Online Appendix to Networks, Barriers, and Trade

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A Data Appendix

To conduct the counterfactual exercises in Sections 4 and 7, we use the World Input-Output Database (Timmer et al., 2015). We use the 2013 release of the data for the final year which has no-missing data — that is 2008. We use the 2013 release because it has more detailed information on the factor usage by industry. We aggregate the 35 industries in the database to get 30 industries to eliminate missing values, and zero domestic production shares, from the data. In Table 3, we list our aggregation scheme, as well as the elasticity of substitution, based on Caliendo and Parro (2015) and taken from Costinot and Rodriguez-Clare (2014) associated with each industry. We calibrate the model to match the input-output tables and the socio-economic accounts tables in terms of expenditure shares in steady-state (before the shock).

For the growth accounting exercise in Section J.2, we use both the 2013 and the 2016 release of the WIOD data. When we combine this data, we are able to cover a larger number of years. We compute our growth accounting decompositions for each release of the data separately, and then paste the resulting decompositions together starting with the year of overlap. To construct the consumer price index and the GDP deflator for each country, we use the final consumption weights or GDP weights of each country in each year to sum up the log price changes of each good. To arrive at the price of each good, we use the gross output prices from the socio-economic accounts tables which are reported at the (country of origin, industry) level into US dollars using the contemporaneous exchange rate, and then take log differences. This means that we assume that the log-change in the price of each good at the (origin, destination, industry of supply, industry of use) level is the same as (origin, industry of supply) level. If there are differential (changing) transportation costs over time, then this assumption is violated.

To arrive at the contemporaneous exchange rate, we use the measures of nominal GDP in the socioeconomic accounts for each year (reported in local currency) to nominal GDP in the world input-output database (reported in US dollars).

	WIOD Sector	Aggregated sector	Trade Elasticity
1	Agriculture, Hunting, Forestry and Fishing	1	8.11
2	Mining and Quarrying	2	15.72
3	Food, Beverages and Tobacco	3	2.55
4	Textiles and Textile Products	4	5.56
5	Leather, Leather and Footwear	4	5.56
6	Wood and Products of Wood and Cork	5	10.83
7	Pulp, Paper, Paper , Printing and Publishing	6	9.07
8	Coke, Refined Petroleum and Nuclear Fuel	7	51.08
9	Chemicals and Chemical Products	8	4.75
10	Rubber and Plastics	8	4.75
11	Other Non-Metallic Mineral	9	2.76
12	Basic Metals and Fabricated Metal	10	7.99
13	Machinery, Enc	11	1.52
14	Electrical and Optical Equipment	12	10.6
15	Transport Equipment	13	0.37
16	Manufacturing, Enc; Recycling	14	5
17	Electricity, Gas and Water Supply	15	5
18	Construction	16	5
19	Sale, Maintenance and Repair of Motor Vehicles...	17	5
20	Wholesale Trade and Commission Trade, ...	17	5
21	Retail Trade, Except of Motor Vehicles and...	18	5
22	Hotels and Restaurants	19	5
23	Inland Transport	20	5
24	Water Transport	21	5
25	Air Transport	22	5
26	Other Supporting and Auxiliary Transport....	23	5
27	Post and Telecommunications	24	5
28	Financial Intermediation	25	5
29	Real Estate Activities	26	5
30	Renting of M&Req and Other Business Activities	27	5
31	Public Admin/Defence; Compulsory Social Security	28	5
32	Education	29	5
33	Health and Social Work	30	5
34	Other Community, Social and Personal Services	30	5
35	Private Households with Employed Persons	30	5

Table 3: The sectors in the 2013 release of the WIOD data, and the aggregated sectors in our data.

B Duality with Multiple Factors and Tariff Revenues

The duality between trade shocks in an open economy and productivity shocks in a closed economy extends beyond the one-factor case. In the multi-factor case with pre-existing tariffs, external productivity shocks (like iceberg shocks) in the open economy translate into productivity shocks *and* shocks to factor prices in the closed economy. In this section, we establish this duality. As an example application, in Appendix C, we show how the model in Galle et al. (2017), which studies the distributional consequences of trade with a Roy model, can be generalized to economies with production networks.

With multiple factors and tariffs, we must use the change in the dual price deflator $\Delta \log \check{P}_{W_c} = \Delta \log \check{P}_{Y_c} = \Delta \log \check{p}_c$ of the dual economy for given changes in factor prices and not the change in real expenditure or welfare for given factor supplies. This requires the choice of a numeraire in the dual closed economy: we use the nominal GDP, which means that we normalize the nominal GDP of the dual closed economy to one. If there are import tariffs, the input-output table should be written gross of any tariffs (that is, including expenditures on tariffs by importers).

Theorem 7 (Exact Duality)**.** *The discrete change in welfare $\Delta \log W_c$ of the original open economy in response to discrete shocks to iceberg trade costs or productivities outside of country c is equal to (minus) the discrete change in the price deflator $-\Delta \log \check{P}_{Y_c}$ of the dual closed economy in response to discrete shocks to productivities $\Delta \log \check{A}_i = -(1/\varepsilon_i) \Delta \log \Omega_{ic}$ and discrete shocks to the productivities of the factors $\Delta \log \check{A}_f = -\Delta \log \Lambda^c_f$. This duality result is global in that it holds exactly for arbitrarily large shocks.*

In other words, shocks to the open economy are equivalent to productivity and factor price shocks in the closed economy. Note that if there are tariffs, tariff revenues imply reductions in factor income shares in the original open economy, which translates into positive shocks to the productivities of the factors in the dual closed economy.

Corollary 7 (First-Order Duality)**.** *A first-order approximation to the change in welfare of the original open economy is:*

$$\Delta \log W_c = -\Delta \log \check{P}_{Y_c} \approx \sum_{i \in M_c + F_c} \check{\lambda}_i \Delta \log \check{A}_i,$$

where applying Hulten's theorem, $\check{\lambda}_i$ is the sales share of producer i when $\in M_c$ and the sales share of factor i in the dual closed economy (which we also sometimes write $\check{\Lambda}_i$).

Corollary 8 (Second-Order Duality)**.** *A second-order approximation to the change in welfare of the original open economy is:*

$$\Delta \log W_c = -\Delta \log \check{P}_{Y_c} \approx \sum_{i \in M_c + F_c} \check{\lambda}_i \Delta \log \check{A}_i - \frac{1}{2} \sum_{i,j \in M_c + F_c} \frac{d^2 \log \check{P}_{Y_c}}{\mathrm{d} \log \check{A}_j \, \mathrm{d} \log A_i} \Delta \log \check{A}_j \Delta \log \check{A}_i,$$

where applying Baqaee and Farhi (2017a),

$$-\frac{d^2 \log \check{P}_{Y_c}}{\mathrm{d} \log \check{A}_j \, \mathrm{d} \log \check{A}_i} = \frac{\mathrm{d} \, \check{\lambda}_i}{\mathrm{d} \log \check{A}_j} = \sum_{k \in N_c} (\theta_k - 1) \check{\lambda}_k Cov_{\check{\Omega}^{(k)}} \left(\check{\Psi}_{(i)}, \check{\Psi}_{(j)} \right).$$

We can re-express the second-order approximation to the change in welfare of the original open economy as:

$$\Delta \log W_c = -\Delta \log \check{P}_{Y_c} \approx \sum_{i \in M_c + F_c} \check{\lambda}_i \Delta \log \check{A}_i + \frac{1}{2} \sum_{k \in N_c} (\theta_k - 1) \check{\lambda}_k Var_{\check{\Omega}^{(k)}} \left(\sum_{i \in M_c + F_c} \check{\Psi}_{(i)} \Delta \log \check{A}_i \right).$$

Corollary 9 (Exact Duality and Nonlinearities with an Industry Structure)**.** *For country c with an industry structure, we have the following exact characterization of the nonlinearities in welfare changes of the original open economy.*

(i) *(Industry Elasticities) Consider two economies with the same initial input-output matrix and industry structure, the same trade elasticities, but with lower elasticities across industries for one than for the other so that $\theta_\kappa \leq \theta'_\kappa$ for all industries κ. Then $\Delta \log W_c = \Delta \log \check{Y}_c \leq \Delta \log W'_c = \Delta \log \check{Y}'_c$ so that negative (positive) shocks have larger negative (smaller positive) welfare effects in the economy with the lower industry elasticities.*

(ii) *(Cobb-Douglas) Suppose that all the elasticities of substitution across industries (and with the factor) are equal to unity ($\theta_\kappa = 1$), then $\Delta \log W_c = -\Delta \log \check{P}_{Y_c}$ is linear in $\Delta \log \check{A}$.*

(iii) *(Complementarities) Suppose that all the elasticities of substitution across industries (and with the factor) are below unity ($\theta_\kappa \leq 1$), then $\Delta \log W_c = -\Delta \log \check{P}_{Y_c}$ is concave in $\Delta \log \check{A}$, and so nonlinearities amplify negative shocks and mitigate positive shocks.*

(iv) *(Substituabilities) Suppose that all the elasticities of substitution across industries (and with the factor) are above unity ($\theta_\kappa \geq 1$), then $\Delta \log W_c = -\Delta \log \check{P}_{Y_c}$ is convex in $\Delta \log \check{A}$, and so nonlinearities mitigate negative shocks and amplify positive shocks.*

(v) *(Exposure Heterogeneities) Suppose that industry κ is uniformly exposed to the shocks as they unfold, so that $Var_{\check{\Omega}_s^{(\kappa)}} \left(\sum_{\iota \in \mathcal{M}_c + \mathcal{F}_c} \check{\Psi}_{(\iota),s} \Delta \log \check{A}_\iota \right) = 0$ for all s where s indexes the*

dual closed economy with productivity shocks $\Delta \log \check{A}_{\iota,s} = s\Delta \log \check{A}_{\iota}$, *then* $\Delta \log W_c = -\Delta \log \check{P}_{Y_c}$ *is independent of* θ_κ. *Furthermore*

$$\Delta \log W_c = -\Delta \log \check{P}_{Y_c} = \sum_{\iota \in \mathcal{M}_c + \mathcal{F}_c} \check{\lambda}_\iota \Delta \log \check{A}_\iota + \int_0^1 \sum_{\kappa \in \mathcal{N}_c} (\theta_\kappa - 1) \check{\lambda}_{\kappa,s} Var_{\check{\Omega}_s^{(\kappa)}} \left(\sum_{\iota \in \mathcal{M}_c + \mathcal{F}_c} \check{\Psi}_{(\iota),s} \Delta \log \check{A}_\iota \right) (1-s) ds.$$

Closed-form Expression for Example in Figure 1

The exact expression for the impact of the trade shock on welfare can be found in closed form by exploiting the recursive structure of the contraction mapping because this example features no reproducibility:

$$\Delta \log W_c = -\frac{1}{1-\sigma} \log \left(\frac{M}{N} \left(\frac{N}{M} \check{\lambda}_E e^{-(1-\theta)\Delta \log \check{A}_E} + 1 - \frac{N}{M} \check{\lambda}_E \right)^{\frac{1-\sigma}{1-\theta}} + \frac{N-M}{N} \right).$$

C Extension to Roy Models

Galle et al. (2017) combine a Roy-model of labor supply with an Eaton-Kortum model of trade to study the effects of trade on different groups of workers in an economy. They prove an extension to the ACR result that accounts for the distributional consequences of trade shocks. In this section, we show how our framework can be adapted for analyzing such models. We generalize our analysis to encompass Roy-models of the labor market, and show how duality with the closed economy can then be used to study the distributional consequences of trade.

Suppose that H_c denotes the set of households in country c. As in Galle et al. (2017), households consume the same basket of goods, but supply labor in different ways.[1] We assume that each household type has a fixed endowment of labor L_h, which are assigned to work in different industries according to the productivity of workers in that group and the relative wage differences offered in different industries.

As usual, let world GDP be the numeraire. Define Λ_f^h to be type h's share of income derived from earning wages f

$$\Lambda_f^h = \frac{\Phi_{hf} \Lambda_f}{\chi_h},$$

[1] Similar methods can also be used when households have different consumption baskets, see Appendix H, although Borusyak and Jaravel (2018) suggest that at least in the US, households have similar imported consumption baskets.

where $\chi_h = \sum_{k \in F} \Phi_{hk} \Lambda_k$. The Roy model of Galle et al. (2017) implies that

$$\frac{\chi_h}{\overline{\chi_h}} = \left(\sum_f \overline{\Lambda}_f^h \left(\frac{w_f}{\overline{w}_f} \right)^{\gamma_h} \right)^{\frac{1}{\gamma_h}} \frac{L^h}{\overline{L}^h},$$

where γ_h is the supply elasticity, variables with overlines are initial values, L^h is the stock of labor h has been endowed with (since we analyze log changes, only shocks to the endowment value are relevant). Galle et al. (2017) show that the above equations can be microfounded via a model where homogenous workers in each group type draw their ability for each job from Frechet distributions, and choose to work in the job that offers them the highest return. The Roy model generalizes the factor market, with $\gamma_h = 1$ representing the case where labor cannot be moved across markets by h. If $\gamma_h > 1$ then h can take advantage of wage differentials to redirect its labor supply and boost its income. When $\gamma \to \infty$, labor mobility implies that all wages in the economy are equalized (and the model collapses to a one-factor model).

Proposition 8 (Exact Duality). *The discrete change in welfare $\Delta \log W_h$ of group $h \in H_c$ of the original open economy in response to discrete shocks to iceberg trade costs or productivities outside of country c is equal to (minus) the discrete change in the price deflator $-\Delta \log \check{P}_{Y_g}$ of the dual closed economy in response to discrete shocks to productivities $\Delta \log \check{A}_i = -(1/\varepsilon_i) \Delta \log \Omega_{ic}$ and discrete shocks to factor wages $\Delta \log \check{A}_f = -\frac{1}{\gamma_h} \Delta \log \Lambda_f^h$. This duality result is global in that it holds exactly for arbitrarily large shocks.*

In the case where $\gamma \to \infty$, we recover the one-factor version of Duality in Theorem 3.

Of course, due to the fact that factor shares Λ_f^h are endogenously respond to factor prices, Theorem 4 can no longer be used to determine how these shares will change in equilibrium. Therefore, we extend those propositions here.

Proposition 9. *The response of the factor prices to a shock* $\mathrm{d} \log A_k$ *is the solution to the following system:*

1. *Product Market Equilibrium:*

$$\Lambda_l \frac{\mathrm{d} \log \Lambda_l}{\mathrm{d} \log A_k} = \sum_{i \in \{H,N\}} \lambda_j (1 - \theta_j) Cov_{\Omega^{(j)}} \left(\Psi_{(k)} + \sum_f \Psi_{(f)} \frac{\mathrm{d} \log w_f}{\mathrm{d} \log A_k}, \Psi_{(l)} \right)$$
$$+ \sum_{h \in H} (\lambda_l^{W_h} - \lambda_l) \left(\sum_{f \in F_c} \Phi_{hf} \Lambda_f \frac{\mathrm{d} \log w_f}{\mathrm{d} \log A_k} \right).$$

2. *Factor Market Equilibrium:*

$$\mathrm{d}\log\Lambda_f = \sum_{h\in H} E_{\Phi^{(h)}}\left[\gamma_h\left(E_{\Lambda^{(h)}}\left(\mathrm{d}\log w_f - \mathrm{d}\log w\right)\right) + \left(E_{\Lambda^{(h)}}(\mathrm{d}\log w)\right) + (\mathrm{d}\log L)\right].$$

Given this, the welfare of the gth group is

$$\frac{\mathrm{d}\log W_h}{\mathrm{d}\log A_k} = \sum_{s\in F}\left(\Lambda_s^h - \Lambda_s^{W_h}\right)\mathrm{d}\log w_s + \lambda_k^{W_h} + \mathrm{d}\log L^h.$$

The product market equilibrium conditions are exactly the same as those in Theorem 4, but now we have some additional equations from the supply-side of the factors (which are no longer endowments). Letting $\gamma_h = 1$ for every $h \in H$ recovers Theorem 4.

D Partial Equilibrium Counterpart to Theorem 5

Proposition 10. *For a small open economy operating in a perfectly competitive world market, the introduction of import tariffs reduces the welfare of that country's representative household by*

$$\Delta W \approx \frac{1}{2}\sum_i \lambda_i \Delta\log y_i \Delta\log\mu_i,$$

where μ_i is the ith gross tariff (no tariff is $\mu_i = 1$), y_i is the quantity of the ith import, and λ_i is the corresponding Domar weight.

Proof. To prove this, let $e(p)W$ be the expenditure function of the household. We have $e(p)W = p\cdot q + \sum_i(\mu_i - 1)p_i y_i$. Differentiate this once to get $c\cdot\mathrm{d}\,p + e(p)\,\mathrm{d}\,W = q\cdot\mathrm{d}\,p + \mathrm{d}\,q\cdot p + \sum_i \mathrm{d}\,\mu_i p_i y_i + \sum_i(\mu_i - 1)\,\mathrm{d}(p_i y_i)$. Theorem 2 implies that this can be simplified to $e(p)\,\mathrm{d}\,W = (q-c)\cdot\mathrm{d}\,p + \sum_i \mathrm{d}\,\mu_i p_i y_i + \sum_i(\mu_i - 1)\,\mathrm{d}(p_i y_i) = \sum_i(\mu_i - 1)\,\mathrm{d}(p_i y_i)$, where the left-hand side is the equivalent variation. Now differentiate this again, and evaluate at $\mu_i = 1$ to get $\sum_i p_i\,\mathrm{d}\,y_i$. Hence the second-order Taylor approximation, at $\mu = 1$, is $\frac{1}{2}\sum_i \mathrm{d}\,\mu_i p_i\,\mathrm{d}\,y_i = \frac{1}{2}\sum_i \mathrm{d}\log\mu_i p_i y_i\,\mathrm{d}\log y_i$, and our normalization implies $p_i y_i$ is equal to its Domar weight. ■

E Some Applications of Theorem 4

In this subsection, we show that Theorem 4 can also be used to answer questions unrelated to welfare, such as for example questions involving the aggregation of trade elasticities or structural transformation in open economies.[2,3]

E.1 Aggregating and Disaggregating Trade Elasticities

We start by defining a class of aggregate elasticities. Consider two sets of producers I and J. Let $\lambda_I = \sum_{i \in I} \lambda_i$ and $\lambda_J = \sum_{j \in J}$ be the aggregate sales shares of producers in I and J, and let $\chi_i^I = \lambda_i / \Lambda_I$ and $\chi_j^J = \lambda_j / \Lambda_J$. Let k be another producer. We then define the following aggregate elasticities capturing the bias towards I vs. J of a productivity shock to m as:

$$\varepsilon_{IJ,m} = \frac{\partial(\lambda_I / \lambda_J)}{\partial \log A_m},$$

where the partial derivative indicates that we allow for this elasticity to be computed holding some things constant.

To shed light on trade elasticities, we proceed as follows. Consider a set of producers $S \subseteq N_c$ in a country c. Let J be denote a set of domestic producers that sell to producers in S, and I denote a set of foreign producers that sell to producers in S. Without loss of generality, using the flexibility of network relabeling, we assume that producers in I and J are specialized in selling to producers in S so that they do not sell to producers outside of S.

Consider an iceberg trade cost modeled as a negative productivity shock $\mathrm{d}\log(1/A_m)$ to some producer m. We then define the trade elasticity as $\varepsilon_{IJ,k} = \partial(\lambda_J/\lambda_I)/\partial \log(1/A_m) = \partial(\lambda_I/\lambda_J)/\partial \log A_m$. As already mentioned, the partial derivative indicates that we allow for this elasticity to be computed holding some things constant. There are therefore different trade elasticities, depending on exactly what is held constant. Different versions of trade elasticities would be picked up by different versions of gravity equations regressions with different sorts of fixed effects and at different levels of aggregation.

There are several possibilities for what to hold constant, ranging from the most partial

[2]Adao et al. (2017) show that economies of the sort that we consider can be represented as economies in which only factors are traded within and across borders, and households have preferences over factors. Theorem 4 can be used to flesh out this representation by locally characterizing its associated reduced-form Marshallian demand for factors in terms of sufficient-statistic microeconomic primitives: the expenditure share of household c on factor f is given by Ψ_{cf}; the elasticities $\partial \log \Psi_{cf} / \partial \log A_i$ holding factor prices constant then characterize its Marshallian price elasticities as well as its Marshallian elasticities with respect to iceberg trade shocks

$$\frac{\partial \log \Psi_{cf}}{\partial \log A_i} = \sum_{k \in N} \frac{\Psi_{ck}}{\Psi_{cf}} (\theta_k - 1) Cov_{\Omega^{(k)}}(\Psi_{(f)}, \Psi_{(i)}).$$

The reduced-form factor demand system is locally stable with respect to a single shock $\mathrm{d}\log A_i$ if, and only, if $\partial \log \Psi_{cf} / \partial \log A_i = 0$, with a similar conditions for a combination of such shocks.

[3]We refer the reader to Appendix K for more examples involving the factor bias of trade, showing adverse trade shocks can reduce the capital share in all countries.

equilibrium to the most general equilibrium. At one an extreme, we can hold constant the prices of all inputs for all the producers in I and J and the relative sales shares of all the producers in S:

$$\varepsilon_{IJ,m} = \sum_{s\in S}\sum_{i\in I}\chi_i^I(\theta_s-1)\frac{\lambda_s}{\lambda_i}Cov_{\Omega^{(s)}}(I_{(i)},\Omega_{(m)}) - \sum_{s\in S}\sum_{j\in J}\chi_j^J(\theta_s-1)\frac{\lambda_s}{\lambda_j}Cov_{\Omega^{(s)}}(I_{(j)},\Omega_{(m)}), \quad (4)$$

where $I_{(i)}$ and $I_{(j)}$ are the ith and jth columns of the identity matrix. An intermediate possibility is to hold constant the wages of all the factors in all countries:

$$\varepsilon_{IJ,k} = \sum_{i\in I}\chi_i^I\Gamma_{ik} - \sum_{j\in J}\chi_j^J\Gamma_{jk}.$$

And at the other extreme, we can compute the full general equilibrium:

$$\varepsilon_{IJ,m} = \sum_{i\in I}\chi_i^I\left(\Gamma_{im} - \sum_{g\in F}\Gamma_{ig}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_m} + \sum_{g\in F}\Xi_{ig}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_m}\right) - \sum_{j\in J}\chi_j^J\left(\Gamma_{jm} - \sum_{g\in F}\Gamma_{jg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_m} + \sum_{g\in F}\Xi_{jg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_m}\right),$$

$\mathrm{d}\log\Lambda_f/\,\mathrm{d}\log A_m$ is given in Theorem 4.

The trade elasticity is a linear combination of microeconomic elasticities of substitution, where the weights depend on the input-output structure. Except at the most microeconomic level where there is a single producer s in S and in the most partial-equilibrium setting where we recover $\epsilon_s - 1$, this means that the aggregate trade elasticity is typically an endogenous object, since the input-output structure is itself endogenous.[4] Furthermore, in the presence of input-output linkages, it is typically nonzero even for trade shocks that are not directly affecting the sales of I to J, except in the most partial-equilibrium setting.

Example: Trade Elasticity in a Round-About World Economy

In many trade models, the trade elasticity, defined holding factor wages constant, is an invariant structural parameter. As pointed out by Yi (2003), in models with intermediate inputs, the trade elasticity can easily become an endogenous object.Consider the two-country, two-good economy depicted in Figure 2. The representative household in each

[4]In Appendix K, we provide necessary and sufficient conditions for the trade elasticity to be constant in the way.

country only consumes the domestic good, which is produced using domestic labor and imports with a CES production function with elasticity of substitution θ. We consider the imposition of a trade cost hitting imports by country 1 from country 2. For the sake of illustration, we assume that the trade cost does not apply to the exports of country 1 to country 2.

The trade elasticity holding factor wages and foreign input prices constant is a constant structural parameter, and given simply by

$$\theta - 1.$$

However, echoing our discussion above, the trade elasticity holding factor wages constant is different, and is given by

$$\frac{\theta - 1}{1 - \Omega_{21}\Omega_{12}},$$

where Ω_{ij} is the expenditure share of i on j, e.g. its intermediate input import share. As the intermediate input shares increase, the trade elasticity becomes larger. Simple trade models without intermediate goods are incapable of generating these kinds of patterns.

Of course, since the intermediate input shares Ω_{ij} are themselves endogenous (depending on the iceberg shock), this means that the trade elasticity varies with the iceberg shocks. In particular, if $\theta > 1$, then the trade elasticity increases (nonlinearly) as iceberg costs on imports fall in all countries since intermediate input shares rise. [5]

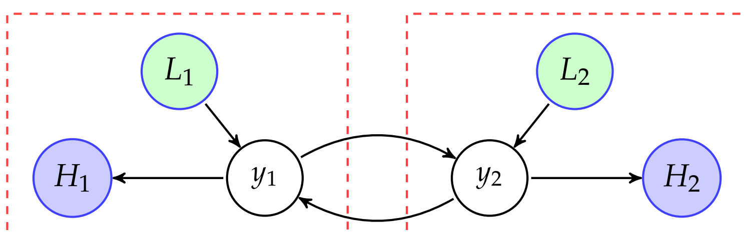


Figure 3: The solid lines show the flow of goods. Green nodes are factors, purple nodes are households, and white nodes are goods. The boundaries of each country are denoted by dashed box.

[5] In Appendix K we show that there it is possible to generate "trade re-switching" examples where the trade elasticity is non-monotonic with the trade cost (or even has the "wrong" sign) in otherwise perfectly respectable economies. These examples are analogous to the "capital re-switching" examples at the center the Cambridge Cambridge Capital controversy.

E.2 Example: Baumol's Cost-Disease and Export-Led Growth

We illustrate the nonlinear effects of trade and productivity shocks on real output discussed in Section 5.1 via a simple example showing how opening up to trade and relying on export-led growth can overcome Baumol's cost disease. Baumol's cost disease is a phenomenon whereby in the presence of complementarities across sectors, the relative sales of sectors with relatively faster productivity growth shrink over time as a result of their higher productivity growth. It reduces down the growth rate of aggregate productivity over time. As discussed in Baqaee and Farhi (2017a), Baumol's cost disease is a manifestation of nonlinearities.

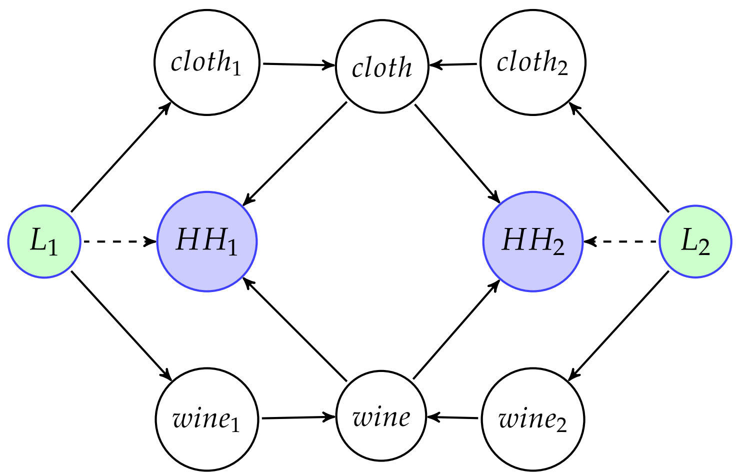


Figure 4: The solid lines show the flow of goods while the dashed lines show the flow of wage payments.

Consider the economy depicted in Figure 4. Countries 1 and 2 produce varieties of wine and cloth. The representative household in each country consumes a composite of foreign and domestic varieties of wine and cloth. We assume that the elasticity of substitution across foreign and domestic varieties of wine or cloth is $\theta > 1$, but that the elasticity of substitution between wine and cloth is $\sigma < 1$. To simplify the algebra, assume that there is no home-bias, so that both households consume the same basket of wine and cloth. Finally, we assume that wine and cloth have the same size at the initial point so that $\lambda^{Y}_{cloth} = \lambda^{Y}_{wine} = 1/2$ and $\lambda^{Y_1}_{cloth_1} = \lambda^{Y_1}_{wine_1} = 1/2$. The relative size of country 1 at the initial point is χ^{Y}_{1}. It is also the share of country 1's varieties in the overall baskets of wine and cloth. It therefore also indexes the degree of openness of country 1: when $\chi^{Y}_{1} = 1$, it is a closed world economy; when $\chi^{Y}_{1} = 0$, country it is a small open economy.

The effect on the real output of country 1 from an increase $\Delta log A_{cloth_1}$ in the productivity of its cloth is given up to the second order by[6]

$$\Delta \log Y_1 \approx \log \frac{\mathrm{d} \log Y_1}{\mathrm{d} \log A_{cloth_1}} \Delta \log A_{cloth_1} + \frac{1}{2} \frac{\mathrm{d}^2 \log Y_1}{\mathrm{d} \log A^2_{cloth_1}} (\Delta \log A_{cloth_1})^2,$$

with

$$\frac{\mathrm{d} \log Y_1}{\mathrm{d} \log A_{cloth_1}} = \lambda^{Y_1}_{cloth_1} = \frac{1}{2}, \quad \frac{\mathrm{d}^2 \log Y_1}{\mathrm{d} \log A^2_{cloth_1}} = \frac{\mathrm{d} \lambda^{Y_1}_{cloth_1}}{\mathrm{d} \log A_{cloth_1}} = \frac{(\chi^Y_1)^2(\sigma - 1)}{2} + \frac{(1 - \chi^Y_1)(\theta - 1)}{4}.$$

The second-order term capture the extent to which large shocks have larger or smaller proportional effects than small shocks, conditional on the size of the sector. Cumulating rates of productivity growth over time is equivalent to increasing the size of the shock. The second-order term therefore captures the strength of Baumol's cost disease. When it is negative, Baumol's cost disease obtains. When it is positive, we have a form of reverse Baumol's cost disease where the sector with faster productivity growth expands instead of shrinking.

As we increase the relative size and openness χ^Y_1 of country 1, the effect on its real output of the productivity of its cloth becomes smaller because Baumol's cost disease becomes stronger. In the small-open economy limit $\chi^Y_1 \to 0$, we have $\mathrm{d}^2 \log Y_1 / \mathrm{d} \log A^2_{cloth_1} = (\theta - 1)/4 > 0$, and so there is reverse Baumol's cost disease. In the large-closed-economy limit $\chi^Y_1 \to 1$, we have $\mathrm{d}^2 \log Y_1 / \mathrm{d} \log A^2_{cloth_1} = (\sigma - 1)/2 < 0$ and so there is Baumol's cost disease.

Of course, if country 1 is small and closed, then it is as if it were a large-closed economy. Opening up to international trade turns it into a small-open economy. Trade can therefore overcome (and indeed overturn) Baumol's cost-disease by allowing export-led growth.

F Generalizing Sections 3 and 5 with Distortions

In this section, we explain how to adapt the results of Sections 3 and 5 in economies with tariffs or other distortions.

[6]Here again, we slightly abuse notation by using derivative symbols since Y_1 is not a function.

Comparative Statics: Ex-Ante Sufficient Statistics

Since any wedge can be represented as a markup, without loss of generality, we assume that all wedges in the economy (including tariffs) have been represented as markups.[7] We use the diagonal matrix of markups/wedges μ. Following Baqaee and Farhi (2017b), we define the *cost-based* HAIO matrix $\tilde{\Omega} = \mu\Omega$ and the corresponding cost-based Leontief inverse matrix $\tilde{\Psi} = (I - \tilde{\Omega})^{-1}$. All the exposures and factor income shares that we defined with the matrix Ω have cost-based analogues which we denote with tildes.

Finally, it is convenient to introduce "fictitious" factors, one for each producer $i \in N$, which collects the revenues $\lambda_i(1 - 1/\mu_i)$ earned by the markup/wedge of this producer. We denote the set of true and fictitious factors to be F^*. For each fictitious factor $f \in F^* - F$, we denote by $\iota(f) \in N$ the good associated with it. Just like for a true factor, we define Φ_{cf} for a fictitious factor to be the share of the income of this factor which accrues to the representative agent of country c. All exposures in gross real output, and in real expenditure or welfare to a fictitious factor are equal to zero, at the country and world levels. But the incomes shares of these factors are not zero. For example $\tilde{\Lambda}_f^{W_c} = \tilde{\Lambda}_f^{W} = 0$, but $\Lambda_f^c \neq 0$ and $\Lambda_f \neq 0$ if the markup/wedge of the corresponding producer is nonzero.

In Theorem 11, we characterize changes in real output $d \log Y_c = \sum_{i \in N} \chi_i^{Y_c} d \log q_{ci}$ exactly as in the model without distortions, where recall that $q_{ci} \geq 0$ for $i \in N_c$ and $q_{ci} \leq 0$ for $i \notin N_c$. We also define the corresponding revenue- and cost-based exposures to goods or factors k as $\lambda_k^{\hat{Y}_c}$ and $\tilde{\lambda}_k^{\hat{Y}_c}$.

However, arguably, real GDP is less interesting in the presence of distortions because the double-deflation method runs into some conceptual problems. Basically, if there are markups or other wedges in the domestic economy, then imported intermediate inputs may not be netted out of real GDP using their shadow value. This means that changes in intermediate imports can affect real GDP holding fixed domestic productivity, domestic factors, *and* the domestic allocation matrix.

To remedy this issue, we define the change in the *gross* real output $d \log \hat{Y}_c = \sum_{i \in N_c} \chi_i^{\hat{Y}_c} d \log q_{ci}$ of a country by treating imports in the same way as factor inputs, where $\chi_i^{\hat{Y}_c} = p_i q_{c_i} / (\sum_{i \in N_c} p_i q_{ci})$. Following the by now usual template, we also define the corresponding revenue- and cost-based exposures to goods or factors k as $\lambda_k^{\hat{Y}_c}$ and $\tilde{\lambda}_k^{\hat{Y}_c}$.

We now state two growth-accounting theorems, one for changes in gross real output and real GDP and the other for changes in real expenditure or welfare at the country and world levels. These theorems offer decompositions into "pure" technology effects and reallocation effects. As for the case without distortions discussed in the main text, we

[7] To represent a wedge on i's ability purchase inputs from k, we can introduce a new producer which buys from k and sells to i at a markup.

slightly abuse notation: first, except for changes in welfare at the country level, these objects are not differentials of corresponding level functions; second except for changes in welfare at the country level, reallocation effects are defined as the changes in the corresponding object with fixed prices (not chained) and holding the allocation matrix constant (this is not necessary for the change in welfare at the country level because it is the differential of a function, which can be evaluated with a constant allocation matrix, along the lines of the exposition in the main text).

Define $\Lambda_i^{M_c}$ to be the expenditures of country c on intermediate imports of good i as a share of the GDP of country c.

Proposition 11 (Output-Accounting)**.** *The change in gross real output of country c to productivity shocks, factor supply shocks, transfer shocks, and shocks to markups/wedges, can be decomposed into:*

(i) For real GDP,

$$\mathrm{d}\log Y_c = \sum_{f\in F_c} \tilde{\Lambda}_f^{\hat{Y}_c}\,\mathrm{d}\log L_f + \sum_{i\in N_c} \tilde{\lambda}_i^{\hat{Y}_c}\,\mathrm{d}\log A_i$$

$$-\sum_{i\in N_c} \tilde{\lambda}_i^{\hat{Y}_c}\,\mathrm{d}\log \mu_i - \sum_{f\in F_c}^{F} \tilde{\Lambda}_f^{\hat{Y}_c}\,\mathrm{d}\log \Lambda_f^{\hat{Y}_c} + \sum_{i\in N-N_c} \left(\Lambda_i^{M_c} - \tilde{\lambda}_i^{\hat{Y}_c}\right)\left(\mathrm{d}\log \Lambda_i^{M_c} - \mathrm{d}\log(q_{ci})\right).$$

(ii) For real gross output,

$$\mathrm{d}\log \hat{Y}_c = \underbrace{\sum_{f\in F_c} \tilde{\Lambda}_f^{\hat{Y}_c}\,\mathrm{d}\log L_f + \sum_{i\in N-N_c} \tilde{\lambda}_i^{\hat{Y}_c}\,\mathrm{d}\log(-q_{ci}) + \sum_{i\in N_c} \tilde{\lambda}_i^{\hat{Y}_c}\,\mathrm{d}\log A_i}_{\Delta Technology}$$

$$\underbrace{-\sum_{i\in N_c} \tilde{\lambda}_i^{\hat{Y}_c}\,\mathrm{d}\log \mu_i - \sum_{f\in F_c}^{F} \tilde{\Lambda}_f^{\hat{Y}_c}\,\mathrm{d}\log \Lambda_f^{\hat{Y}_c}}_{\Delta Reallocation}.$$

For the world, real gross output and real GDP coincide. The change $\mathrm{d}\log \hat{Y}$ *of world gross real output, which coincides with the change in world real output* $\mathrm{d}\log Y$*, can be obtained by simply suppressing the country index c.*

For real GDP, we do not separate the decomposition into a pure technology and reallocation effect. This is due to the presence of the final summand (which disappears in efficient economies because $\Lambda_i^{M_c} = \tilde{\lambda}_i^{\hat{Y}_c}$). Intuitively, when there are markups, changes in imported intermediates can have an effect on real GDP holding fixed technology, domestic factors, and even the allocation matrix, purely because the contribution of intermediate imports to real production is measured by $\tilde{\lambda}_i^{Y_c}$ but it is netted out of GDP using $\Lambda_i^{M_c}$.

Instead, we focus our attention on changes in real gross output. The main differences between Proposition 11 and its equivalent Theorem 1 for economies without distortions are as follows. First the "pure" technology effects use cost-based (and not revenue-based) exposures. Second, because we look at gross real output (and not real output), changes in imports show up as changes in factor supplies via the term $\sum_{i \in N-N_c} \tilde{\lambda}_i^{\hat{Y}_c} \, \mathrm{d}\log(-q_{ci})$. Third, there are non-zero reallocation effects. The term $-\sum_{f \in F_c}^{F} \tilde{\Lambda}_f^{Y_c} \, \mathrm{d}\log \Lambda_f^{Y_c}$ is a weighted average of the changes in the domestic factor income shares. When it is positive, it means that domestic factor shares are reduced on average, which, loosely speaking, means that the domestic share of profits is increasing. It indicates that resources are being reallocated to more distorted parts of the domestic economy, which increases real gross output because these parts of the domestic economy were too small to begin with from a social perspective. Of course, when markups/wedges increase, this mechanically increases the domestic profit share and reduces average domestic factor income shares. This effect must therefore be netted out and this is the role of the term $\sum_{i \in N_c} \tilde{\lambda}_i^{\hat{Y}_c} \, \mathrm{d}\log \mu_i$.

Following Baqaee and Farhi (2017b), we can define the aggregate productivity of country c via the "distorted" Solow residual, which by Proposition 11, is equal to

$$
\begin{aligned}
\mathrm{d}\log \hat{Y}_c - \sum_{f \in F_c} \tilde{\Lambda}_f^{\hat{Y}_c} \, \mathrm{d}\log L_f - \sum_{i \in N-N_c} \tilde{\lambda}_i^{\hat{Y}_c} \, \mathrm{d}\log(-q_{ci}) \\
= \sum_{i \in N_c} \tilde{\lambda}_i^{\hat{Y}_c} \, \mathrm{d}\log A_i - \sum_{i \in N_c} \tilde{\lambda}_i^{\hat{Y}_c} \, \mathrm{d}\log \mu_i - \sum_{f \in F_c}^{F} \tilde{\Lambda}_f^{\hat{Y}_c} \, \mathrm{d}\log \Lambda_f^{\hat{Y}_c}.
\end{aligned}
$$

The distorted Solow residual is a better measure of the aggregate productivity of country c in the sense that it only depends on the evolution of technology and the allocation of resources *inside* country c. In particular, this means that it does not respond directly to external shocks unless those shocks reallocate resources inside country c in a way where the reallocation affects the net output produced by country c.

The distorted Solow residual correctly accounts for the "pure" technology effect of changes in factor inputs by weighing them by their cost-based real gross output exposures (and not by their revenue-based gross real output exposures, as would be the case in the traditional Solow residual).

At the world level, we recover precisely the result of Baqaee and Farhi (2017b) for a closed economy:

$$\mathrm{d}\log Y = \mathrm{d}\log W = \sum_{i\in N}\tilde{\lambda}_i\,\mathrm{d}\log A_i - \sum_{i\in N}\tilde{\lambda}_i\,\mathrm{d}\log\mu_i - \sum_{f\in F}\tilde{\Lambda}_f\,\mathrm{d}\log\Lambda_f.$$

Proposition 12 (Welfare-Accounting, Reallocation)**.** *The change in welfare of country c in response to productivity shocks, factor supply shocks, and transfer shocks can be decomposed into the "pure" effects of changes in technology and the effects of changes in the allocation of resources:*

$$\mathrm{d}\log W_c = \underbrace{\sum_{f\in F}\tilde{\Lambda}_f^{W_c}\,\mathrm{d}\log L_f + \sum_{i\in N}\tilde{\lambda}_i^{W_c}\,\mathrm{d}\log A_i}_{\Delta Technology}$$
$$\underbrace{-\sum_{i\in N}\tilde{\lambda}_i^{W_c}\,\mathrm{d}\log\mu_i - \sum_{f\in F}\tilde{\Lambda}_f^{W_c}\,\mathrm{d}\log\Lambda_f + \sum_{f\in F^*}\Lambda_f^c\,\mathrm{d}\log\Lambda_f + (1/\chi_c^W)\,\mathrm{d}\,T_c}_{\Delta Reallocation}.$$

The change $\mathrm{d}\log W$ *in world real expenditure or welfare can be obtained by simply suppressing the country index c*

The main differences between Theorem 12 and its equivalent Theorem 2 for economies with no distortions are as follows. First, they use cost-based exposures rather than revenue-based exposures. Second, they account for the changes in the contributions to income of the revenues raised by the different markups/wedges, which is reflected in the sum over $f\in F^*$ (and not $f\in F$). Third, they account for the effects of changes in markups/wedges on the country welfare deflator, which is reflected in the term $-\sum_{i\in N}\tilde{\lambda}_i^{W_c}\,\mathrm{d}\log\mu_i$.[8]

Theorems 11 and 12 give a unified framework for growth and productivity accounting in open, closed, distorted, and undistorted economies. They use changes in factor shares (ex-post sufficient statistics). We now supplement them with propagation equations which express changes in factor shares as a function of microeconomic primitives (ex-ante sufficient statistics).

Comparative Statics: Ex-Post Sufficient Statistics

We redefine the $(N+F)\times(N+F)$ "propagation-via-substitution" matrix Γ and the $(N+F)\times F^*$ "propagation-via-redistribution" matrix Ξ:

$$\Gamma_{ij} = \sum_{k\in N}(\theta_k - 1)\frac{\lambda_k/\mu_k}{\lambda_i}Cov_{\Omega^{(k)}}\left(\Psi_{(i)},\tilde{\Psi}_{(j)}\right),$$

[8] At the world level, where we recover once again the result of Baqaee and Farhi (2017b) for a closed economy.

$$\Xi_{if} = \frac{1}{\lambda_i} \sum_{c \in C} (\lambda_i^{W_c} - \lambda_i) \Phi_{cf} \Lambda_f,$$

where we write λ_i and Λ_i interchangeably when $i \in F$ is a factor. The only differences with the case with no distortions are as follows. First, we now use a mix of revenue-based and cost-based columns of the Leontief inverse matrix $\Psi_{(i)}$ and $\tilde{\Psi}_{(j)}$, because the transmission of the transmission of expenditures is governed by Ψ and the transmission of prices by $\tilde{\Psi}$. Second, the sales share of producer k is divided by its markup/wedge μ_k because what matters is its cost, not its revenue. Third, redistribution terms are now also defined for fictitious factors.

Finally, we also define the $(N+F) \times N$ "propagation-via-input-demand-suppression" matrix Σ:

$$\Sigma_{ij} = (1_{\{i=j\}} - \frac{\lambda_j}{\lambda_i} \Psi_{ji}).$$

This matrix will play a role for the characterization of the changes in sales shares and factor shares in response to shocks to markups/wedges, because while these shocks act like negative productivity shocks on prices (for given factor wages) and their associated substitution effects, as captured by the propagation-via-substitution matrix, they also release resources via a reduction in the demand for inputs.

Proposition 13 (Factor Shares and Sales Shares)**.** *The changes in the sales share of goods and factors in response to a productivity shock to producer i are the solution of the following system of linear equations:*[9]

$$\frac{d \log \lambda_j}{d \log A_i} = \Gamma_{ji} - \sum_{g \in F} \Gamma_{jg} \frac{d \log \Lambda_g}{d \log A_i} + \sum_{g \in F^*} \Xi_{jg} \frac{d \log \Lambda_g}{d \log A_i} \quad \textit{for } j \in N,$$

$$\frac{d \log \Lambda_f}{d \log A_i} = \Gamma_{fi} - \sum_{g \in F} \Gamma_{fg} \frac{d \log \Lambda_g}{d \log A_i} + \sum_{g \in F^*} \Xi_{fg} \frac{d \log \Lambda_g}{d \log A_i} \quad \textit{for } f \in F,$$

$$\frac{d \log \Lambda_f}{d \log A_i} = \frac{d \log \lambda_{\iota(f)}}{d \log A_i} \quad \textit{for } f \in F^* - F.$$

The changes in the sales share of goods and factors in response to a markup/wedge shock to producer

[9] When $f \in F^* - F$, if $\mu_{\iota(f)} = 1$, we have $\Lambda_f = 0$, and so $d \log \Lambda_f / d \log A_i$ is not defined. The corresponding equation can then be omitted by using the convention $\Xi_{jf} \, d \log \Lambda_f / d \log A_i = 0$.

i are the solution of the following system of linear equations:[10]

$$\frac{\mathrm{d}\log\lambda_j}{\mathrm{d}\log\mu_i} = \Sigma_{ji} - \Gamma_{ji} - \sum_{g\in F}\Gamma_{jg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log\mu_i} + \sum_{g\in F^*}\Xi_{jg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log\mu_i} \quad \text{for } j\in N,$$

$$\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_i} = \Sigma_{fi} - \Gamma_{fi} - \sum_{g\in F}\Gamma_{fg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log\mu_i} + \sum_{g\in F^*}\Xi_{fg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log\mu_i} \quad \text{for } f\in F,$$

$$\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_i} = \frac{\mathrm{d}\log\lambda_{\iota(f)}}{\mathrm{d}\log\mu_i} + 1_{\{\iota(f)=i\}}\frac{1}{\mu_i - 1} \quad \text{for } f\in F^* - F.$$

More generally, we can use the characterization of the responses of sales shares and factor shares to characterize the responses of the input-output matrix, of the Leontief inverse matrix, and of all the income shares and all the exposures in real output and real expenditure or welfare, at the country and world levels, cost- and revenue-based.

Armed with Theorem 13, it is straightforward to characterize the response of prices and quantities to shocks.[11]

Corollary 10. *(Prices and Quantities) The changes in the wages of factors and in the prices and quantities of goods in response to a productivity shock to producer i are given by:*

$$\frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_i} = \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log A_i},$$
$$\frac{\mathrm{d}\log p_j}{\mathrm{d}\log A_i} = -\tilde{\Psi}_{ji} + \sum_{g\in F}\tilde{\Psi}_{jg}\frac{\mathrm{d}\log w_g}{\mathrm{d}\log A_i},$$
$$\frac{\mathrm{d}\log y_j}{\mathrm{d}\log A_i} = \frac{\mathrm{d}\log\lambda_j}{\mathrm{d}\log A_i} - \frac{\mathrm{d}\log p_j}{\mathrm{d}\log A_k},$$

where $\mathrm{d}\log\Lambda_f / \mathrm{d}\log A_i$ *is given in Theorem 4. The changes in the wages of factors and in the prices and quantities of goods in response to a markup/wedge shock to producer i are given by:*

$$\frac{\mathrm{d}\log w_f}{\mathrm{d}\log\mu_i} = \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_i},$$
$$\frac{\mathrm{d}\log p_j}{\mathrm{d}\log\mu_i} = \tilde{\Psi}_{ji} + \sum_{g\in F}\tilde{\Psi}_{jg}\frac{\mathrm{d}\log w_g}{\mathrm{d}\log\mu_i}$$

[10] When $f\in F^*-F$, if $\mu_{\iota(f)}=1$, we have $\Lambda_f=0$ and so $\mathrm{d}\log\Lambda_f / \mathrm{d}\log\mu_i$ is not defined. The corresponding equation can then be omitted by using the convention $\Xi_{jf}\,\mathrm{d}\log\Lambda_f / \mathrm{d}\log A_i = 0$. When $\mu_i = 1$, $\mathrm{d}\log\mu_i$ is not defined, and so we cannot define the elasticities of sales shares with respect to μ_i, but we can define and compute their semi-elasticities in a straightforward way, but we omit the details for brevity. The same remark applies to Corollary 10.

[11] Recall that prices are expressed in the numeraire where $GDP = GNE = 1$ at the world level.

$$\frac{\mathrm{d}\log y_j}{\mathrm{d}\log \mu_i} = \frac{\mathrm{d}\log \lambda_j}{\mathrm{d}\log \mu_i} - \frac{\mathrm{d}\log p_j}{\mathrm{d}\log \mu_i},$$

where $\mathrm{d}\log \Lambda_f \,/\, \mathrm{d}\log \mu_i$ *is given in Theorem 4.*

G Beyond CES

Following Baqaee and Farhi (2017a), we can extend all the results in this paper to arbitrary neoclassical production functions simply by replacing the input-output covariance operator with the *input-output substitution operator* instead. The only exception to this is the duality result, in say Theorem 3, which make explicit use of the CES functional form.

For a producer k with cost function $\mathbf{C}_k$, the Allen-Uzawa elasticity of substitution between inputs x and y is

$$\theta_k(x,y) = \frac{\mathbf{C}_k d^2\mathbf{C}_k/(dp_x dp_y)}{(d\mathbf{C}_k/dp_x)(d\mathbf{C}_k/dp_y)} = \frac{\epsilon_k(x,y)}{\Omega_{ky}},$$

where $\epsilon_k(x,y)$ is the elasticity of the demand by producer k for input x with respect to the price p_y of input y, and Ω_{ky} is the expenditure share in cost of input y. We also use this definition for final demand aggregators.

The *input-output substitution operator* for producer k is defined as

$$\begin{aligned}\Phi_k(\Psi_{(i)},\Psi_{(j)}) &= -\sum_{x,y\in N+F} \Omega_{kx}[\delta_{xy} + \Omega_{ky}(\theta_k(x,y)-1)]\Psi_{xi}\Psi_{yj},\\ &= \frac{1}{2} E_{\Omega^{(k)}}\left((\theta_k(x,y)-1)(\Psi_i(x)-\Psi_i(y))(\Psi_j(x)-\Psi_j(y))\right),\end{aligned}$$

where δ_{xy} is the Kronecker delta, $\Psi_i(x) = \Psi_{xi}$ and $\Psi_j(x) = \Psi_{xj}$, and the expectation on the second line is over x and y.

In the CES case with elasticity θ_k, all the cross Allen-Uzawa elasticities are identical with $\theta_k(x,y) = \theta_k$ if $x \neq y$, and the own Allen-Uzawa elasticities are given by $\theta_k(x,x) = -\theta_k(1-\Omega_{kx})/\Omega_{kx}$. It is easy to verify that when $\mathbf{C}_k$ has a CES form we recover the input-output covariance operator:

$$\Phi_k(\Psi_{(i)},\Psi_{(j)}) = (\theta_k - 1)Cov_{\Omega^{(k)}}(\Psi_{(i)},\Psi_{(j)}).$$

Even outside the CES case, the input-output substitution operator shares many properties with the input-output covariance operator. For example, it is immediate to verify, that: $\Phi_k(\Psi_{(i)},\Psi_{(j)})$ is bilinear in $\Psi_{(i)}$ and $\Psi_{(j)}$; $\Phi_k(\Psi_{(i)},\Psi_{(j)})$ is symmetric in $\Psi_{(i)}$ and $\Psi_{(j)}$;

and $\Phi_k(\Psi_{(i)}, \Psi_{(j)}) = 0$ whenever $\Psi_{(i)}$ or $\Psi_{(j)}$ is a constant.

All the structural results in the paper can be extended to general non-CES economies by simply replacing terms of the form $(\theta_k - 1)Cov_{\Omega^{(k)}}(\Psi_{(i)}, \Psi_{(j)})$ by $\Phi_k(\Psi_{(i)}, \Psi_{(j)})$.

H Heterogenous Households Within Countries

To extend the model to allow for a set of heterogenous agents $h \in H_c$ within country $c \in C$, we proceed as follows. We denote by H the set of all households. Each household h in country c maximizes a homogenous-of-degree-one demand aggregator

$$C_h = \mathcal{W}_h(\{c_{hi}\}_{i \in N}),$$

subject to the budget constraint

$$\sum_{i \in N} p_i c_{hi} = \sum_{f \in F} \Phi_{hf} w_f L_f + T_h,$$

where c_{hi} is the quantity of the good produced by producer i and consumed by the household, p_i is the price of good i, Φ_{hf} is the fraction of factor f owned by household, w_f is the wage of factor f, and T_h is an exogenous lump-sum transfer.

We define the following country aggregates: $c_{ci} = \sum_{h \in H_c} c_{hi}$, $\Phi_{cf} = \sum_{h \in H_c} \Phi_{hf}$, and $T_c = \sum_{h \in H_c} T_h$. We also define the HAIO matrix at the household level as a $(H + N + F) \times (H + N + F)$ matrix Ω and the Leontief inverse matrix as $\Psi = (I - \Omega)^{-1}$.

All the definitions in Section 2 go through. In addition, we introduce the corresponding household-level definitions for a household h. First, the nominal output and the nominal expenditure of the household are:

$$GDP_h = \sum_{f \in F} \Phi_{hf} w_f L_f, \quad GNE_h = \sum_{i \in N} p_i c_{hi} = \sum_{f \in F} \Phi_{hf} w_f L_f + T_h,$$

where we think of the household as a set producers intermediating the uses by the different producers of the different factor endowments of the household. Second, the changes in real output and real expenditure or welfare of the household are:

$$\mathrm{d}\log Y_h = \sum_{f \in F} \chi_f^{Y_h} \,\mathrm{d}\log L_f, \quad \mathrm{d}\log P_{Y_h} = \sum_{f \in F} \chi_f^{Y_h} \,\mathrm{d}\log w_f,$$

$$\mathrm{d}\log W_h = \sum_{i\in N} \chi_i^{W_h} \,\mathrm{d}\log c_{hi}, \quad \mathrm{d}\log P_{W_h} = \sum_{i\in N} \chi_i^{W_h} \,\mathrm{d}\log p_i,$$

with $\chi_f^{Y_h} = \Phi_{hf} w_f l_f GDP_h$ and $\chi_i^{W_h} = p_i c_{hi}/GNE_h$. Third, the exposure to a good or factor k of the real expenditure and real output of household h is given by

$$\lambda_k^{W_h} = \sum_{i\in N} \chi_i^{W_h} \Psi_{ik}, \quad \lambda_k^{Y_h} = \sum_{f\in F} \chi_f^{Y_h} \Psi_{fk},$$

where recall that $\chi_i^{W_h} = p_i c_{hi}/GNE_h$ and $\chi_f^{Y_h} = \Phi_{hf} w_f L_f/GDP_h$. The exposure in real output to good or factor k has a direct connection to the sales of the producer:

$$\lambda_k^{Y_h} = 1_{\{k\in F\}} \frac{\Phi_{hk} p_k y_k}{GDP_h},$$

where $\lambda_k^{Y_h} = 1_{\{k\in F\}} \Phi_{hk}(GDP/GDP_h)\lambda_k$ the local Domar weight of k in household h and where $\Phi_{hk} = 0$ for $k \in N$ to capture the fact that the household endowment of the goods are zero. Fourth, the share of factor f in the income or expenditure of the household is given by

$$\Lambda_f^h = \frac{\Phi_{hf} w_f L_f}{GNE_h}.$$

The results in Section 3 go through without modification. Theorems 1, 14, and 2, as well as Corollary 11 can be extended to the level of a household h by simply replacing the country index c by the household index h.

The results in Section 5 go through with the following modifications. The $(N+F) \times (N+F)$ propagation-via-substitution matrix Γ must now be defined as

$$\Gamma_{ij} = \sum_{k\in N} (\theta_k - 1) \frac{\lambda_k}{\lambda_i} Cov_{\Omega^{(k)}} \left(\Psi_{(i)}, \Psi_{(j)} \right),$$

and the $(N+F) \times F$ propagation-via-redistribution matrix Ξ as

$$\Xi_{if} = \sum_{h\in H} \frac{\lambda_i^{W_h} - \lambda_i}{\lambda_i} \Phi_{hf} \Lambda_f,$$

where we write λ_i and Λ_i interchangeably when $i \in F$ is a factor.

The results in Section 6.3 go through with the following changes. Theorem 5 go through without modification, and be extended at the household level where $\Delta \log Y_h \approx 0$.

Corollary 5 goes through with some minor modifications. The world Bergson-Samuelson welfare function is now $W^{BS} = \sum_h \overline{\chi}_h^W \log W_h$, changes in world welfare are measured as $\Delta \log \delta$, where δ solves the equation $W^{BS}(\overline{W}_1, \ldots, \overline{W}_H) = W^{BS}(W_1/\delta, \ldots, W_H/\delta)$, where $\overline{W}_h$ are the values at the initial efficient equilibrium. We use a similar definition for country level welfare δ_c, and the same notation for household welfare δ_h. Changes in world welfare are given up to the second order by

$$\Delta \log \delta \approx \Delta \log W + Cov_{\chi_h^W}\left(\Delta \log \chi_h^W, \Delta \log P_{W_h}\right),$$

changes in country welfare are given up to the first order by

$$\Delta \log \delta_c \approx \Delta \log W_c \approx \Delta \log \chi_c^W - \Delta \log P_{W_c},$$

and the change in country welfare up to the first order by

$$\Delta \log \delta_h \approx \Delta \log W_h \approx \Delta \log \chi_h^W - \Delta \log P_{W_h}.$$

Theorems 5 and Corollary 5 go through with some minor modifications. Changes in factor shares are given up to the first order by the system of linear equations

$$\Delta \log \Lambda_f \approx -\sum_{i \in N} \Gamma_{fi} \Delta \log \mu_i - \sum_{g \in F} \Gamma_{fg} \Delta \log \Lambda_g + \sum_{g \in F} \Xi_{fg} \Delta \log \Lambda_g \\ - \sum_{i \in N} \frac{\lambda_i}{\Lambda_f} \Psi_{if} \Delta \log \mu_i + \sum_{i \in N} \Xi_{fi} \Delta \log \mu_i,$$

where the definition of Ξ is extended for $f \in F$ and $i \in N$ by $\Xi_{fi} = \frac{1}{\Lambda_f} \sum_{h \in H} (\Lambda_f^{W_h} - \Lambda_f) \Phi_{hi} \lambda_i$, and Φ_{hi} is the share of the revenue raised by the tariff or other distortion on good i which accrues to household h. Changes in household income shares are given up to the first order by

$$\chi_h^W \Delta \log \chi_h^W = \sum_{g \in F} \Phi_{hf} \Lambda_g \Delta \log \Lambda_g + \sum_{i \in N} \Phi_{hi} \lambda_i \Delta \log \mu_i,$$

and changes in country income shares are given by $\Delta \log \chi_c^W = \sum_{h \in H_c} \chi_h^{W_c} \Delta \log \chi_h^W$. Changes in household real expenditure deflators are given up to the first order by

$$\Delta \log P_{W_h} = \sum_{i \in N} \lambda_i^{W_h} \Delta \log \mu_i + \sum_{g \in F} \Lambda_g^{W_h} \Delta \log \Lambda_g.$$

In Theorem 5, changes in world real output and real expenditure are given up to the second order by

$$\Delta \log Y = \Delta \log W \approx -\frac{1}{2}\sum_{l\in N}\sum_{k\in N}\Delta \log \mu_k \Delta \log \mu_l \sum_{j\in N}\lambda_j\theta_j Cov_{\Omega^{(j)}}(\Psi_{(k)},\Psi_{(l)})$$
$$-\frac{1}{2}\sum_{l\in N}\sum_{g\in F}\Delta \log \Lambda_g \Delta \log \mu_l \sum_{j\in N}\lambda_j\theta_j Cov_{\Omega^{(j)}}(\Psi_{(g)},\Psi_{(l)})$$
$$+\frac{1}{2}\sum_{l\in N}\sum_{h\in H}\chi_h^W \Delta \log \chi_h^W \Delta \log \mu_l (\lambda_l^{W_h}-\lambda_l),$$

changes in the real output of country c are given up to the second order by

$$\Delta \log Y_c \approx -\frac{1}{2}\sum_{l\in N_c}\sum_{k\in N}\Delta \log \mu_k \Delta \log \mu_l \sum_{j\in N}\lambda_j^{Y_c}\theta_j Cov_{\Omega^{(j)}}(\Psi_{(k)},\Psi_{(l)})$$
$$-\frac{1}{2}\sum_{l\in N_c}\sum_{g\in F}\Delta \log \Lambda_g \Delta \log \mu_l \sum_{j\in N}\lambda_j^{Y_c}\theta_j Cov_{\Omega^{(j)}}(\Psi_{(g)},\Psi_{(l)})$$
$$+\frac{1}{2}\sum_{l\in N_c}\sum_{h\in H}\chi_h^W \Delta \log \chi_h^W \Delta \log \mu_l (\lambda_l^{W_h}-\lambda_l)/\chi_c^Y,$$

and changes in the real output are given up to the second order by $\Delta \log Y_h \approx 0$. Corollary 6 goes through unchanged and can also be applied at the level of a household, using Corollary 5.

I Ex-Ante Comparative Statics for Transfer Shocks

In this section, we show how to extend the results in Section 5 to cover shocks to transfers. Define the $(N+F)\times C$ matrix Ξ^T:

$$\Xi_{ic}^T = \sum_{c\in C}\frac{\lambda_i^{W_c}}{\lambda_i}.$$

For some feasible perturbation of transfers $\sum_{i\in C} \mathrm{d}\, T_c = 0$, changes in factor shares solve the following system of linear equations

$$\mathrm{d} \log \Lambda_f = -\sum_{g\in F}\Gamma_{fg}\,\mathrm{d} \log \Lambda_g + \sum_{g\in F}\Xi_{fg}\,\mathrm{d} \log \Lambda_g + \Xi_{fc}^T\,\mathrm{d}\, T_c.$$

Changes in sales shares are then given by

$$\mathrm{d}\log\lambda_j = -\sum_{g\in F}\Gamma_{jg}\,\mathrm{d}\log\Lambda_g + \sum_{g\in F}\Xi_{jg}\,\mathrm{d}\log\Lambda_g + \Xi^T_{jc}\,\mathrm{d}\,T_c.$$

J Terms-of-Trade Decomposition

In this section, we characterize an alternative decomposition of welfare in terms of output and terms-of-trade effects. We then contrast this decomposition with the reallocation decomposition, and provide some notable special cases under which the terms-of-trade decomposition or the reallocation decomposition take especially simple forms.

Proposition 14 (Welfare-Accounting, Terms of Trade)**.** *The change in welfare of country c in response to productivity shocks, factor supply shocks, and transfer shocks can be decomposed into:*[12]

$$\mathrm{d}\log W_c = \underbrace{\frac{\chi_c^Y}{\chi_c^W}\,\mathrm{d}\log Y_c}_{\Delta Output} + \underbrace{\frac{\chi_c^Y}{\chi_c^W}\,\mathrm{d}\log P_{Y_c} - \mathrm{d}\log P_{W_c}}_{\Delta Terms\ of\ Trade} + \underbrace{\frac{1}{\chi_c^W}\,\mathrm{d}\,T_c + \sum_{f\in F}(\Lambda_f^c - \frac{\chi_c^Y}{\chi_c^W}\Lambda_f^{Y_c})\,\mathrm{d}\log\Lambda_f}_{\Delta Transfers\ and\ Net\ Factor\ Payments},$$

where the change in terms of trade $(\chi_c^Y/\chi_c^W)\,\mathrm{d}\log P_{Y_c} - \mathrm{d}\log P_{W_c}$ *is*

$$\sum_{i\in N}(\lambda_i^{W_c} - \frac{\chi_c^Y}{\chi_c^W}\lambda_i^{Y_c})\,\mathrm{d}\log A_i + \sum_{f\in F}(\Lambda_f^{W_c} - \frac{\chi_c^Y}{\chi_c^W}\Lambda_f^{Y_c})\left(-\mathrm{d}\log\Lambda_f + \mathrm{d}\log L_f\right),$$

with $\chi_c^Y/\chi_c^W = GDP_c/GNE_c$. *The change* $\mathrm{d}\log W$ *of world real expenditure can be obtained by simply suppressing the country index c.*

To understand this result, consider for example a unit change in the productivity of producer i on the terms of trade. Intuitively, for given factor wages, the productivity shock affects the terms of trade of country c according to the difference between the country's exposures to producer i in real expenditure and in real output $\lambda_i^{W_c} - (\chi_c^Y/\chi_c^W)\lambda_i^{Y_c}$. The productivity also leads to endogenous changes in the wages of the different factors $\mathrm{d}\log w_f$, which given that factor supplies are fixed, coincide with the changes in their factor income shares $\mathrm{d}\log\Lambda_f$.[13] These changes in factor wages in turn affect the country's

[12]When all factors inside a country are owned by the residents of that country, $\Lambda_f^c = \Lambda_f^{Y_c}$, and so net factor payments are zero. If in addition, there are no transfers so that $T_c = 0$, then $\chi_c^Y = \chi_c^W$ and our decomposition is invariant to changes in the numeraire. Outside of this case, the choice of numeraire influences the breakdown into changes in terms of trade and changes in transfers and net factor payments, but not the sum of the two.

[13]The formula actually still applies with endogenous factor supplies.

terms of trade according to the difference between the country's exposures to producer f in real expenditure and in real output $\Lambda_f^{W_c} - (\chi_c^Y/\chi_c^W)\Lambda_f^{Y_c}$.

At the country level, unlike real output, real expenditure or welfare *does* respond to productivity shocks outside the country in general because these shocks affect the terms of trade (and net factor payments). In particular, while shocks to iceberg trade costs outside a country do not affect its real output or its productivity, they do affect its real expenditure or welfare.

At the world level, there are no terms-of-trade effects (and no transfers or net factor payments). Furthermore, changes in real output and real expenditure or welfare and their corresponding deflators for each country aggregate up to their world counterparts. This implies that changes in the country terms of trade sum up to zero:

$$\sum_{c \in C} \chi_c^W [(\chi_c^Y/\chi_c^W)\, \mathrm{d}\log P_{Y_c} - \mathrm{d}\log P_{W_c}] = \mathrm{d}\log P_Y - \mathrm{d}\log P_W = 0,$$

where $\chi_c^W = GNE_c/GNE$ and $\chi_c^Y = GDP_c/GDP$. Terms-of-trade effects can therefore be interpreted as zero-sum distributive effects. The same goes for transfers and net factor payments.

J.1 Terms of Trade vs. Reallocation

Theorems 14 and 2 provide two decompositions of changes in real expenditure or welfare with different economic interpretations: the terms-of-trade and reallocation decompositions. Both the reallocation effects and the terms-of-trade effects (and the net-factor-payments and transfer effects) can be interpreted as zero-sum distributive effects, and both of them can be written in terms of changes in factor shares. The goal of this section is to compare the two decompositions.

Two Hulten-Like Results

To frame our discussion, it is useful to start by stating two different Hulten-like results for welfare in open economies. We call these "Hulten-like" results because they predict changes in welfare as a function of initial expenditure shares only *without* requiring information on changes in (endogenous) factor shares.

Corollary 11 (Welfare, Two Hulten-like Results)**.** *In the following two special cases, Hulten-like results give changes in the welfare of a country c as exposure-weighted sums of productivity and factor supply shocks (and do not feature changes in factor shares).*

(i) Assume that country c receives no transfers from the rest of the world (balanced trade), there are no cross-border factor holdings, and international prices are exogenous and fixed (small-open economy). Then there are only real output effects, and no terms-of-trade, transfer effects, or net factor payment effects, so that the change in welfare is given by

$$\mathrm{d}\log W_c = \mathrm{d}\log Y_c = \sum_{f\in F_c} \Lambda_f^{Y_c}\,\mathrm{d}\log L_f + \sum_{i\in N_c} \lambda_i^{Y_c}\,\mathrm{d}\log A_i.$$

(ii) Assume either that the world economy is Cobb-Douglas or, if it is not, that we keep the allocation of resources (the allocation matrix) constant. Then there are only "pure" technology effects and no reallocation effects, so that the change welfare is given by:

$$\mathrm{d}\log W_c = \sum_{f\in F} \Lambda_f^{W_c}\,\mathrm{d}\log L_f + \sum_{i\in N} \lambda_i^{W_c}\,\mathrm{d}\log A_i.$$

Corollary 11 follows from Theorems 14 and 2. It shows that in some special cases, we can continue to use exposures to predict the effects of productivity and factor supply shocks on welfare in open economies.

The two Hulten-like results are very different. Focusing on productivity shocks, the elasticities $\mathrm{d}\log W_c/\,\mathrm{d}\log A_i$ of real expenditure to productivity shocks are given by exposures in real output $\lambda_i^{Y_c}$ in case (i) and by exposures in welfare $\lambda_i^{W_c}$ in case (ii).

The intuitions underlying the two Hulten-like results are also very different. The original Hulten theorem applies in a closed economy (e.g. the world) where there are neither terms-of-trade effects nor reallocation effects. In case (i), there are no terms-of-trade effects but there are reallocation effects. In case (ii), there are no reallocation effects, but there are terms-of-trade effects.

More generally, we can interpret the real output effects in Theorem 14 and the "pure" technology effects in Theorem 2 as Hulten-like terms, and the terms-of-trade effects (together with transfers and net factor payments) and reallocation effects as adjustment terms. As we saw earlier, these adjustment terms are zero-sum and depend on changes in factor shares.

Comparing the Terms-of-Trade and Reallocation Decompositions

Both decompositions can be applied at the level of a country and the world. Both decompositions isolate a distributive zero-sum term, which aggregates up to zero at the level of the world economy. These different distributive terms are responsible for departures from two different versions of Hulten's theorem. The main difference between the two decomposition is their economic interpretations.

Beyond their differences in interpretation, the two decompositions have different robustness and aggregation properties, and different data requirements. In these regards, the reallocation decomposition has several advantages.

First the reallocation decomposition is based on general equilibrium counterfactual: "pure" changes in technology coincide with the change in real expenditure that would arise under the feasible counterfactual allocation which keeps the allocation of resources constant. This is not the case for the terms-of-trade decomposition: changes in real output are *not* the changes in real expenditure that would arise under a specified feasible counterfactual allocation.

Second, as discussed in Section 5, this particular general equilibrium counterfactual is extremely useful conceptually and intuitively in order to unpack our counterfactual results. This is because reallocation effects (but not "pure" technology effects) depend *only* on expenditure substitution by the different producers and households in the economy. By contrast, terms-of-trade effects also include technology effects.

Third, the reallocation decomposition is not sensitive to irrelevant changes in the environment, because it does not use changes in real output. This is not the case for the terms-of-trade decomposition: for example, assuming that changes in iceberg trade costs apply to the importers of a good or to its exporter simply produces different representations of the same underlying changes in the economy and is immaterial for changes in welfare, but it does modify the changes in terms of trade of the importers and of the exporter.

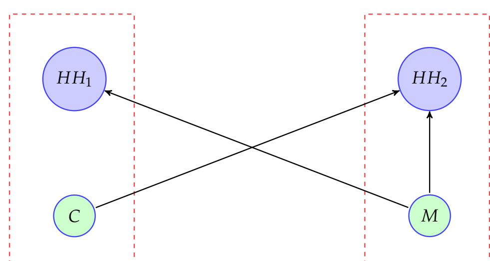


Figure 5: An illustration of the two welfare decompositions in an economy with two countries, two factors, and two goods. Country 1 has an endowment of a commodity good (C), and country 2 has an endowment of the manufacturing good (M). The representative household in country 1 consumes only the manufacturing good, and the representative household in country 2 consumes a CES aggregate of the two goods with an elasticity of substitution θ.

Fourth, the two decompositions have different economic interpretations. It is useful to provide a simple illustrative example. Consider the economy depicted in Figure 5 with two countries, two factors, and two goods. Country 1 has an endowment of a commodity good (C), and country 2 has an endowment of the manufacturing good (M). The representative household in country 1 consumes only the manufacturing good, and the representative household in country 2 consumes a CES aggregate of the two goods with an elasticity of substitution θ:

$$\left(\bar{\omega}_{2C}\left(\frac{y_{2C}}{\bar{y}_{2C}}\right)^{\frac{\theta-1}{\theta}}+\bar{\omega}_{2M}\left(\frac{y_{2M}}{\bar{y}_{2M}}\right)^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}.$$

C and M can either be substitutes ($\theta > 1$) or complements ($\theta < 1$). We denote by λ_2 the sales share of the consumption bundle of producer 2, and by Λ_C and Λ_M the sales shares of C and M (the factor income shares), with $\lambda_2\bar{\omega}_{2C} = \Lambda_C$.

Consider a shock $\mathrm{d}\,\bar{\omega}_{2M} = -\,\mathrm{d}\,\bar{\omega}_{2C} > 0$ which shifts the composition of demand away from C and towards M in country 2. [14] The shock reduces the welfare of country 1 with

$$\mathrm{d}\log W_1 = -\theta\frac{1}{\Lambda_M}\,\mathrm{d}\log\bar{\omega}_{2C} < 0.$$

There are neither real output nor "pure" technology effects, and there are equivalent negative terms-of-trade effects and reallocation effects:

$$\mathrm{d}\log p_C - \mathrm{d}\log p_M = \mathrm{d}\log\Lambda_C - \mathrm{d}\log\Lambda_M = -\theta\frac{1}{\Lambda_M}\,\mathrm{d}\log\bar{\omega}_{2C} < 0.$$

This can be seen as a simple illustration of the Prebisch-Singer hypothesis, whereby demand shifts towards manufacturing as countries develop at the expense of commodity producers.

Consider next a shock $\mathrm{d}\log C > 0$ which increases the endowment of C in country 1. The effect of the shock is different depending on whether C and M are substitutes (complements): it improves (reduces) the welfare of country 1 with

$$\mathrm{d}\log W_1 = (\theta-1)\frac{\bar{\omega}_{2M}}{\Lambda_M}\,\mathrm{d}\log C;$$

there are positive real output effects $\mathrm{d}\log Y_1 = \Lambda_C\,\mathrm{d}\log C > 0$ and less (more) negative

[14]This shock can be modeled as a combination of positive and negative productivity shocks $\mathrm{d}\log A_{2M} = [\theta/(\theta-1)]\,\mathrm{d}\log\mathrm{d}\,\bar{\omega}_{2M}$ and $\mathrm{d}\log A_{2C} = [\theta/(\theta-1)]\,\mathrm{d}\log\mathrm{d}\,\bar{\omega}_{2C}$ for fictitious producers intermediating between C, M, and the representative household of country 2.

terms-of-trade effects

$$\mathrm{d}\log p_C - \mathrm{d}\log p_M = -\Lambda_C \,\mathrm{d}\log C + (\theta - 1)\frac{\bar{\omega}_{2M}}{\Lambda_M}\,\mathrm{d}\log C;$$

there are no "pure" technology effects, and positive (negative) reallocation effects

$$\mathrm{d}\log \Lambda_C - \mathrm{d}\log \Lambda_M = (\theta - 1)\frac{\bar{\omega}_{2M}}{\Lambda_M}\,\mathrm{d}\log C.$$

Finally, consider a shock which increases the endowment of M in country 2. This shock improves the welfare of country 1 as long as goods are not too substitutes with

$$\mathrm{d}\log W_1 = \mathrm{d}\log M - (\theta - 1)\frac{\bar{\omega}_{2M}}{\Lambda_M}\,\mathrm{d}\log M;$$

; there are no real output effects and positive terms-of-trade effects as long as goods are not too substitutes with

$$\mathrm{d}\log p_C - \mathrm{d}\log p_M = \mathrm{d}\log M - (\theta - 1)\frac{\bar{\omega}_{2M}}{\Lambda_M}\,\mathrm{d}\log M;$$

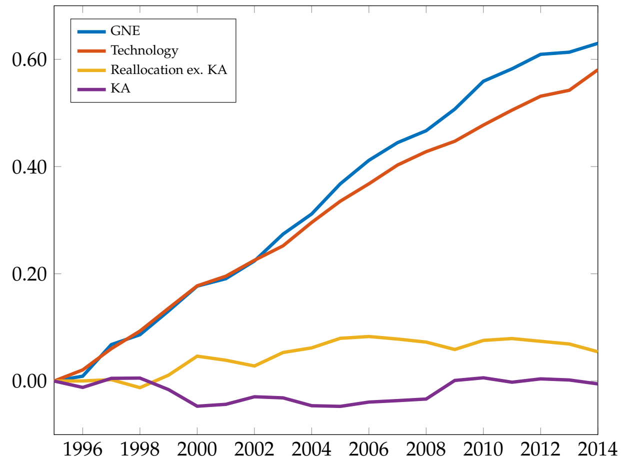
there are positive "pure" technology effects $\mathrm{d}\log M > 0$ and negative (positive) reallocation effects if C and M are substitutes (complements) with

$$\mathrm{d}\log \Lambda_C - \mathrm{d}\log \Lambda_M = -(\theta - 1)\frac{\bar{\omega}_{2M}}{\Lambda_M}\,\mathrm{d}\log M.$$

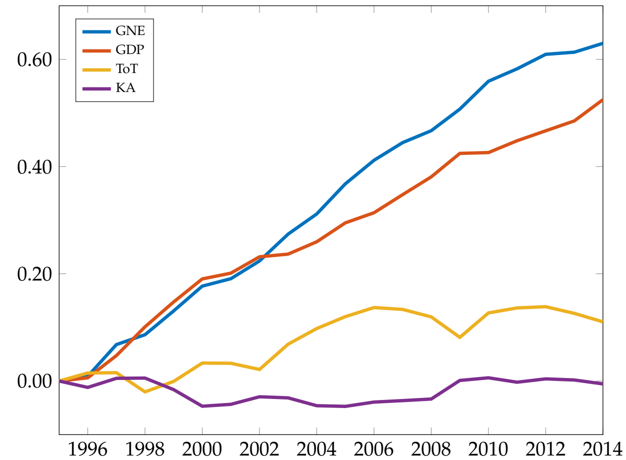
J.2 Application of Welfare-Accounting Formulas

We end this discussion of the welfare-accounting formulas by decomposing the change in real expenditure in different countries over time. We implement our two decompositions: the reallocation decomposition and the terms-of-trade decomposition. We abstract away from distortions. Unlike our previous applications, these decompositions are non-parametric in the sense that they do not require taking a stand on the various elasticities of substitution.

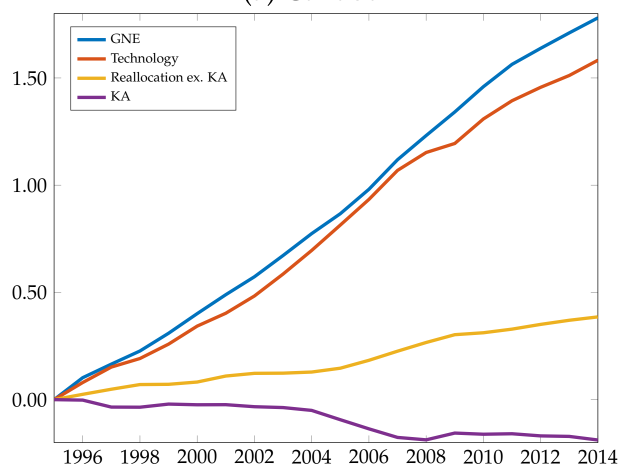
The left column of Figure 6 displays the cumulative change in each component over time of the reallocation decomposition, for a few countries (Canada, China, and Japan). We choose these three countries because they depict a systematic pattern: industrializing countries, like China, and commodities- or services-dependent industrialized countries, like Canada, are experiencing positive reallocation, whereas manufacturing-dependent industrialized countries, like Japan, are experiencing negative reallocation.



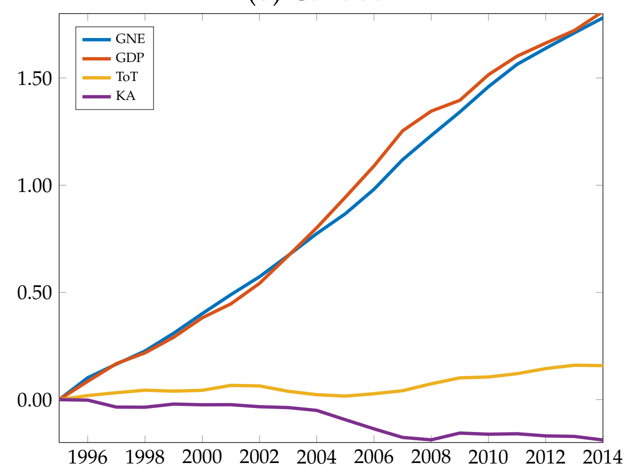
(a) Canada



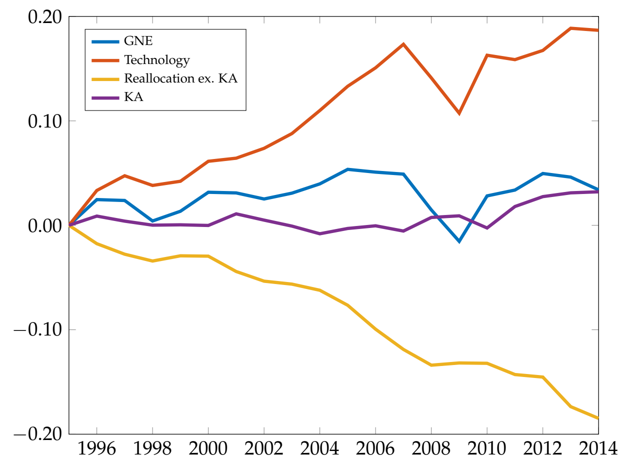
(b) Canada



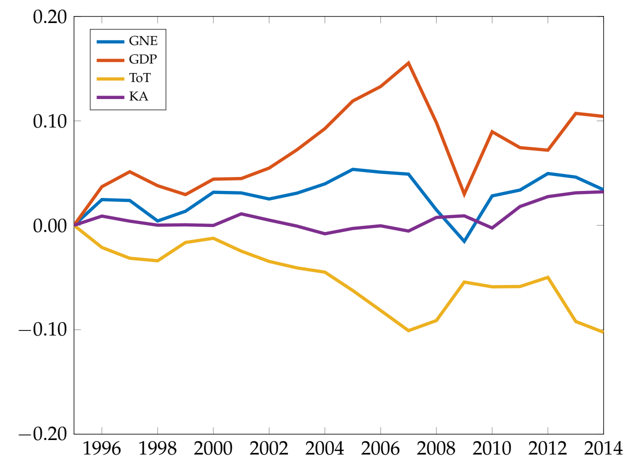
(c) China



(d) China



(e) Japan



(f) Japan

Figure 6: Welfare accounting according to the reallocation decomposition (left column) and according to the terms-of-trade decomposition (right column), for a sample of countries, using the WIOD data.

The right column of Figure 6 displays the terms-of-trade decomposition. Commodity producers like Canada experience large movements in terms of trade due to fluctuations in commodity prices. Even for countries for which terms-of-trade effects are small, reallocation effects are typically large, indicating that these countries cannot be taken to be approximately closed.

Finally, it is interesting to note that the difference between the reallocation effect on the one hand, and the terms of trade effect and the transfer effect on the other hand identifies the following technological residual:[15]

$$\sum_{i\in N}((\chi_c^Y/\chi_c^W)\,\lambda_i^{Y_c}-\lambda_i^{W_c})\,\mathrm{d}\log A_i+\sum_{f\in F}((\chi_c^Y/\chi_c^W)\Lambda_f^{Y_c}-\Lambda_f^{W_c})\,\mathrm{d}\log L_c.$$

This residual is a measure of the difference between country c's technological change and its exposure to world technical change, including the effects of changes in productivities and in factor supplies. For a closed economy, it is always zero. By comparing the two columns of Figure 6, we can see that (and by how much) China and Canada are experiencing faster growth in productivities and factor supplies in their domestic real output than in their consumption baskets, while the pattern is reversed for Japan.

K Stability of the Trade Elasticity

In this section, we prove necessary and sufficient conditions for ensuring that the trade elasticity is constant and stable. We also relate the instability of the trade elasticity to the Cambridge Capital controversy — a mathematically similar issue that arose in capital theory in the middle of the 20th century.

K.1 Necessary and Sufficient Conditions for Constant Trade Elasticity

Recall that the trade elasticity between i and j with respect to shocks to k is defined as

$$\varepsilon_{ij,k}=\frac{\partial(\lambda_i/\lambda_j)}{\partial\log A_k},$$

holding fixed some prices, typically factor prices. We say that a good k is *relevant* for $\varepsilon_{ij,k}$ if

$$\lambda_m Cov_{\Omega^{(m)}}(\Psi_{(k)},\Psi_{(i)}/\lambda_i-\Psi_{(j)}/\lambda_j)\neq 0.$$

[15] That is, we compute $(\partial\log\mathcal{W}_h)(\partial\mathcal{X})\,\mathrm{d}\,\mathcal{X}-(1/\chi_c^W)\,\mathrm{d}\,T_c-(\chi_c^Y/\chi_c^W)\,\mathrm{d}\log P_{Y_c}+\mathrm{d}\log P_{W_c}$.

If k is not relevant, we say that it is irrelevant. For instance, if some producer m is exposed symmetrically to i and j through its inputs

$$\Omega_{ml}(\Psi_{li} - \Psi_{lj}) = 0 \quad (l \in N),$$

then $\varepsilon_{ij,k}$ is not a function of θ_m and m is irrelevant. Another example is if some producer $m \neq j$ is not exposed to k through its inputs

$$\Psi_{mk} = 0,$$

then $\varepsilon_{ij,k}$ is not a function of θ_m and m is irrelevant.

Corollary 12 (Constant Trade Elasticity)**.** *Consider two distinct goods i and j that are imported to some country c. Then consider the following conditions:*

(i) Both i and j are unconnected to one another in the production network: $\Psi_{ij} = \Psi_{ji} = 0$*, and i is not exposed to itself* $\Psi_{ii} = 1$*.*

(ii) The representative "world" household is irrelevant

$$Cov_{\chi}\left(\Psi_{(i)}, \frac{\Psi_{(i)}}{\lambda_i} - \frac{\Psi_{(j)}}{\lambda_j}\right) = 0,$$

which holds if both i and j are only used domestically, so that only household c is exposed to i and j. That is, $\lambda_i^{W_h} = \lambda_j^{W_h} = 0$ *for all* $h \neq c$*. This assumption holds automatically if i and j are imports and domestic goods and there are no input-output linkages.*

(iii) For every relevant producer l, the elasticity of substitution $\theta_l = \theta$*.*

The trade elasticity of i relative to j with respect to iceberg shocks to i is constant, and equal to

$$\varepsilon_{ij,i} = (\theta - 1).$$

if, and only if, (i)-(iii) hold.

The conditions set out in the example above, while seemingly stringent, actually represent a generalization of the conditions that hold in gravity models with constant trade elasticities. Those models oftentimes either assume away the production network, or assume that traded goods always enter via the same CES aggregator.

A noteworthy special case is when i and j are made directly from factors, without any intermediate inputs. Then, we have the following

Corollary 13. *(Network Irrelevance) If some good i and j are only made from domestic factors, then*

$$\sum_{m \in C,N} \lambda_m Cov_{\Omega^{(m)}}(\Psi_{(i)}, \Psi_{(j)}/\lambda_i - \Psi_{(i)}/\lambda_i) = 1.$$

Hence, if all microeconomic elasticities of substitution θ_m are equal to the same value $\theta_m = \theta$ then $\varepsilon_{ij,j} = \theta$.

Suppose that i is domestic goods and j are foreign imports, both of which are made only from factors (no intermediate inputs are permitted). Then a shock to j is equivalent to an iceberg shock to transportation costs. In this case, the trade elasticity of imports j into the country producing i with respect to iceberg trade costs is a convex combination of the underlying microelasticities. Of course, whenever all micro-elasticities of substitution are the same, the weights (which have to add up to one) become irrelevant, and this is the situation in most benchmark trade models with constant trade elasticities. Specifically, this highlights the fact that having common elasticities is not enough to deliver a constant trade elasticity in the presence of input-output linkages as shown in the round-about example of Section E.

K.2 Trade Reswitching

Yi (2003) shows that the trade elasticity can be nonlinear due to vertical specialization, where the trade elasticity can increase as trade barriers are lowered. Building on this insight, we can also show that, at least in principle, the trade elasticity can even have the "sign" due to these nonlinearities. This relates to a parallel set of paradoxes in capital theory.

To see how this can happen, imagine there are two ways of producing a given good: the first technique uses a domestic supply chain and the other technique uses a global value chain. Whenever the good is domestically produced, the iceberg costs of transporting the good are, at most, incurred once — when the finished good is shipped to the destination. However, when the good is made via a global value chain, the iceberg costs are incurred as many times as the good is shipped across borders. As a function of the iceberg cost parameter τ, the difference in the price of these two goods (holding factor prices fixed) is a polynomial of the form

$$B_n \tau^n - B_1 \tau, \tag{5}$$

where B_n and B_1 are some coefficients and n is the number of times the border is crossed. The nonlinearity in τ, whereby the iceberg cost's effects are compounded by crossing the border, drives the sensitivity of trade volume to trade barriers in Yi (2003). The benefits from using a global value chain are compounded if the good has to cross the border many times.

However, this discussion indicates the behavior of the trade elasticity can, in principle, be much more complicated. In fact, an interesting connection can be made between the behavior of the trade elasticity and the (closed-economy) reswitching debates of the 1950s and 60s. Specifically, equation (5) is just one special case. In general, the cost difference between producing goods using supply chains of different lengths is a polynomial in τ – and this polynomial can, in principle, have more than one root. This means that the trade elasticity can be non-monotonic as a function of the trade costs, in fact, it can even have the "wrong" sign, where the volume of trade decreases as the iceberg costs fall. This mirrors the apparent paradoxes in capital theory where the relationship between the capital stock and the return on capital can be non-monotonic, and an increase in the interest rate can cause the capital stock to increase.

To see this in the trade context, imagine two perfectly substitutable goods, one of which is produced by using 10 units of foreign labor, the other is produced by shipping 1 unit of foreign labor to the home country, back to the foreign country, and then back to the home country and combining it with 10 units of domestic labor. If we normalize both foreign and domestic wages to be unity, then the costs of producing the first good is $10(1+\tau)$, whereas the cost of producing the second good is $(1+\tau)^3 + 10$, where τ is the iceberg trade cost. When $\tau = 0$, the first good dominates and goods are only shipped once across borders. When τ is sufficiently high, the cost of crossing the border is high enough that the first good again dominates. However, when τ has an intermediate value, then it can become worthwhile to produce the second good, which causes goods to be shipped across borders many times, thereby inflating the volume of trade.

Such examples are extreme, but they illustrate the point that in the presence of input-output networks, the trade elasticity even in partial equilibrium (holding factor prices constant) can behave quite unlike any microeconomic demand elasticity, sloping upwards when, at the microeconomic level, every demand curve slopes downwards.

Non-Symmetry and Non-Triviality of Trade Elasticities

Another interesting subtlety of Equation (4) is that the aggregate trade elasticities are non-symmetric. That is, in general $\varepsilon_{ij,l} \neq \varepsilon_{ji,l}$. Furthermore, unlike the standard gravity equation, Equation (4) shows that the cross-trade elasticities are, in general, nonzero. Hence, changes in trade costs between k and l can affect the volume of trade between i and j *holding fixed* relative factor prices and incomes. This is due to the presence of global value chains, which transmit shocks in one part of the economy to another independently of the usual general equilibrium effects (which work through the price of factors).

L Proofs

Proof of Theorem 1. For some country h, let N_h be the set of domestically produced goods and let N_{-h} be the set of foreign-produced goods. Let Ω^h be the matrix whose elements are

$$\Omega^h_{ij} = \frac{M_{ij}}{R_i} \quad (i, j \in N_h),$$

where M_{ij} is the value of i's purchases from j and R_i is the revenues of i. Similarly, let Ω^{-h} be given by

$$\Omega^{-h}_{ij} = \frac{M_{ij}}{R_i} \quad (i \in H, j \in N_{-h}).$$

Finally, let

$$\alpha_{ij} = \frac{w_j L_{ij}}{R_i} \quad (i \in H),$$

where $w_j L_{ij}$ are factor payments by i to factor j.

Denoting the vector of domestic prices by p^h, foreign prices by p^{-h}, domestic productivity shocks and wages by w^h and A^h, Shephard's lemma implies that

$$\mathrm{d}\log p^h = (I - \Omega^h)^{-1} \left(\Omega^{-h}\, \mathrm{d}\log p^{-h} + \alpha\, \mathrm{d}\log w^h - \mathrm{d}\log A^h \right).$$

Now, let $\chi^{Y_h}_{(h)}$ be the vector of each domestically produced good's share in GDP, so for domestically produced goods

$$\chi^{Y_h}_i = \frac{F_i + X_i}{GDP},$$

where F_i is final domestic use and X_i is exports. Let $\chi^{Y_h}_{(-h)}$ be the vector of each imported good's share in GDP, so for a foreign good

$$\chi^{Y_h}_i = -\sum_{j \in N_h} \Omega_{ji} \lambda_j, \quad (i \in N_{-h}).$$

By definition,

$$\mathrm{d}\log Y_h = \left(\Lambda^{Y_h}\right)' (\mathrm{d}\log w^h + \mathrm{d}\log L^h) - \mathrm{d}\log P_{Y_h},$$

where L^h is the vector of quantities and $\Lambda^{Y_h}_h$ the local Domar weights of domestic factors.

This can be written as

$$\mathrm{d}\log Y_h = \left(\Lambda^{Y_h}\right)' (\mathrm{d}\log w^h + \mathrm{d}\log L^h) - \chi^{Y_h}_{(h)}\,\mathrm{d}\log p^h + (\chi^{Y_h}_{(-h)})'\,\mathrm{d}\log p^{-h}.$$

Substituting the expression for $\mathrm{d}\log p^h$ gives

$$\begin{aligned}\mathrm{d}\log Y_h = \left(\Lambda^{Y_h}\right)' (\mathrm{d}\log w + \mathrm{d}\log L) - (\chi^{Y_h}_{(h)})'(I-\Omega^h)^{-1}\left(\Omega^{-h}\,\mathrm{d}\log p^{(-h)} + \alpha\,\mathrm{d}\log w - \mathrm{d}\log A\right) \\ + (\chi^{Y_h}_{(-h)})'\,\mathrm{d}\log p^f.\end{aligned}$$

To complete the proof, note that

$$(\chi^{Y_h}_{(h)})'(I-\Omega^h)^{-1}\alpha = \left(\Lambda^{Y_h}\right)',$$

$$(\chi^{Y_h}_{(h)})'(I-\Omega^h)^{-1}\Omega^{-h} = (\chi^{Y_h}_{(-h)})',$$

and

$$(\chi^{Y_h}_{(h)})'(I-\Omega^h)^{-1} = \left(\lambda_{Y_h}\right)',$$

These expressions follow from market clearing. Combining these gives the desired result.

■

Proof of Theorem 2.

$$W_c = \frac{\sum_{f\in F}\Phi_{cf} w_f L_f}{P^{W_c}}.$$

Hence, letting world GDP be the numeraire,

$$\mathrm{d}\log W_c = \sum_f \Lambda^c_f \left(\mathrm{d}\log\Lambda_f\right) - \left(\chi^{W_c}\right)'\mathrm{d}\log p.$$

Use the fact that

$$\mathrm{d}\log p_i = \sum_{j\in N}\Psi_{ij}\,\mathrm{d}\log A_j + \sum_{f\in F}\Psi_{if}\left(\mathrm{d}\log\Lambda_f - \mathrm{d}\log L_f\right)$$

to complete the proof. ■

Proof of Theorem 3. Here we assume that there is only one factor in the domestic economy and normalize its price to one. Define the "fictitious domestic" IO matrix

$$\check{\Omega}_{ij} \equiv \frac{\Omega_{ij}}{\sum_{k\in N_c}\Omega_{ik}},$$

with associated Leontief-inverse matrix

$$\check{\Psi} \equiv (1 - \check{\Omega})^{-1}.$$

Applying Feenstra (1994), for each producer $i \in N_c$, we have

$$d \log p_i = \sum_{j \in C} \check{\Omega}_{ij} d \log p_j + \frac{d \log \lambda_{ic}}{\theta_i - 1},$$

where λ_{ic} is the domestic cost share of producer i. The solution of this system of equations is

$$d \log p_i = \sum_{j \in C} \check{\Psi}_{ij} \frac{d \log \lambda_{jc}}{\theta_j - 1}.$$

From this we can get welfare gains

$$d \log Y^c = - \sum_{i \in C} \check{b}_i d \log p_i = - \sum_{i \in N_c} \sum_{j \in C} \check{b}_i \check{\Psi}_{ij} \frac{d \log \lambda_{jc}}{\theta_j - 1} = - \sum_{j \in N_c} \check{\lambda}_j \frac{d \log \lambda_{jc}}{\theta_j - 1},$$

where

$$\check{\lambda}_i \equiv \sum_{j \in N_c} \check{b}_j \check{\Psi}_{ji}.$$

This can be thought of as hitting the fictitious domestic economy with productivity shocks $-d \log \lambda_{jc} / (\theta_j - 1)$. Since relative domestic prices in the closed and open economy are identical, the relative expenditure shares on domestic goods moves in the same way in both economies. ■

Proof of Theorem 4. This is a special case of Proposition 13 in Appendix F. ■

Proof of Corollary 4. By Shephard's lemma,

$$\mathrm{d} \log p_i = \sum_{j \in N} \Omega_{ij} \, \mathrm{d} \log p_j + \sum_{j \in F} \Omega_{ij} \, \mathrm{d} \log w_j - \mathrm{d} \log A_i.$$

Solve this system of equations in $\mathrm{d} \log p$ to get the desired result. ■

Lemma 15. *In general, for any f and g,*

$$\sum_m \lambda_m Cov_{\Omega^{(m)}}(\Psi_{(f)}, \Psi_{(g)}) = -\lambda_f \lambda_g - Cov_{\chi}(\Psi_{(f)}, \Psi_{(g)}) + \Psi_{gf} \lambda_g + \Psi_{fg} \lambda_f - \lambda_f \mathbf{1}(f = g).$$

Consider a good k which does not use itself directly or indirectly. Then

$$\sum_{m \in C,N} \lambda_m Var_{\Omega^m}(\Psi_{(k)}) = \lambda_k(1-\lambda_k) - Var_\chi(\Psi_{(k)}).$$

Consider two goods which don't rely on each other, then

$$\sum_m \lambda_m Cov_{\Omega^{(m)}}(\Psi_{(f)}, \Psi_{(g)}) = -\lambda_f \lambda_g - Cov_\chi(\Psi_{(f)}, \Psi_{(g)}).$$

Proof of Theorem 5. Proof of Part(1):

The expression for $\mathrm{d}^2 \log Y$ follows from applying part (2) to the whole world. The equality of real GNE and real GDP at the world level completes the proof.

Proof of Part (2):

Denote the set of imports into country c by M_c. Then, we can write:

$$\frac{\mathrm{d}\log Y_c}{\mathrm{d}\log \mu_i} = \sum_{f \in F_c} \Lambda_f^{Y_c} \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_i} + \sum_j \frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log \mu_i} \frac{\left(1-\frac{1}{\mu_j}\right)}{P_{Y_c} Y_c} + \frac{\lambda_i^{Y_c}}{\mu_i} - \frac{\mathrm{d}\log P_{Y_c}}{\mathrm{d}\log \mu_i},$$

where

$$\frac{\mathrm{d}\log P_{Y_c}}{\mathrm{d}\log \mu_i} = \sum_{f \in F_c} \tilde{\Lambda}_f^{Y_c} \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_i} + \sum_{m \in M_c} \tilde{\lambda}_m^{Y_c} \frac{\mathrm{d}\log p_m}{\mathrm{d}\log \mu_i} - \tilde{\lambda}_i^{Y_c} - \sum_{m \in M_c} \Lambda_m^{Y_c} \frac{\mathrm{d}\log p_m}{\mathrm{d}\log \mu_i},$$

and

$$\tilde{\lambda}_i^{Y_c} = \sum_j \chi_j^{Y_c} \tilde{\Psi}_{ji}.$$

Combining these expressions, we get

$$\begin{aligned}\frac{\mathrm{d}\log Y_c}{\mathrm{d}\log \mu_i} &= \sum_{f \in F_c} \left(\Lambda_f^{Y_c} - \tilde{\Lambda}_f^{Y_c}\right) \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_i} + \sum_{m \in M_c} \left(\lambda_m^{Y_c} - \tilde{\lambda}_m^{Y_c}\right) \frac{\mathrm{d}\log p_m}{\mathrm{d}\log \mu_i} \\ &\quad + \sum_{j \in N_c} \lambda_j^{Y_c} \frac{\mathrm{d}\log \lambda_j}{\mathrm{d}\log \mu_i} \left(1 - \frac{1}{\mu_j}\right) + \frac{\lambda_i^{Y_c}}{\mu_i} - \tilde{\lambda}_i^{Y_c}.\end{aligned}$$

At the efficient point,

$$\frac{\mathrm{d}^2 \log Y_c}{\mathrm{d}\log \mu_i \,\mathrm{d}\log \mu_k} = \sum_{f \in F_c} \left(\frac{\mathrm{d}\,\Lambda_f^{Y_c}}{\mathrm{d}\log \mu_i} - \frac{\mathrm{d}\,\tilde{\Lambda}_f^{Y_c}}{\mathrm{d}\log \mu_i} \right) \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}$$

$$
+\sum_{m\in M_c}\left(\frac{\mathrm{d}\,\lambda_m^{Y_c}}{\mathrm{d}\log\mu_i}-\frac{\mathrm{d}\,\tilde{\lambda}_m^{Y_c}}{\mathrm{d}\log\mu_i}\right)\frac{\mathrm{d}\log p_m}{\mathrm{d}\log\mu_k}-\frac{\mathrm{d}\,\tilde{\lambda}_k^{Y_c}}{\mathrm{d}\log\mu_i}
+\lambda_k^{Y_c}\left(\frac{\mathrm{d}\log\lambda_k^{Y_c}}{\mathrm{d}\log\mu_i}-\delta_{ki}\right)+\frac{1}{P_{Y_c}Y_c}\frac{\mathrm{d}\,\lambda_i^{Y_c}}{\mathrm{d}\log\mu_k},
$$

where δ_{ki} is the a Kronecker delta.

Using Lemma 17,

$$
\begin{aligned}
\frac{\mathrm{d}^2\log Y_c}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k}&=-\sum_{f\in F_c}\lambda_i^{Y_c}\Psi_{if}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}-\sum_{m\in M_c}\lambda_i^{Y_c}\Psi_{im}\frac{\mathrm{d}\log p_m}{\mathrm{d}\log\mu_k}-\lambda_i^{Y_c}\left(\Psi_{ik}-\delta_{ik}\right)\\
&-\lambda_k^{Y_c}\delta_{ik}+\frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k}\frac{1}{P_{Y_c}Y_c},\\
&=-\sum_{f\in F_c}\lambda_i^{Y_c}\Psi_{if}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}-\sum_{m\in M_c}\lambda_i^{Y_c}\Psi_{im}\frac{\mathrm{d}\log p_m}{\mathrm{d}\log\mu_k}-\lambda_i^{Y_c}\Psi_{ik}\\
&+\lambda_i^{Y_c}\left(\frac{\mathrm{d}\log p_i}{\mathrm{d}\log\mu_k}+\frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k}\right),\\
&=\lambda_i^{Y_c}\frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k}.
\end{aligned}
$$

■

Lemma 16.

$$
\frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log\mu_k}-\sum_h\overline{\chi}_h\frac{\mathrm{d}\log\tilde{\lambda}_j^{W_h}}{\mathrm{d}\log\mu_k}=\sum_h\frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_j^{W_h}-\lambda_i\left(\Psi_{ij}-\delta_{ij}\right).
$$

Proof. Let μ be the diagonal matrix of μ_i and I_{μ_k} be a matrix of all zeros except μ_k for its kth diagonal element. Then for each h,

$$
\tilde{\lambda}^{W_h}=b^{(h)}+\tilde{\lambda}^{W_h}\mu\Omega.
$$

Hence,

$$
\frac{\mathrm{d}\,\tilde{\lambda}^{W_h}}{\mathrm{d}\log\mu_k}=\frac{b^{(h)}}{\mathrm{d}\log\mu_k}+\frac{\mathrm{d}\,\tilde{\lambda}^{W_h}}{\mathrm{d}\log\mu_k}\mu\Omega+\tilde{\lambda}^{W_h}I_{\mu_k}\Omega+\tilde{\lambda}^{W_h}\mu\frac{\mathrm{d}\,\Omega}{\mathrm{d}\log\mu_k}.
$$

Hence,

$$
\overline{\chi}'\frac{\mathrm{d}\,\tilde{\lambda}}{\mathrm{d}\log\mu_k}=\chi'\frac{b}{\mathrm{d}\log\mu_k}+\chi'\frac{\mathrm{d}\,\tilde{\lambda}}{\mathrm{d}\log\mu_k}\mu\Omega+\chi'\tilde{\lambda}I_{\mu_k}\Omega+\chi'\tilde{\lambda}\mu\frac{\mathrm{d}\,\Omega}{\mathrm{d}\log\mu_k}.
$$

On the other hand,

$$\lambda = \chi' b + \lambda \Omega.$$

Form this, we have

$$\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log\mu_k} = \frac{\mathrm{d}\,\chi'}{\mathrm{d}\log\mu_k} b + \chi' \frac{b}{\mathrm{d}\log\mu_k} + \lambda \frac{\mathrm{d}\,\Omega}{\mathrm{d}\log\mu_k} + \frac{\mathrm{d}\,\lambda}{\mathrm{d}\log\mu_k}\Omega.$$

Combining these two expressions

$$\left(\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log\mu_k} - \overline{\chi}' \frac{\mathrm{d}\log\tilde{\lambda}}{\mathrm{d}\log\mu_k}\right) = \left(\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log\mu_k} - \overline{\chi}' \frac{\mathrm{d}\log\tilde{\lambda}}{\mathrm{d}\log\mu_k}\right)\Omega + \frac{\mathrm{d}\,\chi}{\mathrm{d}\log\mu_k} b - \chi' \tilde{\lambda}^{(h)} I_{\mu_k} \Omega.$$

Rearrange this to get

$$\left(\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log\mu_k} - \overline{\chi}' \frac{\mathrm{d}\log\tilde{\lambda}}{\mathrm{d}\log\mu_k}\right) = \frac{\mathrm{d}\,\chi}{\mathrm{d}\log\mu_k} b\Psi - \chi' \tilde{\lambda}^{(h)} I_{\mu_k} (\Psi - I),$$

or

$$\left(\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log\mu_k} - \overline{\chi}' \frac{\mathrm{d}\log\tilde{\lambda}}{\mathrm{d}\log\mu_k}\right) = \frac{\mathrm{d}\,\chi}{\mathrm{d}\log\mu_k} b\Psi - \lambda I_{\mu_k} (\Psi - I).$$

■

Lemma 17. *At the efficient steady-state*

$$\frac{\mathrm{d}\,\lambda_j^{Y_c}}{\mathrm{d}\log\mu_k} - \frac{\mathrm{d}\,\tilde{\lambda}_j^{Y_c}}{\mathrm{d}\log\mu_k} = -\lambda_k^{Y_c}\left(\Psi_{kj} - \delta_{kj}\right).$$

Proof. Start from the relations

$$\lambda_j^{Y_c} = \chi_j^{Y_c} + \sum_i \lambda_i^{Y_c} \Omega_{ij},$$

and

$$\tilde{\lambda}_j^{Y_c} = \chi_j^{Y_c} + \sum_i \tilde{\lambda}_i^{Y_c} \mu_i \Omega_{ij}.$$

Differentiate both to get

$$\frac{\mathrm{d}\,\lambda_j^{Y_c}}{\mathrm{d}\log\mu_k} - \frac{\mathrm{d}\,\tilde{\lambda}_j^{Y_c}}{\mathrm{d}\log\mu_k} = \sum_i \left(\frac{\mathrm{d}\,\lambda_j^{Y_c}}{\mathrm{d}\log\mu_k} - \frac{\mathrm{d}\,\tilde{\lambda}_j^{Y_c}}{\mathrm{d}\log\mu_k}\right)\Omega_{ij} - \lambda_k^{Y_c}\Omega_{ki}.$$

Rearrange this to get the desired result. ■

Proof of Corollary 5.

$$\frac{\mathrm{d}\log W^{BS}}{\mathrm{d}\log\mu_k} = \sum_{h\in H}\overline{\chi}_h^W \frac{\mathrm{d}\log W_h}{\mathrm{d}\log\mu_k} = \sum_h \overline{\chi}_h^W \left(\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_k} - \frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k}\right).$$

$$\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_k} = \sum_{f\in F_c}\frac{\Lambda_f}{\chi_h}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} + \sum_{i\in N_h}\frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k}\frac{(1-\frac{1}{\mu_i})}{\chi_h}.$$

$$\frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k} = \sum_{f\in F}\tilde{\Lambda}_f^{W_h}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} + \tilde{\lambda}_k^{W_h}.$$

Hence, assuming the normalization $P_Y Y = 1$ gives

$$\begin{aligned}\frac{\mathrm{d}^2\log W^{BS}}{\mathrm{d}\log\mu_k\,\mathrm{d}\log\mu_i} = \sum_h \overline{\chi}_h^W \Bigg(&\sum_f \frac{\mathrm{d}\,\Lambda_f}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{1}{\chi_h^W} + \sum_f\frac{\Lambda_f}{\chi_h^W}\frac{\mathrm{d}^2\log\Lambda_f}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k} \\ &- \sum_f\frac{\Lambda_f}{\chi_h^W}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_i} + \frac{\mathrm{d}\,\lambda_k}{\mathrm{d}\log\mu_i}\frac{1}{\chi_h^W\mu_k} - \frac{\lambda_k}{\chi_h^W\mu_k}\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_i} - \frac{\lambda_k}{\chi_h^W\mu_k}\delta_{ki} \\ &\sum_i\frac{\mathrm{d}^2\,\lambda_j}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k}\frac{1-\frac{1}{\mu_j}}{\chi_h} + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k}\frac{1}{\mu_i\chi_h^W} + \sum_j\frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log\mu_k}\frac{1-\frac{1}{\mu_j}}{\chi_h^W}\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_i} \\ &- \sum_f\frac{\mathrm{d}\,\tilde{\Lambda}_f^{W_h}}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} - \sum_f\tilde{\Lambda}_f^{W_h}\frac{\mathrm{d}^2\log\Lambda_f}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k} - \frac{\mathrm{d}\,\tilde{\lambda}_k^{W_h}}{\mathrm{d}\log\mu_i}\Bigg).\end{aligned}$$

At the efficient point, this simplifies to

$$\begin{aligned}\frac{\mathrm{d}^2\log W^{BS}}{\mathrm{d}\log\mu_k\,\mathrm{d}\log\mu_i} = &\sum_f\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\left(\frac{\mathrm{d}\,\Lambda_f}{\mathrm{d}\log\mu_i} - \sum_h\overline{\chi_h^W}\frac{\mathrm{d}\,\tilde{\Lambda}_f^{W_h}}{\mathrm{d}\log\mu_i}\right) \\ &+ \frac{\mathrm{d}\,\lambda_k}{\mathrm{d}\log\mu_i} - \sum_h\overline{\chi_h^W}\frac{\mathrm{d}\,\tilde{\lambda}_k^{W_h}}{\mathrm{d}\log\mu_i} - \sum_{f,h}\Lambda_f\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_i} \\ &- \lambda_k\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_i} - \lambda_k\delta_{ki} + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k}.\end{aligned}$$

By Lemma 16, at the efficient point,

$$\frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log\mu_i} - \sum_h\overline{\chi}_h^W\frac{\mathrm{d}\,\tilde{\lambda}_j^{W_h}}{\mathrm{d}\log\mu_i} = \sum_h\frac{\mathrm{d}\,\chi_h^W}{\mathrm{d}\log\mu_i}\tilde{\lambda}_j^{W_h} - \lambda_i\left(\Psi_{ij} - \delta_{ij}\right).$$

Whence, we can further simplify the previous expression to

$$
\begin{aligned}
\frac{\mathrm{d}^2 \log W^{BS}}{\mathrm{d}\log \mu_k \,\mathrm{d}\log \mu_i} &= \sum_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}\left(\sum_h \frac{\mathrm{d}\,\chi_h^W}{\mathrm{d}\log \mu_i}\tilde{\Lambda}_f^{W_h} - \lambda_i \Psi_{if}\right) \\
&+ \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i}\tilde{\lambda}_k^{W_h} - \lambda_i(\Psi_{ik} - \delta_{ik}) - \sum_{f,h} \Lambda_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}\frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \\
&- \frac{\lambda_k}{\mathrm{d}\log \chi_h}\mathrm{d}\log \mu_i - \lambda_k \delta_{ki} + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log \mu_k}, \\
&= \sum_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}\left(\sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i}\tilde{\Lambda}_f^{W_h} - \lambda_i \Psi_{if}\right) \\
&+ \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i}\tilde{\lambda}_k^{W_h} - \lambda_i \Psi_{ik} - \sum_{f,h} \Lambda_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}\frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \\
&- \frac{\lambda_k}{\mathrm{d}\log \chi_h}\mathrm{d}\log \mu_i + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log \mu_k},
\end{aligned}
$$

and using $\mathrm{d}\log \lambda_i = \mathrm{d}\log p_i + \mathrm{d}\log y_i$,

$$
\begin{aligned}
&= \sum_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}\left(\sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i}\tilde{\Lambda}_f^{W_h} - \lambda_i \Psi_{if}\right) \\
&+ \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i}\tilde{\lambda}_k^{W_h} - \lambda_i \Psi_{ik} - \sum_{f,h} \Lambda_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}\frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \\
&- \frac{\lambda_k}{\mathrm{d}\log \chi_h}\mathrm{d}\log \mu_i + \lambda_i \frac{\mathrm{d}\log p_i}{\mathrm{d}\log \mu_k} + \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log \mu_k}, \\
&= \sum_{f,h} \chi_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i}\tilde{\Lambda}_f^{W_h}\frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} - \lambda_i \sum_f \Psi_{if}\frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} \\
&+ \sum_h \chi_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i}\tilde{\lambda}_k^{W_h} - \lambda_i \Psi_{ik} - \sum_{f,h} \Lambda_f \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i}\frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} \\
&- \lambda_k \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} + \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log \mu_k} \\
&+ \lambda_i \left(\sum_f \Psi_{if}\frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} + \Psi_{ik}\right), \\
&= \sum_{f,h} \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i}\frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}\left(\chi_h \tilde{\Lambda}_f^{W_h} - \Lambda_f\right) \\
&+ \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log \mu_k} + \sum_h \chi_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i}\tilde{\lambda}_k^{W_h} - \lambda_k \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i},
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log \mu_k} + \sum_h \chi_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i}\left(\tilde{\Lambda}_f^{W_h}\frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} + \tilde{\lambda}_k^{W_h}\right) \\
&- \sum_{f,h}\frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i}\frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}\Lambda_f - \lambda_k \sum_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i}, \\
&= \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log \mu_k} + \sum_h \chi_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i}\frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log \mu_k} \\
&- \left(\sum_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}\Lambda_f\right)\left(\sum_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i}\right) - \lambda_k \sum_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i}, \\
&= \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log \mu_k} + Cov_\chi\left(\frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i}, \frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log \mu_k}\right),
\end{aligned}
$$

since

$$
-\sum_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}\Lambda_f = -\sum_f \frac{\mathrm{d}\,\Lambda_f}{\mathrm{d}\log \mu_k} = \frac{\mathrm{d}\left(1 - \sum_j \lambda_j (1 - \frac{1}{\mu_j})\right)}{\mathrm{d}\log \mu_k} = -\lambda_k
$$

at the efficient point, and

$$
\sum_h \chi_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} = 0.
$$

■

Proof of Theorem 6. From Theorem 5, we have

$$
\mathcal{L} = -\frac{1}{2}\sum_l (d\log \mu_l)\lambda_l d\log y_l.
$$

With the maintained normalization $PY = 1$, we also have

$$
d\log y_l = d\log \lambda_l - d\log p_l,
$$

$$
d\log p_l = \sum_f \Psi_{lf} d\log \Lambda_f + \sum_k \Psi_{lk} d\log \mu_k,
$$

where, from Proposition 13,

$$
\begin{aligned}
d\log \lambda_l = &\sum_k (\delta_{lk} - \frac{\lambda_k}{\lambda_l}\Psi_{kl}) d\log \mu_k - \sum_j \frac{\lambda_j}{\lambda_l}(\theta_j - 1) Cov_{\Omega^{(j)}}(\sum_k \Psi_{(k)} d\log \mu_k - \sum_g \Psi_{(g)} d\log \Lambda_g, \Psi_{(l)}) \\
&+ \frac{1}{\lambda_l}\sum_{g\in F^*}\sum_c \left(\lambda_l^{W_c} - \lambda_l\right)\Phi_{cg}\Lambda_g \frac{d\log \Lambda_g}{d\log \mu_k},
\end{aligned}
$$

and

$$d\log\Lambda_f = -\sum_k \lambda_k \frac{\Psi_{kf}}{\Lambda_f} d\log\mu_k - \sum_j \lambda_j(\theta_j - 1)Cov_{\Omega^{(j)}}(\sum_k \Psi_{(k)} d\log\mu_k - \sum_g \Psi_{(g)} d\log\Lambda_g, \frac{\Psi_{(f)}}{\Lambda_f})$$
$$+ \frac{1}{\Lambda_f} \sum_{g\in F^*} \sum_c \left(\Lambda_i^{W_c} - \Lambda_f\right) \Phi_{cg}\Lambda_g \frac{d\log\Lambda_g}{d\log\mu_k}.$$

We will now use these expressions to replace in formula for the second-order loss function. We get

$$\mathcal{L} = -\frac{1}{2}\sum_l \sum_k (\frac{\delta_{lk}}{\lambda_k} - \frac{\Psi_{kl}}{\lambda_l} - \frac{\Psi_{lk}}{\lambda_k})\lambda_k\lambda_l d\log\mu_k d\log\mu_l + \frac{1}{2}\sum_l \lambda_l d\log\mu_l \sum_f \Psi_{lf} d\log\Lambda_f$$
$$+ \frac{1}{2}\sum_l \sum_j (d\log\mu_l)\lambda_j(\theta_j - 1)Cov_{\Omega^{(j)}}(\sum_k \Psi_{(k)} d\log\mu_k - \sum_g \Psi_{(g)} d\log\Lambda_g, \Psi_{(l)})$$
$$- \frac{1}{2}\sum_l d\log\mu_l \left(\sum_g \sum_c \left(\lambda_l^{W_c} - \lambda_l\right) \Phi_{cg}\Lambda_g d\log\Lambda_g\right)$$

$$\mathcal{L} = -\frac{1}{2}\sum_l \sum_k (\frac{\delta_{lk}}{\lambda_k} - \frac{\Psi_{kl}}{\lambda_l} - \frac{\Psi_{lk}}{\lambda_k})\lambda_k\lambda_l d\log\mu_k d\log\mu_l + \frac{1}{2}\sum_l \lambda_l d\log\mu_l \sum_f \Psi_{lf} d\log\Lambda_f$$
$$+ \frac{1}{2}\sum_l \sum_j (d\log\mu_l)\lambda_j(\theta_j - 1)Cov_{\Omega^{(j)}}(\sum_k \Psi_{(k)} d\log\mu_k - \sum_g \Psi_{(g)} d\log\Lambda_g, \Psi_{(l)})$$
$$- \frac{1}{2}\sum_l \left(\sum_c \left(\lambda_l^{W_c} - \lambda_l\right) \chi_c d\log\chi_c\right) d\log\mu_l$$

We can rewrite this expression as

$$\mathcal{L} = \mathcal{L}_I + \mathcal{L}_X + \mathcal{L}_H$$

where

$$\mathcal{L}_I = \frac{1}{2}\sum_k \sum_l [\frac{\Psi_{kl} - \delta_{kl}}{\lambda_l} + \frac{\Psi_{lk} - \delta_{lk}}{\lambda_k} + \frac{\delta_{kl}}{\lambda_l} - 1]\lambda_k\lambda_l d\log\mu_k d\log\mu_l$$
$$+ \frac{1}{2}\sum_k \sum_l \sum_j d\log\mu_k d\log\mu_l \lambda_j(\theta_j - 1)Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(l)}),$$

$$\mathcal{L}_X = \frac{1}{2}\sum_l \sum_f (\frac{\Psi_{lf}}{\Lambda_f} - 1)\lambda_l \Lambda_f d\log\mu_l d\log\Lambda_f$$
$$- \frac{1}{2}\sum_l \sum_g d\log\mu_l d\log\Lambda_g \sum_j \lambda_j(\theta_j - 1)Cov_{\Omega^{(j)}}(\Psi_{(g)}, \Psi_{(l)}),$$

$$\mathcal{L}_H = -\frac{1}{2}\sum_l \left(\sum_c \left(\lambda_l^{W_c} - \lambda_l \right) \chi_c d\log\chi_c \right) d\log\mu_l,$$

where $d\log\Lambda$ is given by the usual expression.[16] Finally, using Lemma 19, we can write

$$\mathcal{L}_{\mathcal{I}} = \frac{1}{2}\sum_l \sum_k (d\log\mu_l)(d\log\mu_k)\sum_j \lambda_j\theta_j Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(l)}).$$

and

$$\mathcal{L}_X = -\frac{1}{2}\sum_l \sum_g d\log\mu_l d\log\Lambda_g \sum_j \lambda_j\theta_j Cov_{\Omega^{(j)}}(\Psi_{(g)}, \Psi_{(l)}).$$

■

Lemma 18. *The following identity holds*

$$\sum_j \lambda_j \left(\tilde{\Psi}_{jk}\Psi_{jl} - \sum_m \Omega_{jm}\tilde{\Psi}_{mk}\Psi_{ml} \right) = \tilde{\lambda}_k\lambda_l.$$

Proof. Write Ω so that it contains all the producers, all the households, and all the factors *as well as* a new row (indexed by 0) where $\Omega_{0i} = \chi_i$ if $i \in C$ and 0 otherwise. then, letting e_0 be the standard basis vector corresponding to the 0th row, we can write

$$\lambda' = e_0' + \lambda'\Omega,$$

or equivalently

$$\lambda'(I - \Omega) = e_0'.$$

[16] We have used the intermediate step

$$\mathcal{L}_X = \frac{1}{2}\sum_l \sum_k \lambda_k\lambda_l d\log\mu_k d\log\mu_l + \frac{1}{2}\sum_l \sum_f d\log\mu_l d\log\Lambda_f \lambda_l \Psi_{lf}$$
$$- \frac{1}{2}\sum_l \sum_g d\log\mu_l d\log\Lambda_g \sum_j \lambda_j(\theta_j - 1)Cov_{\Omega^{(j)}}(\Psi_{(g)}, \Psi_{(l)}).$$

Let X^{kl} be the vector where $X^{kl}_m = \tilde{\Psi}_{mk}\Psi_{ml}$. Then

$$\sum_j \lambda_j \left(\tilde{\Psi}_{jk}\Psi_{jl} - \sum_m \Omega_{jm}\tilde{\Psi}_{mk}\Psi_{ml} \right) = \lambda'(I-\Omega)X^{kl},$$
$$= e_0'(I-\Omega)^{-1}(I-\Omega)X^{kl}, \quad = e_0'X^{kl} = \tilde{\Psi}_{0k}\Psi_{0l} = \tilde{\lambda}_k\lambda_l.$$

■

Lemma 19. *The following identity holds*

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) = \lambda_l\lambda_k[\frac{\tilde{\Psi}_{lk} - \delta_{lk}}{\lambda_k} + \frac{\Psi_{kl} - \delta_{kl}}{\lambda_l} + \frac{\delta_{lk}}{\lambda_k} - \frac{\tilde{\lambda}_k}{\lambda_k}].$$

Proof. We have

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) =$$
$$\sum_j \lambda_j \mu_j^{-1} \left[\sum_m \tilde{\Omega}_{jm}\tilde{\Psi}_{mk}\Psi_{ml} - \left(\sum_m \tilde{\Omega}_{jm}\tilde{\Psi}_{mk} \right) \left(\sum_m \tilde{\Omega}_{jm}\Psi_{ml} \right) \right],$$

or

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) =$$
$$\sum_j \lambda_j \sum_m \Omega_{jm}\tilde{\Psi}_{mk}\Psi_{ml} - \sum_j \lambda_j \mu_j^{-1} \left(\sum_m \tilde{\Omega}_{jm}\tilde{\Psi}_{mk} \right) \left(\sum_m \tilde{\Omega}_{jm}\Psi_{ml} \right),$$

or

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) =$$
$$\sum_j \lambda_j \sum_m \Omega_{jm}\tilde{\Psi}_{mk}\Psi_{ml} - \sum_j \lambda_j \tilde{\Psi}_{jk}\Psi_{jl}$$
$$+ \sum_j \lambda_j \tilde{\Psi}_{jk}\Psi_{jl} - \sum_j \lambda_j \mu_j^{-1} \left(\sum_m \tilde{\Omega}_{jm}\tilde{\Psi}_{mk} \right) \left(\sum_m \tilde{\Omega}_{jm}\Psi_{ml} \right),$$

or using, Lemma 18

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) = -\tilde{\lambda}_k\lambda_l + \sum_j \lambda_j \tilde{\Psi}_{jk}\Psi_{jl} - \sum_j \lambda_j \left(\tilde{\Psi}_{jk} - \delta_{jk} \right) \left(\Psi_{jl} - \delta_{jl} \right),$$

and finally

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) = \lambda_l \lambda_k [\frac{\tilde{\Psi}_{lk} - \delta_{lk}}{\lambda_k} + \frac{\Psi_{kl} - \delta_{kl}}{\lambda_l} + \frac{\delta_{lk}}{\lambda_k} - \frac{\tilde{\lambda}_k}{\lambda_k}].$$

■

Proof of Proposition 8. The proof closely follows that of Proposition 7. Notably, using the trick by Galle et al. (2017), which builds on Feenstra (1994), we note that

$$\mathrm{d}\log \chi_g = \mathrm{d}\log w_s + \frac{1}{\gamma_g}\,\mathrm{d}\log \Lambda_s^g$$

for any s, and hence,

$$\mathrm{d}\log \chi_g = \sum_f \check{\Lambda}_f^g \left(\mathrm{d}\log w_f + \frac{1}{\gamma_g}\,\mathrm{d}\log \Lambda_f^g \right),$$

where $\check{\Lambda}^g$ and $\check{\lambda}^g$ are the Domar weights under the closed-economy IO matrix. Then we can combine this with the fact that

$$\mathrm{d}\log P_g^c = \sum_f \check{\Lambda}_f^g\,\mathrm{d}\log w_f - \sum_i \check{\lambda}_i\,\mathrm{d}\log \check{A}_i$$

and choosing household g's nominal income as the numeraire to get

$$\mathrm{d}\log W_g = \sum_f \check{\Lambda}_f^g \frac{\mathrm{d}\log \Lambda_s^g}{\gamma_g} + \sum_i \check{\lambda}_i\,\mathrm{d}\log \check{A}_i.$$

■

Proof of Proposition 9. Loglinearizing this, we get

$$\Lambda_f\,\mathrm{d}\log \Lambda_f = \mathrm{d}\log w_f \sum_{h\in H} \Lambda_f^h \chi_h \gamma_h - \sum_{h\in H} (\gamma_h - 1)\chi_h \Lambda_f^h \sum_{l\in F} \delta_{hl}\,\mathrm{d}\log w_l + \sum_{h\in H} \chi_h \Lambda_f^h\,\mathrm{d}\log L^h.$$

We can put this back into familiar notation

$$\Lambda_f\,\mathrm{d}\log \Lambda_f = \mathrm{d}\log w_f \sum_{h\in H} \Phi_{hf}\Lambda_f \gamma_h - \sum_{h\in H} (\gamma_h - 1)\Phi_{hf}\Lambda_f \sum_{l\in F} \frac{\Phi_{hl}\Lambda_l}{\chi_h}\,\mathrm{d}\log w_l + \sum_{h\in H} \Phi_{hf}\Lambda_f\,\mathrm{d}\log L^h.$$

Simplify this to get

$$\mathrm{d}\log\Lambda_f = \mathrm{d}\log w_f \sum_{h\in H}\Phi_{hf}\gamma_h - \sum_{h\in H}(\gamma_h - 1)\Phi_{hf}\sum_{l\in F}\frac{\Phi_{hl}\Lambda_l}{\chi_h}\,\mathrm{d}\log w_l + \sum_{h\in H}\Phi_{hf}\,\mathrm{d}\log L^h.$$

We can beautify this a bit as

$$\mathrm{d}\log\Lambda_f = \sum_{h\in H}\gamma_h E_{\Phi^{(h)}}\left(E_{\Lambda^{(h)}}\left(\mathrm{d}\log w_f - \mathrm{d}\log w\right)\right) + \sum_{h\in H}E_{\Phi^{(h)}}\left(E_{\Lambda^{(h)}}(\mathrm{d}\log w)\right) + \sum_{h\in H}E_{\Phi^{(h)}}\left(\mathrm{d}\log L\right).$$

or

$$\mathrm{d}\log\Lambda_f = \sum_{h\in H}E_{\Phi^{(h)}}\left[\gamma_h\left(E_{\Lambda^{(h)}}\left(\mathrm{d}\log w_f - \mathrm{d}\log w\right)\right) + \left(E_{\Lambda^{(h)}}(\mathrm{d}\log w)\right) + \left(\mathrm{d}\log L\right)\right].$$

or

$$\mathrm{d}\log\Lambda_f = \sum_{h\in H}E_{\Phi^{(h)}}\left[E_{\Lambda^{(h)}}\left(\gamma_h\left(\mathrm{d}\log w_f - \mathrm{d}\log w\right) + (\mathrm{d}\log w)\right) + \left(\mathrm{d}\log L\right)\right].$$

The case with immobile labor is given by $\gamma_h = 1$ for every $h \in H$, in which case $\mathrm{d}\log w_f = \mathrm{d}\log\Lambda_f$. Combine this with demand for the factors to finish the characterization

$$\begin{aligned}\Lambda_l\frac{\mathrm{d}\log\Lambda_l}{\mathrm{d}\log A_k} &= \sum_{i\in\{H,N\}}\lambda_j(1-\theta_j)Cov_{\Omega^{(j)}}\left(\Psi_{(k)} + \sum_f\Psi_{(f)}\frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k}, \Psi_{(l)}\right)\\ &\quad + \sum_{h\in H}(\lambda_l^h - \lambda_l)\left(\sum_{f\in F_c}\Phi_{hf}\Lambda_f\frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k}\right).\end{aligned}$$

This means that we can also redo the welfare accounting and write

$$\mathrm{d}\log W_g = \mathrm{d}\log\chi_g - \mathrm{d}\log P_g^c,$$

where χ_g is the (nominal) income of household g. This can be written as

$$\frac{\mathrm{d}\log W_g}{\mathrm{d}\log A_k} = \sum_{s\in F}\left(\Lambda_s^g - \Lambda_s^{W_g}\right)\mathrm{d}\log w_s + \lambda_k^{W_g}\,\mathrm{d}\log A_k + \mathrm{d}\log L^g,$$

or

$$\frac{\mathrm{d}\log W_g}{\mathrm{d}\log A_k} = \sum_{s\in F}\left(\frac{\Phi_{gs}}{\chi_g}\Lambda_s - \Lambda_s^{W_g}\right)\mathrm{d}\log w_s + \lambda_k^{W_g} + \mathrm{d}\log L^g.$$

■

Proof of Proposition 11. To loglinearize real GDP of country c, let country c's nominal GDP be the numeraire. Then

$$\mathrm{d}\log Y_c = -\,\mathrm{d}\log P_{Y_c},$$

whence

$$\begin{aligned}
&= -\sum_{i\in N_c}\chi_i^{Y_c}\,\mathrm{d}\log p_i + \sum_{i\notin N_c}\lambda_{ci}\,\mathrm{d}\log p_i,\\
&= \sum_{i\in N_c}\chi_i^{Y_c}\sum_{j\in N_c}\tilde{\Psi}_{ij}^{dd}\left(\mathrm{d}\log A_j - \mathrm{d}\log\mu_j\right) - \sum_{f\in F_c}\sum_{i\in N_c} Fcci \sum_{j\in N_c}\tilde{\Psi}_{ij}^{dd}\tilde{\Omega}_{jf}\,\mathrm{d}\log w_f\\
&- \sum_{k\notin N_c}\sum_{i\in N_c}\chi_i^{Y_c}\sum_{j\in N_c}\tilde{\Psi}_{ij}^{dd}\tilde{\Omega}_{jk}\,\mathrm{d}\log p_k + \sum_{i\notin N_c}\Lambda_i^{M_c}\,\mathrm{d}\log p_i.
\end{aligned}$$

where $\tilde{\Omega}^{dd}$ is the domestic-domestic submatrix of the (cost-based) input-output matrix.

This can be further simplified to

$$\begin{aligned}
\mathrm{d}\log Y_c &= \sum_{i\in N_c}\tilde{\lambda}_j^{Y_c}\left(\mathrm{d}\log A_j - \mathrm{d}\log\mu_j\right) - \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c}\,\mathrm{d}\log w_f\\
&- \sum_{k\notin N_c}\tilde{\lambda}_k^{Y_c}\,\mathrm{d}\log p_k + \sum_{k\notin N_c}\lambda_k^{Y_c}\,\mathrm{d}\log p_k,\\
&= \sum_{i\in N_c}\tilde{\lambda}_j^{Y_c}\left(\mathrm{d}\log A_j - \mathrm{d}\log\mu_j\right) - \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c}\,\mathrm{d}\log w_f + \sum_{k\notin N_c}(\Lambda_k^{M_c} - \tilde{\lambda}_k^{Y_c})\,\mathrm{d}\log p_k,
\end{aligned}$$

with

$$\tilde{\lambda}_k^{Y_c} = \sum_{j\in N_c}\chi_j^{Y_c}\tilde{\Psi}_{jk}^{dd}, \quad (k\in N_c),$$

$$\tilde{\lambda}_i^{Y_c} = \frac{p_i c_{ci}}{P_{Y_c}Y_c} + \sum_{j\in N_c}\chi_j^{Y_c}\tilde{\Psi}_{jk}^{dd}\tilde{\Omega}_{ki}, \quad (i\notin N_c),$$

and

$$\tilde{\Lambda}_{cf}^{Y_c} = \sum_{j\in N_c}\chi_j^{Y_c}\tilde{\Psi}_{jk}^{dd}\tilde{\Omega}_{kf}, \quad (f\in F_c).$$

To finish, note that under our choice of numeraire

$$\mathrm{d}\log w_f + \mathrm{d}\log L_f = \mathrm{d}\log\Lambda_f^{Y_c}, \quad (f\in F_c)$$

$$\mathrm{d}\log p_i + \mathrm{d}\log y_i = \mathrm{d}\log\lambda_i^{Y_c}, \quad (i\in N_c),$$

and

$$\mathrm{d}\log p_i + \mathrm{d}\log q_i = \mathrm{d}\log \Lambda_i^{M_c}, \quad (i \notin N_c),$$

where q_i is total quantity of imports of i. Substitute this into the previous expressions to get desired result. ■

Proof of Proposition 13. For each good,

$$\lambda_i = \sum_c \chi_i^{W_c} \chi_c + \sum_i \Omega_{ji} \lambda_j,$$

where χ_c is the share of total income accruing to country c. This means

$$\lambda_i \,\mathrm{d}\log \lambda_i = \sum_c \chi_c \chi_i^{W_c} \,\mathrm{d}\log \chi_i^{W_c} + \sum_j \Omega_{ji}\lambda_j \,\mathrm{d}\log \Omega_{ji} + \sum_j \Omega_{ji} \,\mathrm{d}\,\lambda_j + \sum_c \chi_i^{W_c} \chi_c \,\mathrm{d}\log \chi_c.$$

Now, note that

$$\mathrm{d}\log \chi_i^{W_c} = (1-\theta_c)\left(\mathrm{d}\log p_i - \mathrm{d}\log P_{y_c}\right)$$

$$\mathrm{d}\log \Omega_{ji} = (1-\theta_j)\left(\mathrm{d}\log p_i - \mathrm{d}\log P_j + \mathrm{d}\log \mu_j\right) - \mathrm{d}\log \mu_j$$

$$\mathrm{d}\log \chi_c = \sum_{f \in F_c^*} \frac{\Lambda_f}{\chi_c} \,\mathrm{d}\log \Lambda_f + \sum_{i \in c} \frac{\lambda_i}{\mu_i} \,\mathrm{d}\log \mu_i.$$

$$\mathrm{d}\log p_i = \tilde{\Psi}\left(\mathrm{d}\log \mu - \mathrm{d}\log A\right) + \tilde{\Psi}\tilde{\alpha}\,\mathrm{d}\log \Lambda.$$

$$\mathrm{d}\log P_{y_c} = b'\tilde{\Psi}\left(\mathrm{d}\log \mu - \mathrm{d}\log A\right) + b'\tilde{\Psi}\tilde{\alpha}\,\mathrm{d}\log \Lambda.$$

For shock $\mathrm{d}\log \mu_k$, we have

$$\mathrm{d}\log \chi_i^{W_c} = (1-\theta_c)\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log \Lambda_f - \sum_j \chi_j^{W_c}\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log \Lambda_f\right)\right).$$

$$\mathrm{d}\log \Omega_{ji} = (1-\theta_j)\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log \Lambda_f - \tilde{\Psi}_{jk} - \sum_f \Psi_{jf}\,\mathrm{d}\log \Lambda_f\right) - \theta_j\,\mathrm{d}\log \mu_j.$$

Putting this altogether gives

$$\begin{aligned}\mathrm{d}\,\lambda_l = &\sum_i \sum_c (1-\theta_c)\chi_c \chi_i^{W_c}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log \Lambda_f - \sum_j \chi_j^{W_c}\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log \Lambda_f\right)\right)\Psi_{il} \\ &+ \sum_i \sum_j (1-\theta_j)\lambda_j \mu_j^{-1}\tilde{\Omega}_{ji}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log \Lambda_f - \tilde{\Psi}_{jk} - \sum_f \Psi_{jf}\,\mathrm{d}\log \Lambda_f\right)\Psi_{il}\end{aligned}$$

$$- \theta_k \lambda_k \sum_i \Omega_{ki} \Psi_{il} + \sum_c \chi_c \sum_i \chi_i^{W_c} \Psi_{il} \, \mathrm{d} \log \chi_c.$$

Simplify this to

$$\begin{aligned}
\mathrm{d}\,\lambda_l = & \sum_c (1-\theta_c)\chi_c \left[\sum_i \chi_i^{W_c} \left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if} \, \mathrm{d}\log \Lambda_f \right) \Psi_{il} \right. \\
& \left. - \left(\sum_i \chi_i^{W_c} \left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf} \, \mathrm{d}\log \Lambda_f \right) \right) \left(\sum_i \chi_i^{W_c} \Psi_{il} \right) \right] \\
& + \sum_j (1-\theta_j)\lambda_j \mu_j^{-1} \sum_i \tilde{\Omega}_{ji} \left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if} \, \mathrm{d}\log \Lambda_f \right) \Psi_{il} - \left(\sum_i \tilde{\Omega}_{ji} \Psi_{il} \right) \left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf} \, \mathrm{d}\log \Lambda_f \right) \\
& - \theta_k \lambda_k \left(\Psi_{kl} - \mathbf{1}(l=k) \right) + \sum_c \chi_c \sum_i \chi_i^{W_c} \Psi_{il} \, \mathrm{d}\log \chi_c.
\end{aligned}$$

Simplify this further to get

$$\begin{aligned}
\mathrm{d}\,\lambda_l = & \sum_c (1-\theta_c)\chi_c Cov_{b^{(c)}} \left(\tilde{\Psi}_{(k)} + \sum_f \tilde{\Psi}_{(f)} \, \mathrm{d}\log \Lambda_f, \Psi_{(l)} \right) \\
& + \sum_j (1-\theta_j)\lambda_j \mu_j^{-1} \sum_i \tilde{\Omega}_{ji} \left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if} \, \mathrm{d}\log \Lambda_f \right) \Psi_{il} \\
& - \left(\sum_i \tilde{\Omega}_{ji} \Psi_{il} \right) \left(\sum_i \tilde{\Omega}_{ji} \tilde{\Psi}_{ik} + \sum_i \tilde{\Omega}_{ji} \sum_f \Psi_{if} \, \mathrm{d}\log \Lambda_f \right) \\
& - \theta_k \lambda_k \left(\Psi_{kl} - \mathbf{1}(l=k) \right) + \sum_c \chi_c \sum_i \chi_i^{W_c} \Psi_{il} \, \mathrm{d}\log \chi_c,
\end{aligned}$$

Using the input-output covariance notation, write

$$\begin{aligned}
\mathrm{d}\,\lambda_l = & \sum_c (1-\theta_c)\chi_c Cov_{\chi^{W_c}} \left(\tilde{\Psi}_{(k)} + \sum_f \tilde{\Psi}_{(f)} \, \mathrm{d}\log \Lambda_f, \Psi_{(l)} \right) \\
& + \sum_j (1-\theta_j)\lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}} \left(\tilde{\Psi}_{(k)} + \sum_f \tilde{\Psi}_{(f)} \, \mathrm{d}\log \Lambda_f, \Psi_{(l)} \right) \\
& - (1-\theta_k)\lambda_k(\Psi_{kl} - \mathbf{1}(l=k)) - \theta_k \lambda_k \left(\Psi_{kl} - \mathbf{1}(l=k) \right) + \sum_c \chi_c \sum_i \chi_i^{W_c} \Psi_{il} \, \mathrm{d}\log \chi_c,
\end{aligned}$$

This then simplifies to give from the fact that $\sum_i \chi_i^{W_c} \Psi_{il} = \lambda_l^{W_c}$:

$$\lambda_l \, \mathrm{d}\log \lambda_l = \sum_{j \in N, C} (1-\theta_j) \lambda_j \mu_j^{-1} Cov(\tilde{\Psi}_{(k)} + \sum_f^F \mathrm{d}\log \Lambda_f, \Psi_{(l)}) \\ - \lambda_k \left(\Psi_{kl} - \mathbf{1}(k=l) \right) + \sum_c \chi_c \lambda_l^{W_c} \, \mathrm{d}\log \chi_c.$$

To complete the proof, note that

$$P_{y_c} Y_c = \sum_f w_f L_f + \sum_{i \in c} \left(1 - \frac{1}{\mu_i}\right) p_i y_i.$$

Hence,

$$\mathrm{d}(P_{y_c} Y_c) = \sum_{f \in c} w_f L_f \, \mathrm{d}\log w_f + \sum_{i \in c} \left(1 - \frac{1}{\mu_i}\right) p_i y_i \, \mathrm{d}\log(p_i y_i) + \sum_{i \in c} \frac{\mathrm{d}\left(1 - \frac{1}{\mu_i}\right)}{\mathrm{d}\log \mu_i} p_i y_i \, \mathrm{d}\log \mu_i.$$

In other words, since $P_y Y = 1$, we have

$$\mathrm{d}\,\chi_c = \sum_{f \in c} \Lambda_f \, \mathrm{d}\log w_f + \sum_{i \in c} \left(1 - \frac{1}{\mu_i}\right) \lambda_i \, \mathrm{d}\log \lambda_i + \sum_{i \in c} \frac{\mathrm{d}\left(1 - \frac{1}{\mu_i}\right)}{\mathrm{d}\log \mu_i} \lambda_i \, \mathrm{d}\log \mu_i.$$

Hence,

$$\mathrm{d}\log \chi_c = \sum_{f \in F_c^*} \frac{\Lambda_f}{\chi_c} \mathrm{d}\log \Lambda_f + \sum_{i \in c} \frac{\lambda_i}{\chi_c} \mathrm{d}\log \mu_i.$$

∎

Proposition 20 (Structural Output Loss). *Starting at an efficient equilibrium in response to the introduction of small tariffs or other distortions, changes in the real output of country c are, up to the second order, given by*

$$\Delta \log Y_c \approx -\frac{1}{2} \sum_{l \in N_c} \sum_{k \in N} \Delta \log \mu_k \Delta \log \mu_l \sum_{j \in N} \lambda_j^{Y_c} \theta_j Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(l)}) \\ - \frac{1}{2} \sum_{l \in N_c} \sum_{g \in F} \Delta \log \Lambda_g \Delta \log \mu_l \sum_{j \in N} \lambda_j^{Y_c} \theta_j Cov_{\Omega^{(j)}}(\Psi_{(g)}, \Psi_{(l)}) \\ + \frac{1}{2} \sum_{l \in N_c} \sum_{c \in C} \chi_c^W \Delta \log \chi_c^W \Delta \log \mu_l (\lambda_l^{W_c} - \lambda_l) / \chi_c^Y.$$

Proof. The proof follows along the same lines as Theorem 6. ■