$(\mathrm{X}\times\boxplus^n)\times(\mathrm{Y}\times\boxplus^m) \cong (\mathrm{X}\times\mathrm{Y}\times\boxplus^{n+m})$$

and pullbacks of cycles. This is the advantage of the cubical approach: we do not have to do barycentric subdivision. Now, by the functoriality explained in Remark 2.0.7 we have the diagonal pullback whenever $\mathrm{X}=\mathrm{Y}$:

$$\mathrm{CH}^{j+k}(\mathrm{X}\times\mathrm{X},n+m) \xrightarrow{\Delta^!} \mathrm{CH}^{j+k}(\mathrm{X},n+m);$$

thus a product

$$\cup : \mathrm{CH}^j(\mathrm{X},n)\times\mathrm{CH}^k(\mathrm{X},m) \to \mathrm{CH}^{j+k}(\mathrm{X}\times\mathrm{X},n+m).$$

Remark 2.1.7. Because we have used the functoriality of Remark 2.0.7 there is no a priori reason why the multiplication above can be enhanced to the structure of a strict cdga on the cycle complex. In fact one cannot because of the presence of Steenrod operations, constructed first by Voevodsky, on these higher Chow groups. It is only ever appropriate to think of $z^*(\mathrm{X},\bullet)$ as a $\mathbb{E}_\infty$-$\mathbb{Z}$-algebra!

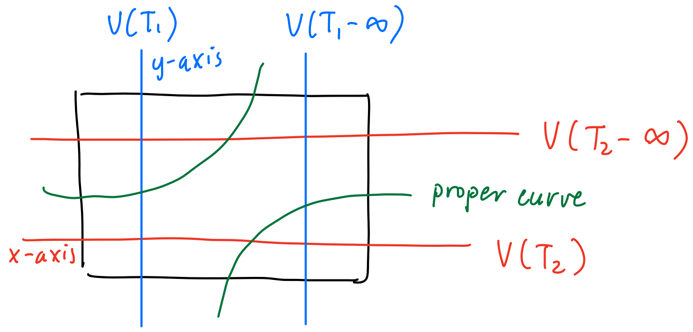
2.1.8. *Step 1: constructing the map* $\mathrm{K}^{\mathrm{M}}_*(\mathrm{F}) \to \mathrm{CH}^*(\mathrm{F};*)$. We now make a map one way; in degree one we need to produce a map

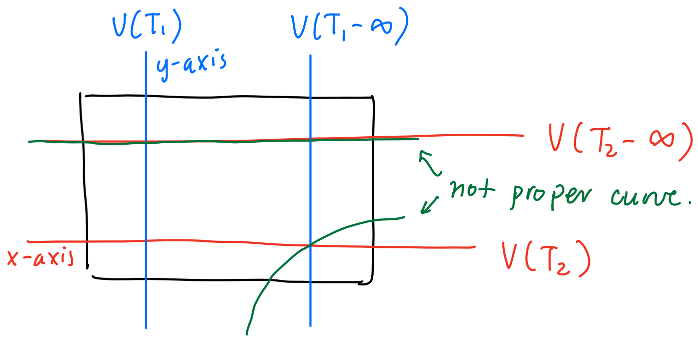
$$c^1 : \mathrm{F}^\times \to \mathrm{CH}^1(\mathrm{F},1).$$

Remark 2.1.9. Let us unpack two cases of the cycle complex:

(1) An element of $z^1_{\boxplus}(\mathrm{F},1)$ is a linear combination of reduced points in $\mathbb{P}^1 \smallsetminus \{1\}$ such that it meets $\{0\}$ and $\{\infty\}$ in codimension 1; but this exactly means that it *does not meet* these two points at all. Therefore, we conclude that a linear combination of closed points of $\mathbb{P}^1 \smallsetminus \{0,1,\infty\}$.
(2) An element of $z^1_{\boxplus}(\mathrm{F},2)$ is a linear combination of codimension 1, integral closed subscheme of $\mathbb{A}^2 \cong (\mathbb{P}^1 \smallsetminus \{1\})^{\times 2}$. There are two conditions that these curves are subject to: 1) it must meet the four faces of at codimension one, which means that they must *meet a points* and it must meet the four codimension two points (the points $(0,0),(0,\infty),(\infty,0),(\infty,\infty)$) at codimension one which means that they *do not meet*

these points at all. We also remind the reader of what is allowed and what is not via picture:





Construction 2.1.10. We construct a map

$$c^1 : \mathrm{F}^\times \to z^1_{\boxplus}(\mathrm{F}, 1)$$

by:

$$c^1(a) = \begin{cases} 0 & \text{if } a = 1 \\ [a] & \text{otherwise.} \end{cases}$$

There is no *a priori* reason why $[a] + [b] = [ab]$, of course. This relation has to happen only after equivalence:

Lemma 2.1.11. *We have the following relations in* $\mathrm{CH}^1(\mathrm{F}, 1)$*:*

$$[a] + [b] = [ab] \qquad a, b, ab \neq 0, 1;$$

$$[a] + [1/a] = 0 \qquad a \neq 0, 1.$$

Proof. For $a, b \neq 0, 1$, we consider the curve $\mathrm{C}(a, b)$ defined by the vanishing of the rational function

$$f(x) = \tfrac{ax - ab}{x - ab}$$

It is an element of $z^1_{\boxtimes}(\mathrm{F}, 2)$: clearly it meets the four faces only at points and it has the following intersection points whenever $ab \neq 1$:

$$(0, 1), (\infty, a), (b, 0), (ab, \infty)$$

and if $ab = 1$, it intersects at

$$(\infty, a), (1/a, 0).$$

The first verifies

$$[a] + [b] - [ab] = 0$$

and the second verifies

$$[a] + [1/a] = 0.$$

□

At this point we get a map

$$\mathrm{K}_1^{\mathrm{M}}(\mathrm{F}) = \mathrm{F}^\times \to \mathrm{CH}^1(\mathrm{F}, 1).$$

To promote the map into a ring map

$$\mathrm{K}_*^{\mathrm{M}}(\mathrm{F}) \to \mathrm{CH}^*(\mathrm{F}, *)$$

we need only verify the Steinberg relation

Lemma 2.1.12. *The following relation holds in the ring* $\mathrm{CH}^*(\mathrm{F}, *)$*:*

$$c^1(a)c^1(1-a) \qquad a \neq 0, 1.$$

Proof. The product $c^1(a)c^1(1-a)$ lies in $z^2_{\boxtimes}(\mathrm{F}, 2)$; to prove that it is zero we need to produce an element of $z^2_{\boxtimes}(\mathrm{F}, 3)$. The latter group is generated by curves in the cube $(\mathbb{P}^1 \smallsetminus \{1\})^{\times 3}$ which meets all six plans at curves and does not meet the vertices at all. Take the curve $\mathrm{C}(a)$ given by

$$f(x) = (1-x, \tfrac{a-x}{1-x}).$$

It only intersects the codimension one faces at the hyperplane $\mathrm{T}_3 = 0$, where it hits the point

$$(a, 1-a, 0).$$

Therefore we do have the desired equality. □

Therefore we get a map

$$c^j : \mathrm{K}_j^{\mathrm{M}}(\mathrm{F}) \to \mathrm{CH}^j(\mathrm{F}, j).$$

2.1.13. *Step 2: constructing the map* $\mathrm{CH}^*(\mathrm{F}; *) \to \mathrm{K}_*^{\mathrm{M}}(\mathrm{F})$. We now want to construct a map backwards

$$\mathrm{CH}^*(\mathrm{F}, *) \to \mathrm{K}_*^{\mathrm{M}}(\mathrm{F}).$$

Assume that F is an algebraically closed field; then a cycle of $z^j_{\boxplus}(\mathrm{F}, j)$ can be described as a coordinate $(a_1, \cdots, a_j)$ such that no coordinate is actually $0, 1$ or ∞. We can then send this to an element $\{a_1, \cdots, a_j\}$ in Milnor K-theory, which is well-defined once we check that the boundary of any curve gets sent to zero; of course we also send the element zero to the element $\{1\}$. From this point of view, it is easy to see that both composites are the identity and we are done. Let us expand this into an extended remark:

Remark 2.1.14 (Algebraically closed field)**.** So assume that F is an algebraically closed field. For simplicitly, we think of an element in $z^1(\mathrm{F}, 2)$ as a curve in $(\mathbb{P}^1 \smallsetminus \{1\})^{\times 2}$ the latter having coordinates $\mathrm{T}_1, \mathrm{T}_2$; we normalize if necessary and think of it as a finite morphism $\mathrm{C} \to (\mathbb{P}^1 \smallsetminus \{1\})^{\times 2}$ from C a smooth F-curve. We remark that by [Ful98, Example 1.2.3], the intersection values are not changed. Anyway, we think of the above map as two rational functions $f, g \in \mathrm{F}(\mathrm{C})^\times$. The properness of interesection conditions translate into:

(1) neither f nor g are constant at 0 or ∞;
(2) if $w \in \mathrm{C}$ such that $f(w) = 0$ or ∞, then $g(w) \notin \{0, \infty\}$ and vice versa.

With this in mind, the intersection of C with, say, $\mathrm{D}^0_{\mathrm{T}_2} \cdot \mathrm{C} = [(f(w), 0)]\nu_w(g)$. Unpacking everything and assuming that each C only intersects each of the divisors once we are asking:

$$(-1)\left([(\infty, g(w_1))]v_{w_1}(f) - [(0, g(w_2))]v_{w_2}(f)\right) + \left(([f(w_3), \infty)]v_{w_3}(g) - [(f(w_4), 0)]v_{w_4}(g)\right) = 0.$$

But this is just Weil reciprocity written additively! To make sense of this we need to discuss symbols in Milnor K-theory.

2.1.15. *Symbols.* A **Steinberg symbol** on a field F consists of an abelian group A, written multiplicatively, and a bilinear map

$$c : \mathrm{F}^\times \otimes_{\mathbb{Z}} \mathrm{F}^\times \to \mathrm{A}$$

such that $c(r, 1-r) = 1$. We can succinctly say that a Steinberg symbol is just an abelian group homomorphism

$$\mathrm{K}_2^{\mathrm{M}}(\mathrm{F}) \to \mathrm{A}.$$

Example 2.1.16. Let m be an integer prime to the characteristic of F. Then Kummer theory provides an short exact sequence on the small étale site of F

$$1 \to \mu_m \to \mathbb{G}_m \to \mathbb{G}_m \to 1.$$

This furnishes a connecting homomorphism

$$\mathrm{F}^\times \to \mathrm{H}^1_{\text{ét}}(\mathrm{F}; \mu_m).$$

We have a map

$$\mathrm{F}^\times \otimes_{\mathbb{Z}} \mathrm{F}^\times \to \mathrm{H}^1_{\text{ét}}(\mathrm{F}; \mu_m) \otimes_{\mathbb{Z}} \mathrm{H}^1_{\text{ét}}(\mathrm{F}; \mu_m) \xrightarrow{\cup} \mathrm{H}^2_{\text{ét}}(\mathrm{F}; \mu_m^{\otimes 2}).$$

A result of Tate (see []) proves that the above map is a Steinberg symbol known as the Galois symbol.

A **higher symbol** or, simply, a **symbol** is an abelian group homomorphism $\mathrm{K}_*^{\mathrm{M}}(\mathrm{F}) \to \mathrm{A}$. The example 2.1.16 above can be promoted to a symbol

$$\mathrm{K}_*^{\mathrm{M}}(\mathrm{F}) \to \mathrm{H}^*_{\text{ét}}(\mathrm{F}; \mu_m^{\otimes *}).$$

For us, the next symbol is most important. To define it we require some preliminaries about basic field theory:

(1) let K be a field equipped with a discrete valuation $\nu : \mathrm{K}^\times \to \mathbb{Z}$; its associated discrete valuation ring is $\mathcal{O} := \nu^{-1}(\mathbb{Z}_{\geqslant 0})$;
(2) a uniformizer or a local parameter is an element π such that $\nu(\pi) = 1$ and we choose one;
(3) having done so any element $x \in \mathrm{K}^\times$ can be uniquely written as $u\pi^i$ for $i \in \mathbb{Z}$;
(4) the residue field of $\mathcal{O}$ is given by $\mathcal{O}/\pi =: \kappa$; if $x \in \mathrm{A}$ we write $\overline{x}$ to be the reduction modulo π

Therefore, the gropups $\mathrm{K}_n^{\mathrm{M}}(\mathrm{K})$ are generated by two flavors of elements: the first are elements of the form $\{\pi, u_2, \cdots, u_n\}$ or elements which are purely $\{u_1, \cdots, u_n\}$ which are units *in* $\mathcal{O}$. The next proposition creates a symbol known as the **Tame symbol** or the **residue map** but it should be thought of as a connecting homomorphism:

Proposition 2.1.17 (Milnor, Serre)**.** *Let $n \geqslant 1$, there exists a unique map of abelian groups*

$$\partial^{\mathrm{M}} : \mathrm{K}_n^{\mathrm{M}}(\mathrm{K}) \to \mathrm{K}_{n-1}^{\mathrm{M}}(\kappa).$$

such that:

$$\partial^{\mathrm{M}}(\{\pi, u_2, \cdots, u_n\}) = \{\overline{u}_2, \cdots \overline{u}_n\},$$

and

$$\partial^{\mathrm{M}}(\{u_1, u_2, \cdots, u_n\}) = 0.$$

*Furthermore, fixing π, we have a **specialization map***

$$s_\pi^{\mathrm{M}} : \mathrm{K}_n^{\mathrm{M}}(\mathrm{K}) \to \mathrm{K}_n^{\mathrm{M}}(\kappa)$$

such that

$$s_\pi(\{\pi^{i_1} u_1, \cdots, \pi^{i_n} u_n\}) = \{\overline{u}_1, \cdots, \overline{u}_n\}.$$

Remark 2.1.18. In the literature, the tame symbol is commonly referred to as the map above for $n = 2$; it takes the form

$$\partial^{\mathrm{M}} : \mathrm{K}_2^{\mathrm{M}}(\mathrm{K}) \to \kappa^\times$$

and is given by

$$\partial^{\mathrm{M}}(\{u_1, u_2\}) = (-1)^{\nu(u_1)\nu(u_2)} \frac{\overline{u_1}^{\nu(a_2)}}{\overline{u_2}^{\nu(u_1)}}.$$

On the other hand when $n = 1$, the map

$$\partial^{\mathrm{M}} : \mathrm{K}^\times \to \mathbb{Z}$$

is just the discrete valuation of K.

The proof of this result is kind of standard; see for example [GS17, Proposition 7.1.4]; let us instead package it into a remark which I learned from Fabien Morel [Mor12, Remark 3.18].

Remark 2.1.19 (Morel)**.** $\operatorname{Spec} \mathcal{O}$ is a Sierpinski space, having a closed point a $\operatorname{Spec} \kappa$ and an open point $\operatorname{Spec} \mathrm{F}$. If this was in manifold theory, we would then have a map

$$\mathbb{V}(\mathcal{N}_\kappa) \smallsetminus \{0\} \to \operatorname{Spec} \mathrm{F},$$

acting as a tubular neighborhood; here $\mathbb{V}(\mathcal{N}_\kappa) \smallsetminus \{0\}$ is just the punctured vector bundle over κ associated to the normal sheaf of the embedding $\operatorname{Spec} \kappa \hookrightarrow \operatorname{Spec} \mathcal{O}$. If we can evaluate Milnor K-theory on $\mathbb{V}(\mathcal{N}_\kappa) \smallsetminus \{0\}$, then we would get a natural map

$$\mathrm{K}_*^{\mathrm{M}}(\mathrm{F}) \to \text{“}\mathrm{K}_*^{\mathrm{M}}(\mathbb{V}(\mathcal{N}_\kappa) \smallsetminus \{0\}).\text{”}$$

Now, $\mathbb{V}(\mathcal{N}_\kappa) \smallsetminus \{0\}$ is isomorphic to $\mathbb{G}_m$, and one can make this isomorphism after choosing the uniformizer π; so we can ask ourselves that $\mathrm{K}_*^{\mathrm{M}}$ of $\mathbb{G}_m$-should be. We set

$$\text{“}\mathrm{K}_*^{\mathrm{M}}(\mathbb{V}(\mathcal{N}_\kappa) \smallsetminus \{0\})\text{”} := \mathrm{K}_*^{\mathrm{M}}(\kappa)[\xi]/(\xi^2 - [-1]\xi) \qquad |\xi| = 1.$$

The relation $\xi^2 = [-1]\xi$ can be explained via $\mathbb{A}^1$-homotopy theory: the reduced diagonal map $\mathbb{G}_m \to \mathbb{G}_m \wedge \mathbb{G}_m$ is $\mathbb{A}^1$-homotopic to the map “$x \mapsto -1 \wedge x$”. This is pictorially similar to decomposing the diagonal divisor in the ruled surface $\mathbb{P}^1 \times \mathbb{P}^1$ to the divisor $\mathbb{P}^1 \times 1 + 1 \times \mathbb{P}^1$.

2.1.20. *Back to Step 2.* To deal with the case that F is not algebraically closed. We will need to use some structure in the Milnor K-groups.

Construction 2.1.21. We have a morphism

$$\operatorname{Spec} \kappa(p) \to (\mathbb{P}^1_{\mathrm{F}} \smallsetminus \{0, 1, \infty\})^j;$$

classifying an element of $z^j_{\boxplus}(\mathrm{F}, j)$ and giving elements $x_1, \cdots x_j \in \kappa(p) \smallsetminus \{0, 1\}$ as the image of $\mathrm{T}_1, \cdots \mathrm{T}_j$; we think of the above map as a *framing.* The field extension $\kappa(p)/\mathrm{F}$ is finite since the point is closed and thus there is the **norm map** (see A)

$$\mathrm{N}_{\kappa(p)/\mathrm{F}} : \mathrm{K}_j^{\mathrm{M}}(\kappa(p)) \to \mathrm{K}_j^{\mathrm{M}}(\mathrm{F}).$$

We set

$$s^j : \mathrm{CH}^j(\mathrm{F}, j) \to \mathrm{K}_j^{\mathrm{M}}(\mathrm{F}) \qquad s^j(\operatorname{Spec} \kappa(p) \to (\mathbb{P}^1_{\mathrm{F}} \smallsetminus \{0, 1, \infty\})^j) = \mathrm{N}_{\kappa(p)/\mathrm{F}}\{x_1, \cdots, x_j\}.$$

We now sketch a proof of the following result, extending Remark 2.1.14:

Lemma 2.1.22. *Any generator of $z^j_{\boxtimes}(\mathrm{F}; j+1)$ has boundary that maps to zero in $\mathrm{K}_n^{\mathrm{M}}(\mathrm{F})$.*

Proof. A generator is an irreducible, reduced curve $\mathrm{C} \subset \mathbb{A}^{j+1}_{\mathrm{F}}$ which means that codimension one faces at points and none of the codimension two faces. We can reinterpret this datum as a map $\mathrm{C} \to (\mathbb{P}^1 \smallsetminus \{1\})^{j+1}$, whence $j + 1$ rational functions

$$g_i \in k(\mathrm{C}) \qquad i = 1, \cdots, j+1$$

such that

(1) no g_j is identically 0 or ∞ and
(2) if $w \in \mathrm{C}$ is such that $g_i(w) = 0$ or ∞, then $g_k(w) \notin \{0, \infty\}$ for $k \neq i$.

In terms of rational functions, the boundary of C is then obtained by □

2.1.23. *Step 3: computing composites.*

Lemma 2.1.24. *Any element in* $\mathrm{CH}^j(\mathrm{F}, j)$ *is equivalent to a sum*

$$\sum n_i[p_i],$$

where each p_i *is an* F*-rational point of* $\mathbb{A}^j_{\mathrm{F}}$.

Appendix A. The Norm map

A good reference form norms in Milnor K-theory is [GS17, Section 7.3], but let us given an informal discussion. Recall that the **field norm** of a finite extension L/K is the map

$$\mathrm{N}_{\mathrm{L/K}} : \mathrm{L} \to \mathrm{K}$$

given as follows: if $\alpha \in \mathrm{L}$, then $\alpha\cdot : \mathrm{K} \to \mathrm{K}$ is a K-linear transformation and $\mathrm{N}_{\mathrm{L/K}}(\alpha) \in \mathrm{K}$ is the determinant of this transformation. Alternatively, if $\mathrm{E}_\alpha(x)$ is the minimal polynomial of α, written as $\mathrm{E}_\alpha(x) = x^d - a_{d-1}x^{d-1} + \cdots (-1)^d a_0 \in \mathrm{K}[x]$, then $\mathrm{N}_{\mathrm{L/K}}(\alpha) = a_0$. The norm map is multiplicative in that it defines a morphism of multiplicative groups

$$\mathrm{N}_{\mathrm{L/K}} : \mathrm{L}^\times \to \mathrm{K}^\times$$

and further satisfies

$$\mathrm{N}_{\mathrm{L/K}} = \mathrm{N}_{\mathrm{M/K}} \circ \mathrm{N}_{\mathrm{L/M}}$$

for intermediate extensions L/M/K. Furthermore, it has the following **base change** property: if F_0/K is any field extension of F, then $\mathrm{L} \otimes_{\mathrm{K}} \mathrm{F}_0$ has finitely many maximal ideals $\{\kappa(\mathfrak{m})\}$ and we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{L}^\times \arrow[r] \arrow[d, "\mathrm{N}_{\mathrm{L/K}}"'] & \bigoplus_{\mathfrak{m}} \kappa(\mathfrak{m})^\times \arrow[d, "\sum_{\mathfrak{m}} d_{\mathfrak{m}} \mathrm{N}_{\kappa(\mathfrak{m})/\mathrm{F}_0}"] \\
\mathrm{K}^\times \arrow[r] & \mathrm{F}_0^\times .
\end{tikzcd}$$

where

$$d_{\mathfrak{m}} = \tfrac{[\mathrm{L}:\mathrm{F}]_{\mathrm{insep}}}{[\kappa(\mathfrak{m}):\mathrm{F}_0]_{\mathrm{insep}}}.$$

We can package all of this into an algebraic structure which is, nowadays, prevalent throughout all of mathematics. If $\mathcal{C}$ is a category (even ∞-category) with finite limits, then we can form the (2,1)-category $\mathbf{Corr}(\mathcal{C})$ whose objects are objects in $\mathcal{C}$ and whose morphisms are given by **spans**:

$$\begin{tikzcd}
& \mathrm{Z} \arrow[dl] \arrow[dr] & \\
\mathrm{X} & & \mathrm{Y}.
\end{tikzcd}$$

Composition is given by the formation of pullbacks in the obvious way and the (2,1)-category structure let us elegantly speak of invertible morphisms between the resulting pullbacks. The point

For example, we can let $\mathrm{Fields}_{\mathrm{F}}$ be the category of finite field extensions of F, we then have a functor

$$\mathbf{Corr}(\mathrm{Fields}_{\mathrm{F}}) \to \mathrm{AbGp},$$

which sends a span

$$\mathrm{K} \leftarrow \mathrm{L} \to \mathrm{M} \mapsto \mathrm{K}^\times \xrightarrow{\mathrm{N}_{\mathrm{L/K}}} \mathrm{L}^\times \to \mathrm{M}^\times.$$

Appendix B. Reciprocity laws

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Department of Mathematics, Harvard University, 1 Oxford St. Cambridge, MA 02138, USA
E-mail address: elmanto@math.harvard.edu
URL: https://www.eldenelmanto.com/