LECTURE 9: IN WHICH WE VISIT BOSTON IN THE 90'S

ELDEN ELMANTO

1. The Geisser-Levine theorem: a retrospective

The starting point is the following theorem of Faltings. We begin with V a complete discrete valuation ring of mixed characteristics $(0, p)$ with residue field κ a perfect field. Let F be its field of fractions. We choose some embedding $\mathrm{F} \hookrightarrow \mathbb{C}_p$ into the p-adic complex numbers; let us write G for the Galois group of $\overline{\mathrm{F}}$ over F.

Theorem 1.0.1. [Fal88] *Let* X *be a smooth proper* F*-scheme. Then there is a* G*-equivariant, natural isomorphism*

$$\mathrm{H}^n_{\text{ét}}(\mathrm{X}_{\overline{\mathrm{F}}}; \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \cong \bigoplus_{i+j=n} \mathrm{H}^i(\mathrm{X}; \Omega^j) \otimes_{\mathrm{F}} \mathbb{C}_p(-j) \qquad \geqslant 0.$$

Here we set $\mathbb{Z}_p(j), j \in \mathbb{Z}$ to be the usual Tate twist, where the first twist is given by:

$$\mathbb{Z}_p(1) := \lim \mu_{p^n} \qquad \mathbb{Z}_p(-1) := (\lim \mu_{p^n})^{\vee},$$

and $\mathrm{R}(j) := \mathrm{R} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(j)$. The above isomorphism is often called the "Hodge-Tate decomposition." Faltings used the methods of his almost étale extensions and almost purity in order to prove the above result. Nowadays it has been generalized by prismatic cohomology.

There is a precursor to this theorem of Faltings, namely the one of Bloch and Kato [BK86]. They essentially proved this result whenever X has good ordinary reduction, so let's consider the following set up:

$$\begin{array}{ccccc} \mathrm{X}_\kappa & \xrightarrow{i} & \mathrm{X} & \xleftarrow{j} & \mathrm{X}_{\mathrm{F}} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}\kappa & \longrightarrow & \operatorname{Spec}\mathrm{V} & \longleftarrow & \operatorname{Spec}\mathrm{F}, \end{array} \tag{1.0.2}$$

where $\mathrm{X} \to \operatorname{Spec}\mathrm{V}$ is proper; we will soon base change everything to the closure and we use notation like $\overline{j} : \mathrm{X}_{\overline{\mathrm{F}}} \to \mathrm{X}$. The idea of Bloch and Kato is to study two different spectral sequences

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p_{\text{ét}}(\mathrm{X}_{\overline{\kappa}}; \overline{i}^*\mathrm{R}^q\overline{j}_*\mathbb{Z}/p^r\mathbb{Z}) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(\mathrm{X}_{\overline{\mathrm{F}}}; \mathbb{Z}/p^r\mathbb{Z});$$

the "vanishing cycles" spectral sequence which interpolates between the vanishing cycles cohomology on the special fiber to the generic fiber. The one thing to note is that the stalks of

$$i^*\mathrm{R}^q j_*\mathbb{Z}/p^r\mathbb{Z}$$

at a point $y \in \mathrm{X}_\kappa$ is given by

$$\mathrm{H}^q_{\text{ét}}(\mathcal{O}^{\text{sh}}_{\mathrm{X},\overline{y}}[\tfrac{1}{p}], \mathbb{Z}/p^r\mathbb{Z})$$

where $\mathcal{O}^{\text{sh}}_{\mathrm{X},\overline{y}}$ is the strict henselization of y in X; so this is a mixed-characteristic local ring. The second one is the slope spectral sequence which we have already seen

$$\mathrm{E}_1^{p,q} = \mathrm{H}^p(\mathrm{X}_{\overline{\kappa}}; \mathrm{W}\Omega^q) \Rightarrow \mathrm{H}^{p+q}_{\text{crys}}(\mathrm{X}_{\overline{\kappa}}/\mathrm{W}(\overline{\kappa})). \tag{1.0.3}$$

One way to understand Theorem 1.0.1 is that there is a certain filtration on $\mathrm{H}^n_{\text{ét}}(\mathrm{X}_{\overline{\mathrm{F}}}; \mathbb{Q}_p)$ which appropriately splits after tensoring with enough scalars; this is actually what happens with the situation in characteristic zero. At that time, the vanishing cycles spectral sequence looks as good as any for a candidate filtration. One reason to like the vanshing cycles spectral sequence, however, is that the E_2 page clearly interpolates between the special and the generic fiber as the étale cohomology groups that appear in the stalks are those of $\mathcal{O}^{\mathrm{sh}}_{X,\overline{y}}[\frac{1}{p}]$. We are not doing anything illegal like taking the mod-p étale cohomology of a mod-p scheme, but at the same time there's still a little bit of characteristic p in this picture.

Bloch-Kato began producing filtrations on the sheaves $i^*R^qj_*\mathbb{Z}/p^r\mathbb{Z}(q) \cong i^*R^qj_*\mu_{p^r}^{\otimes j}$; these are étale cohomology groups in degree j with exactly a power of j; these are of the form:

$$\cdots U^2 i^*R^qj_*\mu_{p^r}^{\otimes j} \subset U^1 i^*R^qj_*\mu_{p^r}^{\otimes j} \subset U^0 i^*R^qj_*\mu_{p^r}^{\otimes j}$$

This filtration stops at a finite stage and is inspired by the following on the level of Milnor K-theory:

Remark 1.0.4. Let F be a discrete valued field with value group V and residue field κ with uniformizer π; we set $U_F := V^\times$, the units in the valuation ring (so that these are exactly elements of valuation zero. Then we have the sequence of groups

$$\cdots \leqslant U^mF \leqslant \cdots \leqslant U^1_F \leqslant U^0_F = F^\times;$$

such that $x \in U^mF$ if and only if it is of the form $1+a\pi^m$, where $a \in V$. Noting that $F^\times = K_1(F)$, This generalizes to the higher Milnor K-groups into a filtration:

$$\cdots \leqslant U^mK_j(F) \leqslant \cdots \leqslant U^1K_j(F) \leqslant U^0K_j(F) = K_j(F)$$

where $U^mK_j(F)$ is the subgroup generated by symbols $\{a_1, \cdots, a_m\}$ where *at least* one of the a_i's is an element of U^mF. By graded commutativity, we may assume that a_m is exactly the element which is in U^mF. A basic calculation in this theory is the following:

Proposition 1.0.5. *The sum* $\partial \oplus s_\pi : K_j(F)/n \to K_{j-1}(\kappa) \oplus K_j(\kappa)$ *gives an exact sequence*

$$0 \to U^1K_j(F) \to K_j(F) \xrightarrow{\partial \oplus s_\pi} K_{j-1}(\kappa) \oplus K_j(\kappa).$$

Now, $U^1K_j(F)$ is "coherent" in nature in that it is killed by a coprime integer. This gives the following result

Proposition 1.0.6. *If n is coprime to the characteristics of κ, then we have an isomorphism*

$$\partial \oplus s_\pi : K_j(F)/n \xrightarrow{\cong} K_{j-1}(\kappa) \oplus K_j(\kappa).$$

In other words, we can describe the mod-n Milnor K-groups of the fraction field using sums of those coming from the residue.

Bloch and Kato was trying to relate the filtration in Remark 1.0.4 to the vanishing cycles sheaves. To do so, they attempted to construct a map (for $r = 1$) of Zariski sheaves on X_κ

$$i^*R^qj_*\mu_p^{\otimes j} \to \Omega^j_{X_\kappa} \oplus \Omega^{j-1}_{X_\kappa}. \tag{1.0.7}$$

Assume, for accuracy and simplicity that X is V itself (or any mixed characteristic dvr), then concretely we want to construct a map:

$$H^j_{\text{ét}}(V[\tfrac{1}{p}]; \mu_p^{\otimes j}) = H^j_{\text{ét}}(F; \mu_p^{\otimes j}) \to \Omega^j_\kappa \oplus \Omega^{j-1}_\kappa.$$

It is actually enough to construct:

$$H^j_{\text{ét}}(V[\tfrac{1}{p}]; \mu_p^{\otimes j}) = H^j_{\text{ét}}(F; \mu_p^{\otimes j}) \to K^M_j(\kappa)/p \oplus K^M_{j-1}(\kappa)/p$$

as I will explain next.

Definition 1.0.8. Let X be a regular $\mathbb{F}_p$-scheme. We consider a map of sheaves on X_{Zar}

$$\mathrm{dlog} : \mathbb{G}_{m,X}^{\otimes j} \to W_r\Omega^j_X$$

given by

$$f_1 \otimes \cdots \otimes f_j \mapsto \mathrm{dlog}[f_1] \wedge \cdots \wedge \mathrm{dlog}[f_j].$$

The **logarithmic Hodge-Witt sheaves** of Deligne-Milne-Illusie is the Zariski-sheafification of the image of the above map and is written as

$$W_r\Omega^j_{\log,X}.$$

Remark 1.0.9. There's a slight "cheat" of notation above. The element $\mathrm{dlog}[f]$ means the following: if R is a $\mathbb{F}_p$-algebra, then we have the element $[f] \in \mathrm{W}(\mathrm{R})$ given by the (multiplicative) Teichmüller section $\mathrm{R} \to \mathrm{W}(\mathrm{R})$. Then we can take a well-defined element $\frac{d[f]}{[f]} \in \widehat{\Omega}^j_{\mathrm{W}(\mathrm{R})}$; we then push it further to the de Rham-Witt form via the canonical saturation map $\widehat{\Omega}^j_{\mathrm{W}(\mathrm{R})} \to \mathrm{W}\Omega^j_{\mathrm{R}}$.

Here is a claim:

Lemma 1.0.10. *For any regular local $\mathbb{F}_p$-algebra* R *the following equality holds in* $\mathrm{W}_r\Omega^j_{\mathrm{R}}$

$$\mathrm{dlog}[f] \wedge \mathrm{dlog}[1-f] = 0 \qquad f \in \mathrm{R}^\times.$$

Proof. We compute directly (in $\widehat{\Omega}^*_{\mathrm{W}(\mathrm{R})}$)

$$\begin{aligned} \mathrm{dlog}[f] \wedge \mathrm{dlog}[1-f] &= \tfrac{1}{[f][1-f]} d[f] \wedge d[1-f] \\ &= \tfrac{1}{[f][1-f]} d[f] \wedge d[f] \\ &= 0. \end{aligned}$$

□

So let us make the following definition:

Definition 1.0.11. If R is a ring, its **naive Milnor K-theory** is defined in the usual way as

$$\mathrm{K}^{\mathrm{M}'}_*(\mathrm{R}) := \mathrm{Tens}^*_{\mathbb{Z}}\mathrm{R}^\times/(f \otimes 1-f, f \neq 0, 1).$$

Remark 1.0.12 (Naive versus improved Milnor K-theory)**.** Gabber, Kerz

In any event, we have a map for any regular $\mathbb{F}_p$-algebra R,

$$\mathrm{K}^{\mathrm{M}'}_j(\mathrm{R}) \to \mathrm{W}_r\Omega^j_{\log,\mathrm{R}}.$$

On the other hand, returning to the original problem at hand, we have a map

$$\mathrm{K}^{\mathrm{M}}_j(\mathrm{F}) \to \mathrm{H}^j_{\text{ét}}(\mathrm{F}; \mu_p^{\otimes j});$$

and if it was an isomorphism then we will be on our way to defining the map to logarithmic forms! This then became the infamous Bloch-Kato conjecture

Theorem 1.0.13 (Bloch-Kato conjecture, Rost-Voevodsky)**.** *If p is coprime to the characteristic of* F*, then the induced map*

$$\mathrm{K}^{\mathrm{M}}_*(\mathrm{F})/p^r \to \mathrm{H}^*_{\text{ét}}(\mathrm{F}; \mu_{p^r}^{\otimes *})$$

is a graded ring isomorphism.

In fact Bloch and Kato verified this result whenever F is a complete, discretely valued field of mixed characteristic and this was enough to prove their version of Faltings' theorem.

1.1. **More on the Bloch-Kato conjecture.**

The above discussion concern only the situation prime to the characteristic, exactly not what this class is about. Yet the ingredients involved are very much close to things we have talked about so far. We see a generalization of the statement of Theorem 1.0.13 which simultaneously 1) lets us consider schemes instead of just fields, 2) lets us vary the weights (thinking of $\mathrm{K}^{\mathrm{M}}_j(\mathrm{F})$ as $\mathrm{H}^j_{\mathrm{mot}}(\mathrm{F}; \mathbb{Z}(j))$) and 3) a version in characteristic p.

Definition 1.1.1 (Lichtenbaum-Milne)**.** The **étale motivic complexes** on Sm_k are étale sheaves $\mathbb{Z}(j)^{\mathrm{et}}$ such that we have equivalences of étale sheaves

$$\mathbb{Z}(j)^{\mathrm{et}}/p^r \simeq \begin{cases} \mu_{p^r}^{\otimes j} & \frac{1}{p} \in k \\ \mathrm{W}_r\Omega^j_{\log}[-j] & p = 0. \end{cases}$$

Let $\mathrm{X} \in \mathrm{Sm}_k$ and let $\lambda : \mathrm{X}_{\text{ét}} \to \mathrm{X}_{\mathrm{Zar}}$ be the usual morphism of sites. Then

Theorem 1.1.2 (Beilinson-Lichtenbaum-Milne conjecture, Rost-Voevodsky and Geisser-Levine)**.** *There is a canonical map of* $\mathbb{Z}(j)^{\mathrm{mot}}_{\mathrm{X}} \to \mathrm{R}\lambda_*\mathbb{Z}(j)^{\mathrm{et}}$ *such that for all primes* p *and for all* $r \geqslant 1$*, the induced map on mod* p^r *reduction factors through an equivalence*

$$\mathbb{Z}/p^r(j)^{\mathrm{mot}}_{\mathrm{X}} \to \tau^{\leqslant j}\mathrm{R}\lambda_*\mathbb{Z}/p^r(j)^{\mathrm{et}}.$$

1.2. **Statement of the Geisser-Levine theorem.** There are many ways to state the Geisser-Levine theorem. In a general and usable form we have:

Theorem 1.2.1 (Geisser-Levine)**.** *Let* $\mathcal{O}$ *be a regular, local* $\mathbb{F}_p$*-algebra, then for all* $r \geqslant 1$ *and* $i, j \geqslant 0$*:*

$$\mathrm{H}^i_{\mathrm{mot}}(\mathcal{O}; \mathbb{Z}/p^r(j)) = \begin{cases} 0 & i \neq j \\ \Omega^j_{\log,\mathcal{O}} & i = j \\ . \end{cases} \tag{1.2.2}$$

1.3. **Overview of the proof.** We are always free to work with mod-p coefficients, as opposed to mod-p^r via usual Bockstein argument. The starting point of the Geisser-Levine theorem is the following algebraic result, due to Bloch-Kato and also Gabber (in an unpublished manuscript).

Theorem 1.3.1 (Bloch-Kato-Gabber)**.** *Let* F *be a field of charateristic* $p > 0$*, then the* dlog *symbol map:*

$$\mathrm{K}^{\mathrm{M}}_j(\mathrm{F})/p \to \Omega^j_{\mathrm{F},\log}$$

is an isomorphism.

Because of the Nesterenko-Suslin-Totaro isomorphism and the fact that for a field F, we have $\mathrm{H}^{>j}_{\mathrm{mot}}(\mathrm{F}; \mathbb{Z}(j)) = 0$ (for purely higher Chow group reasons), we have that

$$\mathrm{H}^j_{\mathrm{mot}}(\mathrm{F}; \mathbb{Z}/p(j)) \cong \mathrm{H}^j_{\mathrm{mot}}(\mathrm{F}; \mathbb{Z}(j))/p \cong \mathrm{K}^{\mathrm{M}}_j(\mathrm{F})/p \cong \Omega^j_{\mathrm{F},\log}.$$

The key point is then to kill groups

$$\mathrm{H}^{<j}(\mathrm{F}; \mathbb{Z}/p(j))$$

for any field F. We note that these groups are almost never zero when F is not characteristic $p > 0$;

Remark 1.3.2 (Weight one)**.** Let X be a smooth F scheme. One of the things that we can compute "by hand" as done in [MVW06, Lecture 4] is weight one motivic cohomology. We have that

$$\mathbb{Z}(1)^{\mathrm{mot}}_{\mathrm{X}} \simeq \mathcal{O}^{\times}_{\mathrm{X}}[-1]. \tag{1.3.3}$$

This is independent of characteristics. But if p is invertible then the Kummer sequence (which is not available in characteristic $p > 0$ étale locally) and Hilbert theorme 90 (which states that étale and Zariski cohomology of $\mathbb{G}_m$-agrees up to degree $\leqslant 1$) gives us

$$\mathrm{H}^0_{\mathrm{mot}}(\mathrm{X}; \mathbb{Z}/p(1)) = \mu_p(\mathrm{X}).$$

the global p-th roots of unity on X.

We will very soon see that Remark 1.3.2 is the basic phenomenon that underlies the difference between motives at p and away from p. In particular, what is going on in characteristic $p > 0$ with mod-p coefficients is a kind of discreteness statement which is, at first glance, surprising but is also pervasive throughout the subject. To formulate this discreteness correctly, we work in the context of **stable motivic homotopy theory**.

1.4. **A rapid primer to stable motivic homotopy theory.** There are, by now, many good expository notes on the subject. However, most do not delve into the actual content of the theory and what the formalism is all about; we hope to rapidly run through it now. Throughout, all schemes are quasicompact and quasiseparated.

Definition 1.4.1. Let B be a scheme.

(1) A **Nisnevich square** is a pullback square of schemes

$$\begin{array}{ccc} \mathrm{U} \times_{\mathrm{X}} \mathrm{V} & \longrightarrow & \mathrm{V} \\ \downarrow & & \downarrow p \\ \mathrm{U} & \xrightarrow{j} & \mathrm{X}. \end{array}$$

where p is étale, j is open and p induces an isomorphism $p : p^{-1}((\mathrm{X} \smallsetminus \mathrm{U})_{\mathrm{red}}) \to (\mathrm{X} \smallsetminus \mathrm{U})_{\mathrm{red}}$.

(2) A functor: $\mathrm{E} : \mathrm{Sm}_{\mathrm{B}}^{\mathrm{op}} \to \mathrm{Spt}$ is said to be an $\mathbb{A}^1$**-invariant Nisnevich sheaf** if it converts all Nisnevich squares in Sm_{B} to cartesian squares and the projection map $\mathrm{X} \times \mathbb{A}^1 \to \mathrm{X}$ induces an equivalence $\mathrm{E}(\mathrm{X}) \xrightarrow{\simeq} \mathrm{E}(\mathrm{X} \times \mathbb{A}^1)$.

(3) For a presheaf E, set

$$\Omega_{\mathbb{P}^1}\mathrm{E}(\mathrm{X}) := \mathrm{fib}(\mathrm{E}(\mathrm{X} \times \mathbb{P}^1) \xrightarrow{\infty^*} \mathrm{E}(\mathrm{X}));$$

which defines a functor $\Omega_{\mathbb{P}^1}\mathrm{E} : \mathrm{Sm}_{\mathrm{B}}^{\mathrm{op}} \to \mathrm{Spt}$. Since ∞^* is split by $\mathbb{P}^1 \times \mathrm{X} \to \mathrm{X}$, we have a direct sum decomposition

$$\mathrm{E}(\mathrm{X} \times \mathbb{P}^1) \simeq \mathrm{E}(\mathrm{X}) \oplus \Omega_{\mathbb{P}^1}\mathrm{E}(\mathrm{X}).$$

(4) Let $\{\mathrm{E}(\bullet)\}$ be a $\mathbb{Z}$-graded collection of presheaves of spectra, then a $\mathbb{P}^1$**-bundle datum at level** j is a map

$$\mathrm{E}(j) \to \Omega_{\mathbb{P}^1}\mathrm{E}(j+1).$$

(5) A **homotopy invariant motivic cohomology theory** is the data of a $\mathbb{Z}$-graded presheaves of spectra $\{\mathrm{E}(\bullet)\}$, $\mathbb{P}^1$-bundle datum at each level j for $j \in \mathbb{Z}$ such that:
   (a) $\mathrm{E}(j)$ is an $\mathbb{A}^1$-invariant Nisnevich sheaf;
   (b) each $\mathbb{P}^1$-bundle datum is an equivalence.

Example 1.4.2. Let $\frac{1}{p} \in \mathcal{O}_{\mathrm{B}}$. Then set

$$\mathrm{E}(j) := \mathrm{R}\Gamma_{\text{ét}}(-; \mu_{p^r}^{\otimes j})[2j].$$

Standard facts from étale cohomology theory tells us that $\mathrm{E}(j)$ is an $\mathbb{A}^1$-invariant Nisnevich sheaf. The theory of chern classes in étale cohomology produces, for each line bundle $\mathcal{L}$ on X, a first Chern class class $c_1(\mathcal{L}) \in \mathrm{H}^2_{\text{ét}}(\mathrm{X}; \mu_{p^r})$. From this we can produce the $\mathbb{P}^1$-bundle datum at all levels; the fact that the map

$$\mathrm{H}^i_{\text{ét}}(\mathrm{X}; \mu_{p^r}^{\otimes j}) \oplus \mathrm{H}^{i-2}_{\text{ét}}(\mathrm{X}; \mu_{p^r}^{\otimes j-1}) \xrightarrow{\pi^* \oplus \pi^* \cup c_1(\mathcal{L})} \mathrm{H}^i_{\text{ét}}(\mathbb{P}^1 \times \mathrm{X}; \mu_{p^r}^{\otimes j})$$

is an equivalence (the projective bundle formula) tells us that we have a homotopy invariant motivic cohomology theory prescribed by $\mathrm{E}(\bullet) := \mathrm{R}\Gamma_{\text{ét}}(-; \mu_{p^r}^{\otimes \bullet})$. We denote this as $\mathrm{H}_{\text{ét}}\mu_{p^r}$

Example 1.4.3. If B is a regular base scheme, then K-theory is $\mathbb{A}^1$-invariant. It is always a Nisnevich sheaf and satisfies the $\mathbb{P}^1$-bundle formula via the decomposition

$$\mathrm{K}(\mathbb{P}^1_{\mathrm{X}}) \simeq \mathrm{K}(\mathrm{X})\{\mathcal{O}\} \oplus \mathrm{K}(\mathrm{X})\{\mathcal{O} - \mathcal{O}(1)\}.$$

Therefore we set

$$\mathrm{KGL}(j) := \mathrm{K},$$

in this situation. Note that weights are irrelevant here. We denote this as KGL

Example 1.4.4. Let B be the spectrum of a field. Set

$$\mathrm{E}(j) := \mathbb{Z}(j)^{\mathrm{mot}}[2j];$$

results of Bloch and Levine says that $\mathrm{E}(j)$ forms a homotopy invariant motivic cohomology theory where the bonding maps are given essentially by (1.3.3) which unpacks to the isomorphism: $\mathrm{Pic}(\mathrm{X}) \cong \mathrm{H}^2_{\mathrm{mot}}(\mathrm{X};\mathbb{Z}(1))$. We denote this by $\mathrm{H}\mathbb{Z}$.

Remark 1.4.5 (The ∞-category $\mathbf{SH}(\mathrm{B})$)**.** We can package everything into an ∞-category, but this maneuver only starts becoming really useful when we speak of symmetric monoidal structures. In particular, homotopy invariant motivic cohomology theories are objects of a symmetric monoidal stable ∞-category $\mathbf{SH}(\mathrm{B})$ called **motivic spectra**. It comes equipped with a symmetric monoidal functor

$$\mathrm{M}_{\mathrm{B}}(-) : \mathrm{Sm}_{\mathrm{B}} \to \mathbf{SH}(\mathrm{B})$$

assigning to $\mathrm{X} \in \mathrm{Sm}_{\mathrm{B}}$ its **relative** **B-motive**; often we suppress B when the context is clear. The **unit motive** over B is defined to be M of B itself:

$$\mathrm{M}(\mathrm{B}) := \mathbb{S}_{\mathrm{B}}$$

Sometimes we insist on pointing X; say $x \hookrightarrow \mathrm{X}$ is a B-point of X, then we set

$$\mathrm{M}(\mathrm{X}, x) := \mathrm{cofib}(\mathbb{S}_{\mathrm{B}} = \mathrm{M}(x) \to \mathrm{M}(\mathrm{X})).$$

The ∞-category $\mathbf{SH}(\mathrm{B})$ modifies the essential image of $\mathrm{M}(-)$ in a certain way because we insist on $\mathbb{A}^1$-invariance, Nisnevich descent and the projective bundle formula; for a standard example:

$$\mathrm{M}(\mathbb{P}^1, 1) \simeq \mathrm{M}(\mathbb{G}_m, 1)[2].$$

More importantly, the ∞-category $\mathbf{SH}(\mathrm{B})$ is characterized by a precise universal property as the place where the motive $\mathrm{M}(\mathbb{P}^1, 1)$ becomes invertible; this is what imposing the projective bundle formula effectively does. Hence it makes sense to speak of $\mathrm{M}(\mathbb{P}^1, 1)^{\otimes -q}$ for any $q \in \mathbb{Z}$.

In any event we set:

$$\mathbb{S}^{p,q} := \mathrm{M}(\mathbb{G}_m, 1)^{\otimes q}[p-q] \qquad p, q \in \mathbb{Z}.$$

If $\mathrm{E} = \{\mathrm{E}(\bullet)\}$ is an $\mathbb{A}^1$-invariant motivic cohomology theory then we get

$$[\mathrm{M}(\mathrm{X}), \mathbb{S}^{p,q} \otimes \mathrm{E}]_{\mathbf{SH}(\mathrm{B})} \simeq \pi_{q-p}(\mathrm{E}(q)(\mathrm{X})).$$

We also write

$$\mathrm{E}_{p,q}(\mathrm{X}) := [\mathrm{M}(\mathrm{X}), \mathbb{S}^{p,q} \otimes \mathrm{E}]_{\mathbf{SH}(\mathrm{B})}.$$

For the most part $\mathbf{SH}(\mathrm{B})$ is merely a bookkeeping device and we try as much as possible to give concrete formulations of the various statements involved.

Remark 1.4.6 (Extensions)**.** A scheme is **essentially smooth** over B if it can be written as a cofiltered limit of smooth B-schemes with affine transition maps; for example field extensions are smooth over the base. Another class of examples we will soon see also include (semi)localizations of smooth schemes at points. Any functor $\mathrm{E} : \mathrm{Sm}_{\mathrm{B}}^{\mathrm{op}} \to \mathrm{Spt}$ can be extended to essentially smooth schemes by left Kan extension and we will implicitly always do this when we speak of values on essentially smooth schemes.

Remark 1.4.7 (Homotopy sheaves)**.** Given an $\mathbb{A}^1$-invariant motivic cohomology theory E, we can consider the following Nisnevich sheaf on Sm_{B}

$$\underline{\pi}_i(\mathrm{E})_{-j} := a_{\mathrm{Nis}}\left(\mathrm{U} \mapsto [\mathrm{M}(\mathrm{U})[i], \mathbb{S}^{j,j} \otimes \mathrm{E}] \simeq [\mathrm{M}(\mathrm{U})[i], \mathrm{E}(j)[-j]]\right).$$

These are called the **homotopy sheaves** of E; they act like cohomology sheaves in usual algebraic geometry in that there is a descent spectral sequence for any $\mathrm{X} \in \mathrm{Sm}_{\mathrm{B}}$:

$$\mathrm{H}^p_{\mathrm{Nis}}(\mathrm{X}; \pi_q(\mathrm{E})_j) \Rightarrow \mathrm{E}_{q-p,j}(\mathrm{X}).$$

We should think of homotopy sheaves as a $\mathbb{Z}$-graded object

$$\underline{\pi}_i(\mathrm{E})_*$$

where, when $i = 0$, extracts the "(j, j)" part of the cohomology theory. So, for example, for any field L, Nesterenko-Suslin and Totaro's theorem says that

$$[\mathrm{M}(\mathrm{L}), \Sigma^{*,*}\mathrm{H}\mathbb{Z}] \cong \pi_0(\mathrm{H}\mathbb{Z})_{-*}(\mathrm{L}) \cong \mathrm{K}^{\mathrm{M}}_*(\mathrm{L}).$$

In this situation, so over a field base, we can then define Milnor K-theory of schemes via

$$\pi_0(\mathrm{H}\mathbb{Z})_{-j}(\mathrm{X}) =: \mathrm{K}^{\mathrm{M}}_j(\mathrm{X}).$$

1.5. **Further structure I: the homotopy t-structure.** In basic algebraic topology and homological algebra, t-structures allow us to speak of homotopy groups and discrete objects. Roughly speaking (see [Lur17, Section 1.2.1] for details), we have two subcategories $\mathcal{C}_{\geqslant 0}$ and $\mathcal{C}_{\leqslant 0}$, the first of which is stable under [1], the second of which is stable under [−1], such that there are no maps from $\mathcal{C}_{\geqslant 0}$ to $\mathcal{C}_{\leqslant 0}[-1]$. Any object in $\mathcal{C}$ decomposes into a cofiber sequence

$$\mathrm{X}_{\geqslant 0} \to \mathrm{X} \to \mathrm{X}_{\leqslant -1}$$

where $\mathrm{X}_{\geqslant 0} \in \mathcal{C}_{\geqslant 0}$ and $\mathrm{X}_{\leqslant -1} \in \mathcal{C}_{\leqslant -1}$. The key point, a result due to Beilinson, Berstein, Deligne and Gabber, is that $\mathcal{C}_{\geqslant 0} \cap \mathcal{C}_{\leqslant 0} =: \mathcal{C}^{\heartsuit}$ is an abelian category; these objects in the **heart** which is one way we can speak of objects being discrete. The embedding $\mathcal{C}^{\heartsuit} \hookrightarrow \mathcal{C}$ lets us think of discrete objects as objects in $\mathcal{C}$.

Example 1.5.1. There is the **standard t-structure** in $\mathbf{D}(\Lambda)$ whose heart is given by Λ-modules; $\mathbf{D}(\Lambda)_{\geqslant 0}$ ($\mathbf{D}(\Lambda)_{\leqslant 0}$) is given by those objects whose cohomology are strictly concentrated in non-positive degrees (in non-negative degrees). On Spt, the **standard t-structure** has, as heart, abelian groups; $\mathrm{Spt}_{\geqslant 0}$ ($\mathrm{Spt}_{\leqslant 0}$) is given by spectra whose homotopy groups lie in non-negative (non-positive) degrees. We call objects in $\mathrm{Spt}_{\geqslant 0}$ and $\mathbf{D}(\Lambda)_{\geqslant 0}$ as **connective**.

The construction of a t-structure on $\mathbf{SH}(k)$ is non-obvious and there is actual geometric content in doing this. For now, let us summarize it:

Theorem 1.5.2 (Morel)**.** *Let k be a perfect field, then there is a t-structure on $\mathbf{SH}(k)$ such that:*

(1) $\mathrm{E} \in \mathbf{SH}(k)_{\geqslant 0}$ if and only if $\underline{\pi}_i(\mathrm{E})_*(\mathrm{L}) = 0$ for all $i < 0$ and for all finitely generated field extensions L;

(2) $\mathrm{E} \in \mathbf{SH}(k)_{\leqslant 0}$ if and only if $\underline{\pi}_i(\mathrm{E})_*(\mathrm{L}) = 0$ for all $i > 0$ and for all finitely generated field extensions L.

In particular, $\pi_0(\mathrm{E})_*$ defines canonically objects in $\mathbf{SH}(k)$ such that

$$[\mathrm{M}(\mathrm{X}), \mathbb{S}^{p,q} \otimes \pi_0(\mathrm{E})_*] \cong \mathrm{H}^{p-q}_{\mathrm{Zar}}(\mathrm{X}, \pi_0(\mathrm{E})_{-q}).$$

Remark 1.5.3 (Unramified sheaves)**.** The phenomena that underlies the above theorem is one of **unramifiedness**: we say that for any field L and any discrete valuation v on L, we have maps $\partial_v : \mathrm{K}^{\mathrm{M}}_j(\mathrm{L}) \to \mathrm{K}^{\mathrm{M}}_{j-1}(\kappa)$. Now if $\mathrm{X} = \operatorname{Spec}\mathrm{R}$ is a smooth, affine k-scheme, we can define the **unramified Milnor K-theory**

$$\mathrm{K}^{\mathrm{M}}_j(\mathrm{R}) := \bigcap_v \left(\mathrm{K}^{\mathrm{M}}_j(k(\mathrm{X})) \xrightarrow{\partial_v} \mathrm{K}^{\mathrm{M}}_{j-1}(\kappa) \right).$$

There many other such examples like unramified étale cohomology. We note that the structure sheaf $\mathcal{O}$, as a Nisnevich sheaf of abelian groups, is unramified on normal, noetherian domains by algebraic Hartog's theorem.

Definition 1.5.4. An **$\mathbb{A}^1$-invariant homotopy module** is an object of $\mathbf{SH}(k)^{\heartsuit}$.

1.6. **Morel's theorem.** We give a discussion of Theorem 1.5.2 emphasizing on the geometry involved. Fix a perfect field k, and an $\mathbb{A}^1$-invariant Nisnevich sheaf:

$$\mathrm{E} : \mathrm{Sm}^{\mathrm{op}}_k \to \mathrm{Spt},$$

which we implicitly extend to essentially smooth schemes. One reason why imposing the $\mathbb{A}^1$-invariance condition is nontrivial is because of the following observation:

Lemma 1.6.1. *Let* $\mathrm{X} \in \mathrm{Sm}_k$ *and let* $\mathcal{O}_{\mathrm{X},x}$ *be the local ring of* X *at a closed point* $x \in \mathrm{X}$*. Then the map*

$$\pi_j \mathrm{E}(\mathcal{O}_{\mathrm{X},x}) \to \pi_j \mathrm{E}(\mathrm{Frac}(\mathcal{O}_{\mathrm{X},x}))$$

is injective for all $j \in *$*.*

In fact, we only need the above lemma for Henselian local rings, which are stalks for the Nisnevich topology. Lemma 1.6.1 reduces many questions about $\mathbb{A}^1$-invariant Nisnevich sheaves to questions about fields. To prove this result, one uses the following lemma due to Gabber [CTHK97] and its enhancement to finite fields by [HK20].

Lemma 1.6.2 (Gabber's presentation lemma)**.** *Let* X *be an affine, smooth connected scheme over a field* k *of dimension* d*,* Z *a closed subscheme of* X *and* $t_1, \cdots, t_j$ *a finite set of closed points of* X*. Then, possibly after shrinking* X *around* $t_1, \cdots, t_j$*, we can find a nonempty open* $\mathrm{V} \subset \mathbb{A}^{d-1}$ *and a map*

$$\varphi = (\psi, v) : \mathrm{U} \to \mathrm{V} \times \mathbb{A}^1$$

such that

(1) φ *is étale;*
(2) $\varphi|_{\mathrm{Z}} : \mathrm{Z} \to \mathrm{V} \times \mathbb{A}^1_{\mathrm{V}}$ *is a closed immersions;*
(3) $\psi|_{\mathrm{Z}} : \mathrm{Z} \to \mathrm{V}$ *is finite;*
(4) $\varphi^{-1}(\varphi(\mathrm{Z})) = \mathrm{Z}$*.*

Remark 1.6.3. Lemma 1.6.2 is an all-purpose result which has had applications in many areas of mathematics. The way that one thinks about it is as follows: first we are actually mostly interested in the semilocalization at around the points $t_1, \cdots, t_j$. The closed subscheme Z should be thought of as "bad" or a loci to avoid. The map $\varphi : \mathrm{X} \to \mathbb{A}^1 \times \mathrm{V}$ is a refined version of noether's normalization theorem and is really constructed from that procedure and ψ being finite around Z tells us that both φ and ψ are kind of a "simultaenous" noether normalization. In practice what one extracts is (1.6.5) which is crucial for all applications.

For any essentially smooth scheme X and $\mathrm{Z} \hookrightarrow \mathrm{X}$ a closed subscheme, possibly not smooth. We set $\mathrm{E}_{\mathrm{Z}}(\mathrm{X})$ to be the fiber of the map

$$\mathrm{E}(\mathrm{X}) \to \mathrm{E}(\mathrm{X} \smallsetminus \mathrm{Z}).$$

This is just a formal expression but should be thought of as "E with supports in Z."

Lemma 1.6.4. *Let* k *be a field. Assume that* $\mathrm{E} : \mathrm{EssSm}_k^{\mathrm{op}} \to \mathrm{Spt}$ *be a finitary Nisnevich sheaf satisfying the following condition: for any* $\mathrm{V} \in \mathrm{Sm}_k$ *with a closed subscheme* $\mathrm{W} \hookrightarrow \mathrm{V}$*, then the diagram*

$$\begin{array}{ccc} \mathrm{E}_{\mathbb{P}^1_{\mathrm{W}}}(\mathbb{P}^1_{\mathrm{V}}) & \xrightarrow{\quad j^* \quad} & \mathrm{E}_{\mathbb{A}^1_{\mathrm{W}}}(\mathbb{A}^1_{\mathrm{V}}) \\ & \searrow^{\infty^*} \quad {}^{\pi^*}\nearrow & \\ & \mathrm{E}_{\mathrm{W}}(\mathrm{V}) & \end{array}$$

commutes. Then for any $j \in \mathbf{Z}$ *and any* R *which is the local ring of a closed point* x *in a smooth* k*-scheme* X*, the map*

$$\pi_j(\mathrm{E}(\mathrm{R})) \to \pi_j(\mathrm{E}(\mathrm{F}))$$

is injective.

Proof. Let $s \in \pi_j(\mathrm{E}(\mathrm{R}))$ which is assumed to vanish when restricted to F. By possibly shrinking X, we may assume that s is defined on X and vanishes away from $\mathrm{Z} \hookrightarrow \mathrm{X}$ which is a closed subscheme of positive codimension. By definition of $\mathrm{E}_{\mathrm{Z}}(\mathrm{X})$ and the vanishing assumption, we have that s lifts to an element $\widetilde{s} \in \pi_j(\mathrm{E}_{\mathrm{Z}}(\mathrm{X}))$. To prove the result, it suffices to produce an open neighborhood U containing x and a closed subscheme Z' with $\mathrm{Z} \cap \mathrm{U} \subset \mathrm{Z}' \cap \mathrm{U}$ such that $\widetilde{s}$ vanishes on $\pi_i(\mathrm{E}_{\mathrm{Z}' \cap \mathrm{U}}(\mathrm{U}))$; because then we have a commutative diagram

$$\begin{array}{ccccc} \pi_j(\mathrm{E}_\mathrm{Z}(\mathrm{X})) & \longrightarrow & \pi_j(\mathrm{E}(\mathrm{X})) & \longrightarrow & \pi_j(\mathrm{E}(\mathrm{X} \smallsetminus \mathrm{Z})) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_j(\mathrm{E}_{\mathrm{Z} \cap \mathrm{U}}(\mathrm{U})) & \longrightarrow & \pi_j(\mathrm{E}(\mathrm{U})) & \longrightarrow & \pi_j(\mathrm{E}((\mathrm{X} \smallsetminus \mathrm{Z}') \cap \mathrm{U})). \end{array}$$

and the vanishing of $\widetilde{s}$ in $\pi_j(\mathrm{E}_{\mathrm{Z} \cap \mathrm{U}}(\mathrm{U}))$ implies that s itself vanishes in U; taking colimits we get that s vanishes on R.

Lemma 1.6.2 produces a Nisnevich square (after possibly shrinking U)

$$\begin{array}{ccc} \mathrm{U} \smallsetminus \mathrm{Z} & \longrightarrow & \mathrm{U} \\ \downarrow & & \downarrow{\scriptstyle \varphi = (v, \psi)} \\ \mathbf{A}^1_\mathrm{V} \smallsetminus (\varphi(\mathrm{Z})) & \longrightarrow & \mathbf{A}^1_\mathrm{V}; \end{array} \tag{1.6.5}$$

Set $\mathrm{Z}' := \psi^{-1}(\psi(\mathrm{Z}))$. By Nisnevich excision, we have that $\pi_i(\mathrm{E}_{\mathrm{Z} \cap \mathrm{U}}(\mathrm{U})) \cong \pi_i(\mathrm{E}_{\mathbf{A}^1_\mathrm{V} \cap \varphi(\mathrm{Z} \cap \mathrm{U})}(\mathbf{A}^1_\mathrm{V}))$. In this case, we have a commutative diagram

$$\begin{array}{ccc} \pi_i(\mathrm{E}_\mathrm{Z}(\mathrm{U})) & \longrightarrow & \pi_i(\mathrm{E}_{\mathrm{Z}'}(\mathrm{U})) \\ {\scriptstyle \cong}\uparrow & & \uparrow \\ \pi_i(\mathrm{E}_{\varphi(\mathrm{Z})}(\mathbf{A}^1_\mathrm{V})) & \longrightarrow & \pi_i(\mathrm{E}_{\mathbf{A}^1_{\psi(\mathrm{F})}}(\mathbf{A}^1_\mathrm{V})); \end{array}$$

The point of the isomorphism on the left, which comes from Nisnevich excision, is that to finish the proof we need only show that the bottom map is zero. The map of interest is the top horizontal map of the following commutative diagram.

$$\begin{array}{ccccc} \mathrm{E}_{\varphi(\mathrm{Z})}(\mathbf{A}^1_\mathrm{V}) & \longrightarrow & \mathrm{E}_{\mathbf{A}^1_{\psi(\mathrm{Z})}}(\mathbf{A}^1_\mathrm{V}) & & \\ {\scriptstyle \simeq}\uparrow & & \uparrow & \nwarrow & \\ & & & & \mathrm{E}_{\psi(\mathrm{Z})}(\mathrm{V}) \\ & & & \nearrow & \\ \mathrm{E}_{\varphi(\mathrm{Z})}(\mathbf{P}^1_\mathrm{V}) & \longrightarrow & \mathrm{E}_{\mathbf{P}^1_{\psi(\mathrm{Z})}}(\mathbf{P}^1_\mathrm{V}) & & \end{array}$$

The triangle commutes because of the hypothesis. However, the bottom composite is zero since, $\varphi(\mathrm{Z})$ does not meet the ∞-section of $\mathbf{P}^1_\mathrm{V}$ and thus the top map is zero as desired.

□

Proof of Lemma 1.6.1. We need only verify the condition of Lemma 1.6.4, but then the upward sloping map in the diagram is an isomorphism so commutativity is trivial. □

The context for Lemma 1.6.1 is the Gersten resolution. We will need a version of this later so let us give a quick sketch. Given E an $\mathbb{A}^1$-invariant Nisnevich sheaf, we can construct a decreasing filtration (functorial on X only *a priori* for flat morphisms)

$$\mathrm{Fil}^{\geqslant j}_{\mathrm{con}} \mathrm{E}(\mathrm{X}) \to \mathrm{E}(\mathrm{X}),$$

in the following way: first we set for any closed, integral subscheme (not necessarily smooth) $\mathrm{Z} \subset \mathrm{X}$, its E-cohomology with supports

$$\mathrm{E}_\mathrm{Z}(\mathrm{X}) := \mathrm{fiber}(\mathrm{E}(\mathrm{X}) \to \mathrm{E}(\mathrm{X} \smallsetminus \mathrm{Z})).$$

Let $\mathcal{S}^j_{\mathrm{X}}$ be the poset of closed immersions of codimension $\geqslant j$; morphisms are determined by inclusions of closed subschemes: $i : \mathrm{Z} \subset \mathrm{Z}'$. For any such immersion, we have an open immersion of smooth schemes $j : \mathrm{X} \smallsetminus \mathrm{Z}' \to \mathrm{X} \smallsetminus \mathrm{Z}$; whence we can contemplate the following diagram of exact triangles

$$\begin{array}{ccccc} \mathrm{E}_{\mathrm{Z}}(\mathrm{X}) & \longrightarrow & \mathrm{E}(\mathrm{X}) & \longrightarrow & \mathrm{E}(\mathrm{X} \smallsetminus \mathrm{Z}) \\ \downarrow{\scriptstyle i_*} & & \downarrow & & \downarrow{\scriptstyle j^*} \\ \mathrm{E}_{\mathrm{Z}'}(\mathrm{X}) & \longrightarrow & \mathrm{E}(\mathrm{X}) & \longrightarrow & \mathrm{E}(\mathrm{X} \smallsetminus \mathrm{Z}') \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{E}_{(\mathrm{X}\smallsetminus\mathrm{Z})\cap\mathrm{Z}'}(\mathrm{X} \smallsetminus \mathrm{Z}) & \longrightarrow & 0 & \longrightarrow & \mathrm{E}_{(\mathrm{X}\smallsetminus\mathrm{Z})\cap\mathrm{Z}'}(\mathrm{X} \smallsetminus \mathrm{Z})[1], \end{array} \tag{1.6.6}$$

with the observation that the fiber of the map $\mathrm{E}(\mathrm{X} \smallsetminus \mathrm{Z}) \to \mathrm{E}(\mathrm{X} \smallsetminus \mathrm{Z}')$ is the exactly the cohomology with supports at $(\mathrm{X} \smallsetminus \mathrm{Z}) \cap \mathrm{Z}'$. We set

$$\mathrm{Fil}^{\geqslant j}_{\mathrm{con}}\mathrm{E}(\mathrm{X}) := \operatorname*{colim}_{\mathrm{Z}\in\mathcal{S}^j_{\mathrm{X}}} \mathrm{E}_{\mathrm{Z}}(\mathrm{X}),$$

whence (assuming that X is irreducible) we get a filtered object in $\mathcal{C}$:

$$\cdots \to \mathrm{Fil}^{\geqslant j}_{\mathrm{con}}\mathrm{E}(\mathrm{X}) \to \cdots \mathrm{Fil}^{\geqslant 1}_{\mathrm{con}}\mathrm{E}(\mathrm{X}) \to \mathrm{Fil}^{\geqslant 0}_{\mathrm{con}}\mathrm{E}(\mathrm{X}) = \mathrm{E}(\mathrm{X})$$

We write, following usual conventions, $\mathrm{X}^{(j)}$ for the codimension j points of X, i.e., points $x \in \mathrm{X}$ whose closure $\overline{\{x\}}$ defines a codimension j integral subscheme. We also extend E to essentially smooth k-schemes via filtered colimits in the usual manner. With these conventions, the graded pieces of the convieau filtration is given, by considerations of the diagram (1.6.6), as

$$\mathrm{gr}^j_{\mathrm{con}}\mathrm{E}(\mathrm{X}) := \mathrm{cofiber}\left(\mathrm{Fil}^{\geqslant j+1}_{\mathrm{con}}\mathrm{E}(\mathrm{X}) \to \mathrm{Fil}^{\geqslant j}_{\mathrm{con}}\mathrm{E}(\mathrm{X})\right) \simeq \bigoplus_{x\in\mathrm{X}^{(j)}} \mathrm{E}_x(\mathrm{X}), \tag{1.6.7}$$

where $\mathrm{E}_x(\mathrm{X})$ is defined as

$$\mathrm{E}_x(\mathrm{X}) := \operatorname*{colim}_{x\in\mathrm{U}} \mathrm{E}_{\overline{\{x\}}}(\mathrm{U}),$$

where the colimit is taken through all opens in X containing the point $x \in \mathrm{U}$; so far this is just a formal expression of local cohomology. The usual yoga of spectral sequences, elaborated, gives us maps of the form:

$$\mathrm{gr}^j_{\mathrm{con}}\mathrm{E}(\mathrm{X}) \to \mathrm{Fil}^{\geqslant j+1}_{\mathrm{con}}\mathrm{E}(\mathrm{X})[1] \to \mathrm{gr}^{j+1}_{\mathrm{con}}\mathrm{E}(\mathrm{X})[1],$$

which are differentials in the following complex:

Definition 1.6.8. A **coniveau** complex for E(X) is one of the form

$$\bigoplus_{x\in\mathrm{X}^{(0)}} \pi_i(\mathrm{E}_x(\mathrm{X})) \to \cdots \to \bigoplus_{x\in\mathrm{X}^{(j)}} \pi_{i-j}(\mathrm{E}_x(\mathrm{X})) \to \bigoplus_{x\in\mathrm{X}^{(j+1)}} \pi_{i-j-1}(\mathrm{E}_x(\mathrm{X})) \cdots.$$

A **coniveau presheaf** is a presheaf of complexes on the small Zariski site of X given by

$$\mathrm{X}^{\mathrm{op}}_{\mathrm{Zar}} \to \mathrm{K}(\mathbb{Z})$$

$$\mathrm{U} \mapsto \left(\bigoplus_{x\in\mathrm{U}^{(0)}} \pi_i(\mathrm{E}_x(\mathrm{U})) \to \cdots \to \bigoplus_{x\in\mathrm{U}^{(j)}} \pi_{i-j}(\mathrm{E}_x(\mathrm{U})) \to \bigoplus_{x\in\mathrm{X}^{(j+1)}} \pi_{i-j-1}(\mathrm{E}_x(\mathrm{U})) \cdots\right).$$

Each term in the complex is equivalent to the direct sum of Nisnevich sheaves of abelian groups

$$\bigoplus_{x\in\mathrm{X}^{(j)}} i_{x*}\pi_{i-j}(\mathrm{E}_x(\mathrm{X}));$$

Each term of the coniveau presheaf is a Nisnevich sheaf of abelian groups which are, in fact, flasque (most easily seen by its description as a sum of skyscraper sheaves).

To get a better handle on the groups $\pi_*(\mathrm{E}_x(\mathrm{U}))$ we can demand something like a "purity isomorphism." So assume that E is actually an $\mathbb{A}^1$-invariant motivic cohomology theory: so we that we have a graded structure $\{\mathrm{E}(n)\}$. For the discussion above, we can plug in E for $\mathrm{E}(n)$ at a particular weight. In the generality stated below, the following result is due to Morel-Voevodsky [].

Theorem 1.6.9. *Let* (X,Z) *be a smooth pair (*$\mathrm{Z} \hookrightarrow \mathrm{X}$ *is a closed immersion and both are smooth k-schemes) of pure codimension j. Then there exists a collection of equivalences*

$$\alpha_{(\mathrm{X},\mathrm{Z})} : \mathrm{E}_{\mathrm{Z}}(n-j)(\mathrm{X})[-2j] \xrightarrow{\simeq} \mathrm{E}(n)(\mathrm{Z}),$$

which are compatible under base change along étale morphisms.

With this result, the complex of Definition 1.6.8 simplies into

$$\bigoplus_{x\in \mathrm{X}^{(j)}} \pi_{i-j}(\mathrm{E}(n)_x(\mathrm{X})) \cong \bigoplus_{x\in \mathrm{X}^{(j)}} \pi_{i+j}(\mathrm{E}(n-j)(\kappa(x)));$$

whence we have a complex (assuming X is irreducible for even greater simplicity)

(1.6.10)

$$\pi_i(\mathrm{E}(n)(k(\mathrm{X}))) \to \cdots \to \bigoplus_{x\in \mathrm{X}^{(j)}} \pi_{i-j}(\mathrm{E}(n-j)(\kappa(x))) \to \bigoplus_{x\in \mathrm{X}^{(j+1)}} \pi_{i-j-1}(\mathrm{E}(n-j-1)(\kappa(x))) \cdots .$$

In this case, we call the complex (1.6.10), the **Gersten complex**; as a presheaf we denote it by

$$\mathrm{C}^*_{\mathrm{E}(n),i} : \mathrm{X}^{\mathrm{op}}_{\mathrm{Zar}} \to \mathrm{K}(\mathbb{Z}).$$

Let $\underline{\pi}_i^{\mathrm{Zar}}(\mathrm{E}(n))$ be the Zariski sheaf associated to $\mathrm{U} \mapsto \pi_i(\mathrm{E}(n))(\mathrm{U})$. Now we can augment $\mathrm{C}^*_{\mathrm{E}(\bullet),i}$ by the map

$$\underline{\pi}_i^{\mathrm{Zar}}(\mathrm{E}(n)) \to \mathrm{C}^*_{\mathrm{E}(n),i};$$

we have seen that locally on X we have a local injection for any local ring of X at a closed point $x \in \mathrm{X}$:

$$\underline{\pi}_i(\mathrm{E}(n))(\mathcal{O}) \hookrightarrow \mathrm{C}^0_{\mathrm{E}(n),i}(\mathcal{O}) = \underline{\pi}_i(\mathrm{E}(n))(\mathrm{Frac}(\mathcal{O})).$$

Lemma 1.6.1 actually leads to:

Theorem 1.6.11 (Gersten exactness)**.** *For any* $\mathbb{A}^1$*-invariant motivic cohomology theory* $\{\mathrm{E}(\bullet)\}$ *and any n and i, the complex* $\mathrm{C}^*_{\mathrm{E}(n),i}$ *is exact and is a resolution of the sheaf* $\underline{\pi}_i(\mathrm{E}(n))$*. In particular* $\mathrm{H}^j_{\mathrm{Zar}}(\mathrm{X};\underline{\pi}_i(\mathrm{E}(n)))$ *is the cohomology of the complex:*

$$[\bigoplus_{x\in \mathrm{X}^{(j+1)}} \pi_{i-j+1}(\mathrm{E}(n-j+1)(\kappa(x)) \to \bigoplus_{x\in \mathrm{X}^{(j)}} \pi_{i-j}(\mathrm{E}(n-j)(\kappa(x)) \to \bigoplus_{x\in \mathrm{X}^{(j-1)}} \pi_{i-j-1}(\mathrm{E}(n-j-1)(\kappa(x))].$$

Example 1.6.12. The Gersten complex associated to algebraic K-theory (which has a purity structure by Quillen's devissage theorem) looks like:

$$\mathrm{U} \mapsto \left(\bigoplus_{x\in \mathrm{U}^{(0)}} \mathrm{K}_i(k(\mathrm{U})) \to \cdots \to \bigoplus_{x\in \mathrm{U}^{(j)}} \mathrm{K}_{i-j}(\kappa(x))) \to \bigoplus_{x\in \mathrm{U}^{(j+1)}} \mathrm{K}_{i-j-1}(\kappa(x)) \cdots \right).$$

The "tail end" of the global sections of the complex is most interesting (we are in the special case of K-theory here):

$$\bigoplus_{x\in \mathrm{X}^{(j+1)}} \kappa(x)^\times \to \bigoplus_{x\in \mathrm{X}^{(j)}} \mathbb{Z} \cong \left(\mathrm{Z}^j(\mathrm{X})\right).$$

This reproves Quillen's theorem:

$$\mathrm{H}^j_{\mathrm{Zar}}(\mathrm{X};\mathrm{K}_j) \cong \mathrm{CH}^j(\mathrm{X}).$$

Sketch proof of Theorem 1.5.2. Let $\mathbf{SH}^{S^1}(k)$ be the stable ∞-category of $\mathbb{A}^1$-invariant Nisnevich sheaves. By standard formalism in sheaf theory, there is a t-structure on $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k; \mathrm{Spt})$ whose non-negative (non-positive) parts is described as those sheaves of Spectra with vanishing negative (positive) homotopy sheaves. The way that this goes should be thought of like this: we can just take $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k; \mathrm{Spt})_{\geqslant 0} \subset \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k; \mathrm{Spt})$ and formally create a t-structure out of this [Lur17, Proposition 1.4.4.11], but this procedure is abstract and we don't necessarily know what the nonpositive part looks like. The computation of the negative part then boils down to a computation of the nonpositive part of the standard t-structure for spectra: this is the idea that any spectrum E which receives no nontrivial map from a connective spectrum are exactly those whose homotopy groups are concentrated in negative degrees.

Now, there is a functor $\mathrm{L}_{\mathbb{A}^1} : \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k; \mathrm{Spt}) \to \mathbf{SH}^{S^1}(k)$ which enforces $\mathbb{A}^1$-invariance. In order to induce a t-structure with a similar description, we need to prove that $\mathbb{A}^1$-localization preserves connectivity; this is Morel's stable connectivity theorem [Mor05, Theorem 6.1.8]. To prove this result, let E be a connective Nisnevich sheaf of spectra, then one first proves that for a 0-dimensional scheme X in Sm_k that

$$\pi_j \mathrm{L}_{\mathbb{A}^1}\mathrm{E}(\mathrm{X}) = 0 \qquad j < 0.$$

This uses an explicit model for $\mathrm{L}_{\mathbb{A}^1}$ and the fact that X is Krull dimension zero [Mor05, Corollary 4.3.3]. Lemma 1.6.1 then tells us that $\mathrm{L}_{\mathbb{A}^1}\mathrm{E}$ induces injections on stalks on homotopy sheaves and thus the required vanishing follows.

An argument with $\mathbb{P}^1$-loops is required to go from $\mathbf{SH}^{S^1}(k)$ to $\mathbf{SH}(k)$, which we will skip. □

1.7. **Reformulating the Geisser-Levine theorem.** Now, as stated in Remark 1.4.7, Milnor K-theory of fields are naturally found as values of a certain $\mathbb{A}^1$-invariant homotopy module, namely the one associated to $\mathrm{H}\mathbb{Z}$. We have a map $\mathrm{H}\mathbb{Z} \to \pi_0(\mathrm{H}\mathbb{Z})_*$. The following is a key point of the Geisser-Levine theorem.

Theorem 1.7.1. *Let k be a perfect field of characteristic $p > 0$, then $\mathrm{H}\mathbb{Z}/p \to \pi_0(\mathrm{H}\mathbb{Z}/p)_*$ is an equivalence in $\mathbf{SH}(k)$.*

Remark 1.7.2. Theorems 1.7.1 and 1.2.1 are equivalent.

1.8. **Further structure II: effective motivic spectra and slice filtrations.** Now we discuss the mechanism by which we can prove Theorem 1.7.1.

Definition 1.8.1. Let B be a base scheme. The ∞-category of **effective motivic spectra** is the full stable subcategory, closed under all colimits, spanned by M(X) for any $\mathrm{X} \in \mathrm{Sm}_{\mathrm{B}}$.

Remark 1.8.2. Effective motivic spectra is the correct analog, in this world, for what it means for a spectrum to be connective. Roughly speaking it says that E can be built, via colimits, just using schemes and not expressions like $\mathrm{M}(\mathbb{P}^1, 1)^{\otimes < 0}$. Be warned, however, that to check that a spectrum is connective one only has to prove that $\pi_{<0}\mathrm{E} = 0$.

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Department of Mathematics, Harvard University, 1 Oxford St. Cambridge, MA 02138, USA
E-mail address: elmanto@math.harvard.edu
URL: https://www.eldenelmanto.com/