Capital Taxation and Ownership When Markets Are Incomplete[*]

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Abstract

This paper is a normative investigation of the theoretical and quantitative properties of optimal capital taxation in the neoclassical growth model with aggregate shocks and incomplete markets. The model features a representative-agent economy with linear taxes on labor and capital. I first allow the government to trade only a real risk-free bond. Taxes on capital are set one period in advance, reflecting inertia in tax codes and preventing replication of the complete-markets allocation. Optimal policy has the following features: labor taxes fluctuate very little; capital taxes are volatile and feature a positive (negative) spike after a negative (positive) shock to the government budget; and capital taxes average to roughly zero across periods. I then consider the implications of allowing the government to trade capital. Optimality calls for a large short position.

1   Introduction

This paper is a normative investigation of optimal capital taxation in the representative-agent neoclassical growth model with uncertainty. The government finances its expenditures by levying linear taxes on labor and capital and issuing risk-free debt. In such environments it has been well known since the work of Judd (1992) and Chari, Christiano, and Kehoe (1994), that, even though the government cannot issue state-contingent debt, state-contingent capital taxes give the government enough instruments to perfectly insulate its budget from aggregate shocks and thereby implement the complete-markets Ramsey outcome. In quantitative versions of such economies, optimal policy has three salient features: (i) labor taxes fluctuate very little; (ii) within-period state-contingent

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capital taxes vary considerably across states; and (iii) within-period averages across states of state-contingent capital taxes are small—in other words, the intertemporal wedge is small.[1]

I explore theoretically and quantitatively how these results are altered when the government can avail itself of only a limited number of instruments to hedge its budget against aggregate shocks, so that markets are truly incomplete; in other words, the complete-markets Ramsey outcome is not attainable. This is important because the large variations of state-contingent capital taxes within a period that are necessary to replicate the complete-markets Ramsey outcome are unlikely to be available. Indeed, such flexible capital taxes run counter to the observed sluggishness of fiscal policy, which probably originates in the administrative and political process that governs adjustments in taxes. In order to capture inertia in fiscal policy, I therefore impose severe restrictions on capital taxes: they must be set one period in advance and for one period.[2]

For my quantitative investigation, I calibrate the model to the U.S. economy and use log balanced growth preferences. In simulations, the period length is more than a mere accounting convention; it controls two important parameters at the same time: the amount of time during which capital taxes are fixed and the maturity of debt. Numerical tractability requires varying both together with the period length, so I perform two sets of simulations: a one-year simulation, which combines a flexible capital tax with short debt maturity; and a five-year simulation, which combines sluggish capital taxes with longer debt maturity.

I find that claim (i) is robust to the form of market incompleteness that I impose: labor taxes are very smooth. In both the one-year and the five-year simulations, the standard deviation of labor taxes is below 2%. The government is successful at spreading the burden of labor taxation across states and dates in order to minimize the corresponding distortions. This is reflected in the relatively small welfare losses over the complete-markets Ramsey outcome, which amount to 0.09% of average consumption.

I show numerically that, *across* periods, capital taxes are very volatile, take both positive and negative values, and average out to a small number. Moreover, they display little persistence. The standard deviation of capital taxes decreases sharply with the period length: it is about 54% for the one-year simulation but only about 11% for the five-year simulation. Average capital taxes are below 5% in absolute value. There is an important difference between these results and claims (ii) and (iii). According to these properties, *within* a period, capital taxes should be very volatile across states and average out to a small number. As a result, the intertemporal wedge should be small in

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[1]The intertemporal wedge is the wedge between agents' intertemporal rate of substitution and the marginal rate of transformation. Chari, Christiano and Kehoe (1994) also use the term "ex ante capital tax rate". In their model, the intertemporal wedge is given by the within period average across states of state-contingent capital taxes, with weights given by the product of the marginal utility of consumption and the marginal product of capital in each state. The reason why the government can implement the complete-markets Ramsey outcome with state-contigent capital taxes is that the government can vary capital taxes across states of the world within a period while keeping the intertemporal wedge constant. This endows the government with enough degrees of freedom to perfectly shift the tax burden across states and to replicate the complete-markets outcome as long as long as it can also trade a risk-free bond.

[2]Without state-contingent capital taxes, the distinction between the intertemporal wedge, ex ante capital taxes, and capital taxes disappears. In my model these are equivalent concepts.

any given period.[3] In contrast, in my model, the intertemporal wedge is large in absolute value in some periods.

Some intuition for these results can be provided through an exact theoretical decomposition of capital taxes into two terms with distinct interpretations: a "hedging" term and an "intertemporal" term.[4] The hedging term, which would be zero under complete markets, balances two effects from increased capital taxes in anticipation of a shock. First is the direct effect in the form of increased revenues in proportion to the marginal product of capital. Second is an opposing indirect effect through the adjustment of capital: lower capital accumulation reduces the revenues from labor and capital taxes. The hedging benefits of capital taxation depend only on the covariance of these two effects with the government's need for funds across realizations of the shock. In the baseline case where the production function (gross of depreciation) is Cobb–Douglas and depreciation is deductible, I show theoretically that these two effects exactly cancel out so that the hedging term is equal to zero. This baseline case turns out to be a good benchmark: in my simulations, the hedging term is always smaller than 1%. The behavior of capital taxes is therefore dominated by the intertemporal term.

This intertemporal term arises from the possibility of manipulating interest rates. Capital taxes have a positive (negative) spike in the period following a bad (good) shock that negatively (positively) affects the government budget. This helps buffer the impact of the shock on the government budget by lowering (increasing) the interest rate on debt issued in that period and increasing (lowering) total tax revenues. It is important that these responses are anticipated. For example, a planned one-time increase in capital taxes in the period after a bad shock increases consumption and decreases marginal utility in the period of the shock. As a consequence, before the shock, interest rates increase and the government is forced to lower capital taxes. The result is a stabilization of the government budget: the debt burden is increased if the bad shock does not materialize and decreased if it does. However, this benefit must be weighed against the distortionary costs imposed on the economy. When the period length is increased, two effects combine to mitigate spikes in capital taxes. First, a smaller positive (negative) spike in capital taxes after a bad (good) shock is required to achieve a given reduction (increase) in the debt burden because this tax rate is imposed for a longer time. Second, the associated distortionary costs increase because consumption is distorted for a longer time. When preferences are quasi-linear, interest rates are pinned down by the discount factor and cannot be manipulated. In this case, the intertemporal term is zero.[5] More

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[3]With log balanced growth preferences, the intertemporal wedge is actually exactly zero at the complete markets Ramsey outcome. This is true more generally when preferences feature constant relative risk aversion and are separable between consumption and leisure.

[4]This decomposition applies only when debt is away from its lower and upper limits. If these debt limits are binding then a third, "debt limits", term arises that imparts a role to capital taxes in relaxing the debt limits.

[5]One might think that even though the interest rate is fixed, the increase (decrease) in total tax revenues resulting from a positive (negative) spike on capital following a bad (good) shock still helps buffer the impact of shocks on the budget of the government. To understand why this does not occur, note that the increase in tax revenues incorporates the offsetting effects resulting from the negative impact on capital accumulation: a lower capital tax base and lower revenues from labor taxation. With an infinite intertemporal elasticity of substitution, these offsetting effects are strong enough to neutralize the direct effect of increased capital taxes on tax revenues.

generally, I am able to extend a theoretical result proved by Zhu (1992) in the context of complete markets; I show that in a stationary equilibrium, the intertemporal term is either zero or takes both signs with positive probability.

I then explore how optimal policy is affected when the government is allowed to trade capital. Preventing the government from trading capital is without loss of generality under complete markets. But in environments with incomplete markets, this arbitrary restriction regains bite. Trading capital provides the government with a powerful instrument for hedging aggregate shocks. Indeed, when preferences are quasi-linear and government expenditure shocks are the only disturbance in the economy, I show theoretically that except in the initial period, the government can perfectly approximate the complete-markets Ramsey outcome by taking a large position in capital, counterbalanced by an equally large opposite position in the risk-free bond. Outside of this benchmark case, government expenditure shocks tend to call for a long position, whereas productivity shocks typically require a short position. The latter follows because low productivity shocks result in low returns to capital and high government need for funds. Hedging the government budget therefore requires a short position. In my calibration, productivity shocks dwarf government expenditure shocks and so the optimal position is short. The magnitude of the short position is very large but decreases with the period length: 400% of the capital stock in the one-year simulation and 150% of the capital stock in the five-year simulation.[6]

I also characterize the optimal holdings of capital by the government in a more general portfolio problem with additional assets. I derive the government's optimal liability structure in a unified framework that resembles the Consumption Capital Asset Pricing Model (CCAPM).

Related literature. An extensive literature on capital taxation with complete markets has emerged from the celebrated zero–capital tax result established by Chamley (1986) and Judd (1985) who showed that, in all steady states of the deterministic economy, taxes on capital are optimally set to zero. This paper adds to this literature by studying the case of incomplete markets.

This paper contributes to the literature pioneered by Barro (1979) on fiscal policy under incomplete markets. Barro considers a deterministic, partial equilibrium environment and associates an exogenous convex deadweight cost to taxation. Variations in the deadweight cost are detrimental; taxes should be smoothed across time. Tax smoothing by the government leads both debt and taxes to move permanently after shocks to present and future government expenditures. Generalizations of this insight to explicitly stochastic models yield a random walk component for taxes and public debt. Most closely related to this paper is the one by Aiyagari et al. (2002), henceforth AMSS, who study fiscal policy in general equilibrium under incomplete markets in a version of the no-capital economy of Lucas and Stokey (1983) with only risk-free debt. AMSS demonstrate that debt and labor taxes inherit a unit root component. I show that this result is robust to the introduction of capital, capital taxation, and a more general asset structure.

This paper is also related to literature studying the optimal liability structure of the government

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[6]Section 6 contains a thorough discussion of this normative result. There, I explain that the prescription of large asset positions is a feature shared by many existing models of government portfolio choice. I give intuition for the role of the period length. I also discuss natural additions to the model that would mitigate these asset positions.

under incomplete markets. The foundational paper is Bohn (1990), who considers a stochastic version of Barro's model with risk-neutral consumers. The literature on the optimal portfolio of the government under incomplete markets has focused entirely on Bohn's model, maintaining the assumption of risk neutrality and adopting an ad hoc deadweight cost for taxes. The model presented here provides microfoundations for Bohn's findings, and I also analyze explicitly the situation where consumers are risk averse.

Angeletos (2002) and Buera and Nicolini (2004) consider economies without capital and where the government can trade only risk-free debt with multiple maturities. They show that the government can generically replicate the complete-markets Ramsey outcome if the number of traded maturities is greater than the cardinality of the state space of the stochastic disturbance in the economy. Shin (2007) points out that fewer maturities might suffice to implement the complete-markets Ramsey outcome if the government actively managed its portfolio: dynamic completeness is enough.[7] These papers characterize the optimal maturity structure of government debt when the complete-markets Ramsey outcome is achievable, and they find that the optimal portfolio typically involves very large positions of opposite signs for different maturities.[8] The large capital positions called for by my model are reminiscent of their finding. However, an important difference is that I explore the optimal liability structure of the government in a setting where the complete-markets Ramsey outcome cannot be implemented.

From a methodological perspective, this paper builds on Kydland and Prescott (1980) and Werning (2005). Previous approaches of optimal policy in incomplete-markets model either adopted a Lagrangian approach (as in AMSS) or developed a recursive representation by incorporating some multipliers in the state space (a method developed by Marcet and Marimon, 1998). My recursive representation of the Ramsey problem has four state variables, which are directly related to the allocation: the present value of government liabilities, capital, past marginal utility, and the state of the Markov shock process.

The rest of the paper is organized as follows. Section 2 introduces the economic environment and sets up the Ramsey problem. Section 3 develops a recursive representation. Section 4 presents the properties of debt and taxes in the case of quasi-linear preferences; I analyze the general case in Section 5. In Section 6 examines capital ownership by the government and characterizes the optimal liability structure of the government. Section 7 contains the numerical analysis.

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[7]A branch of this literature studies imperfect competition and price stickiness. Schmitt-Grohé and Uribe (2004) and Siu (2004) analyze optimal monetary and fiscal policy in quantitative models with nominal bonds and sticky prices but not capital. In this context they find that, for reasonable degrees of price stickiness, optimal inflation displays little volatility. Correia, Nicolini, and Teles (2008) point out that, in such environments, if state-contingent consumption taxes are allowed then the complete-markets Ramsey outcome with flexible prices is always attainable.

[8]Buera and Nicolini (2004) find that in a calibrated version of the US economy with a simple four state process, the government is required to swap bonds of different maturities on the order of a few hundred times total GDP in each period. In Shin (2007), the Markov structure of shocks is such that dynamic trading of short-term debt and a perpetuity are enough to replicate the complete-markets outcome; the optimal positions are still very large (four times GDP) but smaller that in Buera and Nicolini (2004).

2 The Economy

The model is a neoclassical, stochastic production economy. The economy is populated by a continuum of identical, infinite-lived individuals and a government. Time is discrete, indexed by $t \in \{0, 1, ...\}$. The exogenous stochastic disturbances in period t are summarized by a discrete random variable $s_t \in \mathbb{S} \equiv \{1, 2, ..., S\}$: the state at date t. I let $s^t \equiv \{s_0, s_1, ..., s_t\} \in \mathbb{S}^t$ denote the history of events at date t. I assume that s_t follows a Markov process with transition density $P(s'|s)$ and initial distribution $\pi_0 = P(\cdot|s_{-1})$.

In each period t, the economy has two goods: a consumption capital good and labor. Households have access to an identical CRS (for constant returns to scale) technology for transforming capital k_{t-1} and labor l_t into output via the production function $k_{t-1} + F(k_{t-1}, l_t, s_t)$. This formulation allows for capital depreciation, which is subsumed by the production function $F(k_{t-1}, l_t, s_t)$. The production function is smooth in (k_{t-1}, l_t) and satisfies the standard Inada conditions. Notice that this formulation incorporates a stochastic *productivity* shock. Output can be used for private consumption c_t, government consumption g_t, and new capital k_t. Throughout, I will take government consumption $g_t = g(s_t)$ to be an exogenously specified *government expenditure* shock. Hence the resource constraints in the economy are

$$c_t + g_t + k_t \leq F(k_{t-1}, l_t, s_t) + k_{t-1} \qquad \forall t \geq 0 \text{ and } \forall s^t \in \mathbb{S}^t. \tag{1}$$

Households rank consumption and labor streams according to

$$\mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t, s_t), \tag{2}$$

where $\beta \in (0, 1)$ and where u is smooth and concave in (c_t, l_t), increasing in consumption, decreasing in labor, and satisfies the standard Inada conditions. Note that this formulation incorporates a stochastic *preference* shock.

The government raises all revenues through a tax on labor income τ_t^l and a tax on capital income τ_t^k. Except for taxes on capital τ_t^k, the time-t component of the decisions of the households and the government are functions of the history of shocks s^t until t. In contrast, I assume that taxes on capital are predetermined: the government makes decisions on τ_t^k one period in advance. Hence τ_t^k is a function of the history of shocks s^{t-1} up to $t-1$ (see the end of this section for a thorough discussion of the assumption that taxes on capital are set one period in advance). The capital stock k_0 is inelastic and so provides a nondistortionary source of revenue to the government. In order to limit the amount of revenues the government can extract at no cost, I assume that the date-0 tax rate on capital τ_0^k is exogenously fixed.

Incomplete markets and debt limits. Households and the government borrow and lend only in the form of one-period risk-free bonds paying interest r_t. The government budget and debt limit constraints are as follows:

$$(1 + r_t)b_{t-1} \le \tau_t^l l_t w_t + \tau_t^k (F(k_{t-1}, l_t, s_t) - w_t l_t) + b_t \qquad \forall t \ge 0 \text{ and } \forall s^t \in \mathbb{S}^t; \qquad (3)$$

$$\underline{M}(k_t, u_c(c_t, l_t, s_t), s_t) \le u_c(c_t, l_t, s_t)b_t \le \overline{M}(k_t, u_c(c_t, l_t, s_t), s_t), \qquad \forall t \ge 0 \text{ and } \forall s^t \in \mathbb{S}^t. \qquad (4)$$

Here b_t is the amount of government debt outstanding at date t and w_t is the wage rate.

Remark 1 *With this formulation, the base for the tax on capital is $F(k_{t-1}, l_t, s_t) - w_t l_t$. Since depreciation is subsumed by $F(k_{t-1}, l_t, s_t)$, capital depreciation is assumed to be deductible. This is meant to capture the fact that, in practice, most tax codes make for amortization allowances.*

When (3) holds with strict inequality, I let the difference between the right-hand side and the left-hand side be a nonnegative level of lump-sum transfers T_t to the households. The lower debt limit $\underline{M}(k_t, u_c(c_t, l_t, s_t), s_t)$ and the upper debt limit $\overline{M}(k_t, u_c(c_t, l_t, s_t), s_t)$ in (4) influence the optimal government plan. In full generality, I allow the debt limits to depend on the capital stock of the economy and the current marginal utility of consumption. Alternative possible settings for $\underline{M}(k_t, u_c(c_t, l_t, s_t), s_t)$ and $\overline{M}(k_t, u_c(c_t, l_t, s_t), s_t)$ are discussed later. Observe that I define debt and asset limits on $u_c(c_t, l_t, s_t)b_t$ instead of on b_t. This is natural given my definition of debt: b_t is the amount of debt issued at the end of period t. The quantity $u_c(c_t, l_t, s_t)b_t$ is therefore just debt weighted by the state price density.

The representative household operates a firm and supplies and hires labor at wage w_t on a competitive market.[9] The household's problem is to choose stochastic processes $\{c_t, l_t, l_t^d, k_{t-1}, b_{t-1}\}_{t \ge 0}$ that are measurable with respect to s^t and maximize (2) subject to the sequence of budget constraints

$$c_t + b_t \le (1 - \tau_t^l)w_t l_t + (1 - \tau_t^k)(F(k_{t-1}, l_t^d, s_t) - w_t l_t^d) + k_{t-1} - k_t + (1 + r_t)b_{t-1} + T_t, \qquad (5)$$

taking initial debt b_{-1}, initial capital k_{-1}, wages, interest rates, and taxes $\{w_t, r_t, \tau_t^l, \tau_t^k\}_{t \ge 0}$ as given. The labor market clears if $\{l_t\}_{t \ge 0} = \{l_t^d\}_{t \ge 0}$. The household also faces debt limits analogous to (4), which I assume are less stringent than those faced by the government. Therefore, in equilibrium, the household's problem always has an interior solution. The household's first-order conditions require that two Euler equations hold (one for the risk-free rate and the other for the net return on capital) as well as a labor–leisure arbitrage condition and the condition that labor be paid its marginal product:

$$1 = (1 + r_t)\mathbb{E}_t \left\{ \beta \frac{u_{c,t+1}}{u_{c,t}} \right\} \qquad \forall t \ge 0 \text{ and } \forall s^t \in \mathbb{S}^t; \qquad (6)$$

$$1 = \mathbb{E}_t \left\{ \beta \frac{u_{c,t+1}}{u_{c,t}} [1 + (1 - \tau_t^k)F_{k,t+1}] \right\} \qquad \forall t \ge 0 \text{ and } \forall s^t \in \mathbb{S}^t; \qquad (7)$$

$$\tau_t^l = 1 + \frac{u_{l,t}}{w_t u_{c,t}} \qquad \forall t \ge 0 \text{ and } \forall s^t \in \mathbb{S}^t; \qquad (8)$$

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[9]Giving the firms to consumers is just one of the many equivalent ways of resolving the indeterminacy of the objective of competitive firms under incomplete markets.

$$w_t = F_l(k_{t-1}, l_t^d, s_t) \qquad \forall t \geq 0 \text{ and } \forall s^t \in \mathbb{S}^t. \tag{9}$$

Remark 2 *Note for future reference that under CRS, the first-order condition $w_t = F_l(k_{t-1}, l_t^d, s_t)$ implies that $F(k_{t-1}, l_t^d, s_t) - w_t l_t^d = k_{t-1} F_k(k_{t-1}, l_t^d, s_t)$. Revenues from capital taxation can be written as $\tau_t^k k_{t-1} F_k(k_{t-1}, l_t^d, s_t)$.*

Definition 1 *Given b_{-1}, r_0, k_{-1}, τ_0^k, and a stochastic process $\{s_t\}_{t\geq 0}$, a **feasible allocation** is a stochastic process $\{c_t, l_t, k_{t-1}\}_{t\geq 0}$ satisfying the resource constraints (1) such that its time-t elements are measurable with respect to s^t. A **risk-free rate process** $\{r_t\}_{t\geq 0}$, a **wage process** $\{w_t\}_{t\geq 0}$, and a **government policy** $\{\tau_t^l, \tau_t^k, b_t\}_{t\geq 0}$ are sets of stochastic processes such that w_t, τ_t^l, and b_t are measurable with respect to s^t and both τ_t^k and r_t are measurable with respect to s^{t-1}.*

Definition 2 *Given b_{-1}, r_0, k_{-1}, τ_0^k, and a stochastic process $\{s_t\}$, a **competitive equilibrium** is a feasible allocation, a risk-free rate process, a wage process, and a government policy such that (a) $\{c_t, l_t, l_t, k_{t-1}, b_{t-1}\}_{t\geq 0}$ solves the household's optimization problem and (b) the government budget constraints (3) and (4) are satisfied.[10] An allocation $\{c_t, l_t, k_{t-1}\}_{t\geq 0}$ that is part of a competitive equilibrium is a **competitive equilibrium allocation**.*

Definition 3 *The **Ramsey problem** is to maximize consumer welfare (2) over the set of competitive equilibria. A **Ramsey outcome** is a competitive equilibrium that attains the maximum.*

Discussion: Debt limits. By analogy with AMSS, I shall study two kinds of debt limits: *natural* and *ad hoc*. Natural debt limits amount to requiring that debt be less than the maximum debt that could be repaid almost surely. Following AMSS, a debt or asset limit is ad hoc if it is more stringent than the natural one. In this model, natural debt limits—which depend on the capital stock k_t in the economy—are in general difficult to compute. But as mentioned previously, it is easy to see that they are of the form $\underline{M}^n(k_t, u_c(c_t, l_t, s_t), s_t) \leq u_c(c_t, l_t, s_t)b_t \leq \overline{M}^n(k_t, u_c(c_t, l_t, s_t), s_t)$. Requiring debt limits to be weakly tighter than the natural ones rules out Ponzi schemes.

Discussion: Measurability assumption for τ_t^k. It has been known since the work of Judd (1992) and Chari, Christiano and Kehoe (1994) that, with state-contingent taxes on capital, the complete-markets Ramsey outcome can be implemented even when the government can only trade a one-period risk-free bond.[11] The reason is that investment depends only on a within-period average across states of state-contingent capital taxes and not on how the tax is spread across states within a period. The government can use this to its advantage by adjusting taxes on capital to hedge its burden across states without distorting capital, thereby replicating the complete-markets Ramsey outcome.

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[10] This definition imposes labor market clearing because the second and third arguments in $\{c_t, l_t, l_t, k_{t-1}, b_{t-1}\}_{t\geq 0}$ are equal. Similarly, bond market clearing is automatically imposed.

[11] With complete markets, taxes on capital are indeterminate. In fact, the complete-markets Ramsey outcome can be implemented either with state-contingent debt and taxes on capital set one period in advance or with only a risk-free bond but flexible taxes on capital.

Inertia in fiscal policy—captured here by the assumption that taxes must be set one period in advance and for one period—restricts the state-contingency of capital taxes, prevents replication of the complete-markets Ramsey outcome, and requires analyzing optimal taxes on capital in a truly incomplete-markets environment.

In contrast, no such restriction is placed on labor taxes, because they are not the focus of this paper. One may wonder whether this asymmetric treatment of labor and capital taxes doesn't bias the results in favor of labor taxation. In fact, most of the insights that I derive would still be valid if additional restrictions were put on labor taxes. The exact results depend, of course, on the particular form of these restrictions. If, for example, the production function is Cobb–Douglas (with or without depreciation), then the formulas (23) and (24) for taxes on capital are still valid when taxes on labor are also restricted to be set one period in advance. In this case, capital can be factored out from the additional restrictions imposed on the planing problem.

Discussion: Specific form of market incompleteness. The assumptions that I have made require studying an incomplete-markets economy. But there are many different ways to deviate from complete markets, by imposing enough joint restrictions on the set of assets that can be traded by the government as well as on the state contingency of capital taxes. This raises two related questions. First, are the restrictions that I have chosen relevant? Second, are the insights that I derive likely to generalize to other incomplete-markets settings?

I believe that the restrictions chosen for this paper are both natural and realistic. Indeed, inertia in fiscal policy is often mentioned in policy debates and contrasted with the flexibility of monetary policy. The assumption that only one maturity of debt is traded by the government is somewhat less realistic (as governments typically trade bonds corresponding to a set of different maturities) but is often made for analytical convenience and numerical tractability in studies (e.g., AMSS) of incomplete markets. This assumption is relaxed in Section 6, where I allow the government to trade an arbitrarily richer set of state-contingent assets.[12]

In addition, it should be noted that many of the insights derived here would be valid in many different incomplete-markets environments. For example, Propositions 1 and 2 can be shown to apply under many different asset structures. This includes more extreme forms of market incompleteness, when the government is not allowed to issue debt and must balance its budget in every period, and also milder forms of market incompleteness, when the government can trade a large but incomplete set of state-contingent assets including capital.

3   A Recursive Representation for the Ramsey Problem

The following lemma gives necessary and sufficient conditions for competitive equilibria.

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[12]The setup developed here doesn't actually allow for the government to trade debt of different maturities, since the assets that the government can trade have exogenously specified payoffs. However, it can be shown (at the cost of additional notation and analysis that is beyond the scope of this paper) that the results of Propositions 1, 2 and 3—as well as the results of Section 6—still apply if the government is allowed to trade debt of different maturities.

Lemma 1 *A feasible allocation $\{c_t, l_t, k_{t-1}\}_{t\geq 0}$ together with a risk-free rate process $\{r_t\}_{t\geq 0}$, a wage process $\{w_t\}_{t\geq 0}$, and a government policy $\{\tau_t^l, \tau_t^k, b_t\}_{t\geq 0}$ constitute a competitive equilibrium if and only if (1) and (3) hold with equality and (4), (6), (7), (8), and (9) hold with $l_t = l_t^d$.*

Notation. I use prime variable to denote next-period variables and variables with a minus subscript to denote last-period variables. I denote the possible states of the random shock in the current period by s and write X_s for any function $X(k, c_s, l_s, s)$ in state s. For example, I use $u_{c,s}$ to denote the marginal utility $u_c(c_s, l_s, s)$ in state s and use F_s to denote output $F(k, l_s, s)$ in state s. I use $\mathbb{E}\{X_s|s_-\}$ to denote the expectation $\sum_{s\in\mathbb{S}} X_s P(s|s_-)$ of a function X_s of the current state s of the Markov process conditional on the state of the Markov process in the previous period being s_-. Hence, I make a slight abuse of notation in that s denotes both a particular state of the Markov process and a random variable with probability distribution $P(s|s_-)$ conditional on the state of the Markov process in the previous period being s_-.

State variables. With this notation in hand, I can describe the recursive representation that I develop for the Ramsey problem (from $t = 1$ onward). It uses four state variables: the value of the capital stock k inherited from the previous period; the value of government debt from the previous period b; the marginal utility of consumption in the previous period $\theta \equiv u_c(c_-, l_-, s_-)$; and the state of the Markov process in the previous period s_-. This recursive approach is useful for developing intuition, simplifies calculations, and facilitates numerical simulations. The Bellman equation satisfied by the value function of the Ramsey problem can be written as follows.

Bellman equation 1.

$$V(k, b, \theta, s_-) = \max_{\{c_s, l_s, k_s', b_s', \tau^k, \tau_s^l\}} \mathbb{E}\{u_s + \beta V(k_s', b_s', u_{c,s}, s)|s_-\} \tag{10}$$

subject to

$$(1+r)\mathbb{E}\{\beta u_{c,s}|s_-\} = \theta,$$

$$\mathbb{E}\{\beta u_{c,s}[1 + (1-\tau^k)F_{k,s}]|s_-\} = \theta,$$

$$\tau_s^l = 1 + \frac{u_{l,s}}{u_{c,s}F_{l,s}} \qquad \forall s \in \mathbb{S},$$

$$(1+r)b + g_s \leq \tau_s^l l_s F_{l,s} + \tau^k k F_{k,s} + b_s' \qquad \forall s \in \mathbb{S},$$

$$c_s + g_s + k_s' \leq F_s + k \qquad \forall s \in \mathbb{S},$$

$$\underline{M}(k_s', u_{c,s}, s) \leq u_{c,s} b_s' \leq \overline{M}(k_s', u_{c,s}, s) \qquad \forall s \in \mathbb{S}.$$

Six constraints are imposed on the problem and in the following order: (i) the risk-free rate satisfies the usual Euler equation; (ii) the net return on capital satisfies the usual Euler equation; (iii) agents equalize their marginal rates of substitution between leisure and consumption to the net real wage; (iv) the budget constraint of the government is satisfied in each state $s \in \mathbb{S}$; (v) the resource constraint is satisfied in each state $s \in \mathbb{S}$; and (vi) the amount of government debt issued in each state $s \in \mathbb{S}$ satisfies the debt and asset limits.

The initial period must be treated in isolation. There, marginal utility of consumption in the previous period is not defined. One can think of the problem at date $t = 0$, given $(k, b, s_-) = (k_{-1}, b_{-1}, s_{-1})$, as maximizing the right hand side of (10) subject to all six constraints listed above except the first two constraints, which are replaced by $\tau^k = \tau_0^k$ and $r = r_0$.

It will prove convenient to replace b by a new state variable $\tilde{b} = b\theta$ representing debt weighted by the state price density. The corresponding value function is $\tilde{V}(k, \tilde{b}, \theta, s_-) = V(k, b\theta, \theta, s_-)$. In order to write the Bellman equation satisfied by $\tilde{V}$, I first rearrange the constraints. The first constraint is used to substitute r and the third to replace τ_s^l; the fourth constraint is multiplied by $u_c(c_s, l_s, s)$.

Bellman equation 2.

$$\tilde{V}(k, \tilde{b}, \theta, s_-) = \max_{\left\{c_s, l_s, k_s', \tilde{b}_s', \tau^k\right\}} \mathbb{E}\left\{u_s + \beta \tilde{V}(k_s', \tilde{b}_s', u_{c,s}, s) | s_-\right\} \tag{11}$$

subject to

$$\mathbb{E}\left\{\beta u_{c,s}[1 + (1 - \tau^k)F_{k,s}] | s_-\right\} = \theta, \tag{12}$$

$$\tilde{b}\frac{u_{c,s}}{\beta \mathbb{E}\left\{u_{c,s} | s_-\right\}} + g_s u_{c,s} \leq l_s F_{l,s} u_{c,s} + l_s u_{l,s} + \tau^k k F_{k,s} u_{c,s} + \tilde{b}_s' \qquad \forall s \in \mathbb{S}, \tag{13}$$

$$c_s + g_s + k_s' \leq F_s + k \qquad \forall s \in \mathbb{S}, \tag{14}$$

$$\underline{M}(k_s', u_{c,s}, s) \leq \tilde{b}_s' \leq \overline{M}(k_s', u_{c,s}, s) \qquad \forall s \in \mathbb{S}. \tag{15}$$

The constraint set in (11) is not convex. As a result, first-order conditions are necessary but not sufficient for characterizing the solution. The lack of convexity also considerably complicates the task of establishing differentiability of the value function $\tilde{V}$, which is required to characterize the solution by a set of necessary first-order conditions.

All the properties of Ramsey outcomes that I derive can be established either by using a Lagrangian approach or by expanding the Bellman equation (11) over two periods bypassing the latter technical difficulty but at the cost of heavier notation (and thus less intuition). I therefore proceed with the assumption that the value function $\tilde{V}$ is differentiable in $(k, \tilde{b}, \theta)$, which is the case in all my simulations.

4   The Quasi-Linear Case

The problem simplifies drastically when preferences are quasi-linear. In this section I assume that $u(c, l, s) = c + H(l, s)$, where H is a smooth, decreasing, and concave function. I also assume that $H'(0) = \infty$ in order for labor supply to be interior, and I allow for negative consumption.

Two state variables, $\tilde{b}$ (which is equal to b in this case), and s_-, are now sufficient to describe the state of the economy. Intuitively, the reasons for this simplification are twofold. The first reason is that θ is now fixed and equal to 1; it can therefore be dropped as a state variable. The second reason is that, since intertemporal prices are fixed, I can perform a change of timing in the recursive approach: the optimal investment in capital k can now be viewed as being chosen simultaneously

with the tax rate on capital τ^k. As a consequence, the state variable k can be treated as a control variable. To see this formally, note that the objective function (2) of the Ramsey problem can be written as follows, using the resource constraint to substitute $F(k_{t-1}, l_t, s_t) + k_t - k_{t+1} - g_t$ for c_t:

$$\mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t [F_t + k_{t-1} - k_t - g_t + H_t].$$

This expression can be rearranged to yield

$$\frac{1}{\beta} k_0 + \mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t \left[F_t + k_{t-1} \left(1 - \frac{1}{\beta}\right) - g_t + H_t \right].$$

The idea is to write down a new value function $\tilde{V}(\tilde{b}, s_-)$ for the per-period objective function given by $F_s + k(1 - 1/\beta) - g_s + H_s$. The relationship between this new value function and the one defined in (11) is $\tilde{V}(\tilde{b}, s_-) = \max_k \tilde{V}(k, \tilde{b}, 1, s_-) - k/\beta$. The value function $\tilde{V}(\tilde{b}, s_-)$ satisfies the following Bellman equation.

Bellman equation 3.

$$\tilde{V}(\tilde{b}, s_-) = \max_{\{l_s, k, \tilde{b}'_s, \tau^k\}} \mathbb{E} \left\{ F_s + k \left(1 - \frac{1}{\beta}\right) - g_s + H_s + \beta \tilde{V}(\tilde{b}'_s, s) | s_- \right\} \tag{16}$$

subject to

$$\mathbb{E} \left\{ \beta[1 + (1 - \tau^k) F_{k,s}] | s_- \right\} = 1, \tag{17}$$

$$\tilde{b} \frac{1}{\beta} + g_s \le l_s F_{l,s} + l_s H_{l,s} + \tau^k k F_{k,s} + \tilde{b}'_s \qquad \forall s \in \mathbb{S}, \tag{18}$$

$$\underline{M}_s \le \tilde{b}'_s \le \overline{M}_s \qquad \forall s \in \mathbb{S}. \tag{19}$$

Observe that, with quasi-linear preferences, natural debt limits are independent of the capital stock and marginal utility is constant: $\underline{M}^n(k, u_c, s) = \underline{M}^n_s$ and $\overline{M}^n(k, u_c, s) = \overline{M}^n_s$. Consistent with this property, fixed debt debt limits $\underline{M}_s$ and $\overline{M}_s$ are imposed in (16).[13]

4.1 Stochastic Properties of Ramsey Outcomes

In order to derive the first-order conditions necessary for optimality in (16), I attach the multipliers μ to (17), ν_s to (18), and $\nu_{2,s}$ and $\nu_{1,s}$ to the two constraints in (19). The multiplier μ is a function of $\tilde{b}$ and s_-, whereas ν_s, $\nu_{1,s}$, and $\nu_{2,s}$ are functions of $\tilde{b}$, s_-, and s.

The envelope condition is

$$\beta \tilde{V}_b(\tilde{b}, s_-) = -\mathbb{E}\{\nu_s | s_-\}, \tag{20}$$

and the first-order condition for $\tilde{b}'_s$ is

$$\beta \tilde{V}_b(\tilde{b}'_s, s) = -\nu_s + \nu_{1,s} - \nu_{2,s}. \tag{21}$$

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[13]These debt limits can be tighter than the natural ones, but are restricted to be independent of k.

Equation (21) shows that, away from the debt limits, the multiplier ν_s can be interpreted as the marginal value of a unit reduction in debt—in other words, the government's marginal need for funds. At the optimum, this coincides with the marginal cost of increasing taxes in order to generate one unit of government revenues.

Combining equations (20) and (21), and using ν_{s-}, $\nu_{1,s-}$, and $\nu_{2,s-}$ to denote the corresponding multipliers in the previous period, I obtain the following martingale equation:

$$\nu_{s-} = \mathbb{E}\{\nu_s|s_-\} + \nu_{1,s-} - \nu_{2,s-}. \tag{22}$$

In the rest of this section, I will often abuse notation by switching from recursive to sequential notation. Away from the debt-limits, $\nu_{s-} = \mathbb{E}\{\nu_s|s_-\}$. In sequential notation, the process $\{\nu_t\}$ which indexes both the government's marginal need for funds as well as the marginal cost of taxation, is a positive martingale. This reflects the desire to smooth distortionary taxes across states and time. The tax-smoothing intuition has been familiar in incomplete-markets environments since the work of Barro (1979) and AMSS. Under complete-markets, a similar Bellman equation would hold but ν_t would be constant and not a mere martingale.

Equation (21) also shows that debt $\tilde{b}_t$ is a nonlinear function of $-\nu_t + \nu_{1,t} - \nu_{2,t}$ and s_t. The policy functions $\{l_s, k, \tau^k\}$ associated with (16) imply that c_t, l_t, and τ_t^l are functions of $\tilde{b}_{t-1}$, s_{t-1}, and s_t, and that k_{t-1} and τ_t^k are functions of $\tilde{b}_{t-1}$ and s_{t-1}. This martingale component results from the incompleteness of markets. If markets were complete, then debt and taxes would depend only on s_t and s_{t-1}; they would hence inherit the serial correlation properties of the Markov process $\{s_t\}$ as in Lucas and Stokey (1983).

4.2 Taxes on Capital

By manipulating the first-order conditions, it is possible to derive a formula to characterize taxes on capital for $t \geq 1$:

$$\tau^k = \frac{\frac{\mathbb{E}\{-(1-\tau^k)kF_{kk,s}|s_-\}}{\mathbb{E}\{F_{k,s}|s_-\}}}{1 + \frac{\mathbb{E}\{F_{k,s}\nu_s|s_-\}}{\mathbb{E}\{F_{k,s}|s_-\}}} \left[\frac{\text{Cov}\{kF_{k,s}, \nu_s|s_-\}}{\mathbb{E}\{kF_{k,s}|s_-\}} - \frac{\text{Cov}\{kF_{kk,s}, \nu_s|s_-\}}{\mathbb{E}\{kF_{kk,s}|s_-\}} \right]. \tag{23}$$

There are three terms on the right-and side of this equation. The first term has as its numerator the inverse of the elasticity $\mathbb{E}\{-(1-\tau^k)kF_{kk,s}|s_-\}/\mathbb{E}\{F_{k,s}|s_-\}$ of capital k to taxes on capital τ^k. This inverse elasticity factor is standard in the taxation literature. The higher the elasticity, the lower the absolute value of the tax rate.

The second term $\text{Cov}\{kF_{k,s}, \nu_s|s_-\}/\mathbb{E}\{kF_{k,s}|s_-\}$ represents the direct effect of an increase in τ^k: it relaxes the budget constraint of the government (18) in state s in proportion to the tax base $kF_{k,s}$ of τ^k. The more $kF_{k,s}$ is correlated with ν_s, the higher the optimal τ^k, since revenues from capital taxes are higher in states where the need for funds is higher.

The third term $-\text{Cov}\{kF_{kk,s}, \nu_s|s_-\}/\mathbb{E}\{kF_{kk,s}|s_-\}$ reflects the indirect effect of an increase in

τ^k. In each state $s \in \mathbb{S}$, increasing τ^k affects investment k and hence the capital tax base $kF_{k,s}$ and the revenues from labor taxation $l_s F_{l,s} + l_s H_{l,s}$. How adverse these effects are depends on the correlation between $kF_{kk,s}$ and ν_s.[14] The higher the correlation, the lower is τ^k.

Equation (23) makes clear that the government uses capital taxes only to smooth its need for funds across states—a stark difference with labor taxes. At the complete-markets Ramsey outcome, (23) still holds but ν_s is constant across states, so $\tau^k = 0$. This can be seen as a particular case of the classical uniform taxation result of Atkinson and Stilgitz (1972), transposed to this Ramsey setup by Zhu (1992) and by Chari, Christiano and Kehoe (1994), which holds more generally for preferences that exhibit constant relative risk aversion (CRRA) and are separable between consumption and leisure. As the following proposition shows, this zero-tax result carries through in a particular case.

Proposition 1 *If F is Cobb–Douglas, then $\tau_t^k = 0$ for all $t \geq 1$.*

Proof. Suppose that $F(k, l, s) = A(s)k^\alpha l^{1-\alpha}$. Then $kF_{kk,s} = \alpha(\alpha - 1)A(s)k^{\alpha-1}l^{1-\alpha} = (\alpha - 1)F_{k,s}$. This implies that $kF_{kk,s}/\mathbb{E}\{kF_{kk,s}|s_-\} = kF_{k,s}/\mathbb{E}\{kF_{k,s}|s_-\}$ and hence that

$$\frac{\mathrm{Cov}\{kF_{kk,s}, \nu_s|s_-\}}{\mathbb{E}\{kF_{kk,s}|s_-\}} = \frac{\mathrm{Cov}\{kF_{k,s}, \nu_s|s_-\}}{\mathbb{E}\{kF_{k,s}|s_-\}}.$$

By (24) implies $\tau^k = 0$. Since (24) applies from $t = 1$ onward, it follows that $\tau_t^k = 0$ for all $t \geq 1$. ∎

For the Cobb–Douglas benchmark, taxes on capital are zero starting in period 1 on. At $\tau^k = 0$, the hedging benefits from the direct effect of a marginal increase in τ^k are exactly offset by the marginal hedging cost from the indirect effect through the reduction in the capital tax base and the reduction in labor tax revenues.

Remark 3 *Depreciation is subsumed by the function F. Proposition 1 applies to the case where net output (net of depreciation) can be written as a Cobb–Douglas function of capital labor. Moreover, it is assumed that depreciation is deductible. If depreciation is not deductible then it can be shown that capital taxes τ_t^k are equal to zero for all $t \geq 1$ when gross output (gross of depreciation) can be written as a Cobb–Douglas function of capital and labor.*

Remark 4 *Equation (23) and Proposition 1 would still hold under many different asset structures. This includes more extreme forms of market incompleteness, when the government is not allowed to issue debt and must balance its budget in every period, and also milder forms of market incompleteness, when the government can trade more assets as in Section 6.*

Apart from the Cobb–Douglas case, the sign of τ^k is ambiguous: it may be optimal to tax or to subsidize capital. The sign of τ^k will in general depend on how productivity and preference shocks interact with government expenditure shocks as well as on the particular functional form of the production function. Sharp theoretical results are difficult to obtain, because nonconvexities in (16) considerably complicate the task of establishing how l_s covaries with ν_s.

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[14] The formula makes use of the CRS assumption to replace $l_s F_{kl,s}$ by $-kF_{kk,s}$.

When the production function is not Cobb–Douglas, optimal capital taxes are persistent. This might come as a surprise in light of the Chamley–Judd result. This fact reflects the dependence of the government's hedging needs on the level of public debt, which has a random walk component. When public debt is low, the government is free to raise debt when confronted with an adverse shock; debt is then a good shock absorber. In contrast, when public debt is close to the debt limit, the ability of the government to shift the tax burden to the future is limited. Hedging through capital taxes is then more attractive.

However, it is important to emphasize that these deviations from Proposition 1 turn out to be quantitatively small so that the baseline Cobb–Douglas case is a good benchmark. The numerical simulations that I have performed show that when the model is calibrated to the U.S. economy, taxes on capital are very small (less than 1%) when preferences are quasi-linear.[15]

4.3   Long-Run Behavior

The long-run behavior of Ramsey outcomes is similar to AMSS. I refer the reader to that paper for an extensive discussion and only sketch the main properties in this section.

The difference between natural and ad hoc debt limits is marked. Under natural asset limits, the multiplier $\nu_{2,t}$ is zero throughout. The natural asset limit $-\underline{M}_s^n$ is the amount of assets that allows the government to withstand any sequence of shocks with zero taxes. It makes no sense for the government to accumulate more assets than $-\underline{M}_s^n$. If favorable shocks cause government assets to grow beyond $-\underline{M}_s^n$, then it is optimal for the government to pay back the difference to consumers via a lump-sum rebate. In this case, (22) becomes

$$\nu_{s-} = \mathbb{E}\{\nu_s|s_-\} + \nu_{1,s_-}$$

and so the stochastic process $\{\nu_t\}$ is a nonnegative supermartingale. Then, by the supermartingale convergence theorem (see Loève 1977), ν_t converges almost surely to a nonnegative random variable. As in AMSS, there are two possibilities as follows:

(i) If the Markov process $\{s_t\}_{t\geq0}$ has an absorbing state, then ν_t can converge to a strictly positive value; ν_t converges when s_t enters the absorbing state. From then on, labor taxes are constant and capital taxes are zero.

(ii) If the Markov process $\{s_t\}_{t\geq0}$ is ergodic, if $\tilde{V}$ is concave in $\tilde{b}$ and if that the policy functions in (16) are continuous, then ν_t converges almost surely to zero (see Lemma 2 below). In that case, taxes τ_t^k and τ_t^l converge to their respective first-best levels $\tau_t^k = 0$ and $\tau_t^l = 0$. In the limit, the level of government assets converges in state s to $-\underline{M}_s^n$, the level of assets sufficient to finance the worst possible sequence of shocks forever from interest earnings. A technical assumption is necessary to rule out nongeneric cases where the planner is able to achieve the complete-markets Ramsey outcome.

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[15]In the interest of space, these simulations are not included in the paper.

Assumption 1 *For any (b, s) such that $b > \underline{M}_s^n$, the complete-markets Ramsey outcome with initial condition (b, s) is not a competitive equilibrium with incomplete markets.*

Lemma 2 *Consider the case of natural debt and asset limits. Assume that Assumption 1 holds, that the Markov process $\{s_t\}_{t \geq 0}$ is ergodic, that the value function $\tilde{V}$ is continuously differentiable and strictly concave in $\tilde{b}$, and that the policy functions in (16) are continuous. Then ν_t converges to zero almost surely.*

When the lower debt limit is more stringent than the natural one, convergence to the first-best can be ruled out. In that case, the lower debt limit occasionally binds. The result is a nonnegative multiplier $\nu_{2,t}$ in (22), and $\{\nu_t\}$ ceases to be a supermartingale. This fundamentally alters the limiting behavior of the model in the case where the Markov process $\{s_t\}_{t \geq 0}$ has a unique invariant distribution. In particular, rather than converging almost surely, ν_t continues to fluctuate randomly. Away from the debt limits, ν_t behaves like a martingale, and capital taxes do not converge to zero. If, in addition, the range of the policy functions $\tilde{b}'_s$ can be restricted to a compact set, then one can prove the existence of an invariant distribution for government debt.

5 The General Case

When preferences are not quasi-linear, the possibility of manipulating interest rates brings about another motive for taxing capital.

5.1 Stochastic Properties of Ramsey Outcomes

I attach the multipliers μ to (12), ν_s to (13), $\nu_{2,s}$ and $\nu_{1,s}$ to the two constraints in (15), and ψ_s to (14). The first-order condition for $\tilde{b}'_s$ is then

$$\beta \tilde{V}_b(k'_s, \tilde{b}'_s, u_{c,s}, s) = -\nu_s + \nu_{1,s} - \nu_{2,s}.$$

As in the case of quasi-linear preferences, away from debt limits the multiplier ν_t represents the marginal value of a reduction in debt, and it can be interpreted as the marginal need for funds of the government or the marginal cost of taxation.

Using the envelope condition for $\tilde{b}$, one can derive a martingale equation similar to (22):

$$\nu_{s-} = \frac{\mathbb{E}\{\nu_s u_{c,s} | s_-\}}{\mathbb{E}\{u_{c,s} | s_-\}} + \nu_{1,s_-} - \nu_{2,s_-}.$$

Away from the debt limits, the multiplier ν_t is a now a risk-adjusted martingale. As in the quasi-linear case, ν_t would be constant with complete-markets and not a mere martingale. More generally, the stochastic properties of Ramsey outcomes are similar to those discussed in the quasi-linear example; the analysis is only made more difficult by the need to keep track of two extra state variables.

5.2 Taxes on Capital

Capital taxes τ^k can be decomposed as the sum of a "hedging" term, an "intertemporal" term, and a "debt-limits" term:

$$\tau^k = T^h(k, \tilde{b}, \theta, s_-) + T^i(k, \tilde{b}, \theta, s_-) + T^b(k, \tilde{b}, \theta, s_-). \tag{24}$$

This decomposition is useful for three different but related reasons. First, each of these terms has a distinct interpretation. Second, these terms are equal to zero under special configurations of different sets of parameters.[16] Third, these terms have different long-run stochastic properties (see Proposition 3). The expressions given next for these three terms are valid from $t = 1$ onward.

The hedging term. The hedging term $T^h(k, \tilde{b}, \theta, s_-)$ is given by

$$\frac{\frac{\mathbb{E}\{-k(1-\tau^k)F_{kk,s}u_{c,s}|s_-\}}{\mathbb{E}\{F_{k,s}u_{c,s}|s_-\}}}{\frac{\mathbb{E}\{F_{k,s}u_{c,s}\frac{\psi_s}{u_{c,s}}|s_-\}}{\mathbb{E}\{F_{k,s}u_{c,s}|s_-\}} + \frac{\mathbb{E}\{F_{k,s}\nu_s u_{c,s}|s_-\}}{\mathbb{E}\{F_{k,s}u_{c,s}|s_-\}}} \left[\frac{\mathrm{Cov}\{kF_{k,s}u_{c,s}, \nu_s|s_-\}}{\mathbb{E}\{kF_{k,s}u_{c,s}|s_-\}} - \frac{\mathrm{Cov}\{kF_{kk,s}u_{c,s}, \nu_s|s_-\}}{\mathbb{E}\{kF_{kk,s}u_{c,s}|s_-\}} \right].$$

This term reflects the hedging motive discussed in the previous section and would be equal to zero if markets were complete or if the Markov shock were in an absorbing state. Two differences with (23) should be emphasized: first, the formula is adjusted for risk via $u_{c,s}$; second, the multiplier ψ_s on the resource constraint (14) appears (in the quasi-linear case, $\psi_s = 1$). When risk aversion is introduced, the stochastic process $\{\beta^t \psi_t\}_{t\geq 0}$ represents the intertemporal prices the government would be willing to pay for additional resources at different dates. The process $\{\beta^t \psi_t / u_{c,t}\}_{t\geq 0}$ converts these prices in consumption-equivalent units. The presence of ψ_s is natural because taxes on capital affect capital accumulation and hence available resources.

Proposition 2 *If F is Cobb–Douglas, then $T_t^h = 0$ for $t \geq 1$.*

Proof. As noted in the proof of Proposition 1, in the Cobb–Douglas case we have $kF_{kk,s} = (\alpha - 1)F_{k,s}$. This implies that $kF_{kk,s}u_{c,s}/\mathbb{E}\{kF_{kk,s}u_{c,s}|s_-\} = kF_{k,s}u_{c,s}/E\{kF_{k,s}u_{c,s}|s_-\}$ and hence that

$$\frac{\mathrm{Cov}\{kF_{kk,s}u_{c,s}, \nu_s|s_-\}}{\mathbb{E}\{kF_{kk,s}u_{c,s}|s_-\}} = \frac{\mathrm{Cov}\{kF_{k,s}u_{c,s}, \nu_s|s_-\}}{E\{kF_{k,s}u_{c,s}|s_-\}}.$$

This shows, as in the proof of Proposition 1, that $\tau_t^k = 0$ for all $t \geq 1$. ∎

As with quasi-linear preferences, the deviations from Proposition 2 when the production function is not Cobb–Douglas turn out to be quantitatively small so that the baseline Cobb–Douglas case provides a good benchmark. In my numerical simulations with log balanced growth preferences in Section 7, the hedging term is always very small (below 1%). Note that the two remarks (Remarks 3

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[16]The hedging term T^h is equal to zero for $t \geq 1$ when the production function is Cobb–Douglas or when markets are complete. The intertemporal T^i is equal to zero for $t \geq 2$ if preferences are of the form $u(c, l, s) = (c^{1-\sigma} - 1)/(1 - \sigma) + H(l, s)$ and if markets are complete. The intertemporal term T^i is zero for $t \geq 1$ under quasi-linear preferences. The debt-limits term T^b away from the debt limits or if the debt limits do not depend on k (as is the case for natural debt limits with quasi-linear preferences).

and 4) that follow Proposition 1 for the quasi-linear case also apply to Proposition 2 for the general case.

The intertemporal term. The intertemporal term $T^i(k, \tilde{b}, \theta, s_-)$ is given by

$$-\frac{\frac{\mathbb{E}\left\{[1+(1-\tau^k)F_{k,s}]u_{c,s}\left(\frac{\psi_s}{u_{c,s}} - \frac{\psi_{s-}}{\theta}\right)|s_-\right\}}{\beta\mathbb{E}\left\{u_{c,s}F_{k,s}|s_-\right\}}}{\frac{\mathbb{E}\{F_{k,s}u_{c,s}\frac{\psi_s}{u_{c,s}}|s_-\}}{\mathbb{E}\{F_{k,s}u_{c,s}|s_-\}} + \frac{\mathbb{E}\{F_{k,s}\nu_s u_{c,s}|s_-\}}{\mathbb{E}\{F_{k,s}u_{c,s}|s_-\}}}.$$

Here ψ_{s-} is the multiplier on the resource constraint in the previous period. The formula calls for subsidizing capital between t and $t+1$ when resources are expected to be scarcer at $t+1$ than at t—that is, when $\psi_{t+1}/u_{c,t+1}$ is expected to be larger on average than $\psi_t/u_{c,t}$—especially if the net marginal product of capital $1 + (1 - \tau_{t+1}^k)F_{k,t+1}$ or marginal utility $u_{c,t+1}$ is positively correlated with $\psi_{t+1}/u_{c,t+1}$. This term reflects the possibility of manipulating interest rates. A one-time tax capital tax in the period after a bad shock that negatively affects the government budget lowers the interest rate the government must pay on debt issued in that period, increases total tax revenues, and helps smooth the burden of taxation.[17] In order for consumers to accept this lower interest rate, consumption must be temporarily increased (and investment correspondingly reduced) in the period when the shock hits the economy and decreased afterwards. As a result, the multiplier $\psi_t/u_{c,t}$ tends to be higher in the period when the shock hits than in the following periods. The opposite occurs after a good shock that positively affects the government budget (there tends to be a one-time capital subsidy in the period after the shock).

It is important that these responses are anticipated. For example, a planned one-time increase in capital taxes in the period after a bad shock increases consumption and decreases marginal utility in the period of the shock. As a consequence, before the shock, interest rates increase and the government is forced to lower capital taxes. The result is a stabilization of the government budget: the debt burden is increased if the bad shock does not materialize, and decreased if it does. Of course, this stabilization benefit has to be weighed against the distortionary costs imposed on the economy.[18] All in all, in contrast to labor taxes, capital taxes are used not so much to raise revenues on average, but rather to help absorb the variations in present and future government surpluses.

When preferences are quasi-linear, this motive for capital taxation disappears: $\psi_t/u_{c,t} = 1$ for $t \geq 0$ and the intertemporal term is equal to zero from $t = 1$ onward.[19] In contrast, in the

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[17]Here the increase in tax revenues incorporates the offsetting effects resulting from the negative impact on capital accumulation: a lower capital tax base and lower revenues from labor taxation.

[18]This perturbation argument illustrates the trade-off faced by the government when considering a planned increase in capital taxes following a bad shock. As always with such arguments, different choices are possible for the margins along which one decides to perturb the allocation. For example, another way to illustrate the aforementioned trade-off is as follows. The government now keeps the interest before the shock unaffected by holding constant expected marginal utility in the period of the shock. It compensates the decrease in marginal utility resulting from the increase in capital taxes after a bad shock with an increase in marginal utility brought about by a decrease in capital taxes if the bad shock does not occur. As in the perturbation argument presented in the text, the result is an increased debt burden if the bad shock does not materialize and a decreased debt burden if it does.

[19]Not only are interest rates fixed, but also a one-time tax capital tax after a bad shock does not increase total tax revenues. Because the intertemporal elasticity of substitution is infinite, the offsetting effects resulting from the

simulations presented in Section 7 with log balanced growth preferences, the behavior of capital taxes is completely dominated by the intertemporal term. The spikes in capital taxes following aggregate shocks turn out to be large.

This formula (24) for τ^k is valid at the complete-markets Ramsey outcome; the only difference is that ν_s is constant. When state-contingent capital taxes τ^k_s and one-period risk-free debt are used to implement the complete-markets Ramsey outcome, equation (24) describes the within-period average $\mathbb{E}\{\tau^k_s F_{k,s} u_{c,s} | s_-\} / \mathbb{E}\{F_{k,s} u_{c,s} | s_-\}$ of capital taxes (i.e. the intertemporal wedge). Even then, T^i is not equal to zero in general.

The well-known instance of a zero intertermporal wedge at the complete-markets Ramsey outcome is when preferences are CRRA and separable between consumption and leisure: $u(c, l, s) = (c^{1-\sigma} - 1)/(1 - \sigma) + H(l, s)$. In this case, $\psi_t / u_{c,t} = 1 - \nu\sigma$ is constant along the optimal path from $t = 1$ onward, and hence the intertemporal term is equal to zero from period 2 onward. In the implementation of the complete-markets Ramsey outcome with state-contingent capital taxes and no state-contingent debt, the government is able to completely hedge its budget against aggregate shocks by setting a positive (negative) state-contingent capital tax τ^k_s in state s when a bad (good) shock to the government budget hits and imposing $\mathbb{E}\{\tau^k_s F_{k,s} u_{c,s} | s_-\} = 0$ so that the intertemporal wedge is zero in every period. In the absence of state-contingent capital taxes and state-contingent debt, this nondistortionary form of hedging is no longer available. The multiplier $\psi_t / u_{c,t}$ is not constant at the Ramsey outcome and T^i is not equal to zero: the government imposes distortions on the economy by taxing (subsidizing) capital in the period after a bad (good) shock to its budget.

It is possible to give a theoretical characterization of T^i along the lines of Zhu (1992). The only difference between the environment considered there and the one in this paper is that Zhu assumes that the complete-markets Ramsey outcome is achievable. He proves that, under some regularity conditions, if the Ramsey outcome converges to a stationary equilibrium then the intertemporal wedge either (a) is equal to zero with probability one or (b) takes both positive and negative values with positive probability. Proposition 3 shows that these insights generalize to the case of incomplete markets for the intertemporal term T^i: at a stationary equilibrium, T^i cannot be always positive or always negative. The assumptions that follow adapt Zhu's regularity conditions to the environment of my model.

Definition 4 *The Ramsey outcome is a **stationary equilibrium** if the stochastic process $x_t = \{k_t, \tilde{b}_t, u_{c,t}, s_t\}$ is a stationary, ergodic, first-order Markov process on a compact set X^∞—that is, if there exists a probability measure P^∞ on a compact set X^∞ such that, for all t and for any measurable set A, $\Pr\{x_t \in A\} = P^\infty\{A\}$ and $\lim_{j \to \infty} \Pr\{x_{t+j} \in A | x_t\} = P^\infty\{A\}$.*

Assumption 2 *The policy functions in (11), including the multipliers, are continuous. For every closed set $A \subseteq X^\infty$ and $t > 0$, $\Pr\{x_t \in A | x_0 = x\}$ is continuous in x.*

Assumption 3 *$P^\infty\{1 + (1 - \tau^k) F_{k,t} > 0\} = 1$.*

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negative impact on capital accumulation are strong enough to neutralize the direct effect of increased capital taxes on tax revenues.

Proposition 3 *Suppose that the Ramsey outcome is a stationary equilibrium and that Assumptions 2 and 3 hold. Then either (a) $P^\infty \{T_t^i = 0\} = 1$ or (b) $P^\infty \{T_t^i > 0\} > 0$ and $P^\infty \{T_t^i < 0\} > 0$.*

The debt-limits term. The debt-limits term $T^b(k, \tilde{b}, \theta, s_-)$ is equal to zero unless either the maximum or minimum debt limit in the previous period is binding. When this occurs, it imparts to capital taxes the role of relaxing debt limits. This term is given by

$$
\frac{\dfrac{\frac{\nu_{2,s_-}}{\beta}\underline{M}_{k,s_-} - \frac{\nu_{1,s_-}}{\beta}\overline{M}_{k,s_-}}{\beta \mathbb{E}\{u_{c,s}F_{k,s}|s_-\}}}{\dfrac{\mathbb{E}\{F_{k,s}u_{c,s}\frac{\psi_s}{u_{c,s}}|s_-\}}{\mathbb{E}\{F_{k,s}u_{c,s}|s_-\}} + \dfrac{\mathbb{E}\{F_{k,s}\nu_s u_{c,s}|s_-\}}{\mathbb{E}\{F_{k,s}u_{c,s}|s_-\}}}.
$$

For example, if the maximum debt limit is increasing in the economy's capital stock, then it is optimal to subsidize capital when the maximum limit is binding in order to allow for more more debt. This term is zero if the imposed debt limits do not depend on capital, as is the case for the natural debt limits if preferences are quasi-linear.[20] In the general case however, natural debt limits do depend on the capital stock and so the term T^b will be sometimes non-zero.

6 Capital Ownership

So far, I have restricted the government to trading only a risk-free bond with consumers. Prohibiting the government from trading capital is without loss of generality under complete markets. But with incomplete markets, this arbitrary restriction regains bite. Allowing the government to trade capital enables the government to hedge its budget against aggregate shocks.

Remark 5 *Trading capital can also be interpreted as allowing the government to tax capital excess returns. This form of nonlinear capital taxation is nondistortionary and offers the same hedging benefits as a position in capital combined with an opposite position in the risk-free bond.*

6.1 The Optimal Structure of Government Liabilities

Along with capital, I introduce a set of additional assets. For every state s_- of the Markov process in the previous period, these assets are indexed by $i \in \mathbb{I}_{s_-}$. The payoffs of these assets $\{R_s^{i,s_-}, i \in \mathbb{I}_{s_-}\}$ are exogenously specified, and I assume that they are in zero net supply. The government and consumers can trade three kinds of assets: (i) a risk-free bond; (ii) capital, an asset whose return in state s is $1 + (1 - \tau^k)F_{k,s}$; and (iii) $\#\mathbb{I}_{s_-}$ assets as just described. Generically, if the number of traded assets is less than the number of shocks, then markets are truly incomplete and the complete-markets Ramsey outcome is not attainable. I will maintain this assumption throughout.

The planning problem is still recursive with the same state variables k, $\tilde{b}$, θ and s_-, where $\tilde{b}$ should now be interpreted as the value of the government's net liabilities. Denote government

---

[20]Because natural debt limits are independent of capital with quasi-linear preferences, in Section 4 I considered only the case of exogenous debt limits and so $T^b = 0$, which would not have been the case had I allowed debt limits to depend on capital.

holdings of asset $i \in \mathbb{I}_{s_-}$ by x_i and the government holdings of capital by k_g. The value function satisfies a modified version of (11).

Bellman equation 2′.

$$\tilde{V}(k, \tilde{b}, \theta, s_-) = \max_{\left\{ c_s, l_s, k'_s, \tilde{b}'_s, k_g, x_i, \tau^k \right\}} \mathbb{E}\{u_s + \beta \tilde{V}(k'_s, \tilde{b}'_s, u_{c,s}, s) | s_-\} \tag{25}$$

subject to

$$\mathbb{E}\left\{ \beta u_{c,s}[1 + (1 - \tau^k)F_{k,s}] | s_- \right\} = \theta, \tag{26}$$

$$\mathbb{E}\left\{ \beta u_{c,s} R_s^{i,s_-} | s_- \right\} = \theta, \tag{27}$$

$$\sum_{i \in \mathbb{I}_{s_-}} x_i \left(R_s^{i,s_-} - \frac{\theta}{\beta \mathbb{E}\left\{ u_{c,s} | s_- \right\}} \right) u_{c,s} + k_g \left(1 + (1 - \tau^k)F_{k,s} - \frac{\theta}{\beta \mathbb{E}\left\{ u_{c,s} | s_- \right\}} \right) u_{c,s},$$

$$+ \tilde{b} \frac{u_{c,s}}{\beta \mathbb{E}\left\{ u_{c,s} | s_- \right\}} + g_s u_{c,s} \leq l_s F_{l,s} u_{c,s} + l_s u_{l,s} + \tau^k k F_{k,s} u_{c,s} + \tilde{b}'_s \qquad \forall s \in \mathbb{S}, \tag{28}$$

$$c_s + g_s + k'_s \leq F_s + k \qquad \forall s \in \mathbb{S}, \tag{29}$$

$$\underline{M}(k'_s, u_{c,s}, s) \leq \tilde{b}'_s \leq \overline{M}(k'_s, u_{c,s}, s) \qquad \forall s \in \mathbb{S}. \tag{30}$$

It is convenient to label the risk-free rate as

$$R^{s_-} \equiv \frac{\theta}{\beta \mathbb{E}\left\{ u_{c,s} | s_- \right\}}. \tag{31}$$

There are two differences between (11) and (25). First, there is now one Euler equation for each additional asset (27). The second difference is in the budget constraint of the government (28), where the total liability that the government must repay or refinance in state s is now

$$\sum_{i \in \mathbb{I}_{s_-}} x_i (R_s^{i,s_-} - R^{s_-}) + k_g (1 + \left(1 - \tau^k \right) F_{k,s} - R^{s_-}) + \frac{\tilde{b}}{\theta} R^{s_-}.$$

Hence the government faces a nontrivial portfolio decision: it must choose not only the level but also the composition of its liabilities.

Assuming that an interior solution exists and that debt limits do not bind in state s_-, the following set of first-order conditions characterize the optimal asset and liability structure of the government:

$$\mathbb{E}\left\{ \beta R^{s_-} u_{c,s} \nu_s | s_- \right\} = \theta \nu_{s_-}, \tag{32}$$

$$\mathbb{E}\left\{ \beta \left[1 + \left(1 - \tau^k \right) F_{k,s} \right] u_{c,s} \nu_s | s_- \right\} = \theta \nu_{s_-}, \tag{33}$$

$$\mathbb{E}\left\{ \beta R_s^{i,s_-} u_{c,s} \nu_s | s_- \right\} = \theta \nu_{s_-}. \tag{34}$$

These equations form the government counterpart of the standard CCAPM Euler equations (31), (26), and (27). The difference is that the marginal utilities $u_{c,s}$ and θ are now replaced by $u_{c,s} \nu_s$

and $\theta\nu_{s_-}$, incorporating the marginal need for funds of the government. The result is a simple framework for characterizing the optimal portfolio of the government.[21] These equations could also be used to perform tests for the optimality of that portfolio's composition.

Remark 6 *The results on capital taxation still hold when more traded assets are introduced. In particular, taxes are still given by (24). The only difference is in the hedging term where the elasticity of capital to the tax rate must be replaced by:* $\mathbb{E}\{-(k-k_g)(1-\tau^k)F_{kk,s}u_{c,s}|s_-\}/\mathbb{E}\{F_{k,s}u_{c,s}|s_-\}$.

Turning to the initial conditions, the Ramsey problem now requires specifying initial portfolio holdings $k_{g,-1}$ and $\{x_{i,-1}\}_{i\in I_{s_{-1}}}$ together with $(b_{-1}, r_0, k_{-1}, \tau_0^k)$. Given the solution to (25), the problem of the inital period can be treated exactly as explained in Section 3 for the case where the government can only trade a short-term risk-free bond.

6.2 The Quasi-Linear Case

When preferences are quasi-linear, the planning problem is recursive with state variables $\tilde{b}$ and s_-. The value function satisfies a modified version of (25) that is analogous to (16).

Bellman equation 3'.[22]

$$\tilde{V}(\tilde{b}, s_-) = \max_{\{l_s, k, \tilde{b}'_s, k_g, x_i, \tau^k\}} \mathbb{E}\left\{ F_s + k\left(1 - \frac{1}{\beta}\right) - g_s + \beta\tilde{V}(\tilde{b}'_s, s)|s_- \right\} \tag{35}$$

subject to

$$\mathbb{E}\left\{\beta[1 + (1-\tau^k)F_{k,s}]|s_-\right\} = 1, \tag{36}$$

$$\sum_{i\in\mathbb{I}_{s_-}} x_i\left(R_s^{i,s_-} - \frac{1}{\beta}\right) + k_g\left(1 + (1-\tau^k)F_{k,s} - \frac{1}{\beta}\right) + \tilde{b}\frac{1}{\beta} + g_s \leq l_s F_{l,s} + l_s H_{l,s} + \tau^k k F_{k,s} + \tilde{b}'_s \qquad \forall s \in \mathbb{S}, \tag{37}$$

$$\underline{M}_s \leq \tilde{b}'_s \leq \overline{M}_s \qquad \forall s \in \mathbb{S}. \tag{38}$$

An extreme benchmark. The case where government expenditure shocks are the only disturbance in the economy provides a useful benchmark. Consider the Ramsey problem with initial

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[21]This framework can be used to shed some light on capital budgeting rules for the government. Imagine that the government considers whether to undertake a small (marginal) public investment project which requires investing I at some date t where the state is s_-. The payoff of the investment project in period $t+1$ is given by X_s in state is s. If the return X_s/I of the investment project is not spanned by assets traded by the government, then the standard capital budgeting prescription of the CCAPM is altered. The risk-adjusted interest rate that should be used to discount the cash flows of the project is

$$R^{s_-} - \frac{\text{Cov}\{X_s/I, u_{c,s}(\nu_s + \psi_s/u_{c,s})|s_-\}}{\mathbb{E}\{u_{c,s}(\nu_s + \psi_s/u_{c,s})|s_-\}}.$$

The government should therefore use a lower interest rate than predicted by the CCAPM—the CCAPM risk-adjusted interest rate can be derived from the formula above by replacing $\nu_s + \psi_s/u_{c,s}$ by 1—for investment projects that are likely to pay out relatively well in times of stress for public finances.

[22]With quasi-linear preferences, the constraints in (27) become exogenous necessary conditions for an equilibrium—$\mathbb{E}\left\{\beta R_s^{i,s_-}|s_-\right\} = 1$ for all $s \in \mathbb{S}$—which can be dropped from the maximization.

conditions given by $(\tilde{b}_{-1}, r_0, k_{-1}, \tau_0^k)$ and $(k_{g,-1}, \{x_{i,-1}\}_{i \in I_{s_{-1}}})$. The initial period is special and the problem can be solved in two steps. First, solve the *continuation Ramsey problem*—by solving the Bellman equation (35)—over the set of *continuation allocations* of competitive equilibria $\{c_t^{\xi_0}, l_t^{\xi_0}, k_{t-1}^{\xi_0}\}_{t \geq 1}$ after all possible realizations of the state s_0 in period 0 and value of net government liabilities $\tilde{b}_0(s_0)$, where $\xi_0 \equiv (\tilde{b}_0(s_0), s_0)$.[23] Second, solve the problem of the initial period to find $\{c_0(s_0), l_0(s_0)\}_{s_0 \in \mathbb{S}}$ and $\{\tilde{b}_0(s_0)\}_{s_0 \in \mathbb{S}}$, given that the continuation utility is given by $\tilde{V}(\tilde{b}_0(s_0), s_0)$. The supremum of the continuation Ramsey problem is actually not attained when markets are incomplete; Proposition 4 below shows how to construct a sequence of competitive equilibrium continuation allocations with welfare limiting to the supremum. These competitive equilibrium continuation allocations can then be combined with initial period choices for $\{c_0(s_0), l_0(s_0)\}_{s_0 \in \mathbb{S}}$ to form competitive equilibrium allocations that approximate the supremum in the Ramsey problem.

Let $\{c_t^{c,\xi_0}, l_t^{c,\xi_0}, k_{t-1}^{c,\xi_0}\}_{t \geq 1}$ be the continuation allocation of the complete-markets Ramsey outcome: it is obtained recursively from the policy functions in (35) when the matrix of returns of the additional assets that can be traded by the government has full range. Let $\tilde{V}^c(\tilde{b}_0(s_0), s_0)$ be the corresponding value function and let $\{\tilde{b}_{t-1}^{c,\xi_0}\}_{t \geq 1}$ be the corresponding process for the value of net government liabilities. Clearly, welfare is higher under complete markets so that $\tilde{V}(\tilde{b}_0(s_0), s_0) \leq \tilde{V}^c(\tilde{b}_0(s_0), s_0)$. The following proposition shows that this inequality actually holds as an equality.

Proposition 4 *Assume that preferences are quasi-linear, that there are government expenditure shocks but neither productivity shocks nor preference shocks, and that the government can trade capital in addition to short-term bonds. Then, there exists a set of competitive equilibrium continuation allocations $\{c_t^{k_g,\xi_0}, l_t^{k_g,\xi_0}, k_{t-1}^{k_g,\xi_0}\}_{t \geq 1}$ indexed by the government's holding of capital k_g and the starting point ξ_0, such that (a) the corresponding process for the value of net government liabilities coincides with that of the continuation of the complete-markets Ramsey outcome $\{\tilde{b}_{t-1}^{k_g,\xi_0}\}_{t \geq 1} = \{\tilde{b}_{t-1}^{c,\xi_0}\}_{t \geq 1}$ and (b) the limit of these continuation allocations when $|k_g| \to \infty$ is the continuation allocation $\{c_t^{c,\xi_0}, l_t^{c,\xi_0}, k_{t-1}^{c,\xi_0}\}_{t \geq 1}$ of the complete-markets Ramsey outcome starting at ξ_0.*

The intuition for this proposition is that in the absence of productivity and preference shocks, the continuation of the complete-markets Ramsey outcome features constant labor. Hence the return on capital is risk-free: capital and the risk-free bond are perfectly colinear assets. By commanding small deviations from the constant-level labor supply of the complete-markets Ramsey outcome, the government can align the variations of the return on capital with its need for funds. By taking extreme positions in capital that are compensated by opposite positions on the risk-free bond, the government can then leverage these variations and smooth perfectly its need for funds across states. By doing this in every state and date, the government can thus perfectly approximate the continuation of the complete-markets Ramsey outcome for $t \geq 1$. This logic does not apply in period 0: the initial portfolio is an input of the Ramsey problem.[24]

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[23]These continuation allocations would coincide with the full solution to the Ramsey problem if one were to adopt the timeless perspective introduced by Woodford (1999).

[24]Moreover, it is generically not the case that $\mathbb{E}_{-1}\{1 + (1 - \tau_0^k) F_{k,0}\} = 1 + r_0$ at the complete-markets Ramsey outcome: capital and the risk-free bond are not perfectly colinear assets in the initial period.

A comparison with Bohn (1990). Equations (32), (33), and (34) take a very simple form. To facilitate comparison with Bohn (1990), I express the corresponding conditions in an "excess return" format:

$$\mathbb{E}\left\{[1 + (1 - \tau^k)F_{k,s} - R]\nu_s|s_-\right\} = 0, \tag{39}$$

$$\mathbb{E}\left\{[R_s^{i,s_-} - R]\nu_s|s_-\right\} = 0, \tag{40}$$

where $R \equiv 1/\beta$ is the risk-free rate. These equations can be compared to the results in Bohn (1990). Building on Barro (1979), Bohn considers an environment with incomplete markets, no capital, risk-neutral consumers, and where the government must finance an exogenous stream of expenditures using distortionary taxes. Taxes τ—defined at the ratio of tax revenues to GDP—are assumed to impose an ad hoc increasing convex deadweight cost $h(\tau)$. Bohn derives the following formula for the return of any traded asset R_s^{i,s_-}:

$$\mathbb{E}\left\{[R_s^{i,s_-} - R]h'(\tau_s)|s_-\right\} = 0. \tag{41}$$

Equation (40) can be seen as a microfounded version of (41). Some differences are worth noting. In particular, it is not generally true in my model that ν_t is a function of τ_t^l and τ_t^k or even of tax revenues, as a perfect analogy with (41) would require. Suppose that, instead of assuming an exogenous process for government expenditures, there exists a standard utility for government expenditures $v(g_t, s_t)$. In this case, the first-order condition for government expenditures is $v_g(g_t, s_t) - 1 = \nu_t$; ν_t is then a function of g_t and s_t. Even if one were to assume that $v(g_t, s_t)$ is independent of s_t, this discussion suggests that a nonlinear function of government expenditures $v_g(g_t) - 1$ is better suited for approximating the marginal cost of public funds than is an increasing function of taxes or tax revenues $h'(\tau_t)$ as in (41). The logic is straightforward: the marginal distortionary cost of taxation, which cannot in general be expressed as a simple function of tax rates or tax revenues, is equal to the marginal benefit of government expenditures $v_g(g_t) - 1$ at a Ramsey outcome.

Discussion: Large government capital positions. The theoretical result of Proposition 4—that the optimal capital position of the government is infinite—is extreme. However, the simulations of Section 7 illustrate numerically that it generalizes in a milder form to reasonably calibrated environments with productivity shocks and preferences that are not quasi-linear. There I show that, in a business cycle calibration with government expenditure shocks and productivity shocks, the optimal capital position of the government is large and negative (a short position of more than 100% is required in all simulations).

The optimality of extreme positions is not particular to my model. Indeed, it is reminiscent of the findings of Angeletos (2002) and Buera and Nicolini (2004). Both contributions analyze how the government can use different maturities of risk-free debt to implement the complete-markets Ramsey outcome. They find that, generically, if the number of maturities is larger than the number of shocks, then the complete-markets Ramsey outcome can be implemented. However, very large positions in the different maturities are typically required.

Even though a substantial fraction of the welfare gains from optimally trading capital can be reached with smaller positions, these large capital positions put strain on some features of the model. Most importantly, my model takes a simplistic view of asset valuation. The return on capital in the model is much less volatile than stock returns in the data.[25] As a result, large leveraged positions are required to achieve meaningful state contigencies in portfolio returns. If the government were able to take a position in the stock market and if stock market returns were more volatile in my model (but still negatively correlated with the need for funds of the government), then smaller short stock market positions would be sufficient to deliver the same welfare gains.

It is also possible that the differences between the normative properties emphasized here and actual government behavior are especially severe along the dimension of capital ownership. The assumption of a benevolent government with full commitment appears strong given these large capital positions. In practice, political economy considerations might blur the picture and reduce the appeal of large trading positions by the government.

At the very least, the results in this section point to a cost of partial government ownership of the capital stock that is overlooked. In addition to the often mentioned costs stemming from poor management of the corresponding assets, positive capital ownership by the government is bad fiscal hedging policy. In my simulations, the optimal capital position of the government is large and negative, so it follows that a zero government position in capital is better than a positive one.[26]

7   Numerical Simulations

7.1   Numerical method and parameter values

Calibration. I consider the same parameters and functional forms as Chari, Christiano, and Kehoe (1994). Preferences are given by $u(c,l) = (1-\gamma)\log(c) + \gamma\log(1-l)$. Technology is described by a production function $F(k,l,z,t) = k^\alpha(\exp(\rho t + \tilde{z})l)^{1-\alpha} - \delta k$, which incorporates two kinds of labor-augmenting technological change: the variable ρ captures deterministic growth; and the variable $\tilde{z}$ is a zero-mean technological shock that follows a two-state Markov chain with mean $\bar{z}$, standard deviation σ_z, and autocorrelation ρ_z. Government expenditures are given by $g_t = G\exp(\rho t + \tilde{g})$, where G is a constant and $\tilde{g}$ follows a two-state Markov chain with mean $\bar{g}$, standard deviation σ_g and autocorrelation ρ_g. In my baseline calibration, a period corresponds to a year and $\gamma = 0.75$, $\beta = 0.98$, $\alpha = 0.34$, $\rho = 0.016$, $G = 0.07$, $\rho_g = 0.89$, $\sigma_g = 0.07$, $\rho_z = 0.81$ and $\sigma_z = 0.04$. I impose fixed debt limits $\underline{M} = -0.2\overline{GDP}^{fb}$ and $\overline{M} = \overline{GDP}^{fb}$, where $\overline{GDP}^{fb}$ is the mean across states of the first-best level of GDP that would occur if the state were absorbing. Notice that in the absence of shocks, the economy has a balanced growth path along which consumption, capital and government

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[25]This is partly due to the absence of capital gains in my setup (the relative price of installed capital is always equal to one in the absence of adjustment costs). Also, I have only analyzed standard balanced growth preferences with reasonable degrees of risk aversion. Incorporating adjustment costs, habits or long-run risk elements in the model is beyond the scope of this paper.

[26]This is difficult to prove theoretically because the constraint set in (25) is not concave, but I have checked numerically that it holds in my simulations.

spending grow at rate ρ and labor is constant. All the variables reported here are de-trended by removing the corresponding deterministic trend.

Period length. I analyze two series of simulations: one where the period length is one year, and another where the period length is five years. I adjust ρ and β so that they are they keep the same values per unit of time. I adjust the stochastic processes so that the size and the persistence (per unit of time) of the shocks are the same in the two simulations: in the five-year simulation, the values of the states of the two-state Markov chains are left unchanged and the probabilities in the transition matrix are adjusted so that the persistence of the shocks (per unit of time) is unchanged.[27]

In this setup, there are two fundamental reasons why the period length is more than a mere accounting convention. First, it indexes the sluggishness of capital taxes: the time during which capital taxes are held constant as well as the lag after which they can react to information. This matters because capital taxes have a spike after a transition and then revert to a small number; the period length puts a lower bound on the duration of the spike (and of the corresponding distortion imposed on the economy). Second it represents the maturity of government debt. To see why this is important, consider the following thought experiment: an economy with some outstanding government debt coupons experiences a Markov transition to an absorbing state; the transition negatively affects the government budget. As explained in Section 5, the positive spike in capital taxes that follows the transition helps reduce the debt burden by generating tax revenues and by temporarily lowering interest rates. The shorter the maturity of debt, the stronger the buffering effect from the latter channel (temporarily low interest rates). Indeed, given a path of labor tax revenues, a temporary reduction in interest rates raises both the net present value of coupon payments and the net present value of labor tax revenues. The latter increases more than the former, helping to buffer the impact of the shock, when coupon payments are front-loaded (and labor tax revenues are back-loaded).

Ideally, one would want to have the ability to vary the maturity of debt and the level of inertia in capital taxes independently. One would also like to maintain the possibility of making predictions for the yearly properties of the allocation in the five-year calibration. Unfortunately, this would require the introduction of several additional state variables, at the cost of numerical tractability.[28]

Numerical method. I solve Bellman equations (11) and (25). The state space is restricted to be rectangular and bounded. I check numerically that enlarging the rectangle doesn't alter the results. I approximate the value function with cubic splines and use a value iteration algorithm.

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[27]Buera and Nicolini (2004) also investigate the role of the period length with a similar adjustment. See the first simulations in Section 3 of their paper and their Table 1.

[28]A case which remains tractable is that of a perpetuitiy. However, a perpetuitiy is unappealing because the typical maturity of U.S. public debt is much shorter (around five years).

7.2   Results

Because I use log balanced growth preferences, the intertemporal wedge would be exactly zero if state-contingent debt or state-contingent capital taxes were available; all the deviations of the intertemporal wedge from zero can therefore be attributed to the unavailability of such instruments. Moreover, in all the simulations below, the hedging term is always smaller than 1% so that capital taxes are completely dominated by the intertemporal term.[29]

Three simulation segments. In Figure 2, I plot a segment of a simulation of the model with a period length of one year and with government expenditure shocks only. I refer to the low (high) government expenditures state as the good (bad) state corresponding to $s = 1$ ($s = 2$). There are two transitions: from the good state to the bad state and then back to the good state. This delimits three phases.

I begin by focusing on the first transition (from the good state to the bad state). In the period following the transition, capital taxes have a large spike: they jump from -9% to 150%, then fall to 9%, and stay almost constant at that level until the next transition. The interest rate is about 2.1% before the transition. It drops to -1% in the period following the transition, then goes back to 1.9% and stays almost constant at that level. A consequence is that marginal utility of consumption is considerably lower in the period of the transition than in the following periods. Because this effect is anticipated, before the transition, the interest rate is lower and the government is forced to decrease capital taxes. As explained in Section 5, the spike in capital taxes helps absorb the bad shock by increasing tax revenues and decreasing the interest rate on debt.

The labor tax is lowered in the period of the transition from 28.9% to 27.3%. This counteracts the negative wealth effect on labor supply from the lower marginal utility of consumption in that period. In the periods after the transition, labor taxes are then slightly higher than before the transition—a manifestation of their martingale component. The result is a rather smooth path for labor starting at the transition. Observe also that all along the sample path, the variations in labor taxes are extremely small in comparison to the variations in capital taxes.

Debt increases in the period of the transition because of the drop in tax revenues and the increase in government expenditures, but decreases sharply in the period following the transition as a result of the drop in interest rates and the increase in tax revenues brought about by the spike in capital taxes.

Turning now to the second transition (from the bad state to the good state), observe that the effects are almost symmetric. In particular, there is a large negative spike in capital taxes in the period following the transition.

During a phase when government expenditures are high, debt and labor taxes gradually increase: in every period, the absence of a transition to a low government expenditures state is bad news for the government budget. The opposite occurs during a phase when government expenditures are low. This is another illustration of the martingale component in labor taxes.

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[29]With respect to Proposition 2, the hedging term does not always equal to zero because I have assumed a production function such that output gross of depreciation is Cobb–Douglas and depreciation is deductible.

Figure 3 displays the result of a similar experiment with productivity shocks only. I now refer to the high (low) productivity state as the good (bad) state. Similar effects are at work, and in particular, capital taxes have a large positive (negative) spike following a transition from the good state to the bad state.

All in all, this discussion highlights the very different roles of labor and capital taxes: in contrast to labor taxes, capital taxes are not used to raise revenues on average but rather to help buffer the impact of shocks on the government budget.

In Figure 4, I return to a model with government expenditure shocks only, and explore the consequences of increasing the period length to five years. The most notable feature is the large mitigation of the spikes in capital taxes that occur just after the transitions—by a factor of 6.5 from 150% in the one-year simulation to 23% in the five-year simulation. Capital taxes are also smaller in absolute value in the phases between two transitions. Although I do not display the corresponding figure, a similar mitigation of capital taxes occurs when the period length is increased in the model with productivity shocks only. When the period length is increased, two effects contribute prominently to the mitigation of the spikes in capital taxes. First, a smaller positive (negative) spike in capital taxes is required to achieve a given reduction (increase) in the debt burden following a shock that negatively (positively) affects the government budget, because this tax rate is imposed for a longer time. Second, the distortionary costs associated with spikes in capital taxes increase because consumption is distorted for a longer time. A powerful illustration of this logic is the continuous-time limit where the period length is taken to zero: the government can attain the welfare of the complete-markets Ramsey outcome by combining (a) infinite capital taxes (subsidies) during an infinitesimal period of time following a transition from a good (bad) state to a bad (good) state and (b) bounded capital subsidies (taxes) in between transitions when the economy is in the good (bad) state.[30]

The full simulation. Table 1 summarizes the statistical properties of capital and labor taxes in the calibrated economy when the government cannot trade capital. Figure 1 displays the frequency distributions of labor and capital taxes at the stationary equilibrium of the economy.

Both in the one-year and in the five-year simulations, labor taxes are smooth—their standard deviation is less than 2%—persistent—their coefficient of autocorrelation is above 0.85 per period— and average out to about 28%. In contrast, capital taxes are volatile—their standard deviation is 54% in the one-year calibration and 11% in the five-year calibration—hardly display any persistence, and average out to a small number (below 5% in absolute value).[31] The standard deviation of capital taxes decreases sharply with the period length. The intuition for these results is transparent from the discussion above. The welfare gains from completing markets are small: 0.09% of lifetime consumption. This confirms the findings of AMSS for business cycle calibrations.[32]

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[30] When the period length is taken to zero, the spikes in marginal utility, which occur during an infinitesimal period of time following Markov transitions, remain bounded. This explains why capital taxes and subsidies in between transitions remain bounded.

[31] The absence of persistence can be seen almost algebraically: capital taxes are governed by a difference term $\psi_s/u_{c,s} - \psi_{s-}/\theta$, which tends to remove the unit root component in $\psi_s/u_{c,s}$.

[32] The magnitude of the welfare gains is well understood from AMSS. It depends on the size and persistence of the

Allowing the government to trade capital. For an economy with only two government expenditure shocks and a one-year period length, the government can replicate the complete-markets Ramsey outcome with a capital ownership of about 2300% of $\overline{k}^{fb}$.[33] This position drops to 680% of $\overline{k}^{fb}$ when the period length is extended to five years.[34] For an economy with only two productivity shocks and a one-year period length, the government can replicate the complete-markets Ramsey outcome with a capital ownership of about -400% of $\overline{k}^{fb}$. This position moves to -157% of $\overline{k}^{fb}$ when the period length is extended to five years. With both government expenditure shocks and productivity shocks, the optimal position is almost identical to the one that prevails with only productivity shocks. This is because, in this calibration, productivity shocks are the dominant source of variation in the government need for funds.

That a short position is required is easily understood, since the marginal product of capital correlates positively with productivity shocks and hence with government revenues. The magnitude of the position follows because capital is strongly colinear with risk-free debt. A large leveraged position—short in capital and long in the risk-free bond—is required to provide the government with a state-contingent source of revenues that matches the desired variations in present and future government surpluses. An intuition for the reduction in the optimal position as the period length is increased is as follows. In the anticipation of a shock, the government seeks to transfer wealth from bad states to good states in the next period. As the length of the period is increased, the impact of the shock on present and future government surpluses is unchanged and so the size of the desired wealth transfers across states is the same. However, a unit position, long in the bond and short in capital, implies earning a longer-term interest rate for a longer time and paying the marginal product of capital for a longer time, which magnifies the differences in total realized returns across states over a period. A smaller position is therefore required to achieve a given transfer of wealth across states.

8   Conclusion

Refining the normative prescriptions of this paper would require developing a more realistic model for investment—incorporating adjustment costs and time to build—and asset valuation. It would also be interesting to move away from the representative-agent framework used here.[35] Unobservable agent heterogeneity together with the government's concern for redistribution would provide an endogenous reason for the use of distortionary taxes. Finally, the large capital positions called

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shocks, the curvature of the utility function, and the debt limits.

[33] The optimal positions reported here are the average optimal positions in the stationary equilibrium. They turn out to be quite stable.

[34] Buera and Nicolini (2004) find similar effects of the period length on the size of the positions in a portfolio of bonds with different maturities.

[35] Shin (2006) makes an interesting step in that direction. He studies the asymptotic behavior of government debt in a Ramsey model with heterogenous agents and no capital. Shin explores the balancing act between the tax smoothing–induced desire to build a buffer stock of assets for the government and the precautionary savings motive of the agents.

for by the model put strain on the assumption of a benevolent government with full commitment. In this light, incorporating relevant political economy constraints into Ramsey models of optimal taxation appears as a promising research avenue. I leave this for future work.

9 Appendix

Proof of Lemma 1. The consumer's problem is a convex program. The first-order conditions (6), (7), and (8) as well as (5) holding with equality are necessary and sufficient for an optimum in the consumer's problem. It is straightforward to see that, if (1) holds with equality, then the fact that (3) holds with equality implies that (5) holds with equality—a version of Walras' law. ∎

Proof of Lemma 2. It is clear that $\widetilde{V}$ is decreasing in $\widetilde{b}$. Because $\widetilde{V}$ is differentiable, this is equivalent to $\widetilde{V}_b \leq 0$. Because $\beta \widetilde{V}_{b,t} = -\nu_t + \nu_{1,t}$, it follows that $\nu_t - \nu_{1,t} \geq 0$.

Under natural debt limits, (22) becomes $\nu_{s_-} = \mathbb{E}\{\nu_s | s_-\} + \nu_{1,s_-}$ which can be rewritten as $\nu_{s_-} - \nu_{1,s_-} = \mathbb{E}\{\nu_s - \nu_{1,s} | s_-\} + \mathbb{E}\{\nu_{1,s} | s_-\}$. This proves that $\{\widetilde{V}_{b,t}\}_{t \geq 1}$ is a nonnegative supermartingale. Therefore, by the supermartingale convergence theorem (see Loève 1977), $\widetilde{V}_{b,t}$ converges almost surely to a finite nonnegative random variable $\widetilde{V}_{b,\infty}$. Let $\mathbb{S}^e$ be the unique ergodic set of $\{s_t\}_{t \geq 0}$. Consider, for every $s \in \mathbb{S}^e$, the random sets $S^s = \{t, s_t = s\}$. Let $\phi^s(-1) = -1$ and, for all $t \geq 0$, define the sequence of random numbers $\phi^s(t) = \inf\{t > \phi^s(t-1), s_t = s\}$. Because $\widetilde{V}$ is continuously differentiable and strictly concave, this implies that there are finite random variable $\widetilde{b}_{\infty,s}$ such that $\{\widetilde{b}_{\phi^s(t)}\}_{t \geq 0}$ converges to $\widetilde{b}_{\infty,s}$ for every $s \in \mathbb{S}^e$. Because policy functions in (16) are continuous, this implies that every point $\{\widetilde{b}^s\}_{s \in \mathbb{S}^e}$ in the support of $\{\widetilde{b}_{\infty,s}\}_{s \in \mathbb{S}^e}$ is such that, for every states s and s_- in the unique ergodic set of $\{s_t\}_{t \geq 0}$, we have $\widetilde{b}'_s(\widetilde{b}^{s_-}, s_-) = \widetilde{b}^s$ and $\widetilde{V}_{b,s}(\widetilde{b}^s, s) = \widetilde{V}_{b,s_-}(\widetilde{b}^{s_-}, s_-)$. Hence by starting at such a point $(\widetilde{b}^s, s)$, the planner can implement the complete-markets Ramsey outcome. By Assumption 1, this is possible only if $\widetilde{b}^s = -\underline{M}^n_s$ for all s. ∎

Proof of Proposition 3. The proof closely follows Zhu (1992); more details can be found there.

First note that $T_{2,t} > 0$ if and only if

$$\frac{\mathbb{E}_{t-1}\left\{[1 + (1 - \tau^k)F_{k,t}]u_{c,t}\frac{\psi_t}{u_{c,t}}\right\}}{\mathbb{E}_{t-1}\left\{[1 + (1 - \tau^k)F_{k,t}]u_{c,t}\right\}} > \frac{\psi_{t-1}}{u_{c,t-1}},$$

$T_{2,t} < 0$ if and only if

$$\frac{\mathbb{E}_{t-1}\left\{[1 + (1 - \tau^k)F_{k,t}]u_{c,t}\frac{\psi_t}{u_{c,t}}\right\}}{\mathbb{E}_{t-1}\left\{[1 + (1 - \tau^k)F_{k,t}]u_{c,t}\right\}} < \frac{\psi_{t-1}}{u_{c,t-1}},$$

and $T_{2,t} = 0$ if and only if

$$\frac{\mathbb{E}_{t-1}\left\{[1 + (1 - \tau^k)F_{k,t}]u_{c,t}\frac{\psi_t}{u_{c,t}}\right\}}{\mathbb{E}_{t-1}\left\{[1 + (1 - \tau^k)F_{k,t}]u_{c,t}\right\}} = \frac{\psi_{t-1}}{u_{c,t-1}}.$$

Use K_t to denote $\left[1 + \left(1 - \tau^k\right)F_{k,t}\right]u_{c,t}$ and use ξ_t to denote $\psi_t/u_{c,t}$. Since the policy functions in (11) are continuous, it follows that K_t and ξ_t are continuous functions $K(x_t)$ and $\xi(x_t)$ of $x_t =$

$\{k_t, \tilde{b}_t, u_{c,t}, s_t\}$. Call $\pi(x', x)$ the transition function. Let Υ be the operator mapping the space of continuous functions of x into itself defined by:

$$\Upsilon(f)(x) \equiv \frac{\int f(x')K(x')\pi(x', x)}{\int K(x')\pi(x', x)}.$$

Observe that $T_2(x) > 0$ if and only if $\Upsilon(\xi)(x) > \xi(x)$, $T_2(x) < 0$ if and only if $\Upsilon(\xi)(x) < \xi(x)$, and $T_2(x) = 0$ if and only if $\Upsilon(\xi)(x) = \xi(x)$. Thus the sign of T_2 is entirely determined by the sign of $\Upsilon(\xi) - \xi$.

Suppose $P^\infty\{\Upsilon(\xi)(x) \le \xi(x)\} = 1$. Let $\underline{\xi} \equiv \max\{\xi, P^\infty\{\xi(x) \ge \xi\} = 1\}$. Define $A_t = \{x, \xi(x) \ge \Upsilon^t\xi(x)\}$, $B_t = \{x, \Pr\{\xi(x_t) \ge \underline{\xi}|x_0 = x\} = 1\}$, $A = \bigcap_{t=0}^\infty A_t$ and $B = \bigcap_{t=0}^\infty\bigcap_{t=0}^\infty B_t$. Then A_t, B_t, A, and B are closed and also $P^\infty\{A_t\} = P^\infty\{B_t\} = 1$. Hence $A \cap B$ is closed and $P^\infty\{A \cap B\} = 1$. By definition of $\underline{\xi}$, there exists $x \in A \cap B$ such that $\xi(x) = \underline{\xi}$. This implies $\Pr\{\xi(x_t) = \underline{\xi}|x_0 = x\} = 1$. By the ergodicity of x_t, $P^\infty\{\xi(x) = \underline{\xi}\} = \lim_{t\to\infty}\Pr\{\xi(x_t) = \underline{\xi}|x_0 = x\} = 1$. Hence $P^\infty\{\xi(x) = \underline{\xi}\} = 1$. Similarly $P^\infty\{\Upsilon(\xi)(x) \ge \xi(x)\} = 1$ implies that $P^\infty\{\xi(x) = \overline{\xi}\} = 1$, where $\overline{\xi} \equiv \inf\{\xi, P^\infty\{\xi(x) \le \xi\} = 1\}$. ∎

Proof of Proposition 4. For all $t \ge 0$, let $s^{t,s_0} = (s_0, s_1, ..., s_t)$ where s_0 is fixed throughout the proof..The continuation allocation $\{c_t^{c,\xi_0}, l_t^{c,\xi_0}, k_{t-1}^{c,\xi_0}\}_{t\ge 1}$ has the following properties: $l_t^{c,\xi_0}(s^{t,s_0}) = l_{t'}^{c,\xi_0}(s^{t',s_0})$ for all $t \ge 1$, $t' \ge 1$ and (s^{t,s_0}, s^{t',s_0}); $\tilde{b}_t^{c,\xi_0}(s^{t,s_0}) = \tilde{b}_t^{c,\xi_0}(s^{t',s_0})$, and $k_t^{c,\xi_0}(s^{t,s_0}) = k_{t'}^{c,\xi_0}(s^{t',s_0})$ for all $t \ge 0$, $t' \ge 0$ and (s^{t,s_0}, s^{t',s_0}). Moreover, capital taxes are zero in every period $t \ge 1$. Define $X_t^{c,\xi_0}(s^{t,s_0}) \equiv \tilde{b}_{t-1}^{c,\xi_0}(s^{t-1,s_0})/\beta + g(s_t) - \tilde{b}_t^{c,\xi_0}(s^{t,s_0})$ for all $t \ge 1$ and s^{t,s_0}.

Let $R(k,l) \equiv lF_l + lH_l$ denote labor tax revenues. In every period, set capital taxes to zero, the government holdings of capital to k_g, and the government holdings of the additional assets x_i to zero. For every $t \ge 0$ and s^{t,s_0}, solve the following system in $k_t^{k_g,\xi_0}(s^{t,s_0})$ and $\{l_{t+1}^{k_g,\xi_0}(s^{t,s_0}, s_{t+1})\}_{s_{t+1}\in\mathbb{S}}$:

$$\sum_{s_{t+1}\in\mathbb{S}} R(k_t^{k_g,\xi_0}(s^{t,s_0}), l_{t+1}^{k_g,\xi_0}(s^{t,s_0}, s_{t+1}))P(s_{t+1}, s_t) = \sum_{s_{t+1}\in\mathbb{S}} X_{t+1}^{c,\xi_0}(s^{t,s_0}, s_{t+1})P(s_{t+1}, s_t)$$

and for all $s_{t+1} \in \mathbb{S}$,

$$1 + F_k(k_t^{k_g,\xi_0}(s^{t,s_0}), l_{t+1}^{k_g,\xi_0}(s^{t,s_0}, s_{t+1})) - \frac{1}{\beta} + \frac{R(k_t^{k_g,\xi_0}(s^{t,s_0}), l_{t+1}^{k_g,\xi_0}(s^{t,s_0}, s_{t+1}))}{k_g} = \frac{X_{t+1}^{c,\xi_0}(s^{t,s_0}, s_{t+1})}{k_g}.$$

Observe that these equations imply $\sum_{s_{t+1}\in\mathbb{S}} \beta\left[1 + F_k(k_t^{k_g,\xi_0}(s^{t,s_0}), l_{t+1}^{k_g,\xi_0}(s^{t,s_0}, s_{t+1}))\right]P(s_{t+1}, s_t) = 1$. For all $t \ge 1$ and s^{t,s_0}, define $c_t^{k_g,\xi_0}(s^{t,s_0}) = k_{t-1}^{k_g,\xi_0}(s^{t-1,s_0}) + F(k_{t-1}^{k_g,\xi_0}(s^{t-1,s_0}), l_t^{k_g,\xi_0}(s^{t,s_0})) - k_t^{k_g}(s^{t,s_0}) - g(s_t)$. The continuation allocation $\{c_t^{k_g,\xi_0}, l_t^{k_g,\xi_0}, k_t^{k_g,\xi_0}\}_{t\ge 0}$ is a competitive equilibrium continuation allocation. The corresponding process for the value of net government liabilities satisfies $\{\tilde{b}_{t-1}^{k_g,\xi_0}\}_{t\ge 1} = \{\tilde{b}_{t-1}^{c,\xi_0}\}_{t\ge 1}$. For all $t \ge 1$ and s^{t,s_0}, the limit when $|k_g| \to \infty$ of $(c_t^{k_g,\xi_0}(s^{t,s_0}), l_t^{k_g,\xi_0}(s^{t,s_0}), k_{t-1}^{k_g,\xi_0}(s^{t-1,s_0}))$ is $(c_t^{c,\xi_0}(s^{t,s_0}), l_t^{c,\xi_0}(s^{t,s_0}), k_{t-1}^{c,\xi_0}(s^{t,s_0}))$. ∎

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10 Figures and Tables

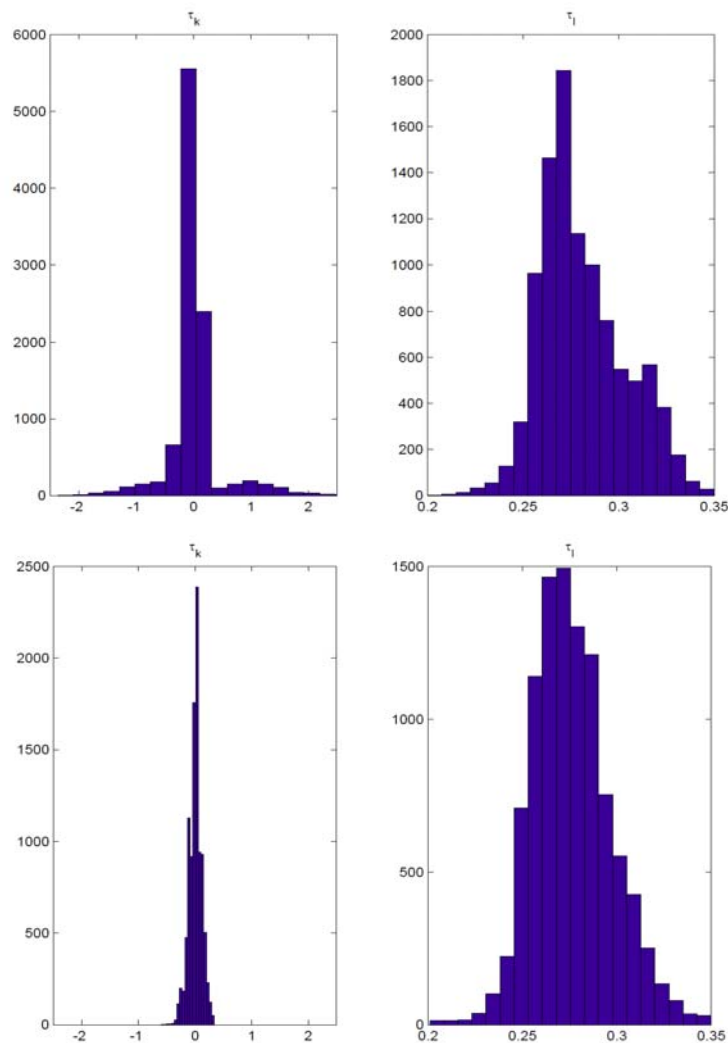|	One-year period length	Five-year period length
Table 1: Summary statistics		
Mean of labor taxes	28.4%	27.6%
Standard deviation of labor taxes	1.8%	1.9%
Per-period autocorrelation of labor taxes	0.85	0.96
Mean of capital taxes	3.7%	0.6%
Standard deviation of capital taxes	53.8%	11.1%
Per-period autocorrelation of capital taxes	0.04	0.11

Figure 1: Frequency distributions of capital and labor taxes in the model with a period length of one year (top pannel) and five years (bottom pannel). The distributions were generated from a simulation with 10000 periods.
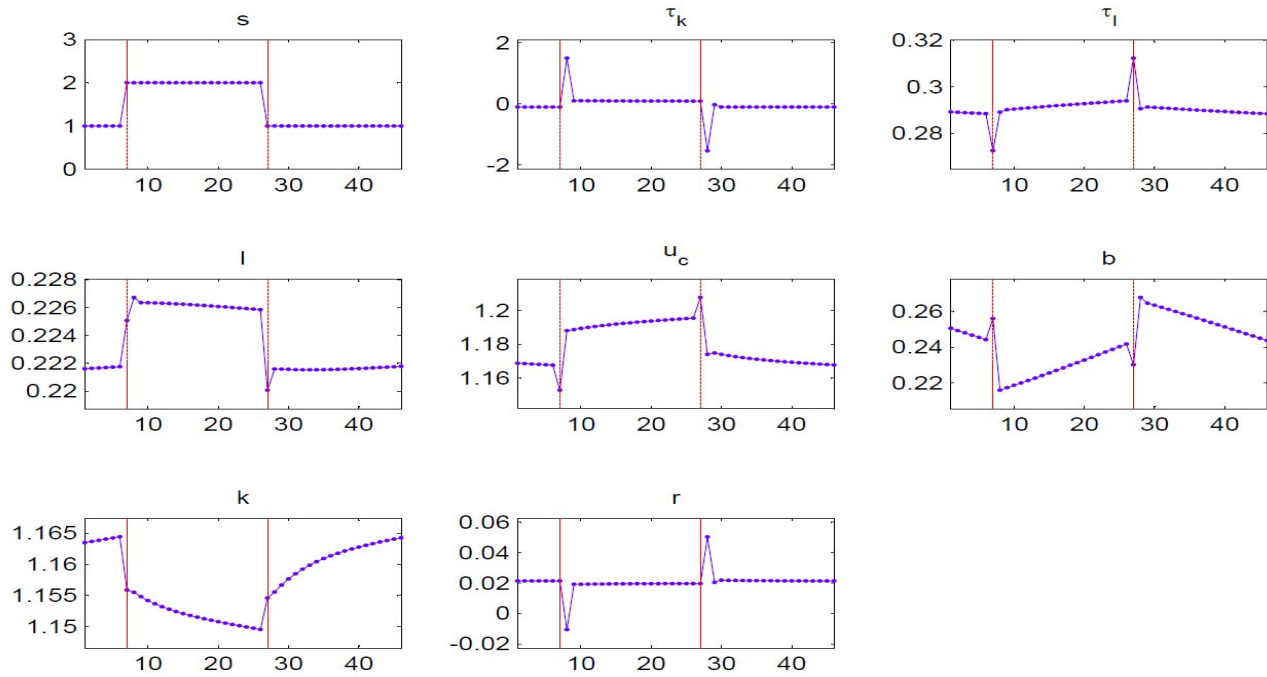
Figure 2: Simulated observations from the model with *government expenditure* shocks only and a *one-year* period length. The high (low) government expenditure state is $s = 2$ ($s = 1$). The x-axis indexes time (in periods). The number of periods during two transitions is set at its average value. The vertical lines indicate the periods when a transition occurs.
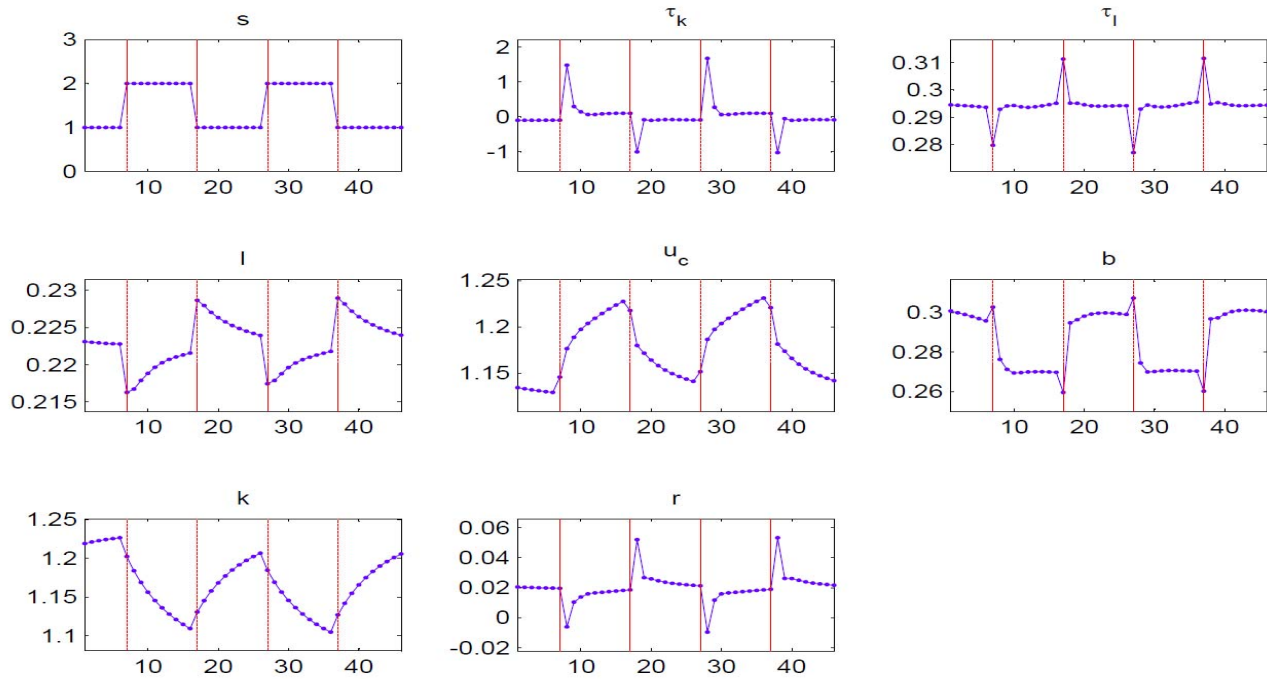

Figure 3: Simulated observations from the model with *productivity* shocks only and a *one year* period length. The low (high) productivity state is $s = 2$ ($s = 1$). The x-axis indexes time (in periods). The number of periods during two transitions is set at its average value. The vertical lines indicate the periods when a transition occurs.
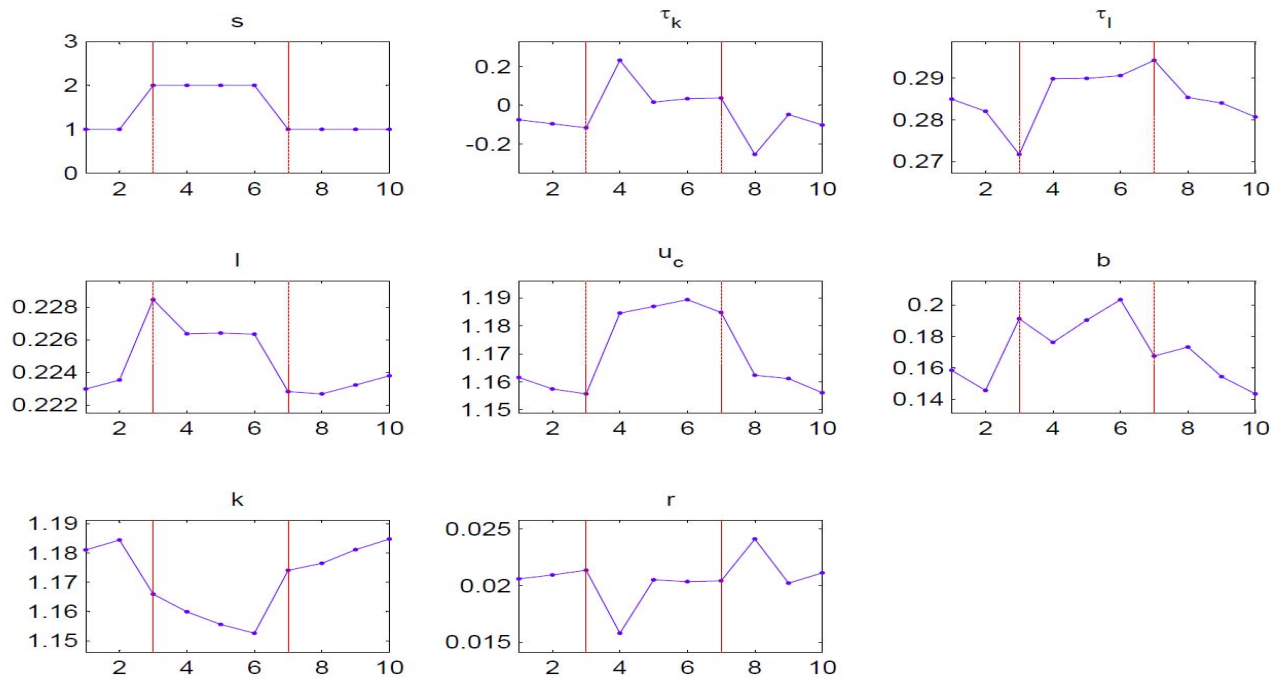
Figure 4: Simulated observations from the model with *government expenditure* shocks only and a *five-year* period length. The high (low) government expenditure state is $s = 2$ $(s = 1)$. The x-axis indexes time (in periods). The number of periods during two transitions is set at its average value. The vertical lines indicate the periods when a transition occurs.