Counting lattice points in triangles and the "Fibonacci staircase"

Dan Cristofaro-Gardiner

Member, Institute for Advanced study

Bard College
October 10, 2013

1. Counting lattice points

Thank you!

Thank you for inviting me to give this talk!

Lattice points

The Cartesian plane $\mathbb{R}^2$ contains a special subset, called the *integer lattice*.

Lattice points

The Cartesian plane $\mathbb{R}^2$ contains a special subset, called the *integer lattice*.

This is the set of points $(x, y) \in \mathbb{R}^2$ such that x and y are both integers. It is denoted $\mathbb{Z}^2$.

Lattice points

The Cartesian plane $\mathbb{R}^2$ contains a special subset, called the *integer lattice*.

This is the set of points $(x, y) \in \mathbb{R}^2$ such that x and y are both integers. It is denoted $\mathbb{Z}^2$.

For example, $(2, 3)$ is an integer lattice point. $(\pi, 1)$ is not.

Counting

Given a region S in $\mathbb{R}^2$, it can be interesting to ask how many lattice points S contains.

Counting

Given a region S in $\mathbb{R}^2$, it can be interesting to ask how many lattice points S contains.

For example, let $S(r)$ be a circle of radius r centered at the origin.

Counting

Given a region S in $\mathbb{R}^2$, it can be interesting to ask how many lattice points S contains.

For example, let $S(r)$ be a circle of radius r centered at the origin. The *Gauss circle problem* asks: *how many integer lattice points are in $S(r)$?*

Counting

Given a region S in $\mathbb{R}^2$, it can be interesting to ask how many lattice points S contains.

For example, let $S(r)$ be a circle of radius r centered at the origin. The *Gauss circle problem* asks: *how many integer lattice points are in $S(r)$?*

Gauss proved that this number is no more than

Counting

Given a region S in $\mathbb{R}^2$, it can be interesting to ask how many lattice points S contains.

For example, let $S(r)$ be a circle of radius r centered at the origin. The *Gauss circle problem* asks: *how many integer lattice points are in $S(r)$?*

Gauss proved that this number is no more than $\pi r^2 + 2\sqrt{2}\pi r$.

Counting

Given a region S in $\mathbb{R}^2$, it can be interesting to ask how many lattice points S contains.

For example, let $S(r)$ be a circle of radius r centered at the origin. The *Gauss circle problem* asks: *how many integer lattice points are in $S(r)$?*

Gauss proved that this number is no more than $\pi r^2 + 2\sqrt{2}\pi r$. It is a famous problem to find a more accurate upper bound.

Triangles

This talk will be about lattice points in triangles.

Triangles

This talk will be about lattice points in triangles.

For example, let T be the triangle with vertices:

$$(3,0),(0,3),(0,0).$$

Triangles

This talk will be about lattice points in triangles.

For example, let T be the triangle with vertices:

$$(3, 0), (0, 3), (0, 0).$$

Question: How many integer lattice points are in T?

Triangles

This talk will be about lattice points in triangles.

For example, let T be the triangle with vertices:

$$(3,0),(0,3),(0,0).$$

Question: How many integer lattice points are in T?

It is helpful to think of T as the region between the positive x-axis, the positive y-axis, and the line $x+y=3$.

Triangles

This talk will be about lattice points in triangles.

For example, let T be the triangle with vertices:

$$(3,0),(0,3),(0,0).$$

Question: How many integer lattice points are in T?

It is helpful to think of T as the region between the positive x-axis, the positive y-axis, and the line $x+y=3$.

Thus, we want to find the number of pairs of nonnegative integers (m,n) such that $m+n\leq 3$.

Example continued

How many pairs are there? We have:

Example continued

How many pairs are there? We have:

$$(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(2,0),(2,1),(3,0).$$

Example continued

How many pairs are there? We have:

$$(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(2,0),(2,1),(3,0).$$

So there are *10* lattice points.

Another triangle

What about the triangle with vertices $(0,0),(0,1)$ and $(1,0)$?

Another triangle

What about the triangle with vertices $(0,0),(0,1)$ and $(1,0)$?
Here, the equation of the slant edge is $x+y=1$.

Another triangle

What about the triangle with vertices $(0,0),(0,1)$ and $(1,0)$?
Here, the equation of the slant edge is $x+y=1$.

There are $3:(0,0),(0,1)$, and $(1,0)$.

Another triangle

What about the triangle with vertices $(0,0), (0,1)$ and $(1,0)$? Here, the equation of the slant edge is $x+y=1$.

There are $3 : (0,0), (0,1)$, and $(1,0)$.

Let t be a positive integer. What about the triangle with vertices $(0,0), (t,0)$ and $(0,t)$?

Another triangle

What about the triangle with vertices $(0,0),(0,1)$ and $(1,0)$? Here, the equation of the slant edge is $x+y=1$.

There are $3 : (0,0),(0,1)$, and $(1,0)$.

Let t be a positive integer. What about the triangle with vertices $(0,0),(t,0)$ and $(0,t)$?

Let's call the number of lattice points in this triangle $P(t)$.

Can we find a nice formula for $P(t)$?

One can compute the first few values of $P(t)$ by hand. They are:

Can we find a nice formula for $P(t)$?

One can compute the first few values of $P(t)$ by hand. They are:

$$P(1) = 3, \quad P(2) = 6, \quad P(3) = 10, \quad P(4) = 15, \quad P(5) = 21.$$

Can we find a nice formula for $P(t)$?

One can compute the first few values of $P(t)$ by hand. They are:

$$P(1) = 3, \quad P(2) = 6, \quad P(3) = 10, \quad P(4) = 15, \quad P(5) = 21.$$

Is there a pattern?

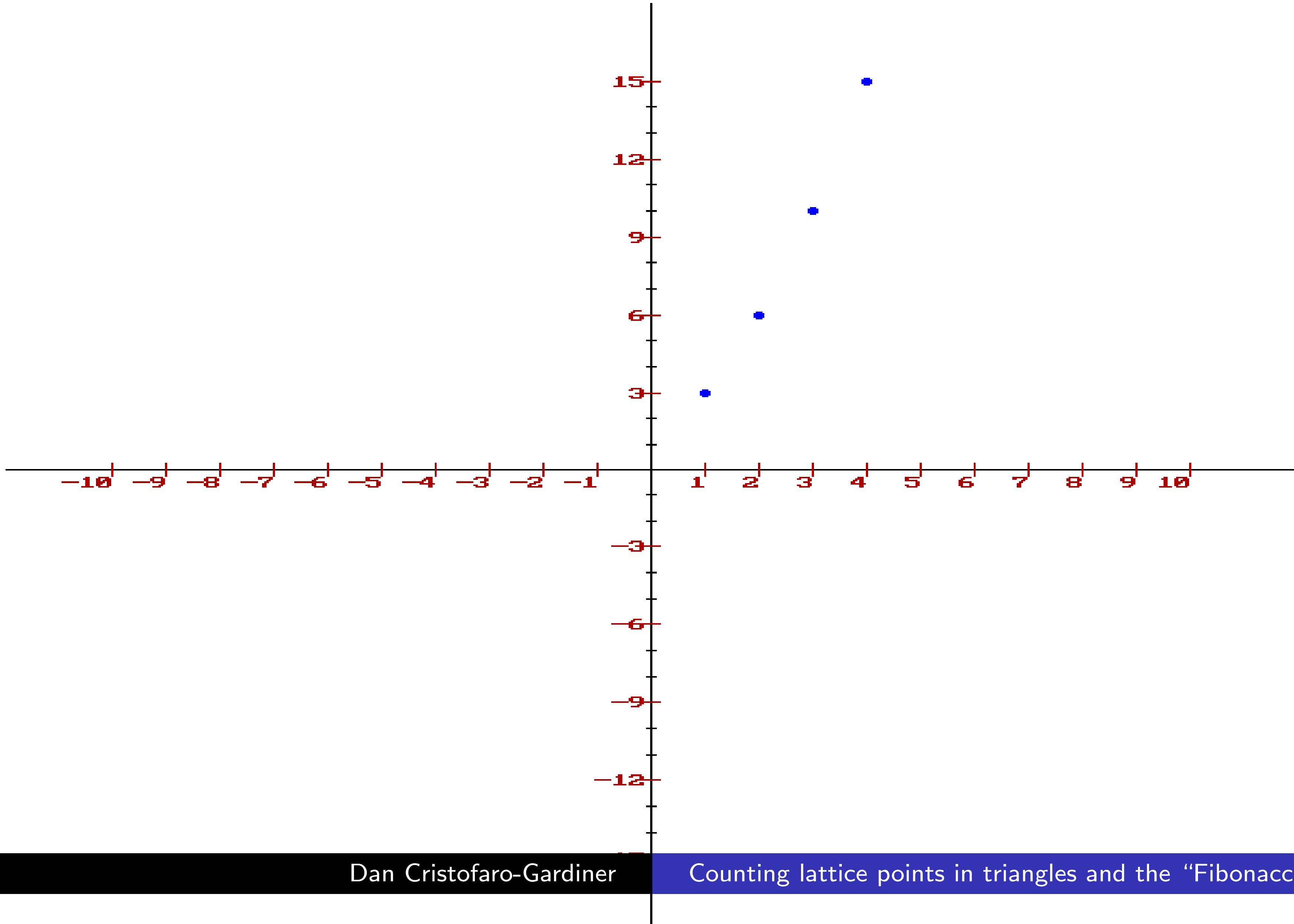
Can we find a nice formula for $P(t)$?

One can compute the first few values of $P(t)$ by hand. They are:

$$P(1) = 3, \quad P(2) = 6, \quad P(3) = 10, \quad P(4) = 15, \quad P(5) = 21.$$

Is there a pattern? Can we find a closed formula for $P(t)$?

The first few points of $P(t)$



A closed formula!

In fact, with a little work, one can show:

A closed formula!

In fact, with a little work, one can show:

Formula

$P(t) = \frac{1}{2}(t+1)(t+2).$

A closed formula!

In fact, with a little work, one can show:

Formula

$P(t) = \frac{1}{2}(t+1)(t+2).$

In particular, $P(t)$ is a *polynomial* in t.

A closed formula!

In fact, with a little work, one can show:

Formula

$P(t) = \frac{1}{2}(t+1)(t+2).$

In particular, $P(t)$ is a *polynomial* in t. *Question:* Do similar formulas hold for other triangles?

A closed formula!

In fact, with a little work, one can show:

Formula

$P(t) = \frac{1}{2}(t+1)(t+2).$

In particular, $P(t)$ is a *polynomial* in t. *Question:* Do similar formulas hold for other triangles?

Pick's theorem

The answer is *yes!*

Pick's theorem

The answer is *yes!*

For example, let $T_{a,b}$ be the triangle with vertices $(0,0),(a,0)$, and $(0,b)$.

Pick's theorem

The answer is *yes!*

For example, let $T_{a,b}$ be the triangle with vertices $(0,0),(a,0)$, and $(0,b)$. Let A be the area of $T_{a,b}$, and let B be the number of lattice points on the boundary of $T_{a,b}$.

Pick's theorem

The answer is *yes!*

For example, let $T_{a,b}$ be the triangle with vertices $(0,0),(a,0)$, and $(0,b)$. Let A be the area of $T_{a,b}$, and let B be the number of lattice points on the boundary of $T_{a,b}$.

Let $P_{a,b}(t)$ be the number of lattice points in the triangle with vertices $(0,0),(ta,0)$, and $(0,tb)$.

Pick's theorem

The answer is *yes!*

For example, let $T_{a,b}$ be the triangle with vertices $(0,0), (a,0)$, and $(0,b)$. Let A be the area of $T_{a,b}$, and let B be the number of lattice points on the boundary of $T_{a,b}$.

Let $P_{a,b}(t)$ be the number of lattice points in the triangle with vertices $(0,0), (ta,0)$, and $(0,tb)$.

Theorem 1 (Pick's theorem)

$$P_{a,b}(t) = At^2 + \frac{1}{2}Bt + 1.$$

Pick's theorem

The answer is *yes!*

For example, let $T_{a,b}$ be the triangle with vertices $(0,0),(a,0)$, and $(0,b)$. Let A be the area of $T_{a,b}$, and let B be the number of lattice points on the boundary of $T_{a,b}$.

Let $P_{a,b}(t)$ be the number of lattice points in the triangle with vertices $(0,0),(ta,0)$, and $(0,tb)$.

Theorem 1 (Pick's theorem)

$P_{a,b}(t) = At^2 + \frac{1}{2}Bt + 1.$

An example

For example, let's use Pick's theorem to compute $P_{a,b}(t)$ where $a = 3$ and $b = 1$. We have $A = \frac{3}{2}$.

An example

For example, let's use Pick's theorem to compute $P_{a,b}(t)$ where $a = 3$ and $b = 1$. We have $A = \frac{3}{2}$.

How about B?

An example

For example, let's use Pick's theorem to compute $P_{a,b}(t)$ where $a = 3$ and $b = 1$. We have $A = \frac{3}{2}$.

How about B?

There are 5 lattice points on the boundary $((0,0),(1,0),(0,1),(0,2)$, and $(0,3))$. Thus $B = 5$.

An example

For example, let's use Pick's theorem to compute $P_{a,b}(t)$ where $a = 3$ and $b = 1$. We have $A = \frac{3}{2}$.

How about B?

There are 5 lattice points on the boundary $((0,0),(1,0),(0,1),(0,2)$, and $(0,3))$. Thus $B = 5$. So Pick's

Theorem says that:

An example

For example, let's use Pick's theorem to compute $P_{a,b}(t)$ where $a = 3$ and $b = 1$. We have $A = \frac{3}{2}$.

How about B?

There are 5 lattice points on the boundary $((0,0),(1,0),(0,1),(0,2)$, and $(0,3))$. Thus $B = 5$. So Pick's

Theorem says that:

$$P_{3,1}(t) = \frac{3}{2}t^2 + \frac{5}{2}t + 1.$$

The Ehrhart polynomial

Pick's theorem holds more generally.

The Ehrhart polynomial

Pick's theorem holds more generally. Specifically, let P be a convex polygon in the plane with integer vertices.

The Ehrhart polynomial

Pick's theorem holds more generally. Specifically, let P be a convex polygon in the plane with integer vertices. Let A be the area of P and let B be the number of integer lattice points on the boundary of P.

The Ehrhart polynomial

Pick's theorem holds more generally. Specifically, let P be a convex polygon in the plane with integer vertices. Let A be the area of P and let B be the number of integer lattice points on the boundary of P.

For a positive integer t, let

$$t \cdot P = \{(tx, ty) | (x, y) \in P\},$$

The Ehrhart polynomial

Pick's theorem holds more generally. Specifically, let P be a convex polygon in the plane with integer vertices. Let A be the area of P and let B be the number of integer lattice points on the boundary of P.

For a positive integer t, let

$$t \cdot P = \{(tx, ty) | (x, y) \in P\},$$

and let $L_P(t)$ be the number of lattice points in $t \cdot P$.

The Ehrhart polynomial

Pick's theorem holds more generally. Specifically, let P be a convex polygon in the plane with integer vertices. Let A be the area of P and let B be the number of integer lattice points on the boundary of P.

For a positive integer t, let

$$t \cdot P = \{(tx, ty) | (x, y) \in P\},$$

and let $L_P(t)$ be the number of lattice points in $t \cdot P$.

Theorem 2 (Pick's theorem)

$L_P(t) = At^2 + \frac{1}{2}Bt + 1.$

The counting function $L_P(t)$ is called the *Ehrhart polynomial* of P.

1. Counting lattice points
2. Pick's formula
3. **Ehrhart theory**
4. Period collapse and number theory
5. Connection with symplectic geometry

The Ehrhart quasipolynomial

In fact, if P is a convex planar polygon with *rational* vertices, then an analogue of Pick's theorem still holds.

The Ehrhart quasipolynomial

In fact, if P is a convex planar polygon with *rational* vertices, then an analogue of Pick's theorem still holds.

Define the counting function $L_P(t)$ as before: $L_P(t)$ counts integer lattice points in the polygon $t \cdot P$ for positive integer t.

The Ehrhart quasipolynomial

In fact, if P is a convex planar polygon with *rational* vertices, then an analogue of Pick's theorem still holds.

Define the counting function $L_P(t)$ as before: $L_P(t)$ counts integer lattice points in the polygon $t \cdot P$ for positive integer t.

Theorem 3

If P is a convex polygon with rational vertices, then $L_P(t)$ is a quadratic quasipolynomial in t

The Ehrhart quasipolynomial

In fact, if P is a convex planar polygon with *rational* vertices, then an analogue of Pick's theorem still holds.

Define the counting function $L_P(t)$ as before: $L_P(t)$ counts integer lattice points in the polygon $t \cdot P$ for positive integer t.

Theorem 3

If P is a convex polygon with rational vertices, then $L_P(t)$ is a quadratic quasipolynomial in t

A quadratic quasipolynomial is a polynomial

$$c_2(t)t^2 + c_1(t)t + c_0(t),$$

where each c_i is a *periodic* function of t.

The Ehrhart quasipolynomial

In fact, if P is a convex planar polygon with *rational* vertices, then an analogue of Pick's theorem still holds.

Define the counting function $L_P(t)$ as before: $L_P(t)$ counts integer lattice points in the polygon $t \cdot P$ for positive integer t.

Theorem 3

If P is a convex polygon with rational vertices, then $L_P(t)$ is a quadratic quasipolynomial in t

A quadratic quasipolynomial is a polynomial

$$c_2(t)t^2 + c_1(t)t + c_0(t),$$

where each c_i is a *periodic* function of t. The minimum common period of the c_i is called the *period* of c_i.

An example of a quasipolynomial

Here is an example:

An example of a quasipolynomial

Here is an example:

Define $c_2(t) = c_1(t) = 1$. Define

An example of a quasipolynomial

Here is an example:

Define $c_2(t) = c_1(t) = 1$. Define

$$c_0(t) := \begin{cases} 1 & \text{if } t \text{ is even,} \\ 0 & \text{if } t \text{ is odd,} \end{cases}$$

Then $c_2(t)t^2 + c_1(t) + c_0(t)$ is a quasipolynomial. It has period 2.

What is the period of the Ehrhart quasipolynomial?

The *denominator* of a rational polygon P is the minimum integer $\mathcal{D}$ such that the vertices of $\mathcal{D} \cdot P$ have integer coordinates.

What is the period of the Ehrhart quasipolynomial?

The *denominator* of a rational polygon P is the minimum integer $\mathcal{D}$ such that the vertices of $\mathcal{D} \cdot P$ have integer coordinates.

Theorem 4

If P is a convex rational polygon, then the period of $L_P(t)$ always divides the denominator of $\mathcal{D}$.

What is the period of the Ehrhart quasipolynomial?

The *denominator* of a rational polygon P is the minimum integer $\mathcal{D}$ such that the vertices of $\mathcal{D} \cdot P$ have integer coordinates.

Theorem 4

If P is a convex rational polygon, then the period of $L_P(t)$ always divides the denominator of $\mathcal{D}$.

This is useful, but one would like to know more!

Period collapse

For example, a basic question that is often asked is the following:

Period collapse

For example, a basic question that is often asked is the following:

Question

For what rational polygons P is the period of P less than the denominator?

Period collapse

For example, a basic question that is often asked is the following:

Question

For what rational polygons P is the period of P less than the denominator?

When the period of P is less than the denominator of P, we say that *period collapse* occurs.

When does period collapse occur?

Aaron Kleinman and I wanted to understand the period collapse question better.

When does period collapse occur?

Aaron Kleinman and I wanted to understand the period collapse question better. We asked the following simplified version.

When does period collapse occur?

Aaron Kleinman and I wanted to understand the period collapse question better. We asked the following simplified version.

Question

Let a be a rational number greater than 1, and consider the triangle T_a with vertices $(0,0),(0,a),(\frac{1}{a},0)$.

When does period collapse occur?

Aaron Kleinman and I wanted to understand the period collapse question better. We asked the following simplified version.

Question

Let a be a rational number greater than 1, and consider the triangle T_a with vertices $(0,0), (0,a), (\frac{1}{a}, 0)$. Can we determine exactly when the period of T_a is 1?

When does period collapse occur?

Aaron Kleinman and I wanted to understand the period collapse question better. We asked the following simplified version.

Question

Let a be a rational number greater than 1, and consider the triangle T_a with vertices $(0,0), (0,a), (\frac{1}{a},0)$. Can we determine exactly when the period of T_a is 1?

Note that if $a = \frac{p}{q}$ in lowest terms, then the denominator of T_a is pq.

Here is the answer:

Here is the answer:

Theorem 5 (CG., Kleinman)

The period of T_a is 1 if and only if $a = \frac{g_{n+1}}{g_n}$, where g_n is the n^{th} odd-index Fibonacci number.

Here is the answer:

Theorem 5 (CG., Kleinman)

The period of T_a is 1 if and only if $a = \frac{g_{n+1}}{g_n}$, where g_n is the n^{th} odd-index Fibonacci number.

The Fibonacci numbers start $1, 1, 2, 3, 5, 8, 13, \ldots$, so the sequence g_n starts

Here is the answer:

Theorem 5 (CG., Kleinman)

The period of T_a is 1 if and only if $a = \frac{g_{n+1}}{g_n}$, where g_n is the n^{th} odd-index Fibonacci number.

The Fibonacci numbers start $1, 1, 2, 3, 5, 8, 13, \ldots$, so the sequence g_n starts

$$g_1 = 1, g_2 = 2, g_3 = 5, g_4 = 13, \ldots$$

In our proof, the Fibonacci numbers come up because they are the integer solutions to the equation

$$x^2 + y^2 - 3xy = -1.$$

Are other interesting sequences related to this problem?

One can ask if other interesting recursive sequences are related to the period collapse problem.

Are other interesting sequences related to this problem?

One can ask if other interesting recursive sequences are related to the period collapse problem. Aaron and I decided to also study triangles $\tilde{T}_a$, defined to have vertices $(0,0),(0,a),(\frac{1}{2a},0)$.

Are other interesting sequences related to this problem?

One can ask if other interesting recursive sequences are related to the period collapse problem. Aaron and I decided to also study triangles $\tilde{T}_a$, defined to have vertices $(0,0), (0,a), (\frac{1}{2a}, 0)$.

Theorem 6 (CG.,Kleinman)

The triangle $\tilde{T}_a$ never has period 1. It has period 2 if and only if (p,q) is a "companion Pell number".

Are other interesting sequences related to this problem?

One can ask if other interesting recursive sequences are related to the period collapse problem. Aaron and I decided to also study triangles $\tilde{T}_a$, defined to have vertices $(0,0), (0,a), (\frac{1}{2a}, 0)$.

Theorem 6 (CG.,Kleinman)

The triangle $\tilde{T}_a$ never has period 1. It has period 2 if and only if (p,q) is a "companion Pell number".

Are other interesting sequences related to this problem?

One can ask if other interesting recursive sequences are related to the period collapse problem. Aaron and I decided to also study triangles $\tilde{T}_a$, defined to have vertices $(0,0),(0,a),(\frac{1}{2a},0)$.

Theorem 6 (CG.,Kleinman)

The triangle $\tilde{T}_a$ never has period 1. It has period 2 if and only if (p,q) is a "companion Pell number".

A similar result holds for the family of triangles with vertices $(0,0),(0,3a),(\frac{1}{2a},0)$.

1. Counting lattice points
2. Pick's formula
3. Ehrhart theory
4. Period collapse and number theory
5. Connection with symplectic geometry

Symplectic geometry

Symplectic geometry is a special geometry that is related to classical mechanics and "parameter spaces".

Symplectic geometry

Symplectic geometry is a special geometry that is related to classical mechanics and “parameter spaces”.

While symplectic geometry has had many applications, many basic questions are poorly understood.

Symplectic geometry

Symplectic geometry is a special geometry that is related to classical mechanics and "parameter spaces".

While symplectic geometry has had many applications, many basic questions are poorly understood. For example, very little is known about when one symplectic shape M "fits inside" another symplectic shape N.

Symplectic geometry

Symplectic geometry is a special geometry that is related to classical mechanics and "parameter spaces".

While symplectic geometry has had many applications, many basic questions are poorly understood. For example, very little is known about when one symplectic shape M "fits inside" another symplectic shape N. This means that there is an injective map from M into N that preserves the "symplectic" geometry on M and N.

Symplectic geometry

Symplectic geometry is a special geometry that is related to classical mechanics and "parameter spaces".

While symplectic geometry has had many applications, many basic questions are poorly understood. For example, very little is known about when one symplectic shape M "fits inside" another symplectic shape N. This means that there is an injective map from M into N that preserves the "symplectic" geometry on M and N.

If M fits inside N symplectically, then we say that M *symplectically embeds* into N and write $M \overset{s}{\hookrightarrow} N$.

When does an ellipsoid fit inside a ball?

Some of the simplest symplectic shapes are *four-dimensional symplectic ellipsoids* $E(a,b)$ and *four-dimensional symplectic balls* $B(c)$.

When does an ellipsoid fit inside a ball?

Some of the simplest symplectic shapes are *four-dimensional symplectic ellipsoids* $E(a, b)$ and *four-dimensional symplectic balls* $B(c)$. To understand when an ellipsoid embeds into a ball, we can define the function

When does an ellipsoid fit inside a ball?

Some of the simplest symplectic shapes are *four-dimensional symplectic ellipsoids* $E(a, b)$ and *four-dimensional symplectic balls* $B(c)$. To understand when an ellipsoid embeds into a ball, we can define the function

$$c(a) := \inf\{\mu : E(1, a) \overset{s}{\hookrightarrow} B(\mu)\}, \quad (1)$$

where the arrow $\overset{s}{\hookrightarrow}$ means that $E(1, a)$ symplectically fits inside $B(\mu)$.

What is $c(a)$?

Aaron and I used our results about period collapse of Ehrhart polynomials to show that $c(a)$ is given by a *Fibonacci staircase* if

What is $c(a)$?

Aaron and I used our results about period collapse of Ehrhart polynomials to show that $c(a)$ is given by a *Fibonacci staircase* if

$$1 \leq a \leq \tau^4,$$

and we showed that $c(a) = \sqrt{a}$ if $a \geq 9$.

What is $c(a)$?

Aaron and I used our results about period collapse of Ehrhart polynomials to show that $c(a)$ is given by a *Fibonacci staircase* if

$$1 \leq a \leq \tau^4,$$

and we showed that $c(a) = \sqrt{a}$ if $a \geq 9$. This was originally shown using different methods by McDuff-Schlenk.

The Fibonacci staircase

4 DUSA MCDUFF AND FELIX SCHLENK

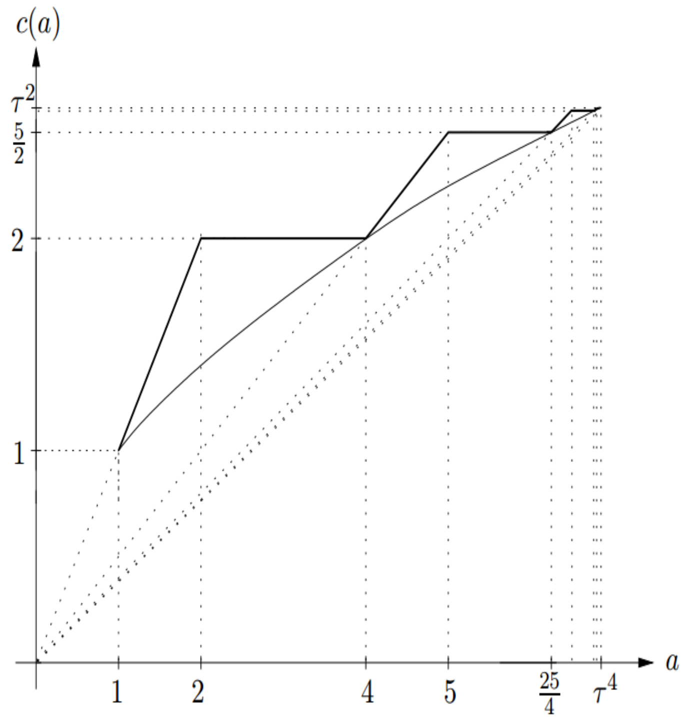


FIGURE 1.1. The Fibonacci stairs: The graph of $c(a)$ on $[1, \tau^4]$.