LECTURE 0: THE ONE IN WHICH WE SET p TO BE ZERO

ELDEN ELMANTO

The goal of this class is to explain the constituent pieces of the following result due to the instructor and Matthew Morrow.

Theorem 0.0.1 (E.-Morrow)**.** *Let k be a field and X a quasicompact, quasiseparated k-scheme. Then there exists functorial complexes*

$$\mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X}) \in \mathbf{D}(\mathbb{Z}) \qquad j \geqslant 0$$

such that

(Descent) *The functor*

$$\mathrm{X} \mapsto \mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X}),$$

defines a Nisnevich sheaf.

(Atiyah-Hirzebruch SS) *There is a spectral sequence*

$$\mathrm{E}_2^{i,j} = \mathrm{H}^{i-j}(\mathbb{Z}(-j)^{\mathrm{mot}}(\mathrm{X})) \Rightarrow \mathrm{K}_{-i-j}(\mathrm{X}), \tag{0.0.2}$$

which is convergent whenever X has finite valuative dimension. It degenerates rationally. From now on write

$$\mathrm{H}^i_{\mathrm{mot}}(\mathrm{X};\mathbb{Z}(j)) := \mathrm{H}^i(\mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X})).$$

(étale comparison) *if p is prime to the characteristic of k, then there is a natural isomorphism*

$$\mathrm{H}^i_{\mathrm{mot}}(\mathrm{X};\mathbb{Z}/p(j)) \cong \mathrm{H}^i_{\mathrm{\acute{e}t}}(\mathrm{X};\mathbb{Z}/p(j)) \qquad i \leqslant j.$$

(p-adic comparison) *if p is zero in k, then there is a cartesian square*

$$\begin{array}{ccc} \mathbb{Z}/p(j)^{\mathrm{mot}}(\mathrm{X}) & \longrightarrow & \mathbb{Z}/p(j)^{\mathrm{syn}}(\mathrm{X}) \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma_{\mathrm{cdh}}(\mathrm{X};\Omega^j_{\mathrm{log}})[-j] & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{eh}}(\mathrm{X};\Omega^j_{\mathrm{log}})[-j]. \end{array} \tag{0.0.3}$$

(Hodge comparison) *If k is characteristic zero, then we have a cartesian square*

$$\begin{array}{ccc} \mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X}) & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{Zar}}(\mathrm{X}, \widehat{\mathrm{L}\Omega^{\geqslant j}_{(-)/k}}) \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma_{\mathrm{cdh}}(\mathrm{X}; z^j(-,\bullet)[-2j]) & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{cdh}}(\mathrm{X}, \widehat{\mathrm{L}\Omega^{\geqslant j}_{(-)/k}}); \end{array} \tag{0.0.4}$$

where $z^j(-,\bullet)$ is the presheaf of Bloch's cycle complex of codimension j and $\widehat{\mathrm{L}\Omega^{\geqslant j}}$ is the j-th step of the Hodge filtration on the Hodge-completed derived de Rham complex.

(Weight zero) *We have an equivalence*

$$\mathbf{Z}(0)^{\mathrm{mot}}(\mathrm{X}) \simeq \mathrm{R}\Gamma_{\mathrm{cdh}}(\mathrm{X};\mathbb{Z}).$$

(Cycles comparison) *We have*

$$\mathrm{H}^{2j}_{\mathrm{mot}}(\mathrm{X};\mathbb{Z}(j)) = \begin{cases} \mathrm{CH}^j(\mathrm{X}) & \text{if } \mathrm{X} \text{ is smooth} \\ \mathrm{Pic}(\mathrm{X}) & j = 1 \\ \mathrm{CH}^{\mathrm{LW}}_0(\mathrm{X}) & \mathrm{X} \text{ is reduced, noetherian surface} \end{cases}$$

where $\mathrm{CH}_0^{\mathrm{LW}}(\mathrm{X})$ *is Levine-Weibel's zero cycles group.*

(Projective bundles) There are natural classes $c_1(\mathcal{O}(1)) \in \mathrm{H}^2_{\mathrm{mot}}(\mathbb{P}^r_{\mathrm{X}};\mathbb{Z}(1))$ *which induces a natural isomorphism*

$$\mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X}) \oplus \cdots \oplus \mathbb{Z}(j-r)^{\mathrm{mot}}(\mathrm{X})[-2r] \xrightarrow{\pi^*\oplus\cdots\oplus\pi^*(-)\cup c_1(\mathcal{O}(1))^r} \mathbb{Z}(j)^{\mathrm{mot}}(\mathbb{P}^r_{\mathrm{X}})$$

(Milnor K*-theory) For any local* k*-algebra* A*, we have an isomorphism*

$$\mathrm{K}^{\mathrm{M}}_j(\mathrm{A}) \cong \mathrm{H}^j_{\mathrm{mot}}(\mathrm{Spec}\,\mathrm{A};\mathbb{Z}(j)).$$

(Weibel vanishing) If X *has finite valuative dimension, then*

$$\mathrm{H}^i_{\mathrm{mot}}(\mathrm{X};\mathbb{Z}(j)) = 0 \qquad i > j + \mathrm{vdim}(\mathrm{X}).$$

(Blowup descent) Let X *be noetherian and suppose that* $\mathrm{Z} \hookrightarrow \mathrm{X}$ *is closed immersion. Then we have a pro-cartesian square:*

$$\begin{array}{ccc} \mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{X}) & \longrightarrow & \{\mathbb{Z}(j)^{\mathrm{mot}}(r\mathrm{Z})\} \\ \downarrow & & \downarrow \\ \mathbb{Z}(j)^{\mathrm{mot}}(\mathrm{Bl}_{\mathrm{Z}}(\mathrm{X})) & \longrightarrow & \{\mathbb{Z}(j)^{\mathrm{mot}}(r\mathrm{E})\}, \end{array}$$

where E *is the exceptional divisor of the blowup* $\mathrm{Bl}_{\mathrm{Z}}(\mathrm{X}) \to \mathrm{X}$.

Theorem 0.0.1 is summarized by saying that there is a good theory of motivic cohomology of singular schemes. It is a culmination of the body of work of many mathematicians including Beilinson, Bloch, Cortiñas, Friedlander, Geisser, Haesemeyer, Kato, Levine, Lichtenbaum, Milne, Rost, Suslin, Voevodsky, Weibel. I will outline a brief history, as I understood it:

(1) The first lucid account, to my knowledge, of motivic cohomology is written in [BMS87]; this is a followup to Beilinson's notes [Bei87] where he laid out some expected properties of motivic cohomology in the final section. It is a provocative paper which I highly encourage everyone to read; it starts with a thought experiment: what if we knew what topological K-theory was before singular cohomology? Before Beilinson's papers appeared, however, Lichtenbaum had made some conjectures on the étale versions of the story [Lic84] in relation to zeta values at non-negative integers.
(2) Around the same time, Milne saw the logarithmic de Rham-Witt sheaves [Mil86] as motivic objects via his investigation of the special values of zeta functions over finite fields.
(3) Spencer Bloch and Kazuya Kato were investigating analogs of the de Rham comparison theorem in the p-adic context; they proposed their famous conjecture in [BK86] and proved some cases of this conjecture.
(4) Spencer Bloch later defined his cycle complexes as a candidate in [Blo86] and the construction of its relationship with algebraic K-theory was sketched in a preprint with Stephen Lichtenbaum. Later, Friedlander and Suslin [FS02] globalized the Bloch-Lichtenbaum construction to smooth schemes over a field;
(5) Marc Levine revisited Bloch's complexes [Lev94] and gave a different method for globalizing the Bloch-Lichtenbaum spectral sequence [Lev01]. The first complete account, to the instructor's knowledge, of the motivic spectral sequence is Levine's machinery of homotopy coniveau tower [Lev06, Lev08].
(6) Around the same time a young mathematician Vladimir Voevodsky had the vision to reproduce the motivic spectral sequence using his newly-minted theory of motivic homotopy theory [Voe02]. He broke down the construction of the motivic spectral sequence into a series of conjectures internal to stable motivic homotopy theory. The required conjectures were solved by Levine in [Lev08].
(7) Thomas Geisser and Marc Levine wrote the massively influential [GL00], describing fully p-adic motivic cohomology for smooth schemes in characteristic $p > 0$; we will

discuss much of this result from a modern viewpoint. One interpretation of this result is to relate Bloch's cycle complexes with its étale counterpart, at the prime p.

(8) Away from the prime, the counterpart to the above result is Rost-Voevodsky's famous theorem which resolves the Bloch-Kato, Beilinson-Lichtenbaum conjectures [Voe11, Voe03] using the machinery of motivic homotopy theory. The influence of Suslin's lectures in Luminy regarding motivic homology cannot be underestimated in this whole program.

(9) In a related but different thread, the work of Cortiñas, Haesemeyer, Weibel and their collaborators studied K-theory in characteristic zero using differential methods, viewing them as motivic objects in characteristic zero [CnHSW08, CnHW08, CnHWW10].

Most of this class will concern the mod-p aspect of the theory, but the characteristic zero part will serve as inspiration. So what makes the above theory "motivic"? More generally what does it mean for something to be "motivic." It is one of those things where I don't really know what it is, but I know it when I see it. Part of my goal in this class is to impart the idea of motives as a "mathematical lifestyle" and not so much a concrete object that one can (or even should) define. Visually, something motivic has two gradings: a cohomological piece and a weight. Perhaps the most familiar instantiation of this the following result:

Theorem 0.0.5. *Let* X *be a smooth, projective* $\mathbb{C}$*-variety, then there is a natural isomorphism*

$$\mathrm{H}^n_{\mathrm{sing}}(\mathrm{X}^{\mathrm{an}};\mathbb{C}) \cong \bigoplus_{p+q=n} \mathrm{H}^p(\mathrm{X};\Omega^q_{\mathrm{X}/\mathbb{C}}).$$

Theorem 0.0.5 holds more generally for the so-called Kähler manifolds, which are complex-analytic manifolds equipped with a certain special closed form: the point is that $\mathbb{CP}^{\mathrm{N}}$ is such a manifold and the Kähler structure is inherited by any submanifold and hence any smooth projective variety has an underlying complex manifold which is also Kähler.

One of the main points of the motivic lifestyle is the presence of *weights* which in Theorem 0.0.5 come in the form of number q appearing above in the wedge powers of the differential forms. Of course when $q = 0$, we are looking at $\mathcal{O}$, the structure sheaf of X and when $q = \dim(\mathrm{X})$ then we are looking at ω_{X}, the canonical sheaf on X. It is entirely not obvious that these are summands of complexified singular cohomology. In fact, the above theorem is extremely surprising, given that the left-hand-side is of topological nature, while the right-hand-side is of algebraic nature. In fact, we have the following "conservativity" style result which is immediate:

Corollary 0.0.6. *Let* $f : \mathrm{X} \to \mathrm{Y}$ *which is a morphism of smooth, projective* $\mathbb{C}$*-varieties which induces an isomorphism on singular cohomology of the underlying analytic space with* $\mathbb{C}$*-coefficients, then there is an isomorphism on the level of the cohomology groups* $\mathrm{H}^p(-;\Omega^q_{-/\mathbb{C}})$.

Our first order of business in this class is to actually prove Theorem 0.0.5 *using methods of characteristic* $p > 0$ *algebraic geometry.* This is another aspect of the motivic lifestyle: one should be able to freely move along characteristics and import ideas, techniques and even *actual proofs* from one to another. It will also serve to introduce some of the main characters involved in the proof of Theorem 0.0.1.

1. The Frölicher/Hodge-to-de Rham spectral sequence

Let us proceed towards the proof of Theorem 0.0.5. In the next class I will give a universal property of the de Rham complex, but for now feel free to think about it as in differential geometry.

Construction 1.0.1. Let $f : \mathrm{X} \to \mathrm{S}$ be a smooth morphism[1] then we have a chain complex of $\mathcal{O}_{\mathrm{X}}$-modules:

$$\Omega^\bullet_{\mathrm{X}/\mathrm{S}} = [0 \to \mathcal{O}_{\mathrm{X}} \xrightarrow{d} \Omega^1_{\mathrm{X}/\mathrm{S}} \to \Omega^2_{\mathrm{X}/\mathrm{S}} \to \cdots \Omega^q_{\mathrm{X}/\mathrm{S}} \to],$$

[1] What follows can be defined more generally, but are usually pathological; what one needs to do instead is to *animate*, a technique which we will find invaluable in the course of our adventure.

called the **relative de Rham complex**. We have a descending filtration

$$\cdots \to \Omega^{\geqslant j}_{X/S} \to \cdots \Omega^{\geqslant 2}_{X/S} \to \Omega^{\geqslant 1}_{X/S} \to \Omega^{\bullet}_{X/S},$$

where $\Omega^{\geqslant j}_{X/S}$ is the **stupid truncation**:

$$[0 \to 0 \to \Omega^{j}_{X/S} \to \Omega^{j+1}_{X/S} \to \cdots],$$

here $\Omega^{j}_{X/S}$ is placed in cohomological degree j so that the associated graded

$$\mathrm{cofiber}(\Omega^{\geqslant j+1}_{X/S} \to \Omega^{\geqslant j}_{X/S}) \simeq \Omega^{j}_{X/S}[-j];$$

ironically this stupid truncation turns out to be a good idea. By the formalism of **descent** we have a filtered object in the derived category of $\mathbb{Z}$

$$\mathrm{Fil}^{\geqslant j}_{\mathrm{Hdg}}(X/S) := R\Gamma(X; \Omega^{\geqslant j}_{X/S}) \to R\Gamma(X; \Omega^{\bullet}_{X/S});$$

this filtration is called the **Hodge filtration**. Its associated gradeds are

$$\mathrm{cofiber}(\mathrm{Fil}^{\geqslant j+1}_{\mathrm{Hdg}}(X/S) \to \mathrm{Fil}^{\geqslant j}_{\mathrm{Hdg}}(X/S)) \simeq R\Gamma(X, \Omega^{j}_{X/S})[-j].$$

We then have a spectral sequence

$$E_1^{i,j} = H^j(X, \Omega^i_{X/S}) \Rightarrow H^{p+q}_{\mathrm{dR}}(X/S) := H^{i+j}(R\Gamma(X; \Omega^{\bullet}_{X/S})).$$

I remark that I have not messed up the grading of the spectral sequence this time. Here's one result that we want to prove:

Theorem 1.0.2. [DI87, Corollaire 2.7] *Let* K *be a field of characteristic zero and* X *a smooth proper* K*-scheme, then the Hodge-to-de Rham spectral sequence for* $\Omega^{\bullet}_{X/\mathbb{Q}}$ *degenerates at the* E_1*-page. Consequently, there is a decreasing filtration*

$$\cdots \mathrm{Fil}^{\geqslant j+1}_{\mathrm{Hdg}} H^n_{\mathrm{dR}}(X/S) \subset \mathrm{Fil}^{\geqslant j}_{\mathrm{Hdg}} H^n_{\mathrm{dR}}(X/S) \subset \mathrm{Fil}^{\geqslant j-1}_{\mathrm{Hdg}} H^n_{\mathrm{dR}}(X/S) \subset \cdots H^n_{\mathrm{dR}}(X/S),$$

whose graded pieces are given by $H^{n-j}(X; \Omega^j)$*.*

To begin the proof, we note that by the technique of **spreading out** we have the following cartesian diagram

$$\begin{array}{ccccccc}
Y & \longrightarrow & \widetilde{Y} & \longrightarrow & \mathcal{X} & \longleftarrow & X \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p_{\mathcal{X}}} & & \downarrow \\
s & \longrightarrow & \mathrm{Spec}\, W_2(\kappa) & \longrightarrow & S & \longleftarrow & \mathrm{Spec}\, K;
\end{array}$$

in which:

(1) the morphism $p_{\mathcal{X}}$ is smooth and proper;
(2) S is smooth over $\mathbb{Z}$;
(3) there is a d such that the dimension of the fibers at any point of $\mathcal{X}$ is bounded above by d;
(4) $s \to S$ is a closed immersion and $s = \mathrm{Spec}\, \kappa$ where κ is a perfect field of dimension $p > d$;
(5) the map $\mathrm{Spec}\, W_2(\kappa) \to S$ is determined by the universal property of $s \to S$.

Remark 1.0.3. We briefly sketch the idea of spreading out. We let K be a field of characteristic zero. Then we can write

$$K = \bigcup_{\alpha \in I} A_\alpha,$$

where each A_α is a subalgebra of K, each of which is finite type over $\mathbb{Z}$. Now for a large enough value of α, we can find a smooth proper A_α-scheme $X_\alpha \to \mathrm{Spec}\, A_\alpha$ for which we have the cartesian square

$$\begin{array}{ccc} X_\alpha & \longleftarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} A_\alpha & \longleftarrow & \operatorname{Spec} K. \end{array} \tag{1.0.4}$$

Intuitively, we are allowed to do this because of the algebraic nature of X: it is defined by finitely many polynomial equations and thus only finitely many scalars in K are used to define X and thus we can choose A_α. This is the framework of **descent** (though of a different flavor from what you might be used to). To imagine that smoothness properties can be maintained recall that being smooth can be described using the Jacobian criterion [Stacks, Tag 01V9]. Properness is a bit trickier; projectivity seems more believable as we can imagine descending the closed immersion $X \hookrightarrow \mathbb{P}^N$. What one then needs to prove is a version of Chow's lemma [Stacks, Tag 01ZZ].

To proceed further we want to, at the cost of enlarging A_α, assume that it is furthermore smooth over $\mathbb{Z}$. We invoke the fact that for any morphism $f : T \to S$ locally of finite presentation, being smooth is an open condition on T [Stacks, Tag 01V9]. Since any finite type $\mathbb{Q}$-scheme is generically smooth, we can spread this smoothness on an open of A_α and conclude that there exists an element $f \in A_\alpha$ for which $A_\alpha[f^{-1}]$ is smooth over $\mathbb{Z}$. At this point we have a smooth proper scheme $\mathfrak{X}$ over $A := A_\alpha[f^{-1}]$ whose base change to K is exactly X.

Now we note that $\mathfrak{X}$ is of dimension d. We can find a large enough prime number $p > d$ such that Spec A has a point of residue characteristic p. We note that since A is finite type over $\mathbb{Z}$, such a point has a residue field which is a finite field of characteristic p. Now we invoke something small about the Witt vectors: the map $W_2(k) \to k$ is a nilpotent immersion; we then have the following lifting problem:

$$\begin{array}{ccc} \operatorname{Spec} A & \longleftarrow & \operatorname{Spec} \kappa \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{Z} & \longleftarrow & \operatorname{Spec} W_2(\kappa), \end{array}$$

which we can solve by the infinitesimal lifting criterion [Stacks, Tag 02H6].

We now invoke the following result:

Lemma 1.0.5. *Let* S *be an affine, noetherian, integral scheme and* $f : X \to S$ *be a smooth, proper morphism. For any cartesian square:*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S, \end{array}$$

(1) *up to possibly shrinking* S*, the sheaves* $R^i f'_* \Omega^j_{X'/S'}$ *are locally free of constant rank* h^{ji}*;*
(2) *up to possibly shrinking* S*, the sheaves* $R^n f_* \Omega *_{X'/S'}$ *are locally free of constant rank* h^n*.*

Proof. Let us sketch a proof of Lemma 1.0.5. First, using that f is proper, the pushforward of a coherent sheaf remains coherent [Stacks, Tag 02O3] [Gro61, Théorème 3.1.2]; the only hypothesis needed here is that S is locally noetherian (not even affine). This proves that $R^i f_* \Omega^j_{X/S}$ is coherent. Using the relative version of the Hodge-to-de Rham spectral sequence, $R^n f_* \Omega^\bullet_{X/S}$ is also coherent.

Next, I claim that, if we are willing to shrink S further, we can assume that $R^i f'_* \Omega^j_{X/S}$ and $R^n f_* \Omega^\bullet_{X/S}$ are locally free of finite type. Indeed, since S = Spec A is an integral scheme, we have a generic point $\eta = \operatorname{Spec} K$. The restriction $R^i f_* \Omega^j_{X/S}|_\eta$ is a finite dimensional vector space over K hence is a free module. Write $\mathrm{K} = \operatorname{colim} \mathrm{A}[\frac{1}{s}]$, we note that we can pick an s for which $\mathrm{R}^i f_* \Omega^j_{\mathrm{X/S}}|_{\operatorname{Spec} \mathrm{A}[\frac{1}{s}]}$ is free: choose a surjection $\mathrm{A}^{\oplus \mathrm{N}} \to \mathrm{R}^i f_* \Omega^j_{\mathrm{X/S}} \to 0$ which exists by the first paragraph, it is an isomorphism after base change to K because we are just looking at vector spaces over K. Then, by part (4) by [Stacks, Tag 05LI], it is an isomorphism at some finite stage. The same argument works for $\mathrm{R}^n f_* \Omega^\bullet_{\mathrm{X/S}}$.

So shrink S appropriately. Now let us use a base change result: we have a canonical comparison map:

$$\mathrm{L}g^* \mathrm{R} f_* \Omega^j_{\mathrm{X/S}} \to \mathrm{R} f'_* \mathrm{L} g'^* \Omega^j_{\mathrm{X/S}}$$

The correct generality for base change is when the square above is tor-independent and the morphism $f : \mathrm{X} \to \mathrm{S}$ is quasicompact and quasiseparated [Stacks, Tag 08IB]. But this concretely gives us

$$g^* \mathrm{R}^i f_* \Omega^j_{\mathrm{X/S}} \simeq \mathrm{R}^i f'_* \Omega^j_{\mathrm{X'/S'}}$$

using that: (1) $\mathrm{L}g^* \simeq g^*$ we are applying it to a locally free sheaf (by the hard work of the previous paragraph!) and (2) $g'^* \Omega^j_{\mathrm{X/S}} \simeq \Omega^j_{\mathrm{X'/S'}}$ by [Stacks, Tag 00RV]. The same argument also works for the de Rham complex. This finishes the proof.

□

So we shrink S further as dictated by Lemma 1.0.5. Therefore we are reduced to the following claim:

Theorem 1.0.6. *Let κ be a perfect field of characteristic $p > 0$ and* X *a smooth proper k-scheme, then the Hodge-to-de Rham spectral sequence for $\Omega^\bullet_{\mathrm{X}/\kappa}$ degenerates at the* E_1*-page.*

Indeed, let us write h^{ji} (resp. h^n) to be the dimension of the K-vector space $\mathrm{H}^j(\mathrm{X}; \Omega^i)$ (resp. $\mathrm{H}^n_{\mathrm{dR}}(\mathrm{X/K})$). It then suffices (and is necessary) to prove that

$$\sum_{i+j=n} h^{ji} = h^n.$$

But Theorem 1.0.6 and Lemma 1.0.5 (apply it to the de Rham complex and the Ω^j over $\mathcal{X}$) gives us the equality

$$\sum_{i+j=n} \dim_\kappa \mathrm{H}^j(\mathrm{Y}; \Omega^i_{\mathrm{Y}/\kappa}) = \dim_\kappa \mathrm{H}^n_{\mathrm{dR}}(\mathrm{Y}/\kappa),$$

which implies the desired equality by Lemma 1.0.5.

Remark 1.0.7. A theorem of Grauert's states that if $f : \mathrm{X} \to \mathrm{S}$ is a proper morphism, S is reduced and locally noetherian and $\mathcal{F}$ is a coherent sheaf on X which is $\mathcal{O}_\mathrm{S}$-flat and the function

$$s \mapsto h^j(\mathrm{X}_s; \mathcal{F}_s)$$

is locally constant, then the pushforward $\mathrm{R}^j f_* \mathcal{F}$ is locally free (see, for example, Vakil's notes Theorem 28.1.5 or see [Gro63, Proposition 7.8.4]). We can certainly use this theorem to prove Lemma 1.0.5 without having to further shrink S.

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Department of Mathematics, Harvard University, 1 Oxford St. Cambridge, MA 02138, USA
E-mail address: elmanto@math.harvard.edu
URL: https://www.eldenelmanto.com/